STRONGLY CONSISTENT ESTIMATION
IN WICKSELL'S CORPUSCLE PROBLEM IN CASE
OF OCCLUDED SPHERES

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Abstract

We consider Wicksell's corpuscle problem for a homogeneous Poisson-grain model of \((d + 1)\)-spheres in \(\mathbb{R}^{d+1}\), where only a single observation of the union set of partially or wholly occluded \(d\)-spheres in a sampling window \(W_n = [0, n]^d\) contained in the \(d\)-dimensional intersection hyperplane is available. In contrast to the existing vast literature we assume that instead of all individual \(d\)-spheres only the boundary of the union set of \(d\)-spheres in \(W_n\) is observable. Using a suitably constructed empirical distribution function \(\hat{G}_n\) for the generic \(d\)-sphere diameter we introduce an empirical distribution function \(\hat{F}_n\) of the corresponding \((d + 1)\)-sphere diameter by means of a discretized Abel transform of \(\hat{G}_n\). Among other asymptotic results we prove strong uniform consistency of \(\hat{F}_n\) as \(n \to \infty\).

1. Introduction and Notation

Wicksell's corpuscle problem can be described as follows: Suppose that a homogeneous system of spherical particles (called \((d + 1)\)-spheres) with identically distributed diameters is contained in some opaque \((d + 1)\)-dimensional body, and one is interested to estimate the distribution function

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(DF) $F$ of the generic diameter. The only information one can get is by making a cross-section of that body with a $d$-dimensional hyperplane and observing the system of $d$-spheres with diameter DF $G$ in some bounded sampling region in the intersecting hyperplane. Wicksell [23] showed in the particular case $d = 2$ that $G$ and $F$ are connected by the following Abel-type integral equation

$$1 - G(t) = \frac{1}{\Delta} \int_{t}^{\infty} \frac{r(1 - F(r))dr}{\sqrt{r^2 - t^2}} = \frac{1}{\Delta} \int_{t}^{\infty} \sqrt{r^2 - t^2} dF(r)$$  \hspace{1cm} (1.1)

for $t \geq 0$, where $\Delta = \int_{0}^{\infty} r dF(r) = \int_{0}^{\infty} (1 - F(r))dr$ denotes the mean $(d+1)$-sphere diameter.

The integral equation (1.1) can be inverted giving an expression of $F$ in terms of $G$,

$$1 - F(r + 0) = \frac{2\Delta}{\pi} \int_{r}^{\infty} \frac{dG(t)}{\sqrt{t^2 - r^2}} \text{ for } r \geq 0.$$  \hspace{1cm} (1.2)

Note that throughout the DFs $G$ and $F$ are assumed to be left-continuous and the above improper Riemann-Stieltjes integrals $\int_{r}^{\infty}$ are understood as

$$\lim_{\epsilon \downarrow 0, x \uparrow \infty} \int_{r+\epsilon}^{x} \text{ for } r \geq 0.$$  

The equation (1.1) is valid for any $d \geq 1$ under the assumption that the marked point process (which need not to be Poisson) formed by the centres and diameters of the $(d+1)$-spheres is stationary with finite intensity and $\Delta < \infty$, see [13, 14]. In practice the most important cases are $d = 2$ (planar section through balls) and $d = 1$ (linear section through disks).

It is a matter of fact that the statistical and numerical inversion of (1.1) is an ill-posed problem. In the stereological literature one can find many attempts to determine a satisfactory estimation of $F$ based on a sample of diameters of the $d$-spheres observed in the intersection plane. Regularization and unfolding techniques via kernel-type estimators for the density of $G$ (which always exists) are the essential ingredients of these methods, see e.g. [2, 9, 19 and 20]. An overview and a thorough discussion on the most relevant statistical procedures can be found in [18]. The asymptotic behaviour including CLTs for estimators of $F$ has been studied in [4] and [22] and in the recent papers [6] and [7] basically under
Condition (Poi). The centres of the \((d + 1)\)-spheres and their diameters are distributed according to a stationary independently marked Poisson process in \(\mathbb{R}^{d+1}\) (with intensity \(\mu > 0\)) and mark (= diameter) \(DF\).

In Heinrich [13] some of these results could be extended to independently marked Brillinger-mixing point processes of \((d + 1)\)-sphere centres.

The main goal of this paper is to present a nonparametric estimation of \(F\) when condition (Poi) is satisfied and only the union set of the \(d\)-spheres in the intersection hyperplane is observable. This means that some \(d\)-spheres are invisible or only partially observable and most of the diameter are not directly measurable, see Figures 1 and 2.

Without loss of generality, let \(H_d = \mathbb{R}^d \times \{0\}\) be the \(d\)-dimensional hyperplane intersecting the system of \((d + 1)\)-spheres. It is well-known that the union set of the \(d\)-spheres in \(H_d\) forms again a stationary Poisson-grain model (PGM) in \(\mathbb{R}^d\) with intensity \(\lambda = \Delta \mu\) and typical grain \(B_d(o, D_0/2)\), where \(B_d(x, r)\) denotes the closed \(d\)-sphere centred at \(x \in \mathbb{R}^d\) with radius \(r > 0\) and the diameter \(DF(t) = P(D_0 < t)\) is connected with \(F\) by (1.2). To be precise we recall, see e.g. [8], that a stationary PGM (also known as Boolean model) in \(\Xi\) with intensity \(\lambda\) and convex compact typical grain \(\Xi_0\) is defined to be the union set

\[
\Xi = \bigcup_{i \geq 1} (\Xi_i + X_i) \quad (1.3)
\]
of independent copies \(\Xi_1, \Xi_2, \ldots\) of the random convex compact set \(\Xi_0 \subset \mathbb{R}^d\), where the grains are shifted by the atoms \(X_1, X_2, \ldots\) of a stationary Poisson process \(\Pi_\lambda(\cdot) = \sum_{i \geq 1} 1(\cdot)(X_i)\) on \(\mathbb{R}^d\) with intensity \(\lambda\) (= mean number of atoms in the unit cube \([0, 1]^d\)).

For numerous applications, statistical inference and more details on the mathematical background of PGM's and point processes we refer to the monographs Hall [8], Stoyan et al. [18] and Molchanov [17]. Throughout in this paper all random elements will be defined on a common probability space \([\Omega, \mathcal{F}, P]\) and \(E\) resp. \(\text{Var}\) denote expectation resp. variance w.r.t. \(P\) and \(|\cdot|\) denotes the \(d\)-dimensional Lebesgue measure.
To obtain consistent estimators $\hat{G}_n(t)$ (for each $t > 0$) from a single observation of the union set of $d$-spheres in a cubic sampling region $W_n = [0, n]^d$ (in $H_d$) we employ the so-called method of exposed tangent points, see [12, 15 and 16]. The desired empirical DF $\hat{F}_n$ is then given by a suitably defined Riemann-Stieltjes sum approximating the integral on the rhs of (1.2) with $G$ replaced by $\hat{G}_n(t)$ on a sufficiently dense grid of $t$-values.

The rest of this paper is organized as follows: The main result (Theorem 2.1) is formulated at the end of Section 2 and proved in Section 5. Section 2 also contains the necessary facts on PGMs and the point process $\Psi_u$ of exposed tangent points and its statistical analysis. In Section 3 uniform strong consistency of a kernel-type estimator of the second-order product density of $\Psi_u$ and of the empirical covariance $\hat{C}_n(x)$ in case of the PGM (1.3) with bounded grains is proved. The $P$-a.s. behaviour of $\hat{G}_n$ is studied in Section 4 and the concluding Section 6 illustrates our estimation methods for $G$ and $F$ with data from planar cross-sections through simulated 3D-PGMs of spheres. Let $c(\cdot)$ and $c_k(\cdot), k = 1, 2, \ldots$, denote positive constants which depend on the quantities in parenthesis.

### 2. Preliminaries and Main Results

We begin by introducing the point process of exposed tangent points $\Psi_u$ associated with the stationary PGM (1.3) in a fixed direction $u \in S^{d-1} = \partial B_d(o, 1)$, see [12, 15 and 17]. For each realization of a random convex body $\Xi_0$ the tangent point $l_u(\Xi_0)$ of $\Xi_0$ in direction $u \in S^{d-1}$ is defined to be the lexicographically smallest point $x \in \partial \Xi_0$ satisfying $\langle -u, x \rangle = \sup_{y \in \Xi_0} \langle -u, y \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product. For example, in the special case of $\Xi_0 = B_d(o, R_0)$ we have $l_u(\Xi_0) = -R_0 u$. Now we are ready to define the above-mentioned point process as random counting measure

$$\Psi_u(\cdot) = \sum_{i \geq 1} 1_{l_i}(X_i + l_u(\Xi_i)) \prod_{j: j \neq i} (1 - 1_{\Xi_j + X_j}(X_i + l_u(\Xi_i))),$$

(2.1)

the support $s(\Psi_u)$ of which contains all those tangent points $l_u(\Xi_i) + X_i$ of the shifted sets $\Xi_i + X_i$ not covered by any other shifted set $\Xi_j + X_j, j \neq i$. The estimation of $G$ and $F$ is illustrated by data from planar cross-sections through simulated 3D-PGMs of spheres.
The point process (2.1) turns out to be simple, stationary (but not necessarily isotropic) with intensity $\lambda_u = \lambda \exp \{-\lambda E | \Xi_0 | \}$. The second-order product density $\varrho_u$ of $\Psi_u$, which is defined by

$$
E \left( \sum_{x_1, x_2 \in s(\Psi_u)} f(x_1, x_2) \right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x_1, x_2) \varrho_u(x_2 - x_1) \, dx_1 \, dx_2
$$

