Absolute regularity and Brillinger-mixing of stationary point processes

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Abstract. We study the following problem: How to verify Brillinger-mixing of stationary point processes in $\mathbb{R}^d$ by imposing conditions on a suitable mixing coefficient? For this, we define an absolute regularity (or $\beta$-mixing) coefficient for point processes and derive, in terms of this coefficient, an explicit condition that implies finite total variation of the $k$th-order reduced factorial cumulant measure of the point process for fixed $k \geq 2$. To prove this, we introduce higher-order covariance measures and use Statulevičius’ representation formula for mixed cumulants in case of random (counting) measures. To illustrate our results, we consider some Brillinger-mixing point processes occurring in stochastic geometry.

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1 Introduction and basic definitions

Point processes (briefly PPs) are adequate models to describe randomly or irregularly scattered points in some Euclidean space $\mathbb{R}^d$ (often $d = 1, 2, 3$ in applications). Statistics of PPs is mostly based on a single observation of a point pattern in some large sampling window, which is assumed to expand unboundedly in all directions; see [18, Chap. 4]. Provided that the underlying PP model is homogeneous (i.e., stationary), the asymptotic behavior of parameter estimators and other empirical characteristics can only be determined under ergodicity and (strong) mixing assumptions, respectively. We encounter a similar situation in statistical physics, where stationary PPs are used to describe limits of configurations of interacting particles given in a “large (expanding) container” (see [13, 16]).

Throughout, let $\Psi := \sum_{i \geq 1} \delta_{x_i} \sim P$ denote a simple stationary PP on $\mathbb{R}^d$ with distribution $P$ defined on the $\sigma$-algebra $\mathcal{N}$ generated by the sets of the form $\{\psi \in \mathcal{N}: \psi(B) = n\}$ for any $n \in \mathbb{N} \cup \{0\}$ and $B \in \mathcal{B}_d^{\leq} (= \text{bounded sets of the Borel-$\sigma$-algebra of $\mathbb{R}^d$})$, where $N$ denotes the family of locally finite

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counting measures \( \psi \) on \( \mathcal{B}^d \) satisfying \( \psi(\{x\}) \leq 1 \) for all \( x \in \mathbb{R}^d \). In other words, \( \Psi \) is a random counting measure with random atoms \( \{X_i, i \geq 1\} \) of multiplicity one that nowhere accumulate. Shortly spoken, \( \Psi \) is a random element defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) taking values in \( [N, \mathcal{N}, \mathbb{P}] \) with \( \mathbb{P} = \mathbb{P} \circ \psi^{-1} \). The stationarity of \( \Psi \sim \mathbb{P} \) means that \( T_x \psi := \sum_{i \geq 1} \delta_{X_i-x} \sim \mathbb{P} \) or, equivalently, that \( P(\{T_x\psi: \psi \in Y\}) = P(Y) \) for any \( Y \in \mathcal{N} \) and all \( x \in \mathbb{R}^d \), where \( T_x \psi(\cdot) = \psi(\cdot + x) \). For an all-embracing and rigorous introduction to the theory of PPs, the reader is referred to [2, 3]. Further, we define the reduced Palm distribution \( P_0^\mathcal{P} \) of \( \Psi \sim \mathbb{P} \) by
\[
P_0^\mathcal{P}(Y) := \frac{1}{\lambda} \int \int f(x) \mathbf{1}_Y(T_x \psi - \delta_o) \psi(dx) \mathbb{P}(d\psi) \quad \text{for any } Y \in \mathcal{N},
\]
where the intensity \( \lambda := \mathbb{E}\psi(E_o) \) is assumed to be positive and finite, and \( f \) can be any nonnegative Borel-measurable function satisfying \( \int f(x) \, dx = 1 \). Here and below, \( \int \) stands for integration over \( \mathbb{R}^d \), and \( E_o \) denotes the half-open unit cube \( [-1/2, 1/2]^d \) centered at the origin \( o = (0, \ldots, 0) \). Note that the left-hand side of (1.1) does not depend on the choice of \( f \) due to the stationarity of \( \Psi \sim \mathbb{P} \) and the shift-invariance of the Lebesgue measure \( \mu_2 \) on \( \mathbb{R}^d \).

The stationary Poisson process \( \Psi \sim \Pi_\lambda \) with intensity \( \lambda > 0 \) is the most important PP model, which is defined by the following two properties:

1. \( \mathbb{P}(\Psi(B) = n) = (\pi!)^{-1}(\lambda \mu_2(B))^n \exp(-\lambda \mu_2(B)) \) for \( n \in \mathbb{N} \cup \{0\} \) and \( B \in \mathcal{B}_d^\mathcal{P} \), and

2. \( \Psi(B_1), \ldots, \Psi(B_k) \) are mutually independent for any pairwise disjoint \( B_1, \ldots, B_k, B \in \mathcal{B}_d^\mathcal{P}, k \geq 2 \).

We recall that a stationary Poisson process \( \Psi \sim \mathbb{P} = \Pi_\lambda \) is characterized by the identity \( P_0^\mathcal{P} = P \) (Sliyvnyak's theorem); see [3, Chap. 13].

Next, we define the absolute regularity or \( \beta \)-mixing coefficient \( \beta(\mathcal{F}_1, \mathcal{F}_2) \) to measure the dependence between two sub-\( \sigma \)-algebras \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of \( \mathcal{F} \) by
\[
\beta(\mathcal{F}_1, \mathcal{F}_2) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J \left| \mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j) \right|,
\]
where the supremum is taken over all pairs of finite partitions \( \{A_1, \ldots, A_I\} \) and \( \{B_1, \ldots, B_J\} \) of \( \Omega \) such that \( A_i \in \mathcal{F}_1 \) for each \( i \) and \( B_j \in \mathcal{F}_2 \) for each \( j \). This measure of dependence has been introduced by Volkonskii and Rozanov [21] (to prove the asymptotic normality of sums of weakly dependent random variables) and later studied and used by many others; see, e.g., [6, 8, 17, 22].

Our first result illustrates that (1.2) is the appropriate mixing coefficient (which is not replaceable by the \( \alpha \)-mixing coefficient, see [1, 17]) to estimate the distance between expectations w.r.t. \( P_0^\mathcal{P} \) and expectations w.r.t. \( \mathbb{P} \). In particular, it yields effective bounds of the total-variation distance between \( P_0^\mathcal{P} \) and \( \mathbb{P} \) on the \( \sigma \)-algebra \( \mathcal{N}(G) = \mathcal{N} \cap \mathcal{N}(G) \) with \( \mathcal{N}(G) = \{ \psi \in \mathcal{N}: \psi(G^c) = 0 \} \) for sets \( G \in \mathcal{B}_d \) being far away from the origin \( o \). For any \( B \in \mathcal{B}_d \), \( \psi_B(\cdot) := \psi((\cdot) \cap B) \), and \( \mathcal{F}_\psi(B) := \{ \psi^{-1}Y: Y \in \mathcal{N}(B) \} \) denotes the sub-\( \sigma \)-algebra of \( \mathcal{F} \) generated by the restriction \( \psi_B \) of the PP \( \psi \) on \( B \in \mathcal{B}_d \).

**Theorem 1.** Assume that the support \( F \) of the function \( f \) in (1.1) is bounded and such that \( F \cap (G \oplus F) = \emptyset \). Then, for any \( \mathcal{N} \)-measurable function \( g|N \rightarrow \mathbb{R} \) and \( p, q \geq 1 \) satisfying \( p + q \leq p \), we have the bound
\[
\left| \int_N g(\psi_G) (P_0^\mathcal{P} - \mathbb{P})(d\psi) \right| \leq \frac{2}{\lambda} \left( \mathbb{E} \left( \sum_{i \geq 1} f(X_i) \right)^p \right)^{1/p} \left( \mathbb{E} \sup_{x \in F} \left| g(T_x \psi_G) \right|^q \right)^{1/q} \left( \beta(\mathcal{F}_\psi(F), \mathcal{F}_\psi(G \oplus F)) \right)^{1-1/p-1/q},
\]
(1.3)
which remains valid for \( p = 1 \) and \( q = \infty \) if \( g(\psi_G) \) is bounded \( P \)-a.s. In particular, for any \( \delta \geq 0 \),

\[
\sup_{Y \in \mathcal{N}(G)} \left| P'_\alpha(Y) - P(Y) \right| \leq \frac{1}{\lambda \nu_d(F)} \left( \beta \left( \mathcal{F}_\psi(F), \mathcal{F}_\psi(G \oplus F) \right) \right)^{\frac{1}{1+\delta}} \left( E(\psi(F))^{1+\delta} \right)^{\frac{1}{1+\delta}}.
\]

(1.4)

2 Factorial moment and cumulant measures and \( B_\alpha \)-mixing

Assume that \( E\psi(E_0)^k < \infty \) for some fixed \( k \in \mathbb{N} \). The \( k \)th-order factorial moment measure \( \alpha^{(k)} \) (on \( [\mathbb{R}^{dk}, \mathcal{B}^{dk}] \)) of \( \psi = \sum_{i \geq 1} \delta_{X_i} \sim P \) is defined by

\[
\alpha^{(k)} \left( \bigtimes_{i=1}^{k} B_i \right) := E \sum_{i_1, \ldots, i_k \geq 1} 1_{B_1}(X_{i_1}) \cdots 1_{B_k}(X_{i_k}) = \int \sum_{x_1, \ldots, x_k \in \text{supp}(\psi)} \prod_{i=1}^{k} 1_{B_i}(x_i) P(\text{d}\psi)
\]

(2.1)

for any \( B_1, \ldots, B_k \in \mathcal{B}_\psi^d \), where the sum \( \sum_{x_k} \) runs over all \( k \)-tuples of pairwise distinct elements.

According to the general relationship between mixed moments and mixed cumulants (see [12] or [17]), the \( k \)th-order factorial cumulant measure is a locally finite signed measure (on \( [\mathbb{R}^{dk}, \mathcal{B}^{dk}] \)) given by

\[
\gamma^{(k)} \left( \bigtimes_{i=1}^{k} B_i \right) := \sum_{j=1}^{k} (-1)^{j-1}(j-1)! \sum_{K = \bigcup_{j=1}^{k} K_j} \prod_{i=1}^{j} \alpha^{(k_i)}(B_{k_i,1} \times \cdots \times B_{k_i,n_i})
\]

(2.2)

for any \( B_1, \ldots, B_k \in \mathcal{B}_\psi^d \), where the inner sum is taken over all decompositions of \( K := \{1, \ldots, k\} \) into \( j \) disjoint nonempty subsets \( K_1, \ldots, K_j \), and \( k_i := \#K_i \) denotes the number of elements of \( K_i := \{k_i,1, \ldots, k_i,n_i\} \). Further, note that \( P = \Pi \lambda \) implies \( \alpha^{(k)} = \lambda^k \nu_{dk} \) for \( k \geq 1 \) and vice versa, and this in turn is equivalent to \( \gamma^{(1)} = \lambda \nu_d \) and \( \gamma^{(k)} = 0 \) for \( k \geq 2 \).

