Central limit theorems for empirical product densities of stationary point processes

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Abstract We prove the asymptotic normality of kernel estimators of second- and higher-order product densities (and of the pair correlation function) for spatially homogeneous (and isotropic) point processes observed on a sampling window $W_n$, which is assumed to expand unboundedly in all directions as $n \to \infty$. We first study the asymptotic behavior of the covariances of the empirical product densities under minimal moment and weak dependence assumptions. The proof of the main results is based on the Brillinger-mixing property of the underlying point process and certain smoothness conditions on the higher-order reduced cumulant measures. Finally, the obtained limit theorems enable us to construct $\chi^2$-goodness-of-fit tests for hypothetical product densities.

Keywords Kernel-type product densities estimators · Empirical pair correlation function · Brillinger-mixing point processes · Reduced cumulant measures · Large domain statistics · $\chi^2$-goodness-of-fit tests

Mathematics Subject Classification (2010) 60G55 · 62M30 · 60F05 · 62G20

1 Introduction

The second-order statistical analysis of stationary and isotropic spatial point processes (short PPs) is mainly based on empirical versions of second-order characteristics such as Ripley's $K$-function and the pair correlation function, see e.g. Baddeley et al. (2005), Cressie (1993), Diggle (2003), Illian et al. (2008), and Stoyan et al. (1995). In comparison with the pair correlation function, the second-order product density contains additional information if the stationary PP is anisotropic. The asymptotic behavior of these estimators has already been studied in, e.g., Jolivet (1984), Heinrich (1988), and Heinrich and Liescher (1997). For Poisson cluster processes Heinrich (1988) proves a central limit theorem (short CLT) for the
empirical second-order product density. Heinrich and Liebscher (1997) prove almost sure convergence (with rates) of kernel-estimators of the second-order product density and the pair correlation function for β-mixing PPs. Jolivet (1984) studies the speed of \(L_p\)-convergence of empirical product densities of any order, and sketches the proof of their asymptotic normality at any fixed argument for Brillinger-mixing point processes by deriving sufficiently sharp bounds of the corresponding cumulants of order \(k \geq 3\). However, the assumptions stated in Jolivet (1984) are definitely not sufficient to get these bounds, which is first noticed when determining the exact asymptotic order of the variances. In the present paper we will provide conditions in addition to Brillinger-mixing which are sufficient to ensure asymptotic normality of the joint distribution of empirical product densities (and likewise of the empirical pair correlation function) taken at finitely many pairwise distinct arguments. This includes a careful study of the corresponding asymptotic covariances. Our multivariate CLTs will be proved by the method of moments (as used also in Jolivet (1981, 1984)), which consists in showing that the \(k\)th-order cumulant - denoted throughout by \(\Gamma_k(\cdot)\) - of the scalar product of a normalized \(q\)-variate estimation vector with any \((a_1, \ldots, a_q) \in \mathbb{R}^q\) disappears asymptotically for all \(k \geq 3\), whereas the second-order cumulant converges to some non-negative quadratic form in \(a_1, \ldots, a_q \in \mathbb{R}^1\).

To begin with we introduce some notation and basic notions. Let \([\mathcal{M}, \mathcal{M}]\) denote the measurable space of all locally finite counting measures on the \(d\)-dimensional Euclidean space \(\mathbb{R}^d\) equipped with its \(\sigma\)-algebra \(\mathcal{B}^d\) of Borel sets. A PP on \(\mathbb{R}^d\) is defined as a measurable mapping \(\Psi\) from a probability space \([\Omega, \mathcal{A}, \mathbb{P}]\) into \([\mathcal{M}, \mathcal{M}]\). Throughout this paper we assume that \(\Psi\) is simple, i.e. \(\mathbb{P}(\Psi((x))) \leq 1\) for all \(x \in \mathbb{R}^d\) \(= 1\), and strictly stationary. Let \(\text{E}, \text{Var}\) and \(\text{Cov}\) denote expectation, variance and covariance, respectively, with respect to \(\mathbb{P}\). Let \(\Psi = \Psi^{-1}\) denote the probability measure on \([\mathcal{M}, \mathcal{M}]\) induced by \(\Psi\) and we will briefly write \(\Psi \sim \mathbb{P}\). If \(\mathbb{E} \Psi^k(B) < \infty\) for all bounded Borel sets \(B \subset \mathbb{R}^d\), then there exist the \(k\)th-order factorial moment measure \(\alpha^{(k)}\) and the \(k\)th-order factorial cumulant measure \(\gamma^{(k)}\) on \([\mathbb{R}^{dk}, \mathcal{B}^{dk}]\) defined by

\[
\alpha^{(k)}\left(\bigtimes_{j=1}^k B_j\right) := \int_{\mathcal{M}} \sum_{\# \neq \#} \prod_{j=1}^k \mathbb{1}_{B_j}(x_j) \mathbb{P}(d\psi)
\]

for any bounded \(B_1, \ldots, B_k \in \mathcal{B}^d\) and

\[
\gamma^{(k)}\left(\bigtimes_{j=1}^k B_j\right) := \sum_{\ell=1}^k (-1)^{\ell-1}(\ell - 1)! \sum_{K_1 \cup \cdots \cup K_{\ell} = [1, \ldots, k]} \prod_{j=1}^{\ell} \alpha^{(#K_j)}\left(\bigtimes_{k_j \in K_j} B_{kj}\right),
\]

respectively. Here, \("x \in \psi"\) means \("x \in \mathbb{R}^d : \psi((x)) > 0"\) and \(\sum_{\# \neq \#}\) indicates summation over tuples of pairwise distinct elements. The inner sum of the second formula is taken over all partitions of the set \([1, \ldots, k]\) into disjoint non-empty subsets \(K_1, \ldots, K_{\ell}\) and \(#K_j\) denotes the cardinality of \(K_j\). By stationarity of the PP \(\Psi \sim \mathbb{P}\) and having intensity \(\lambda > 0\) the \(k\)th-order reduced factorial moment measure \(\alpha^{(k)}_{\text{red}}\) on \([\mathbb{R}^{dk-1}, \mathcal{B}^{dk-1}]\) is uniquely defined by the disintegration formula

\[
\alpha^{(k)}\left(\bigtimes_{j=1}^k B_j\right) = \lambda \int_{B_k} \alpha^{(k-1)}_{\text{red}}\left(\bigtimes_{j=1}^{k-1} (B_j - x)\right)dx
\]

for any bounded \(B_1, \ldots, B_k \in \mathcal{B}^d\),
see Daley and Vere-Jones (2008, p. 238) or Heinrich (2013) for further details. Analogously, the disintegration

\[ \gamma^{(k)} \left( \bigtimes_{j=1}^{k} B_j \right) = \lambda \int_{B_k} \gamma^{(k)}_{\text{red}} \left( \bigtimes_{j=1}^{k-1} (B_j - x) \right) dx \]

provides the \textit{kth-order reduced factorial cumulant measure} \( \gamma^{(k)}_{\text{red}} \) on \( \mathbb{R}^{d(k-1)} \). The \textit{total variation measure} \( |\gamma^{(k)}_{\text{red}}| \) of the signed measure \( \gamma^{(k)}_{\text{red}} \) is defined by

\[ |\gamma^{(k)}_{\text{red}}|(\cdot) = (\gamma^{(k)}_{\text{red}})^+(\cdot) + (\gamma^{(k)}_{\text{red}})^-(\cdot), \]

where the measures \( (\gamma^{(k)}_{\text{red}})^+ \) and \( (\gamma^{(k)}_{\text{red}})^- \) are the positive and the negative part, respectively, of the Jordan decomposition \( \gamma^{(k)}_{\text{red}}(\cdot) = (\gamma^{(k)}_{\text{red}})^+(\cdot) - (\gamma^{(k)}_{\text{red}})^-(\cdot) \).

The \textit{total variation} of \( \gamma^{(k)}_{\text{red}} \) is defined by

\[ \|\gamma^{(k)}_{\text{red}}\| := |\gamma^{(k)}_{\text{red}}|(\mathbb{R}^{d(k-1)}). \]

A stationary PP \( \Psi \sim P \) in \( \mathbb{R}^d \) satisfying \( E\Psi^k([0,1]^d) < \infty \) for some \( k \geq 2 \) is said to be \( B_k \)-mixing if \( \|\gamma^{(j)}_{\text{red}}\| < \infty \) for \( j = 2, \ldots, k \). \( \Psi \sim P \) is called Brillinger-mixing or \( B_\infty \)-mixing if \( \Psi \) is \( B_k \)-mixing for all \( k \geq 2 \), see Brillinger (1975) (for \( d = 1 \)) or Karr (1986, p. 372). Heinrich (1988) and Heinrich and Schmidt (1985) state conditions on several classes of PPs for being \( B_\infty \)-mixing. Recently, Heinrich and Pawlas (2013) obtained bounds of \( \|\gamma^{(k)}_{\text{red}}\| \) in terms of the \( \beta \)-mixing rate of a stationary PP.