(2.2)

for any measurable function $f: \mathbb{R}^d \times \mathbb{R}^d \mapsto [0, \infty)$, exists and takes the following form, see [16], or [17, p. 38]:

$$
\varrho_u(x) = \lambda^2 \exp \{-\lambda E | \Xi_0 \cup (\Xi_0 + x) | \} f_u(x),
$$

where

$$
f_u(x) = P(-x \notin \Xi_0 - l_u(\Xi_0)) P(x \notin \Xi_0 - l_u(\Xi_0)).
$$

(2.3)

Using the well-known formula, see e.g. Stoyan et al. [18],

$$
C(x) = 1 - 2 \exp \{-\lambda E | \Xi_0 | \} + \exp \{-\lambda E | \Xi_0 \cup (\Xi_0 + x) | \}
$$

(2.4)

for the covariance $C(x) = P(o \in \Xi, x \in \Xi)$ of the stationary PGM (1.3), we may write that

$$
\varrho_u(x) = \frac{\lambda^2 (1 - 2C(o) + C(x))}{(1 - C(o))^2} f_u(x).
$$

(2.5)

In case of a spherical typical grain $\Xi_0 = B_d(o, D_0/2)$ with diameter $DF$

$$
G(t) = P(D_0 < t)
$$

it is easily verified, see Heinrich and Werner [12], that

$$
f_u(x) = \begin{cases} 
0, & \text{if } x = 0, \\
1, & \text{if } x \neq 0 \text{ and } \langle u, x \rangle = 0, \\
G(\|x\|^2/\langle u, x \rangle), & \text{if } \langle u, x \rangle \neq 0.
\end{cases}
$$

(2.6)

In particular, $G(t) = f_u(tu)$ for all $t > 0, u \in S^{d-1}$, which together with (2.5) implies that

$$
G(t) = \frac{(1 - C(o))^2 \varrho_u(tu)}{\lambda^2 (1 - 2C(o) + C(tu))} \text{ for all } t > 0 \text{ and } u \in S^{d-1}.
$$

(2.7)
Now we are in a position to derive a plug-in-estimator for $G(t)$ by substituting the covariance $C(x)$ and the intensity $\lambda_u$ in (2.7) by their empirical counterparts

$$\hat{\lambda}_{u,n} = \frac{\Psi_u(W_n)}{|W_n|} \quad (2.8)$$

and

$$\hat{C}_n(x) = \frac{|W_n \cap \Xi \cap (\Xi-x)|}{|W_n|} \text{ for } x \in \mathbb{R}^d, \quad (2.9)$$

whereas the product density $\varrho_u$ in (2.7) is replaced by the following kernel-type estimator

$$\hat{\varrho}_{u,n}(x) = \frac{1}{b_n^d |W_n|} \sum_{x_1, x_2 \in \Psi_u} 1_{W_n}(x_1) k\left(\frac{x_2 - x_1 - x}{b_n}\right), \quad x \in \mathbb{R}^d, \quad (2.10)$$

where $k: \mathbb{R}^d \rightarrow \mathbb{R}$ denotes a kernel function associated with a sequence $(b_n)_{n \geq 1}$ of bandwidths satisfying the below Conditions $(K_L)$ and $(B_\beta)$. To avoid technical difficulties, we choose $W_n = [0, n]^d$ in (2.9), (2.8) and (2.10) although more general convex compact sampling regions $W_n$ which expands unboundedly in all directions (as $n \to \infty$) are possible.

Replacing the model characteristics $\lambda_u$, (2.4) and (2.5) in (2.7) by the above estimators (2.8), (2.9) and (2.10) provides the empirical DF $\hat{G}_n$ with $\hat{G}_n(0) = 0$ and

$$\hat{G}_n(t) := \frac{(1 - \hat{C}_n(o))^2 \hat{\varrho}_{u,n}(tu)}{\hat{\lambda}_{u,n}^2(1 - 2\hat{C}_n(o) + \hat{C}_n(tu))} \quad \text{for } t > 0, \quad (2.11)$$

which turns out to be a uniformly strongly consistent estimation of $G$, see Theorem 4.1.

The asymptotic behaviour of (2.9), (2.8) and (2.10) can be studied by representing the estimators as sum over a weakly dependent random field whose dependence structure is determined by the underlying PGM (1.3). To obtain uniform $P - a.s.$ convergence of (2.9) and (2.10) we will assume that the typical grain $\Xi_0$ is bounded which implies that the corresponding random field is $m$-dependent. Under this additional assumption we can improve earlier results obtained in [10] for kernel-type estimators of the form (2.10) for
β-mixing stationary point processes Ψ (instead of Ψ_u). Moreover, we have to impose certain natural restrictions on the kernel function k and the sequence (b_n)_{n≥1}:

**Condition (K_L).** There exist some R > 0 and L ≥ 0 such that k(x) = 0 for ∥x∥ ≥ R and

\[ \int_{\mathbb{R}^d} k(x) \, dx = 1 \text{ and } |k(x) - k(y)| ≤ L \|x - y\| \text{ for all } x, y ∈ B_d(o, R). \]

**Condition (B_β).** The non-increasing sequence of bandwidths (b_n)_{n≥1} is chosen such that

\[ \lim_{n \to \infty} b_n = 0 \text{ and } \lim_{n \to \infty} n b_n^{1+β} = \infty \text{ for some } 0 < β ≤ \frac{2}{d}. \]

Heinrich and Werner [12] proved multivariate CLTs and established \( \chi^2 \)-goodness-of-fit tests for (2.5) and (2.11) under \( E|ξ_0|^2 < ∞ \).

By means of (2.11) and equation (1.2) we are now in a position to construct a stereological estimator for the (d + 1)-sphere diameter DF F. By (1.2) we may write

\[ F(r + 0) = 1 - \frac{V(r)}{V(0)} \text{ with } V(r) = \int_r^\infty \frac{dG(t)}{\sqrt{t^2 - r^2}}, \quad r ≥ 0. \]  

(2.12)

To estimate the DF F we need reasonable estimators for the improper Riemann-Stieltjes integral \( V(r), r ≥ 0 \). For this purpose we assume additionally that the DF F is continuous [0, B] and \( F(B) = 1 \) for some \( B > 0 \) which in turn implies the continuity of G on [0, B] and \( G(B) = 1 \). Let \( N_n \) be an unboundedly increasing sequence of positive integers which will be specified later. The points

\[ t_{i,n} = ih_n, \quad i = 0, 1, ..., N_n \quad \text{with} \quad h_n = \frac{B}{N_n} \]

define an equidistant partition of [0, B] and the Riemann-Stieltjes sum

\[ V_n(r) := \sum_{i=[rN_n/B]+2}^{N_n} \frac{G(t_{i,n}) - G(t_{i-1,n})}{\sqrt{t_{i,n}^2 - r^2}} \text{ for } r ∈ [0, B] \]  

(2.13)
converges to $V(r)$ as $n \to \infty$. Replacing the values $G(t_{i,n})$ in the latter sum by $\hat{G}_n(t_{i,n})$ taken from (2.11) we may define the announced stereological estimator $\hat{F}_n$ for $F$ as follows

$$
\hat{F}_n(r) := 1 - \frac{\hat{V}_n(r)}{V_n(0)} \quad \text{for} \quad 0 \leq r \leq B,
$$

where

$$
\hat{V}_n(r) := \begin{cases} 
\sum_{i=j+1}^{N_n} \frac{\hat{G}_n(t_{i,n}) - \hat{G}_n(t_{i-1,n})}{\sqrt{t_{i,n}^2 - r^2}}, & r \in [t_{j-1,n}, t_{j,n}), j = 1, \ldots, N_n - 1 \\
0, & r \in [t_{N_n-1,n}, B].
\end{cases}
$$

Our principal result on the asymptotic behaviour of $\hat{F}_n$ is the following

**Theorem 2.7.** Let Condition (Poi) be satisfied and assume in addition that $F(B) = 1$ and

$$
\sup_{0 \leq s < t \leq B} \frac{F(t) - F(s)}{(t - s)^{1+\gamma}} \leq H_\gamma < \infty \quad \text{for some} \quad 0 < \gamma \leq 1/2 \quad \text{and} \quad B > 0. \quad (2.16)
$$

Further, let the kernel function $k$ and the bandwidths $(b_n)_{n \geq 1}$ in (2.10) (defining (2.11)) satisfy the Conditions (K$_L$) and (B$_p$) and put in (2.15) $N_n = \lfloor a_n^{-2/3} \rfloor$ with $a_n = b_n^{d/2} | \log b_n |^{1/2}$. Then we have

$$
\sup_{0 \leq r \leq B} | \hat{F}_n(r) - F(r) | = O(a_n^{1/3}) \quad \text{P - a.s. as} \quad n \to \infty.
$$

3. Rates of Almost Sure Convergence for $\hat{u}_{n,n}(x)$ and $\hat{C}_n(x)$ in Poisson-Grain Models

To establish rates of the uniform P - a.s. convergence for the product density estimator (2.10) and the empirical covariance (2.9) we need the following inequality for large deviations of sums of independent random variables:
Lemma 3.1 (Fuk and Nagaev [5]). Let $Y_1, \ldots, Y_n$ be independent random variables satisfying $E|Y_i|^p < \infty$ for some real $p \geq 2$, $i = 1, \ldots, n$. Then

$$P \left( \sum_{i=1}^{n} (Y_i - EY_i) \geq \tau \right) \leq 2 \exp \left( \frac{-c_1 \tau^2}{\sum_{i=1}^{n} \text{Var}(Y_i)} \right) + c_2 \tau^{-p} \sum_{i=1}^{n} E|Y_i - EY_i|^p$$

for all $\tau > 0$, where $c_1 = 2(p + 2)^{-1} e^{-p}$ and $c_2 = 2(1 + 2/p)^p$.