By the stationarity of \( \psi \sim P \) it follows that both \( \alpha^{(k)} \) and \( \gamma^{(k)} \) are invariant under diagonal shifts, i.e.,

\[
\alpha^{(k)} \left( \bigtimes_{i=1}^{k} B_i \right) = \alpha^{(k)} \left( \bigtimes_{i=1}^{k} (B_i + x) \right) \quad \text{and} \quad \gamma^{(k)} \left( \bigtimes_{i=1}^{k} B_i \right) = \gamma^{(k)} \left( \bigtimes_{i=1}^{k} (B_i + x) \right)
\]

for any \( B_1, \ldots, B_k \in \mathcal{B}_\psi^d \) and all \( x \in \mathbb{R}^d \). This enables us to introduce the (uniquely determined) reduced \( k \)th-order factorial moment (and cumulant) measure \( \alpha^{(k)}_{\text{red}} \) (and \( \gamma^{(k)}_{\text{red}} \)) by disintegration w.r.t. \( \nu_d \), giving

\[
\alpha^{(k)} \left( \bigtimes_{i=1}^{k} B_i \right) = \lambda \int_{B_1} \alpha^{(k)}_{\text{red}} \left( \bigtimes_{i=2}^{k} (B_i - x) \right) dx \quad \text{and} \quad \gamma^{(k)} \left( \bigtimes_{i=1}^{k} B_i \right) = \lambda \int_{B_1} \gamma^{(k)}_{\text{red}} \left( \bigtimes_{i=2}^{k} (B_i - x) \right) dx.
\]

By standard measure-theoretic arguments and using the uniqueness of \( \alpha^{(k)}_{\text{red}} \) and \( \gamma^{(k)}_{\text{red}} \) it follows from (2.1) and (1.1) that \( \alpha^{(k)}_{\text{red}} \) coincides with the \( (k-1) \)st-order factorial moment measure w.r.t. \( P_0^f \) and that \( \gamma^{(k)}_{\text{red}} \) can be expressed by \( \gamma^{(k)} \) as follows:

\[
\gamma^{(k)}_{\text{red}} (B_2 \times \cdots \times B_k) = \frac{1}{\lambda \nu_d(F)} \int_{\mathbb{R}^k} 1_F(x) 1_{B_1}(x_2 - x) \cdots 1_{B_k}(x_k - x) \gamma^{(k)} (dx, x_2, \ldots, x_k)
\]

(2.3)

for any \( F \in \mathcal{B}_X^d \) with \( \nu_d(F) > 0 \). In view of Jordan’s decomposition theorem, the signed measure \( \gamma^{(k)} \) (on \( [\mathbb{R}^{d(k-1)}, \mathcal{B}^{d(k-1)}] \)) can be expressed as the difference of measures \( \gamma^{(k)}_+ \) (positive part) and \( \gamma^{(k)}_- \) (negative part).

part), and the corresponding total-variation measure \( |\gamma_{\text{red}}^{(k)}| \) is then the sum of its positive and negative parts:

\[
\gamma_{\text{red}}^{(k)} = \gamma_{\text{red}}^{(k)+} - \gamma_{\text{red}}^{(k)-} \quad \text{and} \quad |\gamma_{\text{red}}^{(k)}| = \gamma_{\text{red}}^{(k)+} + \gamma_{\text{red}}^{(k)-}.
\]

In view of the corresponding Hahn decomposition, the locally finite measures \( \gamma_{\text{red}}^{(k)+} \) and \( \gamma_{\text{red}}^{(k)-} \) are concentrated on two disjoint Borel sets \( H_{k-1}^+ \) and \( H_{k-1}^- \) with \( H_{k-1}^+ \cup H_{k-1}^- = (\mathbb{R}^d)^{k-1} \). The total variation \( ||\gamma_{\text{red}}^{(k)}||_{TV} \) of \( \gamma_{\text{red}}^{(k)} \) can then be expressed by

\[
||\gamma_{\text{red}}^{(k)}||_{TV} = |\gamma_{\text{red}}^{(k)}|(\mathbb{R}^d)^{k-1} = \gamma_{\text{red}}^{(k)+}(H_{k-1}^+) + \gamma_{\text{red}}^{(k)-}(H_{k-1}^-) = \gamma_{\text{red}}^{(k)}(H_{k-1}^+) - \gamma_{\text{red}}^{(k)}(H_{k-1}^-).
\]

**Definition 1.** (See, e.g., [7,11,15].) A simple stationary PP \( \Psi \sim P \) satisfying \( E\Psi(E_\theta)^k < \infty \) for some integer \( k \geq 2 \) is said to be \( B_k \)-mixing if \( ||\gamma_{\text{red}}^{(j)}||_{TV} < \infty \) for \( j = 2, \ldots, k \). The PP \( \Psi \sim P \) is called Brillinger-mixing if it is \( B_k \)-mixing for all \( k \geq 2 \).

To formulate our main result, we need assumptions on the decay of dependence between the restrictions \( \Psi_{F_a} \) and \( \Psi_{F_{a+r}} \) of the PP \( \Psi \) for large \( r \), where \( F_a := [-a,a]^d \) and \( F_a^c := \mathbb{R}^d \setminus [-a,a]^d \) for \( a > 0 \).

**Theorem 2.** Let \( \Psi \sim P \) be a simple stationary PP on \( \mathbb{R}^d \). Assume that there exists a nonincreasing \( \beta \)-mixing rate \( \beta_{\Psi}([1/2,\infty)) \mapsto [0,1] \) such that

\[
\beta(F_\Psi(F_a),F_\Psi(F_{a+r})) \leq \max\left\{ 1, \frac{a}{r} \right\}^{d-1} \beta_{\Psi}(r) \quad \text{for} \ a, r \geq \frac{1}{2} \quad (2.4).
\]

Then \( \Psi \sim P \) is \( B_k \)-mixing for some \( k \geq 2 \) if, additionally,

\[
E\Psi(E_\theta)^{k+\delta} < \infty \quad \text{and} \quad \int_1^\infty r^{(k-1)d-1} \beta_{\Psi}(r)^{\frac{\delta}{d+1}} \, dr < \infty \quad \text{for some} \ \delta > 0 \quad (2.5).
\]

In the particular cases \( k = 2 \) and \( k = 3 \), condition (2.4) is only needed for \( r \geq a \geq 1/2 \).

**Corollary 1.** Assume that \( E\Psi(E_\theta)^k < \infty \) for all \( k \in \mathbb{N} \). Further, let the \( \beta \)-mixing rate in (2.4) satisfy the bound \( \beta_{\Psi}(r) \leq \exp(-g(r)) \) for \( r \geq 1/2 \), where the function \( g([1/2,\infty)) \mapsto [0,\infty) \) is nondecreasing such that \( g(r)/\log r \to \infty \) as \( r \to \infty \). Then \( \Psi \sim P \) is Brillinger-mixing.

### 3 Higher-order covariance measures and a covariance inequality

In this section, we derive a representation of \( \gamma^{(k)} \) in terms of higher-order covariance measures \( \tilde{\zeta}^{(j)} \). Such representations of higher-order cumulants \( \text{Cum}_n \{ Y_{t_1}, \ldots, Y_{t_k} \} \) (see, e.g., [12]) of (discrete-time) stochastic processes \{\( Y_t, t \in \mathbb{N} \)} in terms of higher-order covariances \( EY_{t_1}Y_{t_2} \cdots Y_{t_k} \) have been introduced in the early 1960s by V.A. Statulevičius, first, to prove large deviations relations for sums of random variables connected in a Markov chain and, later, for other types of weakly dependent random sequences; see [17] for a survey of these results. In [4], the equivalence of the original with the following recursive definition of the \( k \)-th order covariance \( EY_{Y_1} \cdots Y_k \) has been shown: \( EY_1 := EY_1 \), and

\[
\tilde{E}Y_{Y_1} \cdots Y_k := EY_{Y_2} \cdots Y_k - \sum_{j=1}^{k-1} \tilde{E}Y_{Y_2} \cdots Y_j EY_{j+1} \cdots Y_k
\]

for \( k \geq 2 \). By induction on \( k \in \mathbb{N} \) it follows that \( \tilde{E}Y_{Y_2} \cdots Y_k = \tilde{E}Y_k \cdots Y_2Y_1 \).
In analogy to these higher-order covariances of random variables, we introduce the \( k \)-th order (factorial) covariance measure \( \zeta^{(k)} \) of \( \Psi \sim P \) by recursion: \( \zeta^{(1)}(B_1) := \alpha^{(1)}(B_1) = \mathbb{E}\Psi(B_1) \), and

\[
\zeta^{(k)}(B_1 \times \cdots \times B_k) := \alpha^{(k)}(B_1 \times \cdots \times B_k) - \sum_{j=1}^{k-1} \zeta^{(j)}(B_1 \times \cdots \times B_j) \alpha^{(k-j)}(B_{j+1} \times \cdots \times B_k) \tag{3.1}
\]

for any \( B_1, \ldots, B_k \in \mathcal{B}_d^d \) and \( k \geq 2 \). Note that \( \alpha^{(k)} \) and the signed measure \( \gamma^{(k)} \) are completely symmetric in their arguments, while this is not true for the signed measure \( \zeta^{(k)} \), but the relation

\[
\zeta^{(k)} \left( \bigotimes_{i=1}^k B_i \right) = \zeta^{(k)} \left( \bigotimes_{i=1}^{k} B_{i-1} \times B_i \right)
\]

holds. It is easily seen that \( \zeta^{(1)} = \lambda \nu_d \) and \( \zeta^{(k)} = 0 \) for \( k \geq 2 \) yields a further characterization of \( \Psi \sim \Pi_\lambda \).

The total variation of the signed measures \( \zeta^{(k)} \) in case of renewal processes on \( \mathbb{R} \) has been studied in [10]. For such a type of one-dimensional stationary PP, we have \( \beta_\Psi(r) \to 0 \) as \( r \to \infty \) if and only if the distribution of the typical inter-renewal time possesses a convolution power with an absolutely continuous part (see [14]). The rates of decay of \( \beta_\Psi(r) \) have been obtained in [5].