If the \( \ell \)-th-order reduced factorial moment measure \( \alpha^{(\ell)}_{\text{red}} \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^{d(\ell-1)} \), then its Lebesgue density \( \varrho^{(\ell)} \) satisfying

\[ \alpha^{(\ell)}_{\text{red}} \left( \bigtimes_{k=1}^{\ell-1} B_k \right) = \int_{B_1} \cdots \int_{B_{\ell-1}} \varrho^{(\ell)}(t_1, \ldots, t_{\ell-1}) dt_1 \cdots dt_{\ell-1} \]

for bounded \( B_1, \ldots, B_{\ell-1} \in B^d \), is called the \( \ell \)-th-order \textit{reduced product density} - shortly \( \ell \)-th-order \textit{product density} in what follows. For brevity we write \( \varrho = \varrho^{(2)} \) for the second-order product density. If \( \varrho^{(\ell)} \) exists then the signed measure \( \gamma^{(\ell)}_{\text{red}} \) possesses a so-called \( \ell \)-th-order \textit{reduced cumulant density} \( c^{(\ell)} \) - shortly \( \ell \)-th-order \textit{cumulant density} henceforth, satisfying

\[ \gamma^{(\ell)}_{\text{red}} \left( \bigtimes_{k=1}^{\ell} B_k \right) = \int_{B_1} \cdots \int_{B_{\ell-1}} c^{(\ell)}(t_1, \ldots, t_{\ell-1}) dt_1 \cdots dt_{\ell-1} \]

for bounded \( B_1, \ldots, B_{\ell-1} \in B^d \).

The isotropic analogue of the second-order product density, the \textit{pair correlation function} (short PCF), is defined by

\[ g(r) := \frac{\varrho(x)}{\lambda}, \]

where \( r = \|x\|, x \in \mathbb{R}^d \), and \( \lambda \) is the intensity of the stationary PP.

The rest of the paper is organized as follows. Section 2 introduces the estimators for the product densities and the PCF. In Sect. 3 we derive CLTs for the empirical second-order product density. In the following two sections these results are transferred to the empirical PCF and to the empirical higher-order product densities. The Appendix summarizes some facts on “indecomposable integrals” needed for the proofs of the CLTs.
2 Empirical product densities

In this section we will present the estimators for the \( \ell \)th-order product densities \( \ell \geq 2 \) and the PCF and formulate some conditions needed for our asymptotic results in the next sections.

Let \( \rho(W) := \sup\{r \geq 0 : b(x, r) \subset W, x \in \mathbb{R}^d\} \) denote the largest inball radius of the set \( W \subset \mathbb{R}^d \), where \( b(x, r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\} \) is the ball with radius \( r \geq 0 \) centered at \( x \in \mathbb{R}^d \). Let \( \cdot \) denote the Lebesgue measure on \( [\mathbb{R}^d, \mathcal{B}^d] \) and let \( \omega_d = |b(o, 1)| \). The following condition is needed for the precise definition of the kernel-type product density estimators.

**Wbk(\( m \))**

(i) The sequence of observation windows \( (W_n) \) is an increasing sequence of convex and compact sets in \( \mathbb{R}^d \) with \( \rho(W_n) \xrightarrow{n \to \infty} \infty \),

(ii) The sequence of bandwidths \( (b_n) \) is a decreasing sequence of positive real numbers satisfying \( b_n \xrightarrow{n \to \infty} 0 \) and \( b_n^m |W_n| \xrightarrow{n \to \infty} \infty \), and

(iii) The kernel function \( k_m : \mathbb{R}^m \to \mathbb{R}^1 \) is bounded with bounded support, symmetric (i.e., \( k_m(x) = k_m(-x) \) for every \( x \in \mathbb{R}^m \)), and satisfies \( \int_{\mathbb{R}^m} k_m(x) dx = 1 \).

Later on we will use the abbreviation \( \|k_m\| := \left( \int_{\mathbb{R}^m} k_m(x) dx \right)^{1/2} \). The following definition of a kernel-type estimator for the second-order product density goes back to Krickeberg (1982). The speed of \( L_p \)-convergence of this estimator has been studied in Jolivet (1984).

**Definition** Let \( (W_n), (b_n), \) and \( k_d \) satisfy condition **Wbk(d)**. Further, let the PP \( \Psi \sim P \) on \( \mathbb{R}^d \) be stationary and assume that its second-order product density \( q \) exists. Then we define

\[
\hat{q}_n(t) := \frac{1}{b_n^d |W_n|} \sum_{x_1, x_2 \in \Psi} \mathbb{I}_{W_n}(x_1) k_d \left( \frac{x_2 - x_1 - t}{b_n} \right)
\]

as an estimator for \( \lambda q(t) \) at \( t \in \mathbb{R}^d \).

The above definition is generalized for higher-order product density estimators as follows.

**Definition** Let \( (W_n), (b_n), \) and \( k_d(\ell - 1) \) satisfy condition **Wbk(d(\( \ell - 1 \))**). Further, let the PP \( \Psi \sim P \) on \( \mathbb{R}^d \) be stationary and assume that its \( \ell \)-th order product density \( q^{(\ell)} \) exists. Then we define

\[
\hat{q}_n^{(\ell)}(t_1, \ldots, t_{\ell-1}) := \frac{1}{b_n^{d(\ell - 1)} |W_n|} \sum_{x_1, \ldots, x_\ell \in \Psi} \mathbb{I}_{W_n}(x_1) \\
\times k_d(\ell - 1) \left( \frac{x_2 - x_1 - t_1}{b_n}, \ldots, \frac{x_\ell - x_1 - t_{\ell-1}}{b_n} \right)
\]

as an estimator for \( \lambda q^{(\ell)}(t) \) at \( t = (t_1, \ldots, t_{\ell-1}) \in \mathbb{R}^{d(\ell - 1)} \).

Furthermore we consider the following kernel estimator for the PCF.

**Definition** Let \( (W_n), (b_n), \) and \( k_1 \) satisfy condition **Wbk(1)**. Further, let \( \Psi \sim P \) on \( \mathbb{R}^d \) be stationary and isotropic PP with PCF \( g \). Then we define

\[
\hat{g}_n(r) := \frac{1}{b_n |W_n|} \sum_{x_1, x_2 \in \Psi} \mathbb{I}_{W_n}(x_1) \frac{k_1 \left( \frac{\|x_2 - x_1\| - r}{b_n} \right)}{\|x_2 - x_1\|^{d-1}}
\]
as an estimator for $\lambda^2 g(r)$ at $r \in [0, \infty)$.

For a discussion of this and slightly modified estimators for the PCF with regard to bias and variance see Stoyan and Stoyan (2000). It should be noticed that all of the above kernel estimators offer a so-called edge effect problem, namely, to calculate the estimators we need the locations of atoms of $\Psi$ not only inside $W_n$ but also in certain layer outside of $W_n$, where its thickness depends on the support of the kernel functions. In practice this means to replace $W_n$ by a smaller inner parallel set of $W_n$ called “minus-sampling”, see Stoyan et al. (1995). An edge-corrected counterpart of $\hat{Q}_h^{(\ell)}(t)$ for $t = (t_1, \ldots, t_{\ell-1}) \in \mathbb{R}^{d(\ell-1)}$ is the kernel estimator

$$
\hat{Q}_n^{(\ell)}(t) := \frac{1}{b_\ell d(\ell-1)} \sum_{x_1, \ldots, x_\ell \in \Psi} \prod_{i=1}^{\ell} \frac{1}{(W_n - x_i)} \left( \frac{x_2 - x_1 - t_1}{b_n}, \ldots, \frac{x_\ell - x_1 - t_{\ell-1}}{b_n} \right).
$$

Interestingly, one can show that $E\hat{Q}_n^{(\ell)}(t) = E\hat{Q}_n^{(\ell)}(t) = \lambda \int_{\mathcal{R}^{d(\ell-1)}} k_{d(\ell-1)}(y)q^{(\ell)}(b_n, y + t) \, dy$. It turns out that each of our asymptotic results below remains unchanged for so-called edge-corrected kernel estimators, see for details the survey paper Heinrich (2013) and references therein. Likewise, one can define an edge-corrected empirical PCF $\hat{g}_n(r)$ for $r \in [0, \infty)$. In the following conditions we have $\ell \geq 2$, $p \geq 3$, and $s, t, s_1, \ldots, s_{\ell-1}, t_1, \ldots, t_{\ell-1} \in \mathbb{R}^d$.

$\gamma((s_i)_{i=1}^{\ell-1}, p)$

The total variation measures $|\gamma^{(k)}_{red}|$, $k = \ell + 1, \ldots, 2(p-1)$, satisfy

$$
\limsup_{\varepsilon \downarrow 0} \varepsilon^{-d(\ell-1)} |\gamma^{(k)}_{red}| \left( \bigtimes_{i=1}^{\ell-1} b(s_i, \varepsilon) \times \mathbb{R}^{d(k-\ell)} \right) < \infty.
$$

We say that condition $\gamma((s_i)_{i=1}^{\ell-1}, \infty)$ is satisfied if condition $\gamma((s_i)_{i=1}^{\ell-1}, p)$ does so for all $p \geq 3$.