We first prove the uniform strong consistency (including a convergence rate) of the product density estimator (2.10) for a general PGM (1.3) with bounded convex grains.

Theorem 3.L. Let $E$ be the stationary PGM (1.3) with convex compact typical grain $E_0$ satisfying $P (E_0 \subseteq B_d(o, B/2) = 1)$ for some $B > 0$. Let the kernel function $k$ and the bandwidths $(b_n)_{n \geq 1}$ in (2.10) satisfy the Conditions $(K_L)$ and $(B_\beta)$.

Then, for any fixed compact set $K \subset R^d$, we have, as $n \to \infty$,

$$\sup_{x \in K} |\hat{\varphi}_{u,n}(x) - E\hat{\varphi}_{u,n}(x)| = o(a_n) \text{ P - a.s. with } a_n = b_n^{d/2} |\log b_n|^{1/2}. \quad (3.1)$$

Proof of Theorem 3.1. By the very definition of P - a.s. convergence relation (3.1) holds if

$$\sum_{n \geq 1} P \left( \sup_{x \in K} |\hat{\varphi}_{u,n}(x) - E\hat{\varphi}_{u,n}(x)| \geq \tau a_n \right) < \infty \text{ for any } \tau > 0. \quad (3.2)$$

Choose $m$ to be the largest positive integer such that $q := n/2m \geq [2(B + R)] + 1$ (where $R > 0$ stems for Condition $(K_L)$) and $I_m := \{0, 1, \ldots, m-1\}^d$ (and thus $I_2 = \{0, 1\}^d$). The sampling window $W_n = [0, n]^d$ can be subdivided into $2^d m^d$ shifted cubes $W_q = [0, q]^d$ such that

$$W_n = \bigcup_{y \in I_2} \bigcup_{z \in I_m} (W_q + 2z + y) \text{ and } \hat{\varphi}_{u,n}(x) = \sum_{y \in I_2} \sum_{z \in I_m} X_{ny,z}(x).$$
where
\[ X_{n}^{y}(x) := \frac{1}{(n b_{n})^{d}} \sum_{x_{1}, x_{2} \in \Psi_{n}^{(t)}} 1_{W_{q+2z+y}(x_{1})} k\left(\frac{x_{2} - x_{1} - x}{b_{n}}\right). \] (3.3)

Since \( \Xi_{0} \subseteq B_{d}(0, B/2) \) we get the inclusion \( B_{d}(-R_{0}u, R_{0}) \subseteq B_{d}(0, B) \) for any \( u \in S^{d-1} \). This and the fact that the kernel function \( k \) vanishes outside \( B_{d}(0, R) \) together with the choice of \( q \) and the independence properties of the Poisson process \( \mathbb{P}_{\lambda} \) imply that \( X_{n}^{y}(x), z \in I_{m} \), are independent random variables for any \( y \in I_{2} \).

The compact set \( K \) can be covered by \( N \) \( d \)-dimensional cubes \( C_{1}, \ldots, C_{N} \) having edges of length \( h \) and midpoints \( u_{1}, \ldots, u_{N} \) such that \( N \leq c(K) h^{-d} \) with some constant \( c(K) > 0 \).

Let \( \tau > 0 \) be fixed. Then we have
\[
P\left( \sup_{x \in K} |\hat{\xi}_{u,n}(x) - E\hat{\xi}_{u,n}(x)| \geq \tau a_{n} \right) = P\left( \sup_{x \in K} \left| \sum_{y \in I_{2}} \sum_{z \in I_{m}} (X_{n}^{y}(x) - EX_{n}^{y}(x)) \right| \geq \tau a_{n} \right) \\
\leq \sum_{y \in I_{2}} \sum_{i=1}^{N} P\left( \sup_{x \in C_{i}} \left| \sum_{z \in I_{m}} (X_{n}^{y}(x) - EX_{n}^{y}(x)) \right| \geq \frac{\tau a_{n}}{2^{d}} \right) \\
\leq \sum_{y \in I_{2}} \sum_{i=1}^{N} (P_{i,1}^{y} + P_{i,2}^{y}), \tag{3.4}
\]

where
\[
P_{i,1}^{y} = P\left( \left| \sum_{z \in I_{m}} (X_{n}^{y}(u_{i}) - EX_{n}^{y}(u_{i})) \right| \geq \frac{\tau a_{n}}{3 \cdot 2^{d}} \right)
\]
and
\[
P_{i,2}^{y} = P\left( \sup_{x \in C_{i}} \left| \sum_{z \in I_{m}} (X_{n}^{y}(x) - X_{n}^{y}(u_{i})) \right| \geq \frac{\tau a_{n}}{3 \cdot 2^{d}} \right)
\]
provided that the inequality
\[
\sup_{x \in C_{i}} \left| \sum_{z \in I_{m}} E(X_{n}^{y}(u_{i}) - X_{n}^{y}(x)) \right| \leq \frac{\tau a_{n}}{3 \cdot 2^{d}} \quad \text{for } i = 1, \ldots, N, \ y \in I_{2} \tag{3.5}
\]
is satisfied. We first verify (3.5) for a suitably chosen \( h \).
For \( x \in C_i \) we have \( B_d(u_i, R b_n) \cup B_d(x, R b_n) \subset C_{i,n} \equiv C_i \oplus B_d(o, R b_n) \), where \( \oplus \) stands for the Minkowski sum. Condition (K1) entails that \(| X^y_{n,z}(u_i) - X^y_{n,z}(x) |\) is bounded by

\[
\frac{1}{(n b_n)^d} \sum_{x_1, x_2 \in S(\psi_u)} 1_{W_q+2z+y}(x_1) 1_{C_{i,n}}(x_2 - x_1) \left| k \left( \frac{x_2 - x_1 - u_i}{b_n} \right) - k \left( \frac{x_2 - x_1 - x}{b_n} \right) \right|
\]

\[
\leq \frac{1}{b_n} \| x - u_i \| Y^y_{n,z} \leq \frac{h \sqrt{d} L}{2 b_n} Y^y_{n,z} \quad \text{for } x \in C_i \text{ and } i = 1, \ldots, N,
\]

where

\[
Y^y_{n,z} := \frac{1}{(n b_n)^d} \sum_{x_1, x_2 \in S(\psi_u)} 1_{W_q+2z+y}(x_1) 1_{C_{i,n}}(x_2 - x_1). \tag{3.3}
\]

Inserting \( f(x_1, x_2) = 1_{W_q+2z+y}(x_1) 1_{C_{i,n}}(x_2 - x_1) \) in (2.2) and using that \( \psi_u(x) \leq \lambda^2 \) (which follows from (2.4) and (2.5)) gives

\[
\mathbb{E} Y^y_{n,z} = \frac{q^d}{(n b_n)^d} \int_{C_{i,n}} \varrho_u(x) dx \leq \frac{q^d}{(n b_n)^d}\lambda^2 |C_{i,n}| \leq \frac{\lambda^2 q^d}{(n b_n)^d} (h + 2R b_n)^d, \tag{3.7}
\]

and this leads to

\[
\sum_{z \in I_m} \mathbb{E} \sup_{x \in C_i} | X^y_{n,z}(u_i) - X^y_{n,z}(x) | \leq \frac{\lambda^2 h L \sqrt{d}}{2 b_n} \left( \frac{m q (h + 2R b_n)}{n b_n} \right)^d.
\]

Having in mind that \( q = n/2m \) and choosing

\[
h = \varepsilon_0 \tau a_n b_n \quad \text{with } \varepsilon_0 = \frac{1}{3\lambda^2 L \sqrt{d} (2R + 1)^d}, \tag{3.8}
\]

we see that the rhs in the foregoing line is less than \( \tau a_n/6 \cdot 2^d \) for \( n \geq n_0(\tau) = \min \{ k \geq 1 : \varepsilon_0 \tau a_k < 1 \} \). Thus, (3.5) holds for \( n \geq n_0(\tau) \).