For any stationary PP \( \Psi \sim P \), the first-order measures \( \alpha^{(1)} \), \( \gamma^{(1)} \), and \( \zeta^{(1)} \) coincide with \( \lambda \nu_d \), and we have \( \gamma^{(2)} = \zeta^{(2)} \). For \( k = 3 \) and any \( B_1, B_2, B_3 \in \mathcal{B}_d^d \), the above definitions (2.2) and (3.1) give

\[
\gamma^{(3)}(B_1 \times B_2 \times B_3) = \alpha^{(3)}(B_1 \times B_2 \times B_3) - \alpha^{(1)}(B_1) \alpha^{(2)}(B_2 \times B_3) - \alpha^{(1)}(B_2) \alpha^{(2)}(B_1 \times B_3) - \alpha^{(1)}(B_3) \alpha^{(2)}(B_1 \times B_2) + 2 \alpha^{(1)}(B_1) \alpha^{(1)}(B_2) \alpha^{(1)}(B_3),
\]

\[
\zeta^{(3)}(B_1 \times B_2 \times B_3) = \alpha^{(3)}(B_1 \times B_2 \times B_3) - \alpha^{(1)}(B_1) \alpha^{(2)}(B_2 \times B_3) - \alpha^{(1)}(B_2) \alpha^{(2)}(B_1 \times B_3) - \alpha^{(1)}(B_3) \alpha^{(2)}(B_1 \times B_2) + \alpha^{(1)}(B_1) \alpha^{(1)}(B_2) \alpha^{(1)}(B_3),
\]

\[
\gamma^{(3)}(B_1 \times B_2 \times B_3) = \zeta^{(3)}(B_1 \times B_2 \times B_3) - \zeta^{(1)}(B_2) \zeta^{(2)}(B_1 \times B_3). \tag{3.2}
\]

For general \( k \geq 2 \), there are the following representations of \( \zeta^{(k)} \) and \( \gamma^{(k)} \) (see [17, p. 13]) for the case of random processes:

\[
\zeta^{(k)}(B_1 \times \cdots \times B_k) = \sum_{j=1}^{k} (-1)^{j-1} \sum_{0 \leq k_1 < k_2 < \cdots < k_j = k} \prod_{i=1}^{j} \alpha^{(k_i - k_{i-1})}(B_{k_{i-1}+1} \times \cdots \times B_{k_i}) \tag{3.3}
\]

and

\[
\gamma^{(k)}(B_1 \times \cdots \times B_k) = \sum_{j=1}^{k} (-1)^{j-1} \sum_{K_1 \cup \cdots \cup K_j = K} N_j(K_1, \ldots, K_j) \prod_{i=1}^{j} \zeta^{(k_i)}(B_{k_i-1} \times \cdots \times B_{k_i}) \tag{3.4}
\]

for any \( B_1, \ldots, B_k \in \mathcal{B}_d^d \), where the inner sum is taken over all decompositions of \( K = \{1, \ldots, k\} \) into \( j \) disjoint nonempty subsets \( K_1, \ldots, K_j \) and \( K_i = \{k_{i,1}, \ldots, k_{i,k_i}\} \) with \( k_{i,1} < \cdots < k_{i,k_i} \). We always assume that \( k_{1,1} = 1 \). The nonnegative integers \( N_j(K_1, \ldots, K_j) \) depend on all the sets \( K_1, \ldots, K_j \) and are positive if and only if either \( j = 1 \) (since \( N_j(K) = 1 \)) or for any \( i = 2, \ldots, j \), there exists an \( \ell \in \{1, \ldots, j\} \) such that \( k_{\ell,1} < k_{i,1} < k_{\ell,k_\ell} \) (see [17, p. 80] for a detailed description and calculation of these numbers).
After some rearrangement on the right-hand side of (3.3), we are led to the following representation of the signed measure \( \tilde{\zeta}^{(k)} \):

\[
\tilde{\zeta}^{(k)}(B_1 \times \cdots \times B_k) = \sum_{p=0}^{q-1} \sum_{r=q+1}^{k} \tilde{\zeta}^{(p)}(B_1 \times \cdots \times B_p) \Delta_q(B_{p+1} \times \cdots \times B_r) \tilde{\zeta}^{(k-r)}(B_{r+1} \times \cdots \times B_k) \tag{3.5}
\]

with the convention that \( \tilde{\zeta}^{(0)}(B_{k+1} \times B_k) = -1 \) for \( k = 0, 1, \ldots \) and

\[
\Delta_q(B_{p+1} \times \cdots \times B_r) := \alpha^{(r-p)}(B_{p+1} \times \cdots \times B_r) - \alpha^{(q-p)}(B_{p+1} \times \cdots \times B_q) \alpha^{(r-q)}(B_{q+1} \times \cdots \times B_r) \tag{3.6}
\]

for \( 0 \leq p < q < r \leq k \). Formula (3.5) can be proved by induction on \( k \geq 2 \) and \( 1 \leq q \leq k - 1 \) using the above recursive definition of \( \tilde{\zeta}^{(k)} \). The details are left to the reader.

In order to obtain bounds of \( \tilde{\zeta}^{(k)} \), we need estimates of the covariances (3.6). We may rewrite verbatim the proof of Lemma 1 in [22] to our point process setting leading to the subsequent bound of a general covariance-type expression in terms of the \( \beta \)-mixing coefficient (1.2); see also [8].

**Lemma 1.** Let \( \Psi_B, \Psi_{B'} \) be the restrictions of a simple stationary PP \( \Psi \sim P \) to Borel subsets \( B, B' \subset \mathbb{R}^d \). Furthermore, let \( \Psi_B \) and \( \Psi_{B'} \) be independent copies of \( \Psi_B \) and \( \Psi_{B'} \), respectively. Then, for any \( N \otimes N \)-measurable function \( f[N \times N] \to \mathbb{R} \) and for any \( \eta \geq 0 \),

\[
|E f(\Psi_B, \Psi_{B'}) - E f(\Psi_B, \Psi_{B'})| \leq 2\beta(F_{\Psi}(B)F_{\Psi}(B'))^{\frac{2}{1+\eta}} \max\left\{ (E|f(\Psi_B, \Psi_{B'})|^{1+\eta})^{\frac{1}{1+\eta}}, (E|f(\Psi_B, \Psi_{B'})|^{1+\eta})^{\frac{1}{1+\eta}} \right\}.
\]

In combination with Lemma 1, we will use several times the following result.

**Lemma 2.** Under the assumptions of Lemma 1, put \( B = F_{1/2} \cup \bigcup_{i=2}^{q} (F_1 + z_i) \) and \( B' = \bigcup_{j=q+1}^{k} (F_1 + z_j) \) for some \( q = 1, \ldots, k-1 \) and \( z_2, \ldots, z_k \in \mathbb{Z}^d \). If the function \( f[N \times N] \to \mathbb{R} \) admits the estimate \( |f(\Psi_B, \Psi_{B'})| \leq \Psi(F_{1/2}) \Psi(F_1 + z_2) \cdots \Psi(F_1 + z_k) \), then, for any \( \eta \geq 0 \),

\[
\max\left\{ (E|f(\Psi_B, \Psi_{B'})|^{1+\eta})^{\frac{1}{1+\eta}}, (E|f(\Psi_B, \Psi_{B'})|^{1+\eta})^{\frac{1}{1+\eta}} \right\} \leq 2^{(k-1)d} (E\Psi(E_0)^{k(1+\eta)})^{\frac{1}{1+\eta}}.
\]

**Proof.** By Hölder's inequality and the fact that \( \Psi(F_{1/2} \setminus E_0) = 0 \) \( P \)-a.s. we obtain

\[
E|f(\Psi_B, \Psi_{B'})|^{1+\eta} \leq (E\Psi(E_0)^{k(1+\eta)})^\frac{1}{k} \prod_{j=2}^{k} (E\Psi(F_1 + z_j)^{k(1+\eta)})^\frac{1}{k}.
\]

Together with \( E\Psi(F_1 + z_j)^{k(1+\eta)} = E\Psi(F_1)^{k(1+\eta)} \leq 2^{d(1+\eta)} E\Psi(E_0)^{k(1+\eta)} \) for \( j = 2, \ldots, k \), it is easily seen that

\[
E|f(\Psi_B, \Psi_{B'})|^{1+\eta} \leq 2^{(k-1)d(1+\eta)} E\Psi(E_0)^{k(1+\eta)}.
\]

The same upper bound can be shown for \( E|f(\Psi_B, \Psi_{B'})|^{1+\eta} \), which completes the proof of Lemma 2. \( \square \)
4 The special cases of B₂- and B₃-mixing

For any \( z = (z_1, \ldots, z_d) \in \mathbb{Z}^d \), put

\[
E_z := E_0 + z = \sum_{i=1}^d \left( -\frac{1}{2} + z_i, \frac{1}{2} + z_i \right)
\]

and \( |z| := \max\{|z_1|, \ldots, |z_d|\} \).

For \( k \in \{2, 3\} \), condition (2.4) is only needed for \( r \geq a \geq 1/2 \), which means that \( \beta(\mathcal{F}(F_a), \mathcal{F}(F_{a+r})) \leq \beta_\phi(r) \) for \( r \geq a \geq 1/2 \). Since \( \gamma^{(2)}(E_z) = \alpha^{(2)}(E_z) - \lambda \nu_d \) with \( \alpha^{(2)}(E_z) = \int \psi(B) d\nu_d(\psi) \lambda \nu_d(B) = \alpha^{(1)}(B) = \int \psi(B) P(\text{d}\psi) \) for \( B \in \mathcal{B}_d^d \), we may apply (1.3) with \( F = E_0, G = E_z \) for \( |z| \geq 2 \), \( f(x) = 1_{E_0}(x) \),

\[
g(\psi) = \psi(E_z \cap H_2^+) - \psi(E_z \cap H_2^-),\n\]

and \( p = q = 2 + \delta \) and get the estimates

\[
\gamma^{(2)}(E_z) = \gamma^{(2)}(E_z \cap H_2^+) - \gamma^{(2)}(E_z \cap H_2^-) \leq 2 \lambda \left( \mathcal{E}(\psi(E_0))^{2+\delta} \mathcal{E}(\psi(E_0 \cup E_0))^{2+\delta} \right)^{\frac{1}{2+\delta}} \left( \beta(\mathcal{F}(E_0), \mathcal{F}(E_0 \cup E_0)) \right)^{\frac{1}{2+\delta}} \leq \frac{2^{d+1}}{\lambda} \left( \mathcal{E}(\psi(E_0))^{2+\delta} \right)^{\frac{2}{2+\delta}} \left( \beta(\frac{|z|}{2}) \right)^{\frac{1}{2+\delta}} \text{ for } |z| \geq 2.
\]

The last line is a consequence of (2.4) and \( E_z \cup E_0 \subset F_{|z|-1} \cup \partial F_{|z|-1} \), where \( \psi(\partial F_{|z|-1}) = 0 \) \( \mathbb{P} \)-a.s. due to the stationarity of \( \psi \). From

\[
\# \{ z \in \mathbb{Z}^d : |z| = m \} = (2m + 1)^d - (2m - 1)^d \leq 2d(2m + 1)^{d-1} \text{ for } m \in \mathbb{N}
\]

and (2.5) for \( k = 2 \) we obtain immediately that \( \gamma^{(2)}(\mathbb{R}^d) < \infty \). This result has already been proved by slightly different arguments in [8].