$c(s, t)$

The third-order and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ satisfy

$$
\sup_{u \in b(\pm s, \varepsilon)} |c^{(3)}(u, v)| < \infty \quad \text{and} \quad \sup_{u \in b(\pm s, \varepsilon), v \in b(\pm t, \varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u, w, v + w)| \, dw < \infty
$$

for some $\varepsilon > 0$, where $b(\pm s, \varepsilon) := b(s, \varepsilon) \cup b(-s, \varepsilon)$ for $s \in \mathbb{R}^d$.

$c_{\ell}((s_i, t_i))_{i=1}^{\ell-1}$

The cumulant densities up to order $2\ell$ satisfy, for some $\varepsilon > 0$,

$$
\sup_{x_1, \ldots, x_j \in \bigcup_{i=1}^{\ell-1} (b(\pm s_i, \varepsilon) \cup b(\pm t_i, \varepsilon))} |c^{(j)}(x_1, \ldots, x_{j-1})| < \infty
$$

for $j = \ell + 1, \ldots, 2\ell - 1$, and

$$
\sup_{x_i \in b(s_i, \varepsilon), y_i \in b(t_i, \varepsilon)} \int_{\mathbb{R}^d} |c^{(2\ell)}(x_1, \ldots, x_{\ell-1}, z, z + y_1, z + y_2, \ldots, z + y_{\ell-1})| \, dz < \infty.
$$

3 Central limit theorems for the empirical second-order product density

After studying the asymptotic behaviour of mean and covariance of the empirical product density function $t \mapsto \hat{Q}_n(t)$ under mild mixing conditions we derive CLTs for this kernel estimator in the setting of $\mathcal{B}_{\infty}$-mixing PPs. In this section we write $\int$ for $\int_{\mathbb{R}^d}$. 
3.1 Asymptotic representation of mean and covariance

In this section we derive asymptotic representations for the mean and the variance of the estimator for the second-order product density under mild mixing conditions. Similar results for β-mixing PPs have been obtained in Heinrich and Liebscher (1997).

**Definition** The ℓth-order product density \( q^{(ℓ)} \) is said to be (locally) Lipschitz-continuous in \( t = (t_1, \ldots, t_{ℓ-1}) \in \mathbb{R}^{d(ℓ-1)} \) if there exists a constant \( L \geq 0 \) and some \( ε > 0 \) (which may depend on \( t \)) such that \( |q^{(ℓ)}(t) - q^{(ℓ)}(y)| \leq L \sum_{i=1}^{ℓ-1} ||t_i - y_i|| \) for all \( y = (y_1, \ldots, y_{ℓ-1}) \in \mathbb{R}^{d(ℓ-1)} \) satisfying \( y_i \in b(t_i, ε) \) for \( i = 1, \ldots, ℓ - 1 \). For \( ℓ = 2 \) we put \( q = q^{(2)} \).

**Proposition 3.1** Let \( Ψ \sim P \) be a stationary PP on \( \mathbb{R}^d \) with intensity \( λ \) and second-order product density \( q \). Let \( (W_n) \), \( (b_n) \), and \( k_d \) satisfy Condition **Wbk**(d). Then we have

\[
\lim_{n \to \infty} E\tilde{q}_n(t) = λq(t)
\]

in each point of continuity \( t \in \mathbb{R}^d \) of \( q \). If, in addition, \( q \) is Lipschitz-continuous in \( t \in \mathbb{R}^d \), then we have

\[
E\tilde{q}_n(t) = λq(t) + O(b_n) \quad \text{as} \quad n \to \infty.
\]

**Proof** Due to \( E\tilde{q}_n(t) = \int k_d(y) λq(b_n y + t)dy \), the continuity of \( q \) in \( t \) and the boundedness conditions on \( k_d \) yield the first assertion by Lebesgue’s dominated convergence theorem. The Lipschitz-continuity of \( q \) is in \( t \in \mathbb{R}^d \) with Lipschitz constant \( L \) admits for large enough \( n \) the estimate

\[
|E\tilde{q}_n(t) - λq(t)| \leq b_n λL \int |y||k_d(y)|dy = O(b_n).
\]

The following theorem gives an asymptotic representation of the covariance \( \text{Cov}(\tilde{q}_n(s), \tilde{q}_n(t)) \) of the estimated second-order product density in two points \( s, t \in \mathbb{R}^d \). In Heinrich (1988, Theorem 5), the limit \( b_n^d |W_n| \text{Cov}(\tilde{q}_n(s), \tilde{q}_n(t)) \) has already been calculated under mild assumptions on the second-, third- and fourth-order product densities \( p^{(2)} \), \( p^{(3)} \) and \( p^{(4)} \) of the (P-a.s. finite) typical cluster without specific assumptions on cumulant densities as we need for general PPs. Note that Heinrich (1988, Theorem 5) misstates the limiting variance of the second-order product density estimator in the origin \( o \). An extra factor 2 must be added to be correct.

**Theorem 3.2** Let \( Ψ \sim P \) be a stationary fourth-order PP on \( \mathbb{R}^d \) with intensity \( λ \) and second-order product density \( q \). Let \( (W_n) \), \( (b_n) \), and \( k \) satisfy Condition **Wbk**(d). Further, let the third- and fourth-order cumulant densities \( c^{(3)} \) and \( c^{(4)} \) exist and satisfy Condition **C**(s, t). Then we have

\[
b_n^d |W_n| \text{Cov}(\tilde{q}_n(s), \tilde{q}_n(t)) = \begin{cases} 
λq(s)\|k_d\|^2 + O(b_n), & s = ±t \neq o, \\
2λq(o)\|k_d\|^2 + O(b_n), & s = t = o, \\
O(b_n^d), & s \neq ±t
\end{cases}
\]

as \( n \to \infty \) for any point of continuity \( s \in \mathbb{R}^d \) of \( q \). If, in addition, the second-order product density \( q \) is Lipschitz-continuous in \( s \in \mathbb{R}^d \), then

\[
b_n^d |W_n| \text{Cov}(\tilde{q}_n(s), \tilde{q}_n(t)) = \begin{cases} 
λq(s)\|k_d\|^2 + O(b_n), & s = ±t \neq o, \\
2λq(o)\|k_d\|^2 + O(b_n), & s = t = o, \\
O(b_n^d), & s \neq ±t \quad \text{as} \quad n \to \infty.
\end{cases}
\]
Proof By straightforward calculations the term $b_n^{2d} \left| W_n \right|^2 \text{Cov} \left( \tilde{q}_n(s), \tilde{q}_n(t) \right)$ takes the form

$$
\int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{y - x - t}{b_n} \right) \alpha^{(2)}(d(x, y)) + \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{z - x - t}{b_n} \right) \alpha^{(3)}(d(x, y, z))
$$

$$
+ \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{z - y - t}{b_n} \right) \alpha^{(3)}(d(x, y, z)) + \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{x - z - t}{b_n} \right) \alpha^{(3)}(d(x, y, z))
$$

$$
+ \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{y - z - t}{b_n} \right) \alpha^{(3)}(d(x, y, z)) + \int_{(\mathbb{R}^d)^4} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{v - z - t}{b_n} \right)
$$

$$
\times \left[ \alpha^{(4)}(d(x, y, z, v)) - \alpha^{(2)}(d(x, y)) \alpha^{(2)}(d(z, v)) \right],
$$

where the signed measure $\alpha^{(4)}(d(x, y, z, v)) - \alpha^{(2)}(d(x, y)) \alpha^{(2)}(d(z, v))$ expressed in terms of cumulant measures takes the form

$$
\gamma^{(4)}(d(x, y, z, v)) + \gamma^{(1)}(d(x)) \gamma^{(3)}(d(y, z, v)) + \gamma^{(1)}(d(y)) \gamma^{(3)}(d(x, z, v)) + \gamma^{(1)}(d(x, y, z)) \gamma^{(2)}(d(x, z, v))
$$

$$
+ \gamma^{(1)}(d(z)) \gamma^{(3)}(d(x, y, v)) + \gamma^{(1)}(d(v)) \gamma^{(3)}(d(x, y, z)) + \gamma^{(2)}(d(x, y, v)) \gamma^{(2)}(d(x, z, v))
$$

$$
+ \gamma^{(2)}(d(x, v)) \gamma^{(2)}(d(y, z)) + \gamma^{(2)}(d(x, z)) \gamma^{(1)}(d(y)) \gamma^{(1)}(d(v)) + \gamma^{(2)}(d(x, v)) \gamma^{(1)}(d(y)) \gamma^{(1)}(d(v))
$$

$$
\times \gamma^{(1)}(d(z)) + \gamma^{(2)}(d(y, z)) \gamma^{(1)}(d(x)) \gamma^{(1)}(d(v)) + \gamma^{(2)}(d(y, v)) \gamma^{(1)}(d(x)) \gamma^{(1)}(d(v))
$$

$$
(1)
$$

see also Eq. (4.17) in Heinrich (1988). After dividing the above integrals by $b_n^{2d} \left| W_n \right|$ we will see that only the first integral does not converge to zero for $s = t$. We have

$$
\frac{1}{b_n^{2d} \left| W_n \right|} \int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{y - x - t}{b_n} \right) \alpha^{(2)}(d(x, y))
$$