Applying Lemma 3.1 to the sequence \( X^y_{n,z}(u_i), z \in I_m \), and using the elementary inequality

\[
\mathbb{E} | X - \mathbb{E} X |^p \leq 2^p \mathbb{E} | X |^p
\]
we obtain that, for any $p \geq 2$ and $i = 1, \ldots, N$,

$$P_{i,1}^y \leq 2\exp\left\{-\frac{c_1 \tau^2 \alpha_n^2}{9 \cdot 4^d} \sum_{z \in I_m} \text{Var}(X_n^y(u_i)) + c_2 \left(\frac{6 \cdot 2^d}{\tau \alpha_n}\right)^p \sum_{z \in I_m} E|X_n^y(u_i)|^p \right\}. \quad (3.9)$$

By the stationarity of $\Psi_n$, Condition (KL) and the moment inequality $E\Psi^s_u(\cdot) \leq \mathbb{E}\prod_{\lambda}^s(\cdot)$ for any $s > 0$ (which is immediately seen from (2.1)) we get that

$$E\left|X_n^y(x)\right|^p \leq \frac{M^p}{(nb_n)^{p|d|}} \mathbb{E}\left(\sum_{x_1 \in \Psi_u} 1_{W_{q+2z+y}(x_1)^d} \Psi_u(B_d(x + x_1, Rb_n))\right)^p \leq \frac{M^p(|q| + 1)^{p|d|}}{(nb_n)^{p|d|}} \mathbb{E}\Psi^p_u([0, 1]^d) \mathbb{E}\prod_{\lambda}^{2p}([0, 1]^d \oplus B_d(0, Rb_n)) \leq \frac{M^p(2q)^{p|d|}}{(nb_n)^{p|d|}} \mathbb{E}\prod_{\lambda}^{2p}([0, 1]^d \oplus B_d(0, Rb_n)) \leq c_3 \left(\frac{q}{nb_n}\right)^{p|d|}$$

with some constant $c_3 > 0$ depending on $R, \lambda, d, p$ and $M := \sup_{n \in N} \mathbb{E}\left|\Psi_u(x)\right|$. Therefore

$$\sum_{z \in I_m} E\left|X_n^y(x)\right|^p \leq c_3 m^{d/p} \left(\frac{q}{nb_n}\right)^{p|d|} = c_3 \left(\frac{q}{nb_n}\right)^{(p-1)d} \left(2b_n\right)^{-d}. \quad (3.10)$$

As a by-product of the proof of Theorem 3.1 in [12] we can state the following inequality:

$$\text{Var}\left((nb_n)^d X_n^y(x)\right) \leq 2\lambda^2 |W_q| b_n^d \int k^2(y) dy + 4\lambda^3 |W_q| b_n^{2d} (1 + 2\lambda \mathbb{E}|\Xi_0|$$

$$+ \lambda^2 \mathbb{E}|\Xi_0|^2) \left(\int |k(y)|^2 dy\right)^2, \quad (3.11)$$

which is even valid for any function $k \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Hence,

$$\sum_{z \in I_m} \text{Var}(X_n^y(x)) \leq \frac{c_4}{(nb_n)^d}. \quad (3.12)$$
In view of (38) we may take \( N = \lfloor c(K)(\epsilon_0 \tau a_n b_n)^{-d} \rfloor \) which together with (39) leads to

\[
\sum_{y \in I_2} \sum_{i=1}^N P_{i,y} \leq \frac{c_5(\tau)}{(a_n b_n)^d} \exp \left\{ -c_6(\tau) a_n^2 (n b_n)^d \right\} + \frac{c_7(\tau)}{a_n^{p+d} b_n^{(p+1)d} n^{(p-1)d}}. \tag{3.13}
\]

According to Condition (Bp) we may assume that \( n^{-1/(1+\beta)} \leq b_n \leq e^{-1} \). Inserting \( a_n = b_n^{\beta d/2} \left| \log b_n \right|^{1/2} \) on the rhs of (3.13) we see after a short calculation that

\[
\frac{\exp \left\{ -c_6(\tau) a_n^2 (n b_n)^d \right\}}{(a_n b_n)^d} \leq b_n^{c_8(\tau) (n b_n^{1+\beta})^d - 2d} \leq \begin{cases} e^{c_6(\tau) n^{d/2 + 2d}} & \text{if } n b_n^{1+\beta} \geq \sqrt{n} \\ - \frac{c_6(\tau)}{n} n^{(1+\beta) + d} & \text{if } n b_n^{1+\beta} \leq \sqrt{n} \end{cases}
\]

and

\[
\frac{1}{a_n^{p+d} b_n^{(p+1)d} n^{(p-1)d}} \leq \frac{1}{(b_n^{1+p+\beta(p+d)/2})^{2n^{p-1}}} \leq \frac{d}{n^{1+\beta(1+p+\beta(p+d)/2) - d(p-1)}} \leq \frac{1}{n^2}
\]

for \( p \geq d + 2 + \frac{4(1 + \beta + d)}{d\beta} \). Hence, if \( p \) is chosen in this way the rhs of (3.13) is less than \( c_8(\tau)n^{-2} \) for sufficiently large \( n \). Therefore, by (34), the convergence of the series (3.2) is guaranteed whenever \( \sum_{y \in I_2} \sum_{i=1}^N P_{i,y} \) does not exceed the rhs of (3.13) for large enough \( n \).

For \( n \geq n_0(\tau) \) we find from (3.7) and (3.8) that

\[
\sum_{z \in I_m} \mathbb{E} Y_{nz}^z \leq \chi^2(2R + 1)^d 2^{-d} = \frac{1}{3 \cdot 2^d \sqrt{d} L \epsilon_0}
\]

and hence

\[
\max_{1 \leq i \leq N} P_{i,2} \leq P \left( \frac{Lh \sqrt{d}}{2 b_n} \sum_{z \in I_m} Y_{nz}^z \geq \frac{\tau a_n}{3 \cdot 2^d} \right)
\]

\[
\leq P \left( \sum_{z \in I_m} \mathbb{E} (Y_{nz}^z - \mathbb{E} Y_{nz}^z) \geq \frac{1}{3 \cdot 2^d \sqrt{d} L \epsilon_0} \right).
\]
Applying Lemma 3.1 to the rhs of the latter inequality yields

\[ P_{i,2}^r \leq 2 \exp \left\{ -c_1 \left( \frac{3 \cdot 2^d \sqrt{d} L \epsilon_0 \lambda}{\sum_{z \in I_m} \text{Var}(Y_{nz}^r)} \right)^2 \right\} + c_2 \left( 6 \cdot 2^d \sqrt{d} L \epsilon_0 \right)^p \sum_{z \in I_m} \mathbb{E}(Y_{nz}^r)^p \]

for \( i = 1, \ldots, N \). To estimate \( \text{Var}(Y_{nz}^r) \) we use (3.11) for \( x = 0 \) and \( k(y) = 1_{C_{i,n}}(yb_n) \). By \( \int k(y)dy = |C_{i,n}| b_n^{-d} \leq (1 + 2R)^d \) for \( n \geq n_0(\tau) \) we get in analogy to (3.12) and (3.10) that

\[ \sum_{z \in I_m} \text{Var}(Y_{nz}^r) \leq \frac{c_8}{(n b_n)^d} \text{ and } \mathbb{E}(Y_{nz}^r)^p \leq c_9 \left( \frac{q}{n b_n} \right)^{pd}. \] \hspace{1cm} (3.14)

Thus,

\[ \sum_{z \in I_m} \sum_{i=1}^{N} P_{i,2}^r \leq \frac{c_5(\tau)}{(a_n b_n)^d} \exp \{-c_8(\tau)(n b_n)^d\} + \frac{c_9(\tau)}{a_n b_n^{(p+1)d} n^{(p-1)d}}. \] \hspace{1cm} (3.15)

Comparing (3.13) and (3.15) reveals that the rhs of (3.13) is larger than that of (3.15) (at least for large \( n \)) which completes the proof of Theorem 3.1.

Corollary 3.1. If in addition the function (2.3) is Lipschitz continuous on \( K \oplus B_d(0, \varepsilon) \) for some \( \varepsilon > 0 \), i.e., \( |f_u(x) - f_u(y)| \leq L^* \| x - y \| \) for \( x, y \in K \oplus B_d(0, \varepsilon) \), then we have

\[ \sup_{x \in K} |\hat{\varphi}_{u,n}(x) - \varphi_{u}(x)| = o(a_n) \text{ \( \mathbb{P} \)-a.s. as } n \to \infty. \]

Proof of Corollary 3.1. It remains to show

\[ \sup_{x \in K} |\mathbb{E}\hat{\varphi}_{u,n}(x) - \varphi_{u}(x)| = o(a_n) \text{ as } n \to \infty. \]

By means of (2.2) with \( f(x_1, x_2) = (nb_n)^{-d} W_n(x_1) k((x_2 - x_1 - x)/b_n) \) we get

\[ \mathbb{E}\hat{\varphi}_{u,n}(x) - \varphi_{u}(x) = \int k(y)(\varphi_{u}(x + b_n y) - \varphi_{u}(x))dy. \]

Using the estimate

\[ |\varphi_{u}(x) - \varphi_{u}(y)| \leq \lambda^2 (|f_u(x) - f_u(y)| + 2 \lambda \mathbb{E} \| \Xi_0 \oplus B_d(0, 1) \| \| x - y \|). \] \hspace{1cm} (3.16)
which has been proved in Heinrich and Werner [12], Lemma 3.1, and the Lipschitz condition on \( f_u() \) we arrive at

\[
\sup_{x \in K} |E\hat{c}_{u,n}(x) - c_u(x)| = c_{10} b_n \int |k(y)| \|y\| \, dy \text{ if } Rb_n \leq \varepsilon,
\]

where \( c_{10} \) depends on \( \lambda, L^*, B \) and \( d \). Condition \((B_\beta)\) implies that

\[
b_n = a_n b_n^{1-d\beta/2} |\log b_n|^{-1/2} = o(a_n) \text{ as } n \to \infty
\]

proving Corollary 3.1.

The next theorem establishes rates of \( P \)-a.s. convergence of the intensity estimator and the empirical covariance \( \hat{C}_n(x) \), see (2.8) and (2.9).