Next, we derive a bound of \( \gamma^{(3)}(\mathbb{R}^d \times \mathbb{R}^d) = \gamma^{(3)}(H_2^+) - \gamma^{(3)}(H_2^-) \). Using (2.3) for \( k = 3 \) and \( F = E_0 \) and (3.2), we find, for any \( y, z \in \mathbb{Z}^d \),

\[
\lambda \gamma^{(3)}(E_y \times E_z) \leq \int \int \int 1_{E_0}(x) 1_{(E_y \times E_z) \cap H_2^+}(x_2 - x, x_3 - x) \gamma^{(3)}(d(x, x_2, x_3))
\]

\[
= \int \int \int 1_{E_0}(x) 1_{(E_y \times E_z) \cap H_2^+}(x_2 - x, x_3 - x) \zeta^{(3)}(d(x, x_2, x_3))
\]

\[
- \lambda \int \int 1_{E_0}(x) 1_{(E_y \times E_z) \cap H_2^+}(x_2 - x, x_3 - x) \text{ dx}_2 \zeta^{(2)}(d(x, x_3))
\]

\[
=: I_1 - I_2.
\]

The first term \( I_1 \) can be rewritten as

\[
I_1 = \int \int 1_{E_0}(x) 1_{(E_y \times E_z) \cap H_2^+}(x_2 - x, x_3 - x) \alpha^{(3)}(d(x, x_2, x_3)) - \lambda \alpha^{(2)}((E_y \times E_z) \cap H_2^+)
\]

\[
- \lambda \int \int 1_{E_0}(x) 1_{(E_y \times E_z) \cap H_2^+}(x_2 - x, x_3 - x) \alpha^{(2)}(d(x, x_2)) \text{ dx}_3
\]

\[
+ \lambda (\alpha^{(1)} \times \alpha^{(1)})(E_y \times E_z \cap H_2^+)
\]

\[
= E \sum_{i,j,k_1,k_2 \geq 1} 1_{E_0}(X_i) 1_{(E_y \times E_z) \cap H_2^+}(X_j - X_i, X_k - X_i) - \lambda \alpha^{(2)}((E_y \times E_z) \cap H_2^+)
\]

\[ \begin{align*}
- \lambda \int \mathbf{E} \sum_{i,j \geq 1} \mathbf{1}_{E_{x_i}}(X_i) \mathbf{1}_{(E_{x_i} \cap H_{x_i})} (X_j - X_i, x_3 - x_1) \, dx_3 \\
+ \lambda (\alpha^{(1)} \times \alpha^{(1)}) ((E_y \times E_z) \cap H^+_2),
\end{align*} \]

and the second term \( I_2 \) becomes

\[ I_2 = \lambda \int \mathbf{E} \sum_{i,k \geq 1} \mathbf{1}_{E_{x_i}}(x_i) \mathbf{1}_{(E_{x_i} \cap H_{x_i})} (x_2 - x, x_3 - x) \, dx_2 \alpha^{(2)}(d(x, x_3)) \\
- \lambda (\alpha^{(1)} \times \alpha^{(1)}) ((E_y \times E_z) \cap H^+_2) \\
= \lambda \int \mathbf{E} \sum_{i,k \geq 1} \mathbf{1}_{E_{x_i}}(X_i) \mathbf{1}_{(E_{x_i} \cap H_{x_i})} (x_2 - X_i, x_3 - X_i) \, dx_2 \\
- \lambda (\alpha^{(1)} \times \alpha^{(1)}) ((E_y \times E_z) \cap H^+_2).\]

We have now to distinguish different cases according to the norms of \( y \) and \( z \). The set \( S_2 := \{(y, z) \in \mathbb{Z}^d \times \mathbb{Z}^d : |y| \leq |z| \} \) decomposes into three disjoint sets

\[ S_2^{(1)} := \{(y, z) \in S_2 : |y| \leq 1, |z| \leq |y| + 2\}, \quad S_2^{(2)} := \{(y, z) \in S_2 : |y| \geq 2, |z| \leq 2|y|\}, \]

and

\[ S_2^{(3)} := \{(y, z) \in S_2 : |z| \geq \max\{2|y| + 1, |y| + 3\}\}. \]

Since \( S_2^{(1)} \) is finite with cardinality \#\( S_2^{(1)} = 5^d + (3^d - 1)(7^d - 1) \), we need only a uniform bound of (4.2). Replacing \( \gamma^{(3)} \) in (4.2) by \( \alpha^{(3)} + 2\alpha^{(1)} \times \alpha^{(1)} \times \alpha^{(1)} \) and using the fact that \( X_i \in E_o \) and \( (X_j - X_i, X_k - X_i) \in (E_y \times E_z) \cap H^+_2 \) imply \( X_j, X_k \in E_o \cap E_y \subset F_1 + y \) and \( X_k \in E_o \cap E_z \subset F_1 + z \), we get the estimate

\[ I_1 - I_2 \leq \alpha^{(3)}(E_o \times (F_1 + y) \times (F_1 + z)) + 2\lambda^3 \int \int \mathbf{1}_{E_{x_i}}(x_i) \mathbf{1}_{E_{x_i}}(x_2 - x) \mathbf{1}_{E_{x_i}}(x_3 - x) \, dx_3 \, dx_2 \, dx. \]

By applying H\ölder’s inequality and the stationarity of \( \Psi \) (like in the proof of Lemma 2) we obtain that

\[ I_1 - I_2 \leq 2^{2d} E \Psi (E_o)^3 + 2\lambda^3 =: C_1 < \infty. \]

For any pair \( (y, z) \in S_2^{(2)} \), we get the relations

\[ I_1 = \mathbf{E} f(x_{F_1 + y} \cup (F_1 + z)) - \mathbf{E} f(\Psi_{E_o}, \Psi_{(F_1 + y) \cup (F_1 + z)}) \\
- \lambda \int_{F_1 + z} [\mathbf{E} g_{x_3}(x_{F_1 + y}) - \mathbf{E} g_{x_3}(\Psi_{E_o}, \Psi_{(F_1 + y)})] \, dx_3 \]

and

\[ I_2 = \lambda \int_{F_1 + y} [\mathbf{E} h_{x_2}(x_{F_1 + z}) - \mathbf{E} h_{x_2}(\Psi_{E_o}, \Psi_{(F_1 + z)})] \, dx_2, \]

where \( \Psi_B \) and \( \Psi_{B'} \) are defined as in Lemma 1 with \( B = E_o \) and \( B' \in \{F_1 + y, F_1 + z, (F_1 + y) \cup (F_1 + z)\} \).
respectively, and
\[
\begin{align*}
\psi(E_0, \psi(F_1+y) \cup (F_1+z)) := & \sum_{i,j \geq 1} \sum_{k \geq 1} x_i \mathbb{1}_{(E_0 \times E_i) \cap H^+_z} (X_j - X_i, X_k - X_i) \\
& \leq \psi(E_0) \psi(F_1 + y) \psi(F_1 + z), \\
g_{R_F}(\psi(E_0), \psi(F_1+y)) := & \sum_{x_{i,j} \geq 1} \mathbb{1}_{E_0} (X_i) \mathbb{1}_{(E_0 \times E_i) \cap H^+_z} (X_j - X_i, X_3 - X_i) \\
& \leq \psi(E_0) \psi(F_1 + y), \\
h_{R_F}(\psi(E_0), \psi(F_1+z)) := & \sum_{k,l \geq 1} \mathbb{1}_{E_0} (X_i) \mathbb{1}_{(E_0 \times E_i) \cap H^+_z} (x_2 - X_i, X_k - X_i) \\
& \leq \psi(E_0) \psi(F_1 + z).
\end{align*}
\]

Since \( \psi(\partial F_1) = 0 \) P-a.s., the foregoing formulas with \( f, g_{RF}, \) and \( h_{RF} \) remain unchanged when \( F_1 \) is replaced by the open square \( F_1^{\text{int}} = (-1, 1)^d \). In view of \( E_\eta \subset F_{1/2} \) and \( (F_1^{\text{int}} + y) \cup (F_1^{\text{int}} + z) \subset F_{|y| - 1} \), we may apply Lemma 1 and, together with Lemma 2 and (2.4), obtain the following estimates:
\[
|I_1| \leq 2^{2d+1} \beta (|z| - \frac{3}{2}) \frac{\psi(E_0)}{(\psi(E_0)^3 + 3\eta)} \frac{1}{\psi(E_0)^{2+2\eta}} + \frac{\lambda \nu d(F_1)^{2d+1} \beta (|z| - \frac{3}{2}) \frac{\psi(E_0)}{(\psi(E_0)^2 + 2\eta)} \frac{1}{\psi(E_0)^{2+2\eta}}}{\psi(E_0)^{2+2\eta}}
\]
and
\[
|I_2| \leq \lambda \nu d(F_1)^{2d+1} \beta (|z| - \frac{3}{2}) \frac{\psi(E_0)^{2+2\eta}}{(\psi(E_0)^2 + 2\eta)} \frac{1}{\psi(E_0)^{2+2\eta}}.
\]

(4.4)

For \( \eta = \delta / 3 \), the expressions on the right-hand sides are finite, so that
\[
\lambda^{(3)+}_{\text{red}} (E_\eta \times E_\eta) \leq |I_1| + |I_2| \leq C_2 \beta (|z| - \frac{3}{2}) \frac{1}{\psi(E_0)^{2+2\eta}}
\]
for some constant \( C_2 > 0 \).