$$
= \lambda Q(s) \int k_d(y) k_d \left( y + \frac{s - t}{b_n} \right) dy + \lambda \int k_d(y) k_d \left( y + \frac{s - t}{b_n} \right) \left( Q(b_n y + s) - Q(s) \right) dy
$$

$$
= \begin{cases} 
\lambda Q(s) \int k_d^2(x) dx + o(1) & \text{for } s = t \\
0 & \text{for } s \neq t 
\end{cases} \quad \text{as } n \to \infty
$$

by continuity of $Q$ in $s \in \mathbb{R}^d$ and the bounded support of the kernel function $k_d$. Note that in the case $s \neq t$ the integral term disappears for sufficiently large $n$ since $k_d$ has bounded support. Due to the additional assumption of Lipschitz-continuity in $s$ we have $|Q(b_n y + s) - Q(s)| \leq L b_n \| y \|$ implying that the above integral equals $\lambda Q(s) \int k_d^2(x) dx + O(b_n)$ as $n \to \infty$ provided $s = t$. 
Furthermore, again after dividing by $b_n^d |W_n|$ we get for the second integral that
\[
\frac{1}{b_n^d |W_n|} \int_{\mathbb{R}^d/\mathbf{y}} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k_d \left( \frac{x - y - s}{b_n} \right) k_d \left( \frac{x - y - t}{b_n} \right) \alpha^{(2)}(d(x, y))
\]
\[
= \lambda \int \frac{|W_n \cap (W_n - b_n y - s)|}{|W_n|} k_d(y) k_d \left( \frac{y + s + t}{b_n} \right) g(b_n y + s) dy
\]
\[
= \begin{cases} 
\lambda \mathbb{1}_{x \neq -t} \int k_d^2(x) dx + o(1) & \text{for } s = -t \quad \text{as } n \to \infty \\
0 & \text{for } s \neq -t
\end{cases}
\]
in every point of continuity $s \in \mathbb{R}^d$ of $g$. As before, in the case $s \neq -t$ the integral disappears for sufficiently large $n$ since $k_d$ has bounded support. The rate of convergence for this integral under the assumption of Lipschitz-continuity is the same as for the first integral. Next, we show that all the other integrals in the above formula are of the order $O(b_n^2 |W_n|)$ so that they converge to zero after dividing by $b_n^d |W_n|$. For the first integral with respect to the third-order factorial moment measure $\alpha^{(3)}$, it is easily seen that
\[
\int \mathbb{1}_{W_n} k_d \left( \frac{y - s}{b_n} \right) k_d \left( \frac{z - x - t}{b_n} \right) \alpha^{(3)}(d(x, y, z))
\]
\[
= \int \mathbb{1}_{W_n} k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{z - x - t}{b_n} \right) [ \gamma^{(3)}(d(x, y, z)) + \lambda dx \gamma^{(2)}(d(y, z)) + \lambda dy \gamma^{(2)}(d(x, z)) + \lambda dz \gamma^{(2)}(d(x, y)) + \lambda^3 dx dy dz ]
\]  
(2)

The integral with respect to third-order cumulant measure $\gamma^{(3)}$ is equal to
\[
\int \mathbb{1}_{W_n} k_d \left( \frac{y - s}{b_n} \right) k_d \left( \frac{z - t}{b_n} \right) \gamma^{(3)}_{red}(d(y, z)) dy
\]
\[
= b_n^2 |W_n| \lambda \int k_d(y) k_d(z) c^{(3)}(b_n y + s, b_n z + t) dy dz
\]

which has the order $O(b_n^2 |W_n|)$ by our assumption on $c^{(3)}$ in $c(s, t)$. Each of the remaining four integrals in (2) can be rewritten in such a way that, in view of the continuity of $c^{(2)}$ at $s \in \mathbb{R}^d$, their growth rate is seen to be $O(b_n^2 |W_n|)$ as $n \to \infty$. Analogously, one can show that each of the above integrals with respect to the third-order factorial moment measure $\alpha^{(3)}$ can be treated quite similarly and has the asymptotic order $O(b_n^2 |W_n|)$ as $n \to \infty$. For the integral with respect to $\gamma^{(4)}$ resulting from (1) we find that
\[
\int \mathbb{1}_{W_n} \mathbb{1}_{W_n}(z) k_d \left( \frac{y - x - s}{b_n} \right) k_d \left( \frac{z - t - v}{b_n} \right) \gamma^{(4)}(d(x, y, z, v))
\]
\[
= b_n^2 |W_n| \lambda \int \frac{|W_n \cap (W_n - z)|}{|W_n|} k_d(y) k_d(v) c^{(4)}(b_n y + s, z, b_n v + z + t) dy dz dv,
\]

which has the order $O(b_n^2 |W_n|)$ due to the assumption imposed on $c^{(4)}$ in $c(s, t)$. According to the decomposition (1) we have to estimate ten further integrals. Using again the existence
of the cumulant densities $c^{(2)}$ and $c^{(3)}$, the continuity of $c^{(2)}$ at $s \in \mathbb{R}^d$ and Condition $c(s, t)$ we find after rather lengthy calculations that each integral grows not faster than $O(b_n^{k+d} | W_n |)$. Finally, summarizing all of the obtained estimates completes the proof of Theorem 3.2. \hfill \Box

3.2 Central limit theorems

For Poisson cluster processes Heinrich (1988) proved a CLT for the sequence

$$\Delta_n(q, t) := \sqrt{b_n^{d}} | W_n | \left( \hat{\varphi}_n(t) - \mathbb{E} \hat{\varphi}_n(t) \right), \quad t \in \mathbb{R}^d,$$

by applying a CLT for triangular array of $m_n$-dependent random fields. However, these method fails in the setting of general $B_{\infty}$-mixing PPs. For this type of PPs we shall prove a CLT by showing that, for each $k \geq 3$, the $k$th-order cumulant of $\Delta_n(q, t)$ converges to zero as $n \to \infty$.

Jolivet (1984) obtains bounds of the $k$th-order cumulant of the $\ell$th-order product density $\hat{\varphi}_n^{(\ell)}$, $\ell \geq 2$, by applying cumulant techniques developed in Leonov and Shiryaev (1959) and Jolivet (1981). However, Jolivet (1984) studied only the terms of highest order (moment and cumulant measures) but did not consider the convergence of lower-order terms to zero which requires some slight additional assumptions on the cumulant densities and the total variation measures associated with the reduced cumulant measures.

In the following $\frac{d}{n \to \infty}$ denotes convergence in distribution, while $\chi^2_q$ and $\mathcal{N}(0_q, \Sigma_q)$ denote the $\chi^2$-distribution with $q$ degrees of freedom and the $q$-variate normal distribution with zero mean vector $0_q \in \mathbb{R}^q$ and positive-semidefinite $q \times q$-covariance matrix $\Sigma_q$, respectively.

**Theorem 3.3** Let $\Psi \sim P$ be a $B_{\infty}$-mixing PP on $\mathbb{R}^d$ with intensity $\lambda$ and second-order product density $\varphi$. Let $(W_n)$, $(b_n)$, and $k_\delta$ satisfy Condition $W_{\delta}(d)$. Let the $q$-tuple $(u_1, \ldots, u_q) \in (\mathbb{R}^d \setminus \{0\})^q$ be chosen such that $u_i \neq \pm u_j$ for $i \neq j$, and let $u_i$ be a point of continuity of $\varphi$ for every $i = 1, \ldots, q$. Furthermore, let Condition $\gamma(u_i, \infty)$ be satisfied and let the third- and fourth-order cumulant densities exist and satisfy Condition $c(u_i, u_j)$ for all $i, j = 1, \ldots, q$.

Then we have $(\Delta_n(q, u_i))^q_{i=1} \xrightarrow{d \to \infty} \mathcal{N}(0_q, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})^q_{i,j=1}$ is given by $\sigma_{ii} := \lambda q(u_i) \| k_{d} \|^2$ for $i = 1, \ldots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Moreover, it holds that

$$\sum_{i=1}^{q} \frac{(\Delta_n(q, u_i))^2}{\sigma_{ii}} \xrightarrow{d \to \infty} \chi^2_q.$$

**Proof** The asymptotic covariance has already been determined in Theorem 3.2. In order to show normal convergence of $(\Delta_n(q, u_i))^q_{i=1}$ we use the method of Cramér-Wold and consider the linear combination $a_1 \Delta_n(q, u_1) + \ldots + a_q \Delta_n(q, u_q)$ for an arbitrary $q$-tuple $(a_1, \ldots, a_q) \in \mathbb{R}^q \setminus \{0, \ldots, 0\}$. Asymptotic normality of this linear combination will be established by showing that its cumulants of order $k \geq 3$ converge to zero.