**Theorem 3.2.** Let \( \Xi \) be the stationary PGM (1.3) with convex compact typical grain \( \Xi_0 \) satisfying \( P(\Xi_0 \subseteq B_d(o, B/2)) = 1 \) for some \( B > 0 \). Then, for any fixed compact set \( K \subset R^d \) and \( u \in S^{d-1} \), we have

\[
\max_{x \in K} \left\{ |\hat{\lambda}_{u,n} - \lambda_u|, \sup_{x \in K} |\hat{C}_n(x) - C_n(x)| \right\} = o(n^{-d/2} \log n) \text{ } P \text{-a.s. as } n \to \infty.
\]

**Proof of Theorem 3.2.** The idea of this proof is the same as in the foregoing proof of Theorem 3.1. Therefore, we only sketch the essential steps and point out some estimates which differ from those derived in the above proof. First note that the estimators \( \hat{\lambda}_{u,n} \) and \( \hat{C}_n(x) \) are unbiased and that the covariance (2.4) is Lipschitz continuous on \( R^d \). More precisely,

\[
|C(x) - C(y)| \leq \lambda \exp\{-\lambda E|\Xi_0||E|\Xi_0 \cup (\Xi_0 - x)| - E|\Xi_0 \cup (\Xi_0 - y)||
\]

\[
\leq 2\lambda \exp\{-\lambda E|\Xi_0||E|\Xi_0 \oplus B_d(o, 1)||x - y|| \text{ for } x, y \in R^d,
\]

which can be derived from (2.4) in the same way as (3.16). For proving the \( P \)-a.s. convergence \( n^{d/2} \sup_{x \in K} |\hat{C}_n(x) - C(x)|/\log n \longrightarrow 0 \) (resp. \( n^{d/2} |\hat{\lambda}_{u,n} - \lambda_u|/\log n \longrightarrow 0 \)) we again split \( W_n \) into \( 2^d m^d \) \( d \)-cubes \( W_q + 2z + y \) and cover the compact set \( K \) by \( d \)-cubes \( C_i = \left[-\frac{h}{2}, \frac{h}{2}\right]^d + u_i, i = 1, \ldots, N \) such that \( N \leq c(K)h^{-d} \). In contrast to (3.8) we put \( h = \varepsilon_1 n^{-d/2} \log n \) for certain
\( \varepsilon_1 > 0 \). And repeat step by step the proof of Theorem 3.1 with

\[
\bar{X}_{n,z}^\gamma(x) := \frac{|(W_q + 2z + y) \cap \Xi \cap (\Xi - x)|}{|W_n|} \quad \text{resp.} \quad \bar{X}_{n,z}^\gamma(x) := \frac{\Psi_u(W_q + 2z + y)}{|W_n|}
\]

instead of \( X_{n,z}^\gamma(x) \) and the new sequences \( a_n := n^{-d/2} \log n \) and \( b_n := 1 \).

For \( u, x \in C_i, i = 1, \ldots, N \), we can verify that

\[
|x - u| \leq 2 \|B_d(o, B/2) \oplus B_d(o, 1)|\bar{Y}_{n,z}^\gamma
\]

with \( \bar{Y}_{n,z}^\gamma := n^{-d} \prod_{\lambda} ((W_q + 2z + y) \oplus B_d(o, B/2) \oplus C_i) \) and, in analogy to (3.10) and (3.12), that

\[
\sum_{x \in I_m} \text{Var}(\bar{X}_{n,z}^\gamma(x)) \leq \frac{c_{11}}{n^d} \quad \text{and} \quad \sum_{x \in I_m} \mathbb{E} |\bar{X}_{n,z}^\gamma(x)|^p \leq \frac{c_{12}(p,q)}{n^{d(p-1)}} \quad \text{for any} \quad p \geq 2.
\]

Corresponding estimates (with other constants) hold for the random variables \( \bar{Y}_{n,z}^\gamma, z \in I_m \). Finally, applying Lemma 3.1 to the i.i.d. random variables \( \{\bar{Y}_{n,z}^\gamma(x), z \in I_m\} \) and \( \{\bar{Y}_{n,z}^\gamma, z \in I_m\} \) for \( y \in I_2 \) and taking \( p \) large enough, we recognize that

\[
\sum_{n \geq 1} \mathbb{P}\left( \sup_{x \in K} |\hat{C}_n(x) - C(x)| \geq \tau n^{-d/2} \log n \right) + \sum_{n \geq 1} \mathbb{P}\left( |\hat{\lambda}_{u,n} - \lambda_u| \geq \tau n^{-d/2} \log n \right) < \infty
\]

for any \( \tau > 0 \) proving the assertion of Theorem 3.2.

### 4. Rates of Uniform Strong Consistency of the Empirical Diameter Distribution Function \( \hat{G}_n(t) \)

In this section we consider the \( d \)-dimensional PGM (1.3) with spherical grains determined by the common diameter DF \( G \) and study the \( \mathbb{P} - \text{a.s.} \) behaviour of empirical diameter DF \( \hat{G}_n \) defined by (2.11) as \( n \to \infty \). To establish uniform consistency over some interval \([a, b]\) we need the Lipschitz continuity of the product density \( \varrho_u(\cdot) \) in some neighbourhood of the line segment \( \{tu : a \leq t \leq b\} \) as seen from Corollary 3.1. This in turn is a consequence of the Lipschitz continuity of the DF \( G \).
Lemma 4.1. Let $\Xi$ be the stationary PGM (1.3) with typical grain $\Xi_0 = B_d(o, D_0/2)$ satisfying $\Xi D_0^d < \infty$. Further, let the diameter DF $G(r) = P(D_0 < r)$ be Lipschitz continuous on some interval $[a, b]$, i.e.,

$$G(t) - G(s) \leq L_0(t - s) \text{ for } 0 \leq a \leq s < t \leq b < \infty. \quad (4.1)$$

Then, for any $y \in B_d(o, (b - a)/2)$,

$$|f_u(tu + y) - f_u(tu)| \leq L_0 \|y\| \text{ for } t \in [a + \|y\|, b - \|y\|]. \quad (4.2)$$

Proof of Lemma 4.1. An elementary calculation reveals that

$$a \leq \frac{\|tu + y\|^2}{t + \langle y, u \rangle} \leq b \text{ for any } t \in [a + \|y\|, b - \|y\|].$$

From (2.6) and (4.1) we obtain that

$$|f_u(tu + y) - f_u(tu)| \leq L_0 \left| \frac{\|tu + y\|^2}{t + \langle y, u \rangle} - t \right| = L_0 \left| \frac{\|y\|^2 + t\langle y, u \rangle}{t + \langle y, u \rangle} \right|.$$

We complete the proof of (4.3) by noting that, for $t \geq \|y\|$, the function

$$h(x) := \frac{\|y\|^2 + t \cdot x}{t + x}$$

is non-decreasing on $[-\|y\|, \|y\|]$ with $h(-\|y\|) = -\|y\|$ and $h(\|y\|) = \|y\|$.

The following theorem establishes a rate of the P - a.s. convergence of the uniform distance $\sup_{a + \varepsilon \leq t \leq b - \varepsilon} |\hat{G}_n(t) - G(t)|$ to zero (for any $0 < \varepsilon < (b - a)/2$) by combining the results of the preceding Section 3 and the just proved Lemma 4.1.

Theorem 4.1. Let $\Xi$ be the stationary PGM in (1.3) with typical grain $\Xi_0 = B_d(o, D_0/2)$ satisfying $P(D_0 < B) = 1$ for some $B > 0$. Further, let the diameter DF $G(t) = P(D_0 < t)$ satisfy the Lipschitz condition (4.1) on some interval $[a, b] \subseteq [0, B]$ and let the kernel function $k$ defining the product density estimator (2.10) satisfy the Conditions $(K_L)$ and $(B_\beta)$.

Then, for the empirical DF $\hat{G}_n$ defined by (211), we have, as $n \to \infty$,

$$\sup_{a + Rb_n \leq t \leq b - Rb_n} |\hat{G}_n(t) - G(t)| = o(a_n) \text{ P - a.s. with } a_n = b_n^{d\beta/2} |\log b_n|^{1/2}. \quad (4.3)$$
Moreover, if $G$ satisfies (4.1) for $a = 0$, $b = B$, then we have, as $n \to \infty$,

$$\sup_{0 \leq t \leq B} |\hat{G}_n(t) - G(t)| = \begin{cases} o(a_n) & \text{if } d = 1 \text{ or } 0 < \beta < 2/d \\ \mathcal{O}(b_n \log b_n) & \text{if } d \geq 2 \text{ and } \beta = 2/d \end{cases} \quad \text{P-a.s. (4.4)}$$

**Proof of Theorem 4.1.** By Condition $(B_\beta)$, we get $e \leq b_n^{-1} \leq n^{1/(1+\beta)}$ for sufficiently large $n$ so that

$$\frac{\log n}{a_n n^{d/2}} = \frac{\log n}{n^{d/2} b_n \beta / 2} \leq n^{-d/2(1+\beta)} \log n \to 0.$$ 

Therefore, by Theorem 3.2, we get the P-a.s. limits

$$\sup_{t \in [a, b] \cup \{0\}} |\hat{G}_n(tu) - C(tu)| = o(a_n) \quad \text{and} \quad \lambda_{u,n}^2 = \lambda_u^2 = o(a_n) \quad \text{as } n \to \infty.$$ 

From (3.16) and (4.2), it follows that $|E\hat{u}_u(tu) - e_u(tu)|$ is bounded by

$$2\lambda^3 E|\Xi_0 \ominus B_d(o,1)| \int |k(y)| y dy b_n + \lambda^2 \int |k(y)| f_u(tu + b_n y) - f_u(tu) dy \leq c_1 b_n$$

for $a + Rb_n \leq t \leq b - Rb_n$. This together with Theorem 3.1 for $K = \{tu : a \leq t \leq b\}$ implies

$$\sup_{a + Rb_n \leq t \leq b - Rb_n} |\hat{u}_u(tu) - e_u(tu)| = o(a_n) \quad \text{as } n \to \infty.$$ 