In case \((y, z) \in S_{3/2}^{(3)}\), we swap the second and third terms in (4.3) and rewrite \( I_1 \) as follows:
\[
I_1 = \mathbb{E} f(\theta_{E_0, y}(F_1+y), \psi(F_1+z)) - \mathbb{E} [g(\theta_{E_0, y}(F_1+y), \psi(F_1+z)) - \lambda \mathbb{E} g(\psi(F_1+y), \psi(F_1+z))] - \mathbb{E} g(\overline{\psi}_{F_1+y}, \overline{\psi}_{F_1+z})]
\]
where
\[
f(\theta_{E_0, y}(F_1+y), \psi(F_1+z)) := \sum_{i,j \geq 1} \sum_{k \geq 1} \mathbb{1}_{E_0} (X_i) \mathbb{1}_{(E_0 \times E_i) \cap H^+_z} (X_j - X_i, X_k - X_i)
\]
and
\[
g(\psi(F_1+y), \psi(F_1+z)) := \sum_{j,k \geq 1} \mathbb{1}_{(E_0 \times E_i) \cap H^+_z} (X_j, X_k).
\]

In the same manner as above, Lemmas 1 and 2, combined with (2.4), yield the estimate
\[
|I_1| \leq 2^{2d+1} \beta (|z| - |y| - 2) \frac{\psi(E_0)}{(\psi(E_0)^3 + 3\eta)} \frac{1}{\psi(E_0)^{2+2\eta}} + \lambda 2^{d+1} \beta (|z| - |y| - 2) \frac{\psi(E_0)}{(\psi(E_0)^2 + 2\eta)} \frac{1}{\psi(E_0)^{2+2\eta}}
\]
The bound of \( I_2 \) is the same as in (4.4), and therefore, by setting \( \eta = \delta / 3 \) we arrive at
\[
\lambda^{(3)+}_{\text{red}} (E_\eta \times E_\eta) \leq |I_1| + |I_2| \leq C_3 \beta (|z| - |y| - 2) \frac{1}{\psi(E_0)^{2+2\eta}}
\]
for some constant \( C_3 > 0 \).

Using the symmetry of the signed measure $\gamma^{(3)}_{\text{red}}$, we can summarize three cases for the position of $(y, z) \in S_2$ and obtain that

$$\lambda^{(3)+}_{\text{red}}(H^+_2) = \sum_{y, z \in \mathbb{Z}^d} \lambda^{(3)+}_{\text{red}}((E_y \times E_z) \cap H^+_2) \leq 2 \sum_{(y, z) \in S_2} \lambda^{(3)+}_{\text{red}}((E_y \times E_z) \cap H^+_2) \leq 2 \left[ C_1 \# S_2^{(1)} + C_2 \sum_{(y, z) \in S_2^{(2)}} \beta \varphi \left( \left| y \right| - \frac{3}{2} \right)^{\frac{4}{3+d}} + C_3 \sum_{(y, z) \in S_2^{(3)}} \beta \varphi \left( \left| z \right| - \left| y \right| - 2 \right)^{\frac{4}{3+d}} \right].$$

By means of (4.1) some simple rearrangements show that

$$\sum_{(y, z) \in S_2^{(2)}} \beta \varphi \left( \left| y \right| - \frac{3}{2} \right)^{\frac{4}{3+d}} \leq \sum_{m=2}^{\infty} 2d(2m + 1)^{d-1}(2m + 2)d(4m + 1)^{d-1} \beta \varphi \left( m - \frac{3}{2} \right)^{\frac{4}{3+d}},$$

$$\sum_{(y, z) \in S_2^{(3)}} \beta \varphi \left( \left| y \right| - \frac{3}{2} \right)^{\frac{4}{3+d}} \leq \sum_{n=3}^{\infty} \beta \varphi (n - 2)^{\frac{4}{3+d}} + (3d - 1) \sum_{n=4}^{\infty} \beta \varphi (n - 3)^{\frac{4}{3+d}}$$

$$+ \sum_{m=2}^{\infty} 2d(2m + 1)^{d-1} \sum_{n=2m+1}^{\infty} 2d(2n + 1)^{d-1} \beta \varphi (n - m - 2)^{\frac{4}{3+d}}.$$

By condition (2.5) for $k = 3$ it is not difficult to see that $\gamma^{(3)+}_{\text{red}}(H^+_2) \leq C_4 \sum_{n \geq 1} n^{2d-1} \beta \varphi (n)^{4/(3+d)} < \infty$ for some constant $C_4 > 0$ depending on $d$, $\lambda$, $\delta$, and $\mathbf{E} \psi(E_0)^{3+\delta}$. In the same way, we can prove that $\gamma^{(3)-}_{\text{red}}(H^-_2) < \infty$ and thus $|\gamma^{(3)}_{\text{red}}|([\mathbb{R}^d \times \mathbb{R}^d]) < \infty$, completing the proof of Theorem 2 for $k = 2, 3$.

5 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $f : \mathbb{R}^d \to [0, \infty]$ be Borel-measurable with bounded support $F$ and $\int f(x) \, dx = 1$. Since $F \cap (G \oplus F) = \emptyset$, we have $o \notin G$, implying $(T_x \psi - \delta_o)_G = (T_z \psi)_G$ for all $\psi \in N$. By applying the Campbell–Mecke formula (see [3, Chap. 13]) to the stationary PP $\tilde{\Psi} \sim P$ we get the equality

$$\int \int f(x)g((T_x \psi - \delta_o)_G) \psi(dx) \, P(d\psi) = \lambda \int f(x) \, dx \int g(\psi_G) \, P_\psi^I(d\psi),$$

which, combined with the simple Campbell formula

$$\mathbb{E} \sum_{i \geq 1} f(X_i) = \int \int f(x) \, \psi(dx) \, P(d\psi) = \lambda \int f(x) \, dx = \lambda,$$

yields the relation

$$\lambda \int g(\psi_G) \left( P_\psi^I - P \right) (d\psi)$$

$$= \int \int f(x)g((T_x \psi - \delta_o)_G) \psi(dx) \, P(d\psi) - \lambda \int \int f(x)g((T_z \psi)_G) \, dx \, P(d\psi)$$

$$= \mathbb{E} h(\tilde{\Psi}_F, \psi_{G \oplus F}) - \mathbb{E} h(\tilde{\Psi}_F, \tilde{\psi}_{G \oplus F}),$$
where the $\mathcal{N} \otimes \mathcal{N}$-measurable function $h|\mathcal{N}(F) \times \mathcal{N}(G \otimes F)| \to \mathbb{R}$ is defined by

$$h(\psi_F, \psi_{G \otimes F}) := \sum_{i \geq 1} f(X_i) 1_{F}(X_i) g((T_x \psi)_G).$$

The independence of the restricted PPs $\tilde{\psi}_F$ and $\tilde{\psi}_{G \otimes F}$, Fubini's theorem, and the stationarity of $\psi \sim P$ allow us to write

$$E h(\tilde{\psi}_F, \tilde{\psi}_{G \otimes F}) = \int_{\mathcal{N}(F)} \int_{\mathcal{N}(G \otimes F)} \int_{F} f(x) g((T_x \psi)_G) \varphi(dx) P(d\psi) P(d\varphi)$$

$$= \int_{\mathcal{N}} \int_{F} f(x) \varphi(dx) P(d\varphi) \int_{\mathcal{N}} g(\psi_G) P(d\psi) = \lambda \int_{\mathcal{N}} g(\psi_G) P(d\psi).$$

A straightforward application of Lemma 1 yields the estimate

$$\lambda \left| \int_{\mathcal{N}} g(\psi_G) (P_0^F - P)(d\psi) \right| \leq 2 \max \left\{ \left( E |h(\psi_F, \psi_{G \otimes F})|^{1+\eta} \right)^{\frac{1}{1+\eta}}, \left( E |h(\tilde{\psi}_F, \tilde{\psi}_{G \otimes F})|^{1+\eta} \right)^{\frac{1}{1+\eta}} \right\} \times (\beta(\mathcal{F}_F, \mathcal{F}(G + F)))^{1-\frac{1}{1+\eta}}$$

for any $\eta \geq 0$.

Further, for any $p, q \in [1, \infty]$ satisfying $1/p + 1/q = 1/(1 + \eta)$, we employ Hölder's inequality to show that

$$\left( E |h(\psi_F, \psi_{G \otimes F})|^{1+\eta} \right)^{\frac{1}{1+\eta}} \leq \left( E \left( \sum_{i \geq 1} f(X_i) \right)^{1+\eta} \sup_{x \in F} |g((T_x \psi)_G)|^{1+\eta} \right)^{\frac{1}{1+\eta}}$$

$$\leq \left( E \left( \sum_{i \geq 1} f(X_i) \right)^p \right)^{\frac{1}{p}} \left( E \sup_{x \in F} |g((T_x \psi)_G)|^q \right)^{\frac{1}{q}}.$$

Likewise, we get the same upper bound for $(E |h(\tilde{\psi}_F, \tilde{\psi}_{G \otimes F})|^{1+\eta})^{1/(1+\eta)}$. This immediately provides the desired estimate (1.3). To prove (1.4), we consider the Hahn decomposition $N^+(G) \cup N^-(G) = N(G)$ of the signed measure $P_0^F(\cdot \cap N(G)) = P(\cdot \cap N(G))$. Inserting $g(\psi) = 1_{N^+(G)}(\psi) - 1_{N^-(G)}(\psi)$ on both sides of inequality (1.3), we can take $p = 1 + \delta, q = \infty$ (since $|g((T_x \psi)_G)| \leq 1$ for $x \in F$), and $f(x) = 1_F(x) / \nu_0(F)$ on the right-hand side, whereas the left-hand side equals $2 \sup_{Y \in N(G)} |P_0^F(Y) - P(Y)|$. Hence, (1.4) is shown, and the proof of Theorem 1 is finished.