Applying the Lemma formulated in the Appendix and using the notation given there the $k$th-order cumulant of $a_1 \Delta_n(q, u_1) + \ldots + a_q \Delta_n(q, u_q)$ for $k \geq 2$ can be expressed as
follows:
\[
\Gamma_k \left( \sum_{i=1}^{q} a_i \Delta_n(q, u_i) \right) = \Gamma_k \left( \sum_{i=1}^{q} (b_n^d | W_n |)^{-1/2} a_i \Psi^{(2)} (f_i) \right) = (b_n^d | W_n |)^{-k/2} k! \sum_{k_1 + \ldots + k_q = k, k_1, \ldots, k_q \geq 0} \frac{a_1^{k_1} \ldots a_q^{k_q}}{k_1! \ldots k_q!} \varphi_{k_1, \ldots, k_q}
\]

with \( f_i(x_1, x_2) = \mathbb{1}_{W_n}(x_1) k_d \left( \frac{x_2 - x_1 - u_i}{b_n} \right) \) for \( i = 1, \ldots, q \).

The main issue is to determine the growth rate (in dependence on \( n \to \infty \)) of the sum \( \varphi_{k_1, \ldots, k_q} \) for \( k_1, \ldots, k_q \) \( \geq 0 \) with \( \sum_{i=1}^{q} k_i = k \). Since \( \varphi_{k_1, \ldots, k_q} \) consists only of indecomposable integrals it can be seen by disintegration and substitution that the largest asymptotic order of integrals with respect to the product (measure) of at least two factorial cumulant measures is \( O(b_n^d | W_n |) \) due to \( \| \gamma_{\text{red}}^{(j)} \| < \infty \) for \( j \geq 2 \) and the boundedness assumptions on the kernel function. Together with the factor \( (b_n^d | W_n |)^{-k/2} \) this yields the asymptotic order \( O((b_n^d | W_n |)^{1-k/2}) \) for these terms. For an integral taken with respect to just one factorial cumulant measure, disintegration and the finiteness of the corresponding total variation yield the asymptotic order \( O(|W_n|) \) as \( n \to \infty \). For example, by disintegration the integral
\[
\int \mathbb{1}_{W_n}(x_1) \prod_{i=2}^{k} k_d \left( \frac{x_i - x_1 - u_1}{b_n} \right) \gamma_{\text{red}}^{(k)} (d(x_1, \ldots, x_k))
\]
is easily seen to be equal to
\[
|W_n| \lambda \int \prod_{i=2}^{k} k_d \left( \frac{x_i - u_1}{b_n} \right) \gamma_{\text{red}}^{(k)} (d(x_2, \ldots, x_k)).
\]
(3)

The asymptotic order \( O(|W_n|) \) of this term is insufficient for our purposes. Using the boundedness conditions on the kernel function \( k_d \), we obtain the upper bound
\[
C \cdot |W_n| \int \mathbb{1}_{b(u_1, b_n R)(x_2)} |\gamma_{\text{red}}^{(k)} (d(x_2, \ldots, x_k))| = C \cdot |W_n| |\gamma_{\text{red}}^{(k)} (b(u_1, b_n R) \times (\mathbb{R}^{d})^{k-2})
\]
for the absolute value of the term (3), where \( C \) and \( R \) do not depend on \( n \). Now Condition \( \gamma (u_1, \infty) \) yields the asymptotic order \( O(b_n^d | W_n |) \) of this term. For an integral taken with respect to \( \gamma^{(2)} \) the continuity of the second-order product density in \( u_i \), \( i = 1, \ldots, q \), (which implies the continuity of \( c^{(2)} \) in these points due to \( g(\cdot) = c^{(2)}(\cdot) + \lambda \) and Lebesgue’s dominated convergence theorem again yield the asymptotic order \( O(b_n^d | W_n |) \) as \( n \to \infty \).

Altogether we arrive at \( \Gamma_k (a_1 \Delta_n(q, u_1) + \ldots + a_q \Delta_n(q, u_q)) = O((b_n^d | W_n |)^{1-k/2}) \) as \( n \to \infty \). Hence, the cumulants \( \Gamma_k (\cdot) \) of order \( k \geq 3 \) converge to zero as \( n \to \infty \), which in combination with Theorem 3.2 proves the desired \( q \)-variate CLT. The weak convergence of the quadratic form
\[
\sum_{i=1}^{q} \frac{(\Delta_n(q, u_i))^2}{\sigma_{ii}}
\]
to a \( \chi^2 \)-distributed random variable with \( q \) degrees of freedom is an immediate consequence of the Continuous Mapping Theorem, see e.g. Daley and Vere-Jones (2008).
Remark 3.4 Theorem 3.3 remains valid if Condition $\gamma(u_i, \infty)$ is replaced by $\gamma(u_i, p)$ and the additional assumption $(b_n^p | W_n|^{p-2}) \rightarrow_{n \to \infty} \infty$ for some integer $p \geq 3$. Condition $\gamma(u_i, p)$ is relevant only for the total variation measures up to order $2(p-1)$. This weakening of condition $\gamma(u_i, \infty)$ is compensated by the slightly stronger assumption on the bandwidth $b_n$.

We now use the asymptotic formula of the expectation $E\widehat{\gamma}_n(.)$ in Proposition 3.1 to replace the centering sequence $E\widehat{\gamma}_n(.)$ by $\lambda\theta(.)$, and apply a simple variance-stabilizing transformation.

Corollary 3.5 Let $\Psi \sim P$ be a $B_\infty$-mixing $PP$ on $\mathbb{R}^d$ with intensity $\lambda$ and second-order product density $\theta$. Let $(W_n), (b_n)$, and $k_d$ satisfy Condition Wbk(d) and $b_n^{d+2} | W_n| \rightarrow_{n \to \infty} 0$. Let the $q$-tuple $(u_1, \ldots, u_q) \in (\mathbb{R}^d \setminus \{0\})^q$ be chosen such that $u_i \neq \pm u_j$ for $i \neq j$, and let $\theta$ be Lipschitz-continuous in $u_i$ for all $i = 1, \ldots, q$. Let Condition $\gamma(u_i, \infty)$ be satisfied and let the cumulant densities $\kappa^{(3)}$ and $\kappa^{(4)}$ exist and satisfy Condition c($u_i, u_j$) for all $i, j = 1, \ldots, q$. Then we have

$$\left(\frac{2 b_n^d | W_n|}{\|k_d\|} \left(\sqrt{\widehat{\gamma}_n(u_i)} - \sqrt{\lambda \theta(u_i)}\right)\right)^q \rightarrow_{n \to \infty} \mathcal{N}(0_q, I_q),$$

where $I_q$ is the $q \times q$ identity matrix. Moreover, it holds that

$$\frac{4 b_n^{d} | W_n|}{\|k_d\|^2} \sum_{i=1}^{q} \left(\sqrt{\widehat{\gamma}_n(u_i)} - \sqrt{\lambda \theta(u_i)}\right)^2 \rightarrow_{n \to \infty} \chi^2_q.$$

Proof The asymptotic normality of the sequence $((b_n^d | W_n|)^{1/2} (\widehat{\gamma}_n(u_i) - \lambda \theta(u_i)))_{i=1}^q$ is an immediate consequence of Theorem 3.3, the second part of Proposition 3.1, and $b_n^{d+2} | W_n| \rightarrow_{n \to \infty} 0$. The claim is established based on the weak consistency of the estimated product density and on variance-stabilization by a simple square-root transformation.

Corollary 3.5 can be formulated according to the modified assumptions in Remark 3.4. Note that due to the condition $b_n^{d+2} | W_n| \rightarrow_{n \to \infty} 0$ in Corollary 3.5 the integer $p \geq 3$ must be chosen large enough such that the condition $(b_n^d | W_n|^{p-2}) \rightarrow_{n \to \infty} \infty$ in Remark 3.4 can be met.

4 Central limit theorems for the empirical PCF

All results given in Sect. 3 carry over to the estimator for the PCF. In this section we assume that the PP is both $B_\infty$-mixing and isotropic. We formulate the main results in Sect. 3.2 for the sequence

$$\Delta_n(g, r) := \sqrt{b_n} | W_n| \left(\widehat{g}_n(r) - E\widehat{g}_n(r)\right), \quad r \geq 0.$$

The proofs resemble those given for the empirical second-order product density and will be therefore widely omitted. In this section we write $\int$ for $\int_{\mathbb{R}}$.

Theorem 4.1 Let $\Psi \sim P$ be a $B_\infty$-mixing, isotropic $PP$ on $\mathbb{R}^d$ with intensity $\lambda$ and PCF $\varphi$. Let $(W_n), (b_n)$, and $k_1$ satisfy Condition Wbk(1). Let the $q$-tuple $(r_1, \ldots, r_q) \in (0, \infty)^q$
be chosen such that $r_i \neq r_j$ for $i \neq j$, and let $r_i$ be a point of continuity of $g$ for every $i = 1, \ldots, q$. Furthermore, let condition $\gamma(u, \infty)$ be satisfied for all $u \in \bigcup_{i=1}^{q} \partial b(0, r_i)$, and let the third- and fourth-order cumulant densities exist and satisfy Condition $c(u, v)$ for all $u, v \in \bigcup_{i=1}^{q} \partial b(0, r_i)$.