Finally, the above asymptotic relations immediately imply that

$$\frac{1}{a_n} \sup_{a + Rb_n \leq t \leq b - Rb_n} \left| \frac{(1 - \hat{C}_n(o))^2 \hat{u}_u(tu)}{\lambda_{u,n}^2 (1 - 2\hat{C}_n(o) - \hat{C}_n(tu))} - \frac{(1 - C(o))^2 e_u(tu)}{\lambda_u^2 (1 - 2C(o) - C(tu))} \right| \to 0 \quad \text{P-a.s.}$$

Hence, (4.3) is proved. To prove the second assertion we distinguish several cases. For $d = 1$ we have $\langle u, y \rangle = u \cdot y$ with $u \in \{-1, 1\}$ giving

$$|f_u(tu + y) - f_u(tu)| = |G(\langle t - uy \rangle) - G(t)| \leq L_0 |y| \quad \text{for } 0 \leq t \leq B$$

provided (4.1) holds for $a = 0$, $b = B$. Thus, the supremum in (4.3) can be taken over the whole interval $[0, B]$. The case $d \geq 2$ is somewhat more
intricate. We first realize that the supremum in (4.3) can be extended over $t \in [Rb_n, B^*]$ for any $B^* \geq B$, if $F_d$ is Lipschitz continuous on $[0, \infty)$. Thus, we only need to prove that

$$\sup_{0 \leq t \leq Rb_n} \int_{B_d(o, R)} |f_u(tu + b_n y) - f_u(tu)| \, dy \leq c_{14}(R)b_n |\log b_n|^2. \quad (4.5)$$

Using (2.6), the identity $\|tu + y\|^2/|\langle u, tu + y \rangle| = |t + \langle u, y \rangle| + (\|y\|^2 - \langle u, y \rangle^2)/|t + \langle u, y \rangle|$ and the Lipschitz condition (4.1) for $a = 0, b = B$, we get that

$$\int_{B_d(o, R)} |f_u(tu + b_n y) - f_u(tu)| \, dy \leq 3L_0R |B_d(o, R)| b_n + b_n^{-d} \int_{B_d(o, Rb_n)} G \left( \frac{\|y\|^2 - \langle u, y \rangle^2}{|t + \langle u, y \rangle|} \right) \, dy$$

for $0 \leq t \leq Rb_n$. Since the integral on the rhs does not depend on $u \in S^{d-1}$, we may take $u = (0, \ldots, 0, 1)$ and so by introducing $d$-dimensional spherical coordinates it can be rewritten as follows

$$\omega_{d-1} \int_0^\pi \int_0^{Rb_n} \rho^{d-1} G \left( \frac{\rho^2 \sin^2 \phi}{|t + \rho \cos \phi|} \right) \cos \phi d\rho d\phi,$$

where $\omega_{d-1}$ denotes the surface area of $S_{d-2}$. By splitting $\int_0^\pi = \int_0^{\pi/2 - b_n} + \int_{\pi/2 - b_n}^{\pi/2 + b_n} + \int_{\pi/2 + b_n}^\pi$, we find that the latter integral is less than

$$c_{15}(R)b_n^{d+1} + \int_{b_n}^{\pi/2} \int_0^{Rb_n} \rho^{d-1} G \left( \frac{\rho^2 \cos^2 \phi}{|t + \rho \sin \phi|} \right) \, d\rho d\phi$$

$$+ \int_{b_n}^{\pi/2} \int_0^{Rb_n} \rho^{d-1} G \left( \frac{\rho^2 \cos^2 \phi}{|t - \rho \sin \phi|} \right) \, d\rho d\phi.$$

Let $I_n^{(1)}(t)$ resp. $I_n^{(2)}(t)$ denote the first resp. second integral in the last line.

Then, by $G(t) \leq L_0 t$,

$$I_n^{(1)}(t) \leq \int_{b_n}^{\pi/2} \int_0^{Rb_n} \rho^{d-1} G \left( \frac{\rho \cos \phi}{\sin \phi} \right) \, d\rho d\phi \leq \frac{L_0 R^{d+1}}{d + 1} b_n^{d+1} |\log (\sin b_n)| \quad (4.6)$$
for $0 \leq t \leq R_b$. On the other hand, by $G(t) \leq 1$,

$$I_n^{(2)}(t) \leq \frac{\pi b_n^{d+1}}{2d} + \int_{b_n}^{R_b} \int_{b_n^{d+1}/d}^{R_b} \rho^{d-1} G\left(\frac{\rho^2 \cos \phi}{|t - \rho \sin \phi|}\right) d\rho d\phi.$$ 

For brevity let $J_n(t)$ denote the integral in the last line. For $0 \leq t < b_n^{(2d+1)/d}/\pi$, we apply Jordan's inequality $\sin x/x \geq 2/\pi$ for $0 \leq x \leq \pi/2$ and so, in analogy to (4.6), we get that

$$J_n(t) \leq L_0 R_{d+1} b^{-d+1}_n |\log (\sin b_n)|.$$ 

For $b_n^{(2d+1)/d}/\pi \leq t \leq R_b$, we may write

$$J_n(t) \leq t^d \int_{b_n}^{R_b/t} \int_{b_n^{(d+1)/d}/t}^{R_b/t} \tau^{d-1} F_d\left(\frac{t \tau^2 \cos \phi}{|1 - \tau \sin \phi|}\right) d\tau d\phi.$$ 

Next, we divide the domain of integration $[b_n, \pi/2] \times [b_n^{(d+1)/d}/t, R_b t^{-1}]$ into three parts according to $|1 - \tau \sin \phi| \leq b_n$, $\tau \leq (1 - b_n)/\sin \phi$ and $\tau \geq (1 + b_n)/\sin \phi$ and replace $F_d(t)$ by $\min \{L_0 t, 1\}$. An elementary, but somewhat lengthy evaluation of the three integrals yields the estimate

$$J_n(t) \leq c_{16}(R)b_n^{d+1} |\log b_n|^2,$$

which combined with the above estimates confirms (4.5). Together with the first part of the proof we finally obtain (4.4).

**Remark.** The case $d = 1$ is of particular interest for non-parametric estimation of the service-time DF $G$ in an $M/G/\infty$-queueing model. It is obvious (see Hall [8]) that the busy periods of this queueing system form a stationary PGM in $\mathbb{R}^1$ with $\Xi = [0, D_0]$, where $D_0$ stands for the typical service time. An observation of the idle and busy periods (called 'spacings' and 'clumps' in [8]) including their endpoints in $[0, n]$ enables us to compute the empirical service time DF (2.11). An alternative approach to statistical inference of $M/G/\infty$-queueing is discussed in Bingham and Pitts [1].
5. Proof of the Main Theorem

We first show the Lipschitz continuity of the common diameter DF $G$ of the $d$-spheres in the stationary PGM induced in the intersection hyperplane $H_d = \mathbb{R}^d \times \{0\}$ under some smoothness and moment conditions on $F$. It should be mentioned that $G$ always possesses a possibly unbounded probability density function

$$g(r) = \frac{r}{\Delta} \int_r^\infty \frac{dF(t)}{\sqrt{t^2 - r^2}} \text{ for } r \geq 0,$$

which is seen from (1.1) and the subsequent short calculation using Fubini's theorem

$$\Delta \int_0^x g(r)dr = \int_0^x \int_r^\infty \frac{r}{\sqrt{t^2 - r^2}} dF(t)dr$$

$$= \int_0^x \int_0^t \frac{r}{\sqrt{t^2 - r^2}} dr dF(t) + \int_x^\infty \int_0^x \frac{r}{\sqrt{t^2 - r^2}} dr dF(t)$$

$$= \int_0^x t dF(t) + \int_x^\infty (t - \sqrt{t^2 - x^2}) dF(t) = \Delta G(x) \text{ for any } x > 0.$$

**Lemma 5.1.** Let the DF $F$ in relation (1.1) satisfy the Hölder condition

$$\sup_{0 < \gamma \leq \frac{1}{2}} \frac{F(t) - F(s)}{(t-s)^{1+\gamma}} \leq H_{\gamma, \delta} < \infty \text{ for some } 0 < \gamma \leq \frac{1}{2}, \delta > 0, \quad (5.2)$$

and $0 \leq a < b < \infty$. Then the DF $G$ defined by (1.1) is Lipschitz continuous on $[a, b]$, i.e.,

$$G(t) - G(s) \leq L_{\gamma, \delta} (t-s) \text{ for } a \leq s < t \leq b,$$

where

$$L_{\gamma, \delta} = \frac{1}{\Delta} \left(1 + H_{1+2\gamma, \delta} \frac{\delta/2}{\sqrt{2}\gamma} \min \left\{ \sqrt{b}, M_{\gamma, \delta}^{1+2\gamma} \right\} + \frac{\Delta}{2\delta} \right)$$

with

$$M_{\gamma} := \int_0^\infty t^{(1+2\gamma)/2\gamma} dF(t).$$
Proof of Lemma 5.1. The Lipschitz continuity of $G$ expressed by (5.3) is equivalent to the uniform boundedness of the density (5.1) by $L_{\gamma, \delta}$ on $[a, b]$.