**Proof of Theorem 2.** We have to show that

$$|\gamma^{(k)}_{\text{red}}|_{(\mathbb{R}^d)^{k-1}} = \gamma^{(k)}_{\text{red}}(H^+_{k-1}) - \gamma^{(k)}_{\text{red}}(H^-_{k-1}) < \infty$$

for some fixed $k \geq 4$, where $H^+_{k-1} \cup H^-_{k-1}$ is the Hahn decomposition of the signed measure $\gamma^{(k)}_{\text{red}}$. Due to the complete symmetry of $\gamma^{(k)}_{\text{red}}$, we have

$$\gamma^{(k)}_{\text{red}}(H^+_{k-1}) = \sum_{z_2, \ldots, z_k \in \mathbb{Z}^d} \gamma^{(k)}_{\text{red}}((E_{z_2} \times \cdots \times E_{z_k}) \cap H^+_{k-1})$$

$$\leq (k-1)! \sum_{z_2, \ldots, z_k \in \mathbb{Z}^d} \gamma^{(k)}_{\text{red}}((E_{z_2} \times \cdots \times E_{z_k}) \cap H^+_{k-1}).$$

where $S_{k-1} := \{(z_2, \ldots, z_k) \in (\mathbb{Z}^d)^{k-1} : 0 \leq |z_2| \leq \cdots \leq |z_k|\}$. Let us fix $(z_2, \ldots, z_k) \in S_{k-1}$ and put $E^+_k := (E_{z_2} \times \cdots \times E_{z_k}) \cap H^+_{k-1}$ for notational ease. Our next aim is to derive an upper bound for $\gamma^{(k)}(E^+_k)$. Using (3.4), we can express $\gamma^{(k)}(E^+_k)$ in terms of higher-order covariance measures $\zeta^{(j)}$:

$$\lambda \gamma^{(k)}_\text{red}(E^+_k) = \int \cdots \int 1_{E_o}(x_1) 1_{E^+_k}((x_2 - x_1, \ldots, x_k - x_1)) \gamma^{(k)}(d(x_1, \ldots, x_k))$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \sum_{K_1 \cup \cdots \cup K_j = K} N_j(K_1, \ldots, K_j) I_j(K_1, \ldots, K_j),$$

(5.1)

where

$$I_j(K_1, \ldots, K_j) := \int \cdots \int 1_{E_o}(x_1) 1_{E^+_k}((x_2 - x_1, \ldots, x_k - x_1)) \prod_{i=1}^{j} \zeta^{(\kappa_i)}(d(x_{k_i+1}, \ldots, x_{k_i+p})).$$

Since $x_1 \in E_o$ and $(x_2 - x_1, \ldots, x_k - x_1) \in E^+_k$, it follows that $x_i \in E_o \oplus E_{z_i} \subset F_1 + z_i$ for $i = 2, \ldots, k$, and, together with (5.1), we arrive at

$$\lambda \gamma^{(k)}_\text{red}(E^+_k) \leq |\gamma^{(k)}|(E_o \times (F_1 + z_2) \times \cdots \times (F_1 + z_k)).$$

Obviously, $\alpha^{(j)}((F_1 + z_{k_1}) \times \cdots \times (F_1 + z_{k_j})) \leq \mathbb{E} \Psi(F_1 + z_{k_1}) \cdots \Psi(F_1 + z_{k_j})$, and, using Hölder's inequality and the stationarity of $\Psi \sim P$, we get that

$$\alpha^{(j)}((F_1 + z_{k_1}) \times \cdots \times (F_1 + z_{k_j})) \leq 2^{jd} \mathbb{E} \Psi(E_o)^j \leq (2^{kd} \mathbb{E} \Psi(E_o)^k)^j.$$

(5.2)

Inserting the latter estimate into (2.2) gives

$$|\gamma^{(k)}|(E_o \times (F_1 + z_2) \times \cdots \times (F_1 + z_k)) \leq k! 2^{kd} \mathbb{E} \Psi(E_o)^k.$$

Thus, each summand of the sum $\sum_{(z_2, \ldots, z_k) \in S_{k-1}} \gamma^{(k)}_\text{red}(E^+_k)$ is finite, and, consequently, it suffices to show that

$$\sum_{(z_2, \ldots, z_k) \in S_{k-1}} |I_j(K_1, \ldots, K_j)| < \infty$$

for any decomposition of $K = \{1, \ldots, k\}$ into $j \in \{1, \ldots, k-1\}$ disjoint nonempty subsets $K_1, \ldots, K_j$ such that $N_j(K_1, \ldots, K_j) > 0$.

Let $z_1 = 0$ and $m(z_2, \ldots, z_k) := \max\{|z_j| - |z_{j-1}|, j = 2, \ldots, k\}$ be the largest gap in the sequence $0 = |z_1| \leq |z_2| \leq \cdots \leq |z_k|$. If $|z_k| \geq 2k - 1$, then the maximal gap $m(z_2, \ldots, z_k)$ is at least 3. Let $q \in \{1, \ldots, k-1\}$ be such that $|z_{q+1}| - |z_q| = m(z_2, \ldots, z_k)$, i.e., the largest gap occurs between $|z_q|$ and $|z_{q+1}|$. We start with the case $j = 1$.

Making use of formula (3.5) with (3.6), we may express $I_1(K)$ as

$$I_1(K) = \sum_{p=0}^{q-1} \sum_{r=0}^{k} \int \int \int \int 1_{E_o}(x_1) 1_{E^+_k}((x_2 - x_1, \ldots, x_k - x_1))$$

$$\times \zeta^{(k-r)}(d(x_{r+1}, \ldots, x_k)) \Delta_q(d(x_{p+1}, \ldots, x_r)) \zeta^{(p)}(d(x_1, \ldots, x_p)).$$
\[
\sum_{p=0}^{q-1} \sum_{r=q+1}^{k} \int \int \left[ Ef(\Psi_{B_p}, \Psi_{B'_r}; x_1, \ldots, x_p, x_{r+1}, \ldots, x_k) \right. \\
- Ef(\Psi_{B_p}, \Psi_{B'_r}; x_1, \ldots, x_p, x_{r+1}, \ldots, x_k) \right] \tilde{\zeta}(k-r)\left(d(x_{r+1}, \ldots, x_k)\right) \tilde{\zeta}(p)\left(d(x_1, \ldots, x_p)\right),
\]
where \(B_p = \bigcup_{r=p}^{q+1} (F_1 + z_t)\), \(B'_r = \bigcup_{r=q+1}^{k} (F_1 + z_t)\), \(\Psi_{B_p}\) and \(\Psi_{B'_r}\) are copies of \(\Psi_{B_p}\) and \(\Psi_{B'_r}\), respectively, assumed to be independent, and
\[
f(\Psi_{B_p}, \Psi_{B'_r}; x_1, \ldots, x_p, x_{r+1}, \ldots, x_k) \\
:= \sum_{i_1, \ldots, i_q \geq 1} \sum_{i_1, \ldots, i_r \geq 1} 1_{E_0}(x_1) 1_{E_k}(x_2 - x_1, \ldots, x_p - x_1, X_{i_p+1} - x_1, \ldots, X_{i_q} - x_1, \\
X_{i_q+1}, \ldots, X_{i_r} - x_1, x_{r+1} - x_1, \ldots, x_k - x_1) \\
\leq \sum_{t=p+1}^{k} \Psi(F_1 + z_t).
\]
The latter inequality holds \(P\)-a.s. if \(F_t\) is replaced by \(F_t^{\text{int}} = (-1, 1)^d\). Thus, we can apply Lemma 1 for \(B = B_p \subset F_{|z_{q+1}|1}^{\text{int}}\) and \(B' = B'_r \subset F_{|z_{r+1}|1}^{\text{int}}\), and, together with assumption (2.4) and Lemma 2 (with obvious modifications for \(p \geq 1\)), we obtain the inequality
\[
\left| Ef(\Psi_{B_p}, \Psi_{B'_r}; x_1, \ldots, x_p, x_{r+1}, \ldots, x_k) - Ef(\Psi_{B_p}, \Psi_{B'_r}; x_1, \ldots, x_p, x_{r+1}, \ldots, x_k) \right| \\
\leq 2^{(r-p)d+1}\left(E\Psi(E_0)^{r-p}1(1+q)v_\eta\right) \max \left\{ 1, \frac{|z_q| + 1}{|z_{q+1}| - |z_q| - 2} \right\} \beta_\Psi \left(|z_{q+1}| - |z_q| - 2\right)^{\frac{r}{v_\eta}} \\
\times 1_{E_0}(x_1) \prod_{j=2}^{p} 1_{E_{q_j}}(x_j - x_1) \prod_{j=r+1}^{k} 1_{E_{q_j}}(x_j - x_1)
\]
for any \(\eta \geq 0\), where the right-hand side (with \(0 \leq p \leq r \leq k\)) is finite for \(\eta = \delta/k\). From (3.3) and (5.2) we get that the total-variation measures \(|\tilde{\zeta}(p)|\) and \(|\tilde{\zeta}(k-r)|\) for \(0 \leq p \leq r \leq k\) satisfy the estimates
\[
|\tilde{\zeta}(p)|\left(\sum_{j=1}^{p} (F_1 + z_j)\right) \leq 2^{(d+1)p} \Psi(E_0)^p
\]
and
\[
|\tilde{\zeta}(k-r)|\left(\sum_{j=r}^{k} (F_1 + z_j)\right) \leq 2^{(d+1)(k-r)-1} \Psi(E_0)^{k-r}.
\]
Combining the previous estimates with \(\eta = \delta/k\) and applying again Hölder's inequality, we find that
\[
|I_1(K)| \leq 2^{(k+1)d} \left(E\Psi(E_0)^{k+\delta}v_\Psi\right) \max \left\{ 1, \frac{|z_q| + 1}{|z_{q+1}| - |z_q| - 2} \right\} \beta_\Psi \left(|z_{q+1}| - |z_q| - 2\right)^{\frac{r}{v_\Psi}} \\
\times (2|z_q| + 1)^{d(q-1)} ((2|z_k| + 1)^d - (2|z_k| - 1)^d)^{k-q} \leq 2d (2|z_k| + 1)^{d(k-2)+d-1}
\]
for any \((z_2, \ldots, z_k) \in S_{k-1}\) satisfying \(|z_k| \geq 2k - 1\) and \(m(z_2, \ldots, z_k) = |z_{q+1}| - |z_q| \geq 3\). The number of such \((k-1)\)-tuples \((z_2, \ldots, z_k)\) is at most
\[
(2|z_q| + 1)^{d(k-1)} ((2|z_k| + 1)^d - (2|z_k| - 1)^d)^{k-q} \leq 2d (2|z_k| + 1)^{d(k-2)+d-1}
\]
for \(2 \leq q \leq k - 1\), where the latter bound is justified by \(|z_q| < |z_{q+1}| \leq |z_k|\) and (4.1).
Therefore, first fixing the largest gap \( m(z_2, \ldots, z_k) = m \) and having in mind that \( |z_\ell| \leq (\ell - 1)m \) for \( \ell = 2, \ldots, k \) and then summing up over all \( m \geq 3 \) yield that

\[
\sum_{(z_2, \ldots, z_k) \in S_{k-1}; |z_k| \geq 2k-1} |I_1(K)| \leq 2^{k+1}d \left( E\Psi(E_0)^{k+\delta} \right)^{\frac{k+\delta}{k+\delta}} \sum_{m=3}^{\infty} 2d(2(k-1)m + 1)^{(k-1)d-1} \times \max \left\{ 1, \frac{(k-2)m + 1}{m-2} \right\}^{d-1} \beta_\varphi(m-2)^{\frac{k-\delta}{k+\delta}} \\
\leq C_5(k, d, \delta) \sum_{m=1}^{\infty} m^{(k-1)d-1} \beta_\varphi(m)^{\frac{k-\delta}{k+\delta}},
\]

where, by (2.5), the series in the last line converges, and the constant \( C_5(k, d, \delta) \) depends only on \( d \geq 1, k \geq 2, \) and \( E\Psi(E_0)^{k+\delta} < \infty. \)