Then we have \( (\Delta_n(g, r_i))_{i=1}^{q} \xrightarrow{n \to \infty} \mathcal{N}(0_q, \Sigma_q) \), where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^{q}$ is given by

\[
\sigma_{ii} := 2\lambda^2 \frac{g(r_i)}{d \omega_d r_i^{d-1}} \|k_1\|^2
\]

for $i = 1, \ldots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Moreover, as a consequence, we have

\[
\sum_{i=1}^{q} \frac{\left(\Delta_n(g, r_i)\right)^2}{\sigma_{ii}} \xrightarrow{n \to \infty} \chi_q^2.
\]

Proof: We only refer to the use of condition $\gamma(u, \infty)$, $u \in \bigcup_{i=1}^{q} \partial b(0, r_i)$. Again, this condition is needed for integrals like

\[
\int_{(\mathbb{R}^d)^k} \mathbb{1}_{W_n(x_1)}(x_1) \left( \frac{\|x_2 - x_1\| - r_1}{b_n} \right) \prod_{(0\ldots)} \gamma^{(k)}(d(x_1, \ldots, x_k)),
\]

where the product $\prod_{(0\ldots)}$ contains only functions $\mathbb{1}_{W_n(x_j)}$ and $k_1 \left( \frac{\|x_m - x_j\| - r_1}{b_n} \right)$, where $j, m \in \{2, \ldots, k\}$ with $j \neq m$ and $i = 2, \ldots, q$. By disintegration combined with the boundedness conditions on the kernel function $k_1$ we achieve the upper bound

\[
C \cdot |W_n| \int_{(\mathbb{R}^d)^{k-1}} \mathbb{1}_{[b_n R, b_n R + r_1 \ (\|x_2\|)] \gamma^{(k)}_{\text{red}}(d(x_2, \ldots, x_k))}
\]

for the absolute value of the term (4), where $C$ and $R$ are constants. Note that there are $N = \mathcal{O}(e^{-(d-1)})$ points $u_1, \ldots, u_N \in \partial b(o, r_1)$ such that $\bigcup_{i=1}^{N} b(u_i, 2\varepsilon) \supseteq \partial b(o, r_1 + b_n R) \setminus b(o, r_1 - b_n R)$. Therefore, we are in a position to use Condition $\gamma(u, \infty)$ for all $u \in \partial b(o, r_1)$ in order to get the order $\mathcal{O}(b_n |W_n|)$ of the above term. Other integrals of this kind can be treated in the same manner.

Just as in Remark 3.4, condition $\gamma(u, \infty)$ in the above theorem can be replaced by Condition $\gamma(u, p)$ and $b_n^{p} |W_n|^{p-2} \xrightarrow{n \to \infty} \infty$ for some $p \geq 3$.

Corollary 4.2 Let $\Psi \sim P$ be a $B_\infty$-mixing and isotropic PP on $\mathbb{R}^d$ with intensity $\lambda$ and PCF $g$. Let $(W_n)$, $(b_n)$, and $k_1$ satisfy Condition Wbk(1) and let $b_n |W_n| \xrightarrow{n \to \infty} 0$. Let the $q$-tuple $(r_1, \ldots, r_q) \in (0, \infty)^q$ be chosen such that $r_i \neq r_j$ for $i \neq j$, and let $g$ be Lipschitz-continuous in $r_i$ for all $i = 1, \ldots, q$. Let Condition $\gamma(u, \infty)$ be satisfied for all $u \in \bigcup_{i=1}^{q} \partial b(0, r_i)$, and let the third- and fourth-order cumulant densities exist and satisfy Condition $c(u, v)$ for all $u, v \in \bigcup_{i=1}^{q} \partial b(0, r_i)$. Then we have

\[
\left( \frac{\sqrt{2 d \omega_d b_n |W_n| r_i^{d-1}}}{\|k_1\|} \left( \sqrt{g_n(r_i)} - \sqrt{\lambda^2 g(r_i)} \right) \right)_{i=1}^{q} \xrightarrow{n \to \infty} \mathcal{N}(0_q, I_q)
\]
and consequently it holds that

\[
\frac{2d}{\|k_1\|^2} \sum_{i=1}^{q} r_i^{d-1} \left( \sqrt{g_n(r_i)} - \sqrt{\lambda^2 g(r_i)} \right)^2 \xrightarrow{n \to \infty} \chi_q^2.
\]

5 Central limit theorems for empirical higher-order product densities

This section extends the results on the empirical second-order product density in Sect. 3 to product densities \(q^{(\ell)}\) of order \(\ell \geq 2\). We consider the sequence

\[
\Delta_n(Q^{(\ell)}, t) := \sqrt{b_n^{(\ell-1)d}|W_n|} \left( \bar{Z}_n(t) - E_{Q_n^{(\ell)}}(t) \right) \quad \text{for} \quad t \in \mathbb{R}^{d(\ell-1)}.
\]

Now we write \(\int\) for \(\sum_{\mathbb{R}^{d(\ell-1)}}\).

**Proposition 5.1** Let \(\Psi \sim P\) be a stationary PP on \(\mathbb{R}^d\) with intensity \(\lambda\) and \(\ell\)-th-order product density \(q^{(\ell)}\) for some \(\ell \geq 2\). Let \((W_n), (b_n),\) and \(k_{d(\ell-1)}\) satisfy condition \(\text{Wbk}(d(\ell-1))\). Then we have

\[
\lim_{n \to \infty} E_{Q_n^{(\ell)}}(t) = \lambda q^{(\ell)}(t)
\]

in each point of continuity \(t \in \mathbb{R}^{d(\ell-1)}\) of \(q^{(\ell)}\). If, in addition, \(q^{(\ell)}\) is Lipschitz-continuous in \(t \in \mathbb{R}^{d(\ell-1)}\), then

\[
E_{Q_n^{(\ell)}}(t) = \lambda q^{(\ell)}(t) + O(b_n) \quad \text{as} \quad n \to \infty.
\]

**Theorem 5.2** Let \(\Psi \sim P\) be a stationary \(2\ell\)-th-order PP on \(\mathbb{R}^d\) with intensity \(\lambda\) and \(\ell\)-th-order product density \(q^{(\ell)}\) for some fixed \(\ell \geq 2\). Let \((W_n), (b_n),\) and \(k_{d(\ell-1)}\) satisfy condition \(\text{Wbk}(d(\ell-1))\). In addition, let the cumulant densities \(c^{(j)}\) for \(j = \ell + 1, \ldots, 2\ell\) exist and satisfy Condition \(c_{\ell}(s_i, t_i)_{i=1}^{\ell-1}\). Then we have

\[
\lim_{n \to \infty} b_n^{(\ell-1)d}|W_n| \text{Cov}(\bar{Z}_n^{(\ell)}(s), \bar{Z}_n^{(\ell)}(t)) = \begin{cases}
\lambda q^{(\ell)}(s) \int k_{d(\ell-1)}^2(x) dx, & s = t, \\
0, & s \neq t,
\end{cases}
\]

in each point of continuity \(s = (s_1, \ldots, s_{\ell-1}) \in (\mathbb{R}^d \setminus \{o\})^{\ell-1}\) of \(q^{(\ell)}\) satisfying \(s_i \neq \pm s_j\) for all \(1 \leq i < j \leq \ell - 1\).

**Remark 5.3** Note that, by definition, the \(\ell\)-th-order product density \(q^{(\ell)}(s_1, \ldots, s_{\ell-1})\) is completely symmetric and invariant on the set \(T_\ell(s) = \{(s_1 - s_i, \ldots, -s_i, \ldots, s_{\ell-1} - s_i) : 1 \leq i \leq \ell - 1\}\). This implies that there exist up to \(\ell! - 1\) points \(t \in \mathbb{R}^{d(\ell-1)} \setminus \{s\}\) such that \(q^{(\ell)}(s) = q^{(\ell)}(t)\) and the limit in Theorem 5.2 for any continuity point \(s\) equals \(\lambda q^{(\ell)}(s) I(k_{d(\ell-1)})\), where the integral \(I(k_{d(\ell-1)})\) over \(k_{d(\ell-1)}\) depends on the mapping \(s \mapsto t \in T_\ell(s)\) and the permutation of the components of \(t\). However, the restrictions imposed on the continuity point \(s \neq (o, \ldots, o)\) in Theorem 5.2 ensure that \(q^{(\ell)}(s) \neq q^{(\ell)}(t)\) for \(t \neq s\).

**Theorem 5.4** Let \(\Psi \sim P\) be a \(B_{\infty}\)-mixing PP on \(\mathbb{R}^d\) with intensity \(\lambda\) and \(\ell\)-th-order product density \(q^{(\ell)}\) for some fixed \(\ell \geq 2\). Let \((W_n), (b_n),\) and \(k_{d(\ell-1)}\) satisfy Condition \(\text{Wbk}(d(\ell-1))\). Let \(u_1, \ldots, u_q \in \mathbb{R}^{d(\ell-1)}\) with \(u_i = (u_{ik})_{k=1}^{\ell-1} \in \mathbb{R}^{d(\ell-1)}\) be chosen such that \(u_i \neq u_j\) for all \(i, j \in \{1, \ldots, q\}\), \(u_{ij} \neq o\) and \(u_{ij} \neq \pm u_{ik}\) for all \(i \in \{1, \ldots, q\}\) and \(j, k \in \{1, \ldots, \ell - 1\}\) with \(j \neq k\). Let \(u_i\) be a point of continuity of \(q^{(\ell)}\) for every \(i = 1, \ldots, q\).
Furthermore, let Condition $\gamma((u_{ik})^{\ell-1}_{k=1}, \infty)$ be satisfied. In addition, let the cumulant densities $c^{(j)}$ for $j = \ell + 1, \ldots, 2\ell$ exist and satisfy Condition $c_\ell((u_{ik}, u_{jk})^{\ell-1}_{k=1})$ for all $i, j = 1, \ldots, q$.