Using the identity
\[
\frac{z}{\sqrt{z^2 - s^2}} = 1 + s^2 \int_s^\infty \frac{dx}{(x^2 - s^2)^{3/2}} \text{ for } z > s \geq 0
\]
and $F(s + 0) = F_s$ for $s \in [a, b]$ ($F$ is even Hölder continuous on $[a, b + \delta]$), we find by applying Fubini's theorem that
\[
\int_s^\infty \frac{z}{\sqrt{z^2 - s^2}} dF(z) = 1 - F(s) + s^2 \int_s^\infty \frac{F(x) - F(s)}{(x^2 - s^2)^{3/2}} dx.
\]

We now show that the improper integral on the rhs exists for $s \in [a, b]$. As a consequence of (5.2) combined with the inequality $s^v(1 - F(s)) \leq \min \{b^v, \int_a^b x^v dF(x)\}$ for $s \in [a, b], v > 0$, we may estimate as follows

\[
I_1(s) := s^2 \int_s^{s + \delta} \frac{F(x) - F(s)}{(x^2 - s^2)^{3/2}} dx \leq \int_s^{s + \delta} \frac{\sqrt{s}(F(x) - F(s))}{2\sqrt{2}(x - s)^{3/2}} dx \leq H_{\gamma, \delta}^{1+2\gamma} \int_s^{s + \delta} \sqrt{s}(F(x) - F(s))^{\gamma/(1+2\gamma)} \frac{(x - s)^{(1+\gamma)/2}}{2\sqrt{2}(x - s)^{3/2}} dx \leq \frac{1+\gamma}{\sqrt{2}\gamma} \min \left\{ \frac{\gamma}{\gamma}, M_{\gamma}^{1+2\gamma} \right\}.
\]

To estimate the remaining integral we use that $\sqrt{s}(1 - F(s)) \leq \int_s^\infty \sqrt{x} dF(x) \leq \sqrt{\Delta} \leq \frac{\sqrt{5(1 - F(s))}}{\sqrt{2\delta}} \leq \frac{\sqrt{\Delta}}{\sqrt{2\delta}}$. 

\[
I_2(s) := s^2 \int_s^\infty \frac{F(s) - F(x)}{(x^2 - s^2)^{3/2}} dx \leq s^2(1 - F(s)) \int_0^\infty \frac{dx}{x^{3/2}(x + 2s)^{3/2}} \leq \frac{\sqrt{5(1 - F(s))}}{\sqrt{2\delta}} \leq \frac{\sqrt{\Delta}}{\sqrt{2\delta}}.
\]
The above estimates and
\[ \Delta g(s) \leq \int_{s}^{\infty} \frac{z}{\sqrt{z^2 - s^2}} dF(z) = 1 - F(s) + I_1(s) + I_2(s) \]
provide the estimate \( g(s) \leq L_{\gamma, \delta} \) for \( a \leq s \leq b \) which is exactly the same as (5.3).

**Lemma 5.2.** Let the DF \( F \) satisfy the Hölder condition (5.2) for \( a = 0 \) and \( b = \delta \). Then
\[ \frac{g(r)}{\sqrt{r}} \leq \frac{1}{\Delta} \left( \sqrt{\frac{r}{\delta}} + H_{\gamma, \delta} \frac{\delta^{\gamma}}{\sqrt{2\gamma}} \right) \text{ for } 0 < r \leq \delta. \] (5.4)

**Proof of Lemma 5.2.** Since
\[ \frac{1}{\sqrt{z^2 - r^2}} = \int_{z}^{\infty} \frac{s}{(s^2 - r^2)^{3/2}} ds \text{ for } z > r \geq 0 \] (5.5)
and \( F(r + 0) = F(r) \) for \( r \in [0, 2\delta] \) we may write again by using Fubini's theorem
\[ \int_{r}^{t} \frac{dF(z)}{\sqrt{z^2 - r^2}} = \frac{F(t) - F(r)}{\sqrt{t^2 - r^2}} + \int_{r}^{t} \frac{s(F(s) - F(r))}{(s^2 - r^2)^{3/2}} ds \] (5.6)
for \( r \in (0, 2\delta) \) and \( t > r \).

The Hölder condition (5.2) for \( a = 0 \) and \( b = \delta \) and the identity (5.5) give
\[ \int_{r}^{\infty} \frac{dF(z)}{\sqrt{z^2 - r^2}} = \int_{r}^{r+\delta} \frac{s(F(s) - F(r))}{(s^2 - r^2)^{3/2}} ds + \int_{r+\delta}^{\infty} \frac{s(F(s) - F(r))}{(s^2 - r^2)^{3/2}} ds \leq H_{\gamma, \delta} \int_{r}^{r+\delta} \frac{ds}{(s - r)^{1-\gamma}(s + r)^{1/2}} + \int_{r+\delta}^{\infty} \frac{s}{(s^2 - r^2)^{3/2}} ds \leq H_{\gamma, \delta} \frac{\delta^{\gamma}}{\sqrt{2r}} + \frac{1}{\sqrt{(r + \delta)^2 - r^2}}. \]

Thus, the desired estimate (5.4) is immediately seen from (5.1).

Now we are ready to prove our main result.

**Proof of Theorem 2.1.** In the first step we show that
\[ \sup_{0 \leq r \leq B} |V_n(r) - V(r)| = O(h_n^{1/2}) \text{ as } n \to \infty, \] (5.7)
where the Riemann-Stieltjes sum \( V_n(r) \) is defined by (2.13) with \( h_n = \)
\[ B/|a_n^{-2/3}| \text{ and } t_{j,n} = jh_n. \] To simplify the notation write \( i(r) = \lfloor rN_n/B \rfloor + 2 \) and \( t_j \) instead of \( t_{j,n} \).

Then, by an obvious splitting of the integral (2.12),

\[
|V_n(r) - V(r)| \leq \sum_{j=i(r)+1}^{N_n} \left( \int_{t_{j-1}}^{t_j} \frac{dG(t)}{\sqrt{t^2 - r^2}} - \frac{G(t) - G(t_{j-1})}{\sqrt{t_j^2 - r^2}} \right) + \frac{G(t_{i(r)}) - G(t_{i(r)} - 1)}{\sqrt{t_{i(r)}^2 - r^2}} + \int_r^{t_{i(r)}} \frac{dG(t)}{\sqrt{t^2 - r^2}}. \tag{5.8}
\]

Note that each summand on the rhs of (5.8) is non-negative. We first treat the case \( 0 \leq r \leq \min \{1/2, B\} \). By the Hölder condition (2.16) we see from Lemma 5.2 that

\[ g(t) \leq c(\gamma, B)\sqrt{t} \text{ for } 0 \leq t \leq 1 \text{ and } g(t) = 0 \text{ for } t \geq B. \tag{5.9} \]

Thus,

\[
\int_r^{t_{i(r)}} \frac{dG(t)}{\sqrt{t^2 - r^2}} \leq c(\gamma, B) \int_r^{t_{i(r)}} \frac{\sqrt{t}}{(t-r)(t+r)} \, dt \\
\leq 2c(\gamma, B)\sqrt{t_{i(r)} - r} \\
\leq 2\sqrt{2} c(\gamma, B)h_n^{1/2}.
\]

Further, since \( ti_{i(r)-1} \geq r \) and \( t \mapsto (t^2 - r^2)^{-1/2} \) is strictly increasing for \( t > r \), we obtain that

\[
\frac{G(t_{i(r)}) - G(t_{i(r)} - 1)}{\sqrt{t_{i(r)}^2 - r^2}} \leq \int_r^{t_{i(r)} - 1} \frac{dG(t)}{\sqrt{t^2 - r^2}} \leq 2\sqrt{2} c(\gamma, B)h_n^{1/2}.
\]

In order to estimate the sum \( \sum_{j=i(r)+1}^{N_n} (\ldots) \) on the rhs of (5.8) we split up the range of \( j \)

\[
\sum_{j=i(r)+1}^{N_n} \int_{t_{j-1}}^{t_j} \left( \frac{1}{\sqrt{t^2 - r^2}} - \frac{1}{\sqrt{t_j^2 - r^2}} \right) g(t) dt \\
\leq c(\gamma, B) \sum_{j=i(r)+1}^{i(r)+1} \int_{t_{j-1}}^{t_j} \left( \frac{1}{\sqrt{t^2 - r^2}} - \frac{1}{\sqrt{t_j^2 - r^2}} \right) \sqrt{t} \, dt
\]
Here we have used (5.9) (since \((i(r) + \lfloor \sqrt{N_n} \rfloor + 1)h_n \leq r + 3h_n + \sqrt{Bh_n} \leq 1\) for sufficiently large \(n\)) and the Lipschitz condition
\[G(t + h) - G(t) \leq L_t h \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad h > 0,
\]
which follows from Lemma 5.1 and (2.16). Again using the monotonicity of \(t \mapsto (t^2 - r^2)^{-1/2}\), we find that
\[\sum_{j=i(r)+\lfloor \sqrt{N_n} \rfloor + 1}^{N_n} \int_{t_{j-1}}^{t_j} \left( \frac{1}{\sqrt{t^2 - r^2}} - \frac{1}{\sqrt{t_j^2 - r^2}} \right) dt \leq h_n \left( \frac{1}{i(r)}(i(r) + \lfloor \sqrt{N_n} \rfloor)^2 - r^2 \right)^{-1/2}
\]
\[\leq h_n \left( \frac{1}{\sqrt{Bh_n}} \left( 2r + \sqrt{Bh_n} \right) \right)^{-1/2}
\]
\[\leq \left( \frac{h_n}{B} \right)^{1/2}.
\]
Analogously, since \(t \mapsto \sqrt{t^2 - r^2} \left( (t^2 - r^2)^{-1/2} - (t_j^2 - r^2)^{-1/2} \right)\) is strictly decreasing for \(t \in [t_{j-1}, t_j]\),
\[\sum_{j=i(r)+\lfloor \sqrt{N_n} \rfloor + 1}^{N_n} \int_{t_{j-1}}^{t_j} \left( \frac{1}{\sqrt{t^2 - r^2}} - \frac{1}{\sqrt{t_j^2 - r^2}} \right) \sqrt{t} dt
\]
\[\leq \left( \frac{1}{\sqrt{Bh_n}} \left( 2r + \sqrt{Bh_n} \right) \right)^{-1/2}
\]
\[\left( \frac{1}{\sqrt{2h_n}} \right)^{1/2} \sum_{j=i(r)+\lfloor \sqrt{N_n} \rfloor + 1}^{N_n} \frac{1}{\sqrt{t_{i(r)} r - r (t_i(r) + j - 1) - r}} \leq \left( \frac{1}{\sqrt{2h_n}} \right)^{1/2} \sum_{j=1}^{\lfloor \sqrt{N_n} \rfloor + 1} \frac{1}{j \sqrt{j + 1}},
\]
where the last line is a consequence of \(t_i(r) + j - 1 - r = h_n (\lfloor r/h_n \rfloor + 1 + j) - r \geq jh_n\) for \(j \geq 1\). Summarizing all above estimates obtained after (5.9) we get from (5.8) that
\[\sup_{0 \leq r \leq \min \{1/2, B\}} | V_n(r) - V(r) | = O(h_n^{1/2}) \quad \text{as} \quad n \to \infty.
\]
Now, let $r \geq 1/2$ (provided that $B \geq 1/2$). From (5.10) it is easily seen that

