Second-order properties of the point process of nodes in a stationary Voronoi tessellation

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In this paper we derive representation formulae for the second factorial moment measure of the point process of nodes and the second moment of the number of vertices of the typical cell associated with a stationary normal Voronoi tessellation in \(\mathbb{R}^d\). In case the Voronoi tessellation is generated by a stationary Poisson process with intensity \(\lambda > 0\) the corresponding pair correlation function \(g_{V, \lambda}(r)\) can be expressed by a weighted sum of \(d+2\) (numerically tractable) multiple parameter integrals. The asymptotic variance of the number of nodes in an increasing cubic domain as well as the second moment of the number of vertices of the typical Poisson Voronoi cell are calculated exactly by means of these parameter integrals. The existence of a \((d-1)\)st-order pole of \(g_{V, \lambda}(r)\) at \(r = 0\) is proved and the exact value of \(\lim_{r \to 0^+} r^{d-1} g_{V, \lambda}(r)\) is determined. In the particular cases \(d = 2\) and \(d = 3\) the graph of \(g_{V, 1}(r)\) including its local extreme points, the points of level 1 of \(g_{V, 1}(r)\) and other characteristics are computed by numerical integration. Furthermore, an asymptotically exact confidence interval for the intensity of nodes is obtained.

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In this paper we derive representation formulae for the second factorial moment measure of the point process of nodes and the second moment of the number of vertices of the typical cell associated with a stationary normal Voronoi tessellation in $\mathbb{R}^d$. In case the Voronoi tessellation is generated by a stationary Poisson process with intensity $\lambda > 0$ the corresponding pair correlation function $g_{V,\lambda}(r)$ can be expressed by a weighted sum of $d+2$ (numerically tractable) multiple parameter integrals. The asymptotic variance of the number of nodes in an increasing cubic domain as well as the second moment of the number of vertices of the typical Poisson Voronoi cell are calculated exactly by means of these parameter integrals. The existence of a $(d-1)$-order pole of $g_{V,\lambda}(r)$ at $r=0$ and the exact value of $\lim_{r \to 0} r^{d-1} g_{V,\lambda}(r)$ is determined. In the particular cases $d=2$ and $d=3$ the graph of $g_{V,1}(r)$ including its local extreme points, the points of level 1 of $g_{V,1}(r)$ and other characteristics are computed by numerical integration. Furthermore, an asymptotically exact confidence interval for the intensity of nodes is obtained.

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{A tessellation of the $d$-dimensional Euclidean space $\mathbb{R}^d$ is a subdivision of this space into countably many compact sets $\{C_i, i \in \mathbb{N}\}$ (called cells or crystals), where two such cells have no common interior points and the number of cells intersecting any bounded subset of $\mathbb{R}^d$ is finite. Usually the cells are assumed to be convex so that each $C_i$ becomes a convex $d$-polytope. In this case the boundary $\partial C_i$ consists of $s$-polytopes with $s \in \{0, 1, \ldots, d-1\}$ called $s$-faces in what follows. As usual 0-faces, 1-faces, and $(d-1)$-faces are called vertices, edges, and facets, respectively. The union of all vertices forms the countable set of nodes of the tessellation.

Many real-life tessellations are random, see e.g. Stoyan et al. (1995), Okabe et al. (2000). There are various stochastic-geometric mechanisms to generate a random tessellation $\{C_i, i \in \mathbb{N}\}$ which can be defined over a hypothetical probability space $[\Omega, \mathcal{A}, P]$ such that the above properties are satisfied $P$-almost surely. The random tessellation is said to be stationary if its distribution is invariant under translation of the cells.

Throughout the present paper we are concerned with stationary Voronoi (or Dirichlet, or Thiessen) tessellations (briefly VT’s) generated by stationary point processes. It should be mentioned that some authors use the term VT merely relative to a homogeneous Poisson point process (henceforth such a VT will be called Poisson Voronoi tessellation (briefly PVT)). A mathematically rigorous and self-contained presentation of the basic material (mean-value relationships, PVT’s, stochastic- and integral-geometric tools) on random VT’s can be found in Møller (1994). For further details and the historical background the reader is also referred to the monographs of Okabe et al. (2000) and Stoyan et al. (1995) (in particular Chapter 10 and references therein) and the papers by Miles (1970), Miles and Maillardet (1982), Møller (1989) and others. Because there are only a few explicit

}\end{figure}
formulae of distributional characteristics of PVT’s (see e.g. Muche and Stoyan (1992), Mecke and Muche (1995), Muche (1996/98), Heinrich (1998), Schlather (2000), Calka (2003 a,b), Muche (2005)), large-scale simulation studies are often the only way to obtain approximately the desired parameters or non-parametric characteristics, see e.g. Brakke (1985 a), Hahn and Lorz (1994), Heinrich and Schülle (1995).

The main goal of this paper is to prove a structural formula for the second factorial moment measure of the point process of nodes in stationary VT’s and to derive (in some sense) a closed-form representation for the pair correlation function (briefly: PCF) of the nodes in a stationary PVT in terms of $d + 2$ multiple parameter integrals. In this way it is possible to study the analytic properties and to determine the graph of this function by numerical integration. In the planar case the numerical integration is done by standard procedures, whereas already the three-dimensional case requires additional analytical considerations and advanced numerical methods for the evaluation of higher-dimensional integrals, see Heinrich et al. (1998) for details.