Then we have $(\Delta_n(Q^{(\ell)}, u_{i}))_{i=1}^{q} \overset{d}{\underset{n \to \infty}{\to}} N(0_q, \Sigma_q)$, where the covariance matrix $\Sigma_q = (\sigma_{ij})_{i,j=1}^{q}$ is given by

$$\sigma_{ii} := \lambda_{q}(\ell)(u_{i})\|k_d(\ell-1)\|^2$$

for $i = 1, \ldots, q$, and $\sigma_{ij} = 0$ for $i \neq j$. Moreover, it holds that

$$\sum_{i=1}^{q} \frac{(\Delta_n(Q^{(\ell)}, u_{i}))^2}{\sigma_{ii}} \overset{d}{\underset{n \to \infty}{\to}} \chi_2^2.$$ 

**Proof** The proof of Theorem 5.4 parallels the one of Theorem 3.3 for the second-order product density estimator. Again, we apply the Lemma given in the Appendix. Using the notation given there the 4th-order cumulant of $a_1 \Delta_n(Q^{(\ell)}, u_1) + \ldots + a_q \Delta_n(Q^{(\ell)}, u_q)$ for $k \geq 2$ can be represented as follows:

$$\Gamma_k \left( \sum_{i=1}^{q} a_i \Delta_n(Q^{(\ell)}, u_{i}) \right) = \Gamma_k \left( \sum_{i=1}^{q} (b_n^{(\ell-1)d} |W_n|)^{-1/2} a_i \Psi^{(\ell)}(f_i) \right)$$

$$= (b_n^{(\ell-1)d} |W_n|)^{-k/2} k! \sum_{k_1 + \ldots + k_q = k} a_1^{k_1} \ldots a_q^{k_q} \mu_{k_1} \ldots \mu_{k_q}$$

with functions $f_i(x_1, \ldots, x_{\ell}) = \mathbb{1}_{W_n}(x_1) \ k_d(\ell-1) \left( \frac{x_2-x_1-u_{11}}{b_n}, \ldots, \frac{x_{\ell}-x_1-u_{1(\ell-1)}}{b_n} \right)$ for $i = 1, \ldots, q$ and any $q$-tuple $(a_1, \ldots, a_q) \in \mathbb{R}^q \setminus \{(0, \ldots, 0)\}$.

Since $\mu_{k_1} \ldots \mu_{k_q}$ consists only of indecomposable integrals, all of the integrals with respect to the product of two or more factorial cumulant measures is asymptotically bounded by $O((b_n^{(\ell-1)d} |W_n|)^{-k/2})$ due to $\gamma^{(j)} \| \gamma^{(j)} \| \leq \infty$ for $j \geq 2$ and the boundedness assumptions on the kernel function. Together with the factor $(b_n^{(\ell-1)d} |W_n|)^{-k/2}$ this yields the asymptotic order $O((b_n^{(\ell-1)d} |W_n|)^{-k/2})$ for these terms. For an integral taken with respect to just one factorial cumulant measure like

$$\int_{(\mathbb{R}^d)^k} \mathbb{1}_{W_n}(x_1) k_d(\ell-1) \left( \frac{x_2-x_1-u_{11}}{b_n}, \ldots, \frac{x_{\ell}-x_1-u_{1(\ell-1)}}{b_n} \right) \prod_{i=1}^{\ell} \gamma^{(k)}(d(x_1, \ldots, x_k)),$$ 

(5)

where the product $\prod_{i=1}^{\ell} \gamma^{(k)}(d(x_1, \ldots, x_k))$ contains only functions $k_d(\ell-1) \left( \frac{x_{m_1-1}-x_{j}-u_{11}}{b_n}, \ldots, \frac{x_{m_{\ell-1}-1}-x_{j}-u_{1(\ell-1)}}{b_n} \right)$ and $\mathbb{1}_{W_n}(x_j)$ with $i, j, m_1, \ldots, m_{\ell-1} \neq 1$, Condition $\gamma((u_{ik})^{\ell-1}_{k=1}, \infty)$ is needed. By disintegration and by using the conditions on $k_d(\ell-1)$ we first achieve the upper bound

$$C \cdot |W_n| \int_{(\mathbb{R}^d)^k} \mathbb{1}_{b(u_1, b_n R)}(x_2) \ldots \mathbb{1}_{b(u_1, b_n R)}(x_{\ell}) \gamma_{\text{red}}^{(k)}(d(x_2, \ldots, x_k))$$

$$= C \cdot |W_n| \gamma_{\text{red}}^{(k)}((b(u_1, b_n R))^{\ell-1} \times (\mathbb{R}^d)^{k-\ell})$$

for the absolute value of the integral (5), where $C$ and $R$ do not depend on $n$. Now, we apply of Condition $\gamma((u_{ik})^{\ell-1}_{k=1}, \infty)$ and get the asymptotic order $O(b_n^{(\ell-1)d} |W_n|)$ for the integral
(5). For an integral taken with respect to $\gamma^{(\ell)}$ the continuity of the $\ell$th-order product density in $u_1, \ldots, u_\ell$ and Lebesgue’s dominated convergence theorem yield the asymptotic order $O(b_n^{(\ell-1)d}|W_n|)$ as $n \to \infty$.

In summary, we get that $\Gamma_k(a_1 \Delta_n(\Theta^{(\ell)}, u_1) + \ldots + a_\ell \Delta_n(\Theta^{(\ell)}, u_\ell)) = O((b_n^{(\ell-1)d}|W_n|)^{1-k/2})$ as $n \to \infty$. Hence, the cumulants of order three and beyond converge to zero as $n \to \infty$, whereas Theorem 5.2 yields the limit for $k = 2$. Both results and the moment convergence theorem confirm the asserted $q$-variate CLT. Finally, the weak convergence of

$$\sum_{i=1}^{q} \frac{(\Delta_n(\Theta^{(\ell)}, u_i))^2}{\sigma_{ii}}$$

to a $\chi^2$-distributed random variable with $q$ degrees of freedom results from the Continuous Mapping Theorem, see e.g. Daley and Vere-Jones (2008).

As before, Condition $\gamma((u_{ik})_{k=1}^{q}, \infty)$ in the above theorem can be replaced by Condition $\gamma((u_{ik})_{k=1}^{q}, p)$ and $(b_n^{(\ell-1)d}|W_n|^{p-2})_{n \to \infty} \to \infty$ for some $p \geq 3$. The following corollary generalizes Corollary 3.5.

**Corollary 5.5** Let $\Psi \sim P$ be a $B_\infty$-mixing PP on $\mathbb{R}^d$ with intensity $\lambda$ and $\ell$th-order product density $\rho^{(\ell)}$, $\ell \geq 2$. Let $(W_n)$, $(b_n)$, and $k_{d(\ell-1)}$ satisfy Condition $Wbk(d(\ell-1))$ and let $b_n^{2\ell+2(\ell-1)d}|W_n|_{n \to \infty} \to 0$. Let $u_1, \ldots, u_q \in \mathbb{R}^{d(\ell-1)}$ with $u_i = (u_{ik})_{k=1}^{\ell-1} \in \mathbb{R}^{d(\ell-1)}$ be chosen such that $u_i \neq u_j$ for all $i, j \in \{1, \ldots, q\}$, $u_{ij} \neq u_{k}\neq u_{ik}$ for all $i \in \{1, \ldots, q\}$ and $j, k \in \{1, \ldots, \ell-1\}$ with $j \neq k$. Let $\Theta^{(\ell)}$ be Lipschitz-continuous in $u_i$ for all $i = 1, \ldots, q$. Let Condition $\gamma((u_{ik})_{k=1}^{\ell-1}, \infty)$ be satisfied and let the cumulant densities up to order $2\ell$ exist and satisfy Condition $C_L((u_{ik}, u_{jk})_{k=1}^{\ell-1})$ for all $i, j = 1, \ldots, q$. Then we have

$$\left(\frac{2\sqrt{b_n^{(\ell-1)d}|W_n|}}{\|k_{d(\ell-1)}\|} \lambda^{\rho^{(\ell)}(u_i)} \sqrt{\lambda^{\rho^{(\ell)}(u_i)}} \right)_{i=1}^{q} \xrightarrow{d \ n \to \infty} \mathcal{N}(0_q, I_q)$$

and, as a consequence,

$$\frac{4b_n^{(\ell-1)d}|W_n|}{\|k_{d(\ell-1)}\|^2} \sum_{i=1}^{q} \left(\frac{\sqrt{\lambda^{\rho^{(\ell)}(u_i)}}}{\lambda^{\rho^{(\ell)}(u_i)}} - \frac{\sqrt{\lambda^{\rho^{(\ell)}(u_i)}}}{\lambda^{\rho^{(\ell)}(u_i)}}\right)^2 \xrightarrow{d \ n \to \infty} \chi^2_q.$$  

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**Appendix: Indecomposable integrals**

The notion *indecomposable integrals* can be traced back to the early times of advanced cumulant theory initiated by Leonov and Shiryaev (1959) (see Krickeberg (1982) for an upgrade to random PPs) and has already been used in Jolivet (1981) to prove CLTs for some class of unbiased estimators of Palm functionals on stationary PPs. The main result given at the end of this Appendix coincides with Lemma 5.1 in Heinrich and Klein (2011) and states that the $k$th-order cumulant of linear combinations of certain multiple sums stretched over atoms of a PP can be represented as sum of integrals that are indecomposable in the sense
that they cannot be written as a product of at least two “simpler” integrals. The rigorous
definition of decomposability of integrals will be given below after some preliminaries.