$$
\sum_{j=i(r)+1}^{N_h} \left( \int_{t_{j-1}}^{t_j} \frac{dG(t)}{\sqrt{t^2 - r^2}} - \frac{G(t_j) - G(t_{j-1})}{\sqrt{t_j^2 - r^2}} \right)
\leq \frac{L_{\gamma} h_n}{\sqrt{t_{i(r)}^2 - r^2}} \leq \frac{h_n}{\sqrt{h_n(2r + h_n)}} \leq L_{\gamma} h_n^{1/2}.
$$

Since $g(t) < L$ for $t \in [0, B]$, we have, for $r \geq 1/2$,

$$
\frac{G(t_{i(r)}) - G(t_{i(r)-1})}{\sqrt{t_{i(r)}^2 - r^2}} \leq \int_{t_{i(r)}}^{t_{i(r)+1}} \frac{dG(t)}{\sqrt{t^2 - r^2}} \leq \frac{L_{\gamma}}{2} \sqrt{\frac{t_{i(r)}^2 - r^2}{2r}} \leq L_{\gamma} h_n^{1/2}.
$$

Hence, from (5.8), we get the estimate

$$
|V_n(r) - V(r)| \leq 2L_{\gamma} h_n^{1/2} \text{ for } r \geq 1/2
$$

which together with (5.11) proves (5.7).

In the second step we will accomplish the proof of Theorem 2.1 by showing that

$$
\sup_{0 \leq r \leq B} |\hat{V}_n(r) - V_n(r)| = o(a_n^{1/3}) \ P - \text{a.s. as } n \to \infty. \tag{5.12}
$$

From (2.13) and (2.15) we immediately obtain that $\hat{V}_n(r) = V_n(r) = 0$ for $B - h_n \leq r \leq B$ and

$$
|\hat{V}_n(r) - V_n(r)| \leq \frac{\hat{G}_n(B) - G(B)}{\sqrt{B^2 - r^2}} + \frac{\hat{G}_n(t_{i(r)-1}) - G(t_{i(r)-1})}{\sqrt{t_{i(r)}^2 - r^2}}
$$

$$
+ \left| \sum_{i=i(r)}^{N_n-1} \left( \hat{G}_n(t_i) - G(t_i) \right) \left( \frac{1}{\sqrt{t_i^2 - r^2}} - \frac{1}{\sqrt{t_{i+1}^2 - r^2}} \right) \right|
$$

for $0 \leq r < B - h_n$. By our Hölder condition (2.16) on $F$ we may choose in Theorem 4.1 the boundary points $a = 0$ and $b = B + \delta$ (for any $\delta > 0$) so that, for $0 \leq r < B - h_n$,

$$
\frac{|\hat{G}_n(B) - G(B)|}{\sqrt{B^2 - r^2}} \leq \frac{|\hat{G}_n(B) - G(B)|}{\sqrt{Bl_n}} = o(a_n/\sqrt{h_n}) \ P - \text{a.s. as } n \to \infty.
$$
On the other hand, since \( t_i(r)_{-1} \geq h_n(\lfloor r/h_n \rfloor + 1) \geq \max \{ r, h_n \} \) and \( b_n = o(h_n) \) as \( n \to \infty \), relation (4.3) of Theorem 4.1 (for \( \alpha = 0 \)) yields

\[
\frac{|\hat{G}_n(t_{i(r)_{-1}}) - G_n(t_{i(r)_{-1}})|}{\sqrt{t_{i(r)}^2 - r^2}} \leq \frac{1}{\sqrt{2}h_n} \sup_{h_n \leq t \leq B} |\hat{G}_n(t) - G(t)|
\]

\[
= o(a_n/h_n) \quad \mathbb{P} \text{- a.s. as } n \to \infty.
\]

Finally, for any \( r > 0 \),

\[
\left| \sum_{j=i(r)}^{N_n-1} \left( \hat{G}_n(t_j) - G(t_j) \right) \left( \frac{1}{\sqrt{t_j^2 - r^2}} - \frac{1}{\sqrt{t_{j+1}^2 - r^2}} \right) \right| \leq \frac{1}{\sqrt{t_{i(r)}^2 - r^2}} \sup_{2h_n \leq t \leq B} |\hat{G}_n(t) - G(t)|
\]

\[
= o(a_n/h_n) \quad \mathbb{P} \text{- a.s. as } n \to \infty,
\]

which together with the other two relations confirms (5.12). In view of (2.12) and (2.14) and by combining (5.7) and (5.12) with \( h_n = B/\lfloor a_n^{-2/3} \rfloor = O(a_n^{2/3}) \) we arrive at

\[
\sup_{0 \leq r \leq B} |\hat{F}_n(r) - F(r)| = \sup_{0 \leq r \leq B} \left| \frac{\hat{V}_n(r)}{\hat{V}_n(0)} - \frac{V(r)}{V(0)} \right| = O(a_n^{1/3}) \quad \mathbb{P} \text{- a.s. as } n \to \infty,
\]

which is just the assertion of Theorem 2.1.

6. Two Examples of Planar Cross-Sections Through Simulated 3D-PGMs with Spherical Grains and Concluding Remarks

In order to prove uniform strong consistency of the empirical DFs (2.11) and (2.14) it seems that the assumption of boundedness of the generic \( d \)-sphere diameter \( D_0 \) could be replaced by a weaker moment condition, say \( ED_0^p < \infty \) for some \( p > 2 \). A further open question concerns the asymptotic normality of the stereological estimator (2.14). In other words, does there exist a normalizing sequence \( c_n \uparrow \infty \) such that \( c_n(\hat{F}_n(t) - F(t)) \) converges weakly (as \( n \to \infty \)) to a mean zero Gaussian random variable with variance \( \sigma^2(t) > 0 \) for \( 0 < t < B \). In particular the calculation of \( \sigma^2(t) > 0 \) seems to be the main difficulty. Another unsolved estimation problem arises when \( F \) has discontinuities. How to estimate the magnitude and the location of the jumps?
Regardless of the above-proved strong consistency of the estimators (2.11) and (2.14), it should be stressed that $\hat{F}_n$ as well as $\hat{G}_n$ still keep a lot of intrinsic instability.

Figure 1. Section through 3-spheres with uniformly distributed diameter and intensity $\mu = 2.5$:

\[ F(t) = t, \Delta = 0.5 \text{ and } \lambda = \mu \Delta = 1.25, \]
\[ G(t) = \frac{t^2}{1 + \sqrt{1 - t^2}} + t^2 \log \left( \frac{1 + \sqrt{1 - t^2}}{t} \right) \]
for $0 \leq t \leq 1$.

Figure 2. Section through 3-spheres with Rayleigh-distributed diameter and intensity $\mu = 2.5$:

\[ F(t) = G(t) = 1 - \exp \left\{ -\pi t^2 \right\} \text{ for } t \geq 0, \]
\[ \Delta = 0.5 \text{ and } \lambda = \mu \Delta = 1.25. \]
STRONGLY CONSISTENT ESTIMATION IN WICKSELL'S ...

Figure 1 and Figure 2 show planar cross-sections through spatial PGMs of spheres with intensity $\mu = 2.5$ and uniformly resp. Rayleigh-distributed diameters restricted to the sampling window $W_n = [0, 10] \times [0, 10]$. The PGMs of disks induced in the intersection plane has intensity $\lambda = 2.5 \Delta$ (see Section 1) with $\Delta = 0.5$ in both cases. $\hat{G}_n(t)$ and $\hat{F}_n(t)$ are computed according to (2.11) and (2.14), respectively. We realize that especially the empirical DF $\hat{F}_n(t)$ need not to be a monotone. However, large-scale simulation experiments confirm that the stereological estimator $\hat{F}_n(t)$ becomes already stable for moderate window sizes. It could be observed that the stabilization is indeed faster if the empirical volume fraction $\hat{C}_n(o)$ is comparatively small.

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References