Next, we consider the terms \( I_j(K_1, \ldots, K_j) \) for \( j \geq 2 \) with decompositions \( K_1, \ldots, K_j \) of \( K = \{1, \ldots, k\} \) satisfying \( N_j(K_1, \ldots, K_j) > 0. \) These terms are multiple integrals over some subset of \( E_0 \times (E_{z_2} \otimes E_0) \times \cdots \times (E_{z_k} \otimes E_0) \) w.r.t. products of higher-order covariance measures (3.3). Let \( q \in \{1, \ldots, k-1\} \) be the (largest) index such that \( |z_{q+1}| - |z_q| = m \) is the maximal gap in the sequence \( 0 = |z_1| \leq |z_2| \leq \cdots \leq |z_k| \). Then there exists an (ordered) index set \( K_q = \{k_{q1}, \ldots, k_{qk_{q+1}}\} \) such that \( |z_{k_{q+1}}| - |z_{k_{q+1}}| \geq m \) for at least one \( r \in \{1, \ldots, k_{q+1} - 1\}. \) This is obvious if \( q = q+1 \) belong to the same index set. Otherwise, we distinguish two cases. First, \( q+1 \in K_q \) with \( k_{q+1} \geq 2 \) and \( k_{q+1} < q+1, \) so that \( |z_{q+1}| - |z_{k_{q+1}}| \geq m, \) where \( k_{q+1} \) is the largest index in \( K_q \) less than \( q+1. \) Second, \( q+1 \) coincides with the smallest index \( k_{p1} \) in \( K_p \) for some \( p \in \{2, \ldots, j\}. \) Due to the positivity of \( N_j(K_1, \ldots, K_j) \) (see [17, p. 80]), there exists an index set \( K_p \) with \( k_{p1} \geq 2 \) such that \( k_{p1} < q+1 < k_{q+1}, \) implying that \( |z_{k_{q+1}}| - |z_{k_{p1}}| \geq m, \) where \( k_{q+1} \) is the largest (smallest) index in \( K_q \) less (greater) than \( q+1. \)

In this way, we have found a covariance measure \( \hat{\zeta}(\kappa) \) occurring in \( I_j(K_1, \ldots, K_j) \) to which the same arguments as to \( \hat{\zeta}(k) \) in \( I_1(K) \) can be applied. Hence, taking into account that

\[
|\hat{\zeta}(\kappa)\left((F_1 + z_{k_1}) \times \cdots \times (F_1 + z_{k_j})\right)| \leq 2^{j-1}2^{jd} E\Psi(E_0)^{\delta}
\]

for any \( \{k_1, \ldots, k_j\} \subset \{2, \ldots, q\}, \) we obtain the estimate

\[
|I_j(K_1, \ldots, K_j)| \leq C_6(k, d, \delta) \left( E\Psi(E_0)^{k+\delta} \right)^{\frac{k+\delta}{k+\delta}} \beta_\varphi(m-2)^{\frac{1}{k+\delta}}.
\]

Finally, repeating the above counting procedure and using (2.5) lead to

\[
\sum_{(z_2, \ldots, z_k) \in S_{k-1}; |z_k| \geq 2k-1} |I_j(K_1, \ldots, K_j)| \leq C_7(k, d, \delta) \sum_{m=1}^{\infty} m^{(k-1)d-1} \beta_\varphi(m)^{\frac{1}{k+\delta}} < \infty,
\]

where the constant \( C_7(k, d, \delta) \) depends only on \( d \geq 1, k \geq 2, \) and \( E\Psi(E_0)^{k+\delta} < \infty. \)

In the same way, we can show that \( -\hat{\tau}_{red}(H_{k-1}^-) < \infty, \) which terminates the proof. \( \square \)

### 6 Some examples from stochastic geometry

**Example 1.** \( m \)-dependent stationary PP \( \Psi \sim P \) (i.e., \( F_\Psi(F_a) \) and \( F_\Psi(F_{a+m}) \) are independent for some fixed \( m = 0 \) and any \( a > 0 \)) is \( B_k \)-mixing if \( E\Psi(E_0)^k < \infty. \) Special cases of \( m \)-dependent PPs are Poisson cluster processes and dependently thinned Poisson processes with bounded cluster diameter and thinning procedures of bounded reach, respectively; see Example 4 below. Note that in Theorem 2 we can take \( \beta_\varphi(m) = 0 \) and \( \delta = 0. \)
Example 2. Voronoi-tessellation $V(\Psi) = \bigcup_{i \geq 1} \partial C_i(\Psi)$ generated by a simple stationary PP $\Psi = \sum_{i \geq 1} \delta_{X_i}$ in $\mathbb{R}^d$, where $\partial C_i(\Psi)$ denotes the boundary of the cell $C_i(\Psi)$ formed by all points in $\mathbb{R}^d$ that are closest to the atom $X_i$, i.e., $C_i(\Psi) = \{ x \in \mathbb{R}^d : \| x - X_i \| < \| x - X_j \|, \ j \neq i \}$ (see [19]). Let $F_{V(\Psi)}(F)$ denote the $\sigma$-algebra generated by the random closed set $V(\Psi) \cap F$ (see [6] for details). In case the $X_i$s are atoms of a Poisson process $\Psi \sim \Pi_\lambda$, the following bound is shown in [6]:

$$
\beta(F_{V(\Psi)}(F_0), F_{V(\Psi)}(F_0^c + r)) \leq \begin{cases} 
 c_0 \left( \frac{\lambda}{r} \right)^{d-1} \exp\left\{ -\lambda c_4 a^{d-1} r \right\} & \text{if } r \geq c_0 a, \\
 c_4 \left( \frac{\lambda}{r} \right)^{d-1} \exp\left\{ -\lambda c_2 r^d \right\} & \text{if } r \leq c_0 a 
\end{cases}
$$

for $a, r \geq 1/2$, giving $\beta_\Psi(r) = c_0 \lambda r^{d-1} \exp\{ -\lambda c_4 r \}$ according to (2.4) with constants $c_0, c_1, \ldots, c_5 > 0$ depending only on the dimension $d \geq 1$. Hence, the stationary PP of the cell vertices and other PPs associated with the cells $C_i(\Psi)$ (e.g., circumcenters of the $(d-1)$-facets or Cox processes supported by $V(\Psi)$) are Brilliogner-mixing. Furthermore, the exponential decay of $\beta_\Psi(r)$ holds also for Poisson cluster processes with typical cluster diameter $D_0$ satisfying $E \exp\{ h D_0 \} < \infty$ for some $h > 0$ (see [6]).

Example 3. Germ-grain models $\Xi = \bigcup_{i \geq 1} (X_i + \Xi_0)$ defined by a stationary PP $\Psi = \sum_{i \geq 1} \delta_{X_i}$ in $\mathbb{R}^d$ with intensity $\lambda > 0$ and a sequence $\{ \Xi_i, i \geq 1 \}$ (independent of $\Psi$) of independent copies of a compact set $\Xi_0 \subset \mathbb{R}^d$, called typical grain. In [9], the subsequent bound of the $\beta$-mixing coefficient between two $\alpha$-algebras generated by the random closed set $\Xi$ on $F_0$ and $R_{\alpha+\tau}$, respectively, is derived:

$$
\beta(F_{\Xi}(F_0), F_{\Xi}(F_0^c)) \leq \beta(F_{\Psi}(F_{\alpha+\tau}), F_{\Psi}(F_{\alpha+\tau})) + \lambda d \left( 1 + \frac{4a}{r} \right)^{d-1} \left( 3 + \frac{4a}{r} \right)^{d-1} E \| \Xi_0 \|^d 1\left( \| \Xi_0 \| \geq \frac{r}{4} \right)
$$

(6.1)

for $a, r \geq 1/2$, where $\| \Xi_0 \| := \sup\{ \| x - y \| : x, y \in \Xi_0 \}$ denotes the diameter of the typical grain $\Xi_0$. Taking into account condition (2.4) with $\beta$-mixing rate $\beta_\Psi(r)$, we easily see from (6.1) that

$$
\beta(F_{\Xi}(F_0), F_{\Xi}(F_0^c)) \leq \max \left\{ 1, \frac{4a}{r} \right\}^{d-1} \left( \frac{r}{2} \right)^{d-1} \beta_\Psi \left( \frac{r}{2} \right) + \lambda d \left( 1 + \frac{4a}{r} \right)^{d-1} E \| \Xi_0 \|^d 1\left( \| \Xi_0 \| \geq \frac{r}{4} \right)
$$

(6.2)

for $a, r \geq 1/2$.

Note that (6.2) provides the $\beta$-mixing rate of a cluster PP $\Psi_{\alpha} := \sum_{i \geq 1} \sum_{j \geq 1} \delta_{X_i + \Xi_0}$ if $\Xi_0 = \{ Y_1, \ldots, Y_N \}$ consists of (P-a.s.) finitely many random points with typical cluster diameter $D_0 = \| \Xi_0 \|$. Further, Cox processes $\Psi_{\alpha}$ are frequently used PP models (see, e.g., [2, 3] for a general definition), in particular, so-called interrupted Poisson processes supported by a random set $\Xi$ or its boundary $\partial \Xi$ (see [8, 19]). For example, the atoms of a Poisson process $\Psi = \sum_{i \geq 1} \delta_{P_i}$ independent of the germ-grain model $\Xi$ are only counted when they lie in $\Xi$, i.e., $\Psi_{\alpha} = \sum_{i \geq 1} \mathbb{1}_{X_i} (P_i) \delta_{P_i}$. Due to (6.2) and the properties of $\Phi$, it is clear that the $\beta$-mixing rate $\beta_\Psi(r)$ satisfies (2.5) if $\beta_\Psi(r)$ does and

$$
\int_1^\infty r^{(k-1) d - 1} \left( \frac{E \| \Xi_0 \|^d 1\left( \| \Xi_0 \| \geq \frac{r}{4} \right)}{r^{d-1}} \right)^{\frac{r}{d} + \frac{q}{d}} dr
$$

$$
\leq \int_1^\infty r^{-\frac{q}{d} - 1} \left( \frac{E \| \Xi_0 \|^d 1\left( \| \Xi_0 \| \geq \frac{r}{4} \right)}{r^{d-1}} \right)^{\frac{q}{d} + \frac{q}{d}} dr \leq \frac{k + \delta}{\delta \varepsilon} \left( \frac{d}{q} \right)^{\frac{q}{d} + \frac{q}{d}} \left( \frac{E \| \Xi_0 \|^d 1\left( \| \Xi_0 \| \geq \frac{r}{4} \right)}{r^{d-1}} \right)^{\frac{q}{d} + \frac{q}{d}} < \infty
$$

(6.3)

for $q = d + (k-1)(k+\delta)/\delta$ with some $\delta > 0$, where $\varepsilon > 0$ can be chosen arbitrarily small. Hence, since

Figure 1. Exposed tangents points in a Boolean model with discs.

\[ \mathbb{E} \psi_c(\mathcal{F}_0)^{k+\delta} < \infty \] obviously holds, both assumptions (2.5) and (6.3), i.e., \( \mathbb{E} \| \Xi_0 \|^d(k+\delta-1)/\delta + \varepsilon < \infty \), imply that the stationary Cox PP \( \psi_{c0} \) turns out to be \( \mathcal{B}_k \)-mixing.