For fixed \(k \in \mathbb{N}, p_1, \ldots, p_k \in \mathbb{N}\) and Borel-measurable functions \(f_i : (\mathbb{R}^d)^{p_i} \to \mathbb{R}^1, i \in I = \{1, \ldots, k\}\), we define

\[
\Psi^{(p_i)}(f_i) := \sum_{x_1, \ldots, x_{p_i} \in \Psi} f_i(x_1, \ldots, x_{p_i}).
\]  

(6)

Provided that \(E[|\Psi^{(p_i)}(f_i)|^k] < \infty\) for all \(i \in I\) we derive a representation of the mixed
moment \(M(\Psi^{(p_1)}(f_1), \ldots, \Psi^{(p_k)}(f_k)) := \mathbb{E}[\prod_{i=1}^k \Psi^{(p_i)}(f_i)]\) as a sum of the subsequently
defined integrals:

For arbitrary \(T \subseteq I, q \in \{1, \ldots, p_T\}\) with \(p_T := \sum_{i \in T} p_i, r \in \{1, \ldots, q\}\), and decompositions \(P_T = \{P_1, \ldots, P_q\}\) of \(\{1, \ldots, p_T\}\) and \(Q = \{Q_1, \ldots, Q_r\}\) of \(\{1, \ldots, q\}\) we define the integral

\[
I_{P_T, Q}(f_i : i \in T) := \int \int \cdots \int \prod_{b=1}^{q} \prod_{a \in P_b} 1_{\{x_a = z_b\}} f_i(x_1, \ldots, x_{p_i}) \times f_{i_2}(x_{p_i+1}, \ldots, x_{p_i+p_{i_2}}) \\
\cdots \cdots \cdots f_{i_{q_T}} \left( x_{\sum_{j=1}^{q_T-1} p_{i_j}+1}, \ldots, x_{p_{i_T}} \right) \prod_{c=1}^{r} \gamma^{(q_c)}(dz_{Q_c}),
\]

(7)

where \(\{i_1, \ldots, i_{q_T}\} = T\) with \(1 \leq i_1 < i_2 < \ldots < i_{q_T} \leq k\) and \(z_{Q_c} = (z_{q})_{q \in Q_c}\). The
elements of a set \(P_b \in P_T\) are the indices of the arguments of the functions \(f_{i_1}, \ldots, f_{i_{q_T}}\)
that are identical and distinct from all the arguments in any other set \(P_c \neq P_b\). In (7) this is
indicated by the product term \(\prod_{b=1}^{q} \prod_{a \in P_b} 1_{\{x_a = z_b\}}\). For \(T = I\) the integral (7) takes the
form

\[
I_{P_T, Q}(f_1, \ldots, f_k) = \int \int \cdots \int \prod_{b=1}^{q} \prod_{a \in P_b} 1_{\{x_a = z_b\}} f_1(x_1, \ldots, x_{p_1}) \cdots f_k(x_{\sum_{i=1}^{k-1} p_i+1}, \ldots, x_{p_k}) \prod_{c=1}^{r} \gamma^{(q_c)}(dz_{Q_c}).
\]

Using the very definition of higher-order factorial moment measures, see Sect. 1, and an
inversion formula that expresses these factorial moment measures by factorial cumulant
measures, see Leonov and Shiryaev (1959) and Krickeberg (1982), we are in a position to express
the mixed moment \(M(\Psi^{(p_1)}(f_1), \ldots, \Psi^{(p_k)}(f_k))\) as follows:

\[
M(\Psi^{(p_1)}(f_1), \ldots, \Psi^{(p_k)}(f_k))
\]

\[
= \sum_{q=1}^{p_1} \sum_{P_1 \cup \ldots \cup P_q = (\mathbb{R}^d)^q} \int \prod_{b=1}^{q} \prod_{a \in P_b} 1_{\{x_a = z_b\}} \times f_1(x_1, \ldots, x_{p_1}) \cdots f_k(x_{\sum_{i=1}^{k-1} p_i}, \ldots, x_{p_k}) a^{(q)}(dz_1, \ldots, z_q)) \\
= \sum_{q=1}^{p_1} \sum_{r=1}^{q} \sum_{P_r \cup \ldots \cup P_q = (\mathbb{R}^d)^q} \int \prod_{b=1}^{q} \prod_{a \in P_b} 1_{\{x_a = z_b\}} \times f_1(x_1, \ldots, x_{p_1}) \cdots f_k(x_{\sum_{i=1}^{k-1} p_i+1}, \ldots, x_{p_k}) \prod_{c=1}^{r} \gamma^{(q_c)}(dz_{Q_c}).
\]
With the above-introduced notation we may write

\[ M(\Psi^{(p_1)}(f_1), \ldots, \Psi^{(p_k)}(f_k)) = \sum_{q=1}^{p_1} \sum_{p_1 \cup \ldots \cup p_q} \sum_{r=1}^{q} \sum_{Q_1 \cup \ldots \cup Q_r} I_{p_1, Q}(f_1, \ldots, f_k). \]

Let \( T = \{T_1, T_2\} \) be a decomposition of \( I = [1, \ldots, k] \). An integral \( I_{p_1, Q}(f_1, \ldots, f_k) \) is decomposable with respect to the decomposition \( T = \{T_1, T_2\} \) if there exist a decomposition \( \mathcal{P}^{(1)} \) of \( \{1, \ldots, p_{T_1}\} \), a decomposition \( \mathcal{P}^{(2)} \) of \( \{1, \ldots, p_{T_2}\} \), \( q_1 \in \{1, \ldots, p_{T_1}\} \) and \( q_2 \in \{1, \ldots, p_{T_2}\} \) with \( q_1 + q_2 = q \), and decompositions \( Q^{(1)} \) of \( \{1, \ldots, q_1\} \) and \( Q^{(2)} \) of \( \{1, \ldots, q_2\} \) such that

\[ I_{p_1, Q}(f_1, \ldots, f_k) = I_{p_{T_1}, Q^{(1)}}(f_i : i \in T_1) \cdot I_{p_{T_2}, Q^{(2)}}(f_i : i \in T_2). \]

An integral is called decomposable if there exists a nontrivial decomposition of \( I \) such that this integral is decomposable with respect to this decomposition of \( I \). An integral which is not decomposable with respect to any non-trivial decomposition is called indecomposable.

The subsequent lemma is the key result to prove Theorem 3.3 and Theorem 5.4.

**Lemma.** Let there be given a PP \( \Psi \sim P \) on \( \mathbb{R}^d \) and \( j, k \in \mathbb{N} \). Further, for fixed \( p_1, \ldots, p_j \in \mathbb{N} \) and \( i = 1, \ldots, j \), let the mapping \( f_i : (\mathbb{R}^d)^{p_i} \to \mathbb{R} \) be Borel-measurable such that the random sum \( \Psi^{(p_i)}(f_i) \) defined by (6) has a finite kth-order moment, i.e. \( \mathbb{E}|\Psi^{(p_i)}(f_i)|^k < \infty \) for \( i = 1, \ldots, j \). Then we have

\[
\Gamma_k \left( \sum_{i=1}^{j} a_i \Psi^{(p_i)}(f_i) \right) = k! \sum_{k_1 + \ldots + k_j = k} \frac{a_1^{k_1} \cdots a_j^{k_j}}{k_1! \cdots k_j!} \mu_{k_1, \ldots, k_j}^{*}
\]

for any \( a_1, \ldots, a_j \in \mathbb{R} \), where

\[
\mu_{k_1, \ldots, k_j}^{*} := \left( \sum_{r=1}^{q} \sum_{Q_r} \sum_{r=1}^{q} I_{p_1, Q}(f_1, \ldots, f_j) \right)^{*}
\]

and \( p_{k_1, \ldots, k_j} = \sum_{i=1}^{j} p_i k_i \). The sum \( \left( \sum \cdot \cdot \cdot \right)^{*} \) is taken only over indecomposable integrals.

The proof of this Lemma is given in Heinrich and Klein (2011).

**References**