From the view point of statistics of germ-grain models (see [15]), the family of PPs \( \psi_c \) of exposed tangent points associated with the germ-grain model \( \Xi \) in direction (of a unit vector) \( u \) contains a lot of information on \( \Xi_0 \) and \( \psi_0 \). Assuming additionally that \( \Xi_0 \) is convex and \( \mathbf{0} \in \Xi_0 \), the PP \( \psi_c \) is defined by

\[ \psi_c := \sum_{i \geq 1} \delta \ell(\mathbf{u}, \Xi_i) + X_i \prod_{j \neq i} \left( 1 - 1_{\Xi_i + X_i}(\ell(\mathbf{u}, \Xi_j) + X_j) \right), \]

where \( \ell(\mathbf{u}, \Xi_i) \) denotes the lexicographically smallest tangent point of the convex grain \( \Xi_i \) in direction \( u \). This means that the atoms of \( \psi_c \) are those tangent points of the shifted grains \( \Xi_i + X_i \) that are not covered by any other shifted grain \( \Xi_j + X_j, j \neq i \) (see Fig. 1). Note that the PP \( \psi_c \) turns out to be stationary (but not isotropic even if \( \psi_0 \) and \( \Xi_0 \) are isotropic).

The definition of \( \psi_c \) reveals that the \( \beta \)-mixing coefficient on the left-hand side of (6.2) can be replaced by \( \beta(\mathcal{F}_{\psi_c}(\mathcal{F}_0, \mathcal{F}_{\psi_c}(\mathcal{F}_0^r + r))) \). Together with the obvious fact that the moments of \( \psi_c(\mathcal{F}_0) \) do not exceed the moments of \( \psi(\mathcal{F}_0) \), we arrive at the conclusion that \( \psi_c \) is \( \mathcal{B}_k \)-mixing for any \( u \) if \( \psi \) fulfills (2.5) and \( \mathbb{E} \| \Xi_0 \|^d(k+\delta-1)/\delta + \varepsilon \) exists for some \( \delta > 0 \) and \( \varepsilon > 0 \).

The best studied and most used germ-model is the so-called Boolean model \( \Xi \), where the germs form a Poisson process \( \Psi \sim \Pi_\Lambda \). The random union set \( \Xi \) is P-a.s. closed if \( \mathbb{E} \| \Xi_0 \|^4 < \infty \) (see, e.g., [15, 19] for more on this basic model of stochastic geometry). Since, in this special case, \( \beta(\mathcal{F}_0) = 0 \) for \( r > 0 \) and all moments of \( \psi_0(\mathcal{F}_0) \) exist, the number \( \delta > 0 \) in (2.5) can be taken arbitrarily large, which relaxes the moment assumption on \( \| \Xi_0 \| \) to \( \mathbb{E} \| \Xi_0 \|^{kd+\varepsilon} < \infty \) for an arbitrarily small \( \varepsilon > 0 \) in order to ensure \( \mathcal{B}_k \)-mixing of \( \psi_{c0} \) and \( \psi_c \). It is worth noting that, for Boolean models, the intensity \( \lambda_u \) of \( \psi_u \sim P_u \) can be simply expressed by \( \lambda_u = \lambda \exp\{-\lambda E\nu_u(\Xi_0)\} \) and that the Lebesgue density \( \nu_u^{(k)} \) of the \( k \)-th order factorial moment measure (2.1) (with \( P_u \) instead of \( P \)) exists for any \( k \geq 2 \) and takes the form

\[ \nu_u^{(k)}(x_1, \ldots, x_k) = \lambda^k \prod_{p=1}^k \mathbb{E} \left( \prod_{q=1}^k (1 - 1_{\Xi_0(u)}(x_q - x_p)) \right) \times \exp \left\{ \lambda \int \left[ \prod_{r=1}^k (1 - 1_{\Xi_0(u)}(x - x_r)) - 1 \right] \, dx \right\}, \]
where $\Xi_0(u) := -\Xi_0 + \ell(u, \Xi_0)$. This formula allows us to check the $B_k$-mixing property directly by showing that, indeed, $E|\Xi_0|^{kd} < \infty$ is sufficient. Furthermore, $\vartheta_k(\cdot)$ is uniformly bounded by $\lambda_k$ for $k \geq 2$, which is significant for so-called sub-Poisson processes.

Example 4. $(\pi(x))$-thinning of point processes: Let $\{\pi(x), x \in \mathbb{R}^d\}$ be a stationary random field on $[\Omega, \mathcal{F}, P]$ taking values in $[0, 1]$ and independent of the stationary PP $\Psi = \sum_{i \geq 1} \delta_{X_i}$ in $\mathbb{R}^d$ (see [19]). Define the 0–1-valued random mark field $\{M(x), x \in \mathbb{R}^d\}$ with finite-dimensional distributions $P(M(x_1) = 1, \ldots, M(x_k) = 1) = E[\pi(x_1) \cdots \pi(x_k)]$ for any $x_1, \ldots, x_k \in \mathbb{R}^d$ and $k \in \mathbb{N}$. In this way, we obtain the so-called $\pi(x)$-thinned stationary PP $\Psi_\pi = \sum_{i \geq 1} \delta_{X_i} M(X_i)$. This thinning procedure means that, for a given realization of the probabilities $\pi(x) = p(x), x \in \mathbb{R}^d$, the atom $X_i$ survives with probability $p(X_i)$ independently of the survival of the other atoms $X_j, j \neq i$. As special cases, we mention $\pi(x) = 1(\xi(x) \in B)$ or $\pi(x) = (\xi(x) - a)1(a \leq \xi(x) \leq b)/(b - a)$ for some stationary random field $\{\xi(x), x \in \mathbb{R}^d\}$ and certain fixed $B \in \mathcal{B}$ and $a, b \in \mathbb{R}$. As particular case of geostatistical marking of PPs, we deduce from Lemma 5.1 in [8] (with $\sigma$-algebra $\mathcal{F}_n(F)$ generated by $\{\pi(x), x \in F\}$) that

$$
\beta(F_{\psi_a}(F_a), F_{\psi_{a+r}}(F_{a+r})) \leq \beta(F_{\psi}(F_a), F_{\psi}(F_{a+r})) + \beta(F_{\pi}(F_a), F_{\pi}(F_{a+r}))
$$

for $a, r \geq 1/2$, which gives $\beta_{\psi_a}(r) \leq \beta_{\psi}(r) + \beta_{\pi}(r)$ for the corresponding $\beta$-mixing rates. This enables us to check $B_k$-mixing of $\psi_{\pi}$. On the other hand, this property of $\psi_{\pi}$ holds for any $B_k$-mixing PP $\psi$ if, additionally, $\int_{(\mathbb{R}^d)^{k-1}} \sum_{j=1}^k \pi(\alpha) \pi(\xi_j) \nu(d\xi_j) < \infty$ for $j = 2, \ldots, k$.

Example 5. Generalized Stoyan soft-core process I and II: As in Example 3, let $\Psi = \sum_{i \geq 1} \delta_{X_i}$ be a simple stationary PP in $\mathbb{R}^d$ independently marked by a sequence of random vectors $\{(\varepsilon_i, U_i), i \geq 1\}$ with independent components, where the first ones are independent copies of a compact set $\Xi_0 \subset \mathbb{R}^d$ containing $0$, and the second ones are independently uniformly distributed in $(0, 1)$. Then we are in a position to define two types of dependently thinned PP generalizing two thinning procedures suggested in [20]:

$$
\Psi_{th, 1} := \left( \sum_{i \geq 1} \delta_{X_i} \prod_{j \neq i} (1 - 1_{\varepsilon_i + X_j}(X_j)) \right) \quad \text{and} \quad \Psi_{th, 2} := \left( \sum_{i \geq 1} \delta_{X_i} \prod_{j \neq i} 1_{[U_{i,j}]}(U_j) \right).
$$

To be precise, in the first model, an atom $X_i$ of $\Psi$ survives if and only if no other atom $X_j$ (of $\Psi$) lies in $\Xi_i + X_i$, whereas in the second model, $X_i$ survives iff either no other atom $X_j$ lies in $\Xi_i + X_i$ or all atoms $X_j \in \Xi_i + X_i, j \neq i$, have marks $U_j$ greater than or equal to $U_i$. In [20], $\Psi_{th, 1}$ and $\Psi_{th, 2}$ were introduced and studied in the special case of a random ball $\Xi_0 = b(o, R_0)$ centered at the origin with the aim to generalize Matérn’s hard-core processes I and II for which $P(H_0 = \text{const} > 0) = 1$ (see, e.g., [19]). Note that both Stoyan’s soft-core PPs inherit the isotropy of $\Psi$, whereas a noncircular set $\Xi_0$ can generate a high degree of anisotropy in $\Psi_{th, i}, i = 1, 2$, even if $\Psi \sim \Pi_\Lambda$.

Finally, it is easily checked that the $\beta$-mixing coefficients $\beta(F_{\psi_{\omega_i}}(F_a), F_{\psi_{\omega_i}}(F_{a+r}))$, $i = 1, 2$, have the same bound as $\beta(F_{\Xi}(F_a), F_{\Xi}(F_{a+r}))$ in (6.2) with all consequences mentioned above. In case of $\Psi \sim \Pi_\Lambda$, this implies that each of the soft-core Poisson processes $\Psi_{th, 1}$ and $\Psi_{th, 2}$ (with intensities $\lambda_1 = \lambda \exp(-\lambda \nu_\Xi(\Xi_0))$ and $\lambda_2 = E\{1(1 - \exp(-\lambda \nu_\Xi(\Xi_0)))/\nu_\Xi(\Xi_0)\}$, respectively) turns out to be Brillinger-mixing whenever $E|\Xi_0|^n < \infty$ for any $n \in \mathbb{N}$, and they are $m$-dependent (as defined in Example 1) if $P(|\Xi_0| \leq \text{const}) = 1$.

References