The paper is organized as follows. After the introductory Sect. 1 which contains, among others, formulae for the intensity of the nodes in a stationary (Poisson) VT, we derive in Sect. 2 (Theorems 2.1 and 2.2) and Sect. 3 (Theorem 3.3) the announced main results. In Sect. 4 we are concerned with the PCF of the nodes in a planar VT which can be represented by a weighted sum of four two-fold parameter integrals. In Sect. 5 we study the behaviour of this function in any dimension $d \geq 2$ as $r \downarrow 0$ and $r \uparrow \infty$ (Theorem 5.1) and derive a formula for the second factorial moment for the number of vertices of the typical Voronoi cell (Theorem 5.3). In Sect. 6 we construct an (asymptotically exact) confidence interval for the intensity of nodes in a $d$-dimensional PVT (Theorem 6.1).

Before passing to the rigorous definition of a VT we list some notation and give some necessary facts from point process theory needed in the sequel:

$$\|x\| = \text{Euclidean norm of } x \in \mathbb{R}^d,$$

$$b(x, r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\} : \text{(closed) } d \text{-ball with center } x \text{ and radius } r,$$

$$\mathcal{B}^d (\mathcal{B}^d_o) = \sigma \text{-field (ring) of (bounded) Borel sets in } \mathbb{R}^d,$$

$$|B| = d \text{-dimensional Lebesgue measure of } B \in \mathcal{B}^d_o,$$

$$\partial B (\mathcal{B}^d \cup \mathcal{B}^d_o) = \text{boundary (complement) of } B \in \mathcal{B}^d,$$

$$S^{d-1} = \partial b(a, 1) : (d - 1) \text{-dimensional unit sphere},$$

$$\omega_d = |b(a, 1)| = \pi^{d/2} / \Gamma \left( \frac{d}{2} + 1 \right)$$

$$\Theta_{d-1} = [-\pi, \pi] \times [0, \pi]^{d-2} : \text{domain of spherical coordinates on } S^{d-1},$$

$$S(x_0, \ldots, x_d) = d \text{-simplex with vertices } x_0, \ldots, x_d \in \mathbb{R}^d,$$

$$c(x_0, \ldots, x_d) = \text{center of the uniquely determined circumball of } S(x_0, \ldots, x_d),$$

$$r(x_0, \ldots, x_d) = \text{radius of the uniquely determined circumball of } S(x_0, \ldots, x_d),$$

$$(\text{provided } x_1 - x_0, \ldots, x_d - x_0 \text{ are linearly independent}),$$

$$N = \text{set of locally finite counting measures } \psi \text{ on } \mathbb{R}^d,$$

$$\mathcal{N} = \sigma \text{-field generated by the sets } \{\psi \in N : \psi(B) = n\}, \ n \in \mathbb{N}, B \in \mathcal{B}^d_o,$$

$$Y(x_0, \ldots, x_d) = \{\psi \in N : \psi(b(c(x_0, \ldots, x_d), r(x_0, \ldots, x_d))) = 0\} (\in \mathcal{N}),$$

$$1_Y (\cdot) = \text{indicator function of the set } Y,$$

$$\delta_\mu (B) = 1_B (x) : \text{Dirac measure}.$$

A point process $\Psi = \sum_{x \in \mathcal{N}} \delta_x, \sim P$ on $\mathbb{R}^d$ (with distribution $P = P \circ \Psi^{-1}$) is defined to be an $(\mathcal{A}, \mathcal{N})$-measurable mapping from $[\Omega, \mathcal{A}, P]$ into the measurable space $[N, \mathcal{N}]$. We always assume that $\Psi$ is simple, i.e. $P (\Psi (\{ x \}) \leq 1$ for all $x \in \mathbb{R}^d) = 1$.

If $E\Psi^k (B) < \infty$ for all $B \in \mathcal{B}^d_o$, then there exist the k-th factorial moment measure $\alpha^{(k)}$ defined on $([\mathbb{R}^d]^k, (\mathcal{B}^d)^k)$ by

$$\alpha^{(k)} (B_1 \times \cdots \times B_k) = \int_N \prod_{x_1, \ldots, x_k \in \psi} 1_{B_1} (x_1) \cdots 1_{B_k} (x_k) P (d\psi).$$

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Here the loose writing “\(x \in \psi\)” means “\(x \in \mathbb{R}^d : \psi(\{x\}) > 0\)” and the sum \(\sum_{x \in \psi}^k\) is taken over all \(k\)-tuples \((x_1, \ldots, x_k) \in (\mathbb{R}^d)^k\) of pairwise distinct atoms of \(\psi \in \mathcal{N}\).

If \(\Psi \sim P\) is additionally stationary and has a positive intensity \(\lambda = \mathbb{E}\Psi([0, 1]^d)\), the following disintegration of \(\alpha^{(k)}\) is possible, see e.g. Daley and Vere-Jones (1988):

\[
\alpha^{(k)}(B_1 \times \cdots \times B_k) = \lambda \int_{B_k} \alpha^{(k)}_{red}(\{(B_1 - x) \times \cdots \times (B_{k-1} - x)\}) \, dx ,
\]

where \(\alpha^{(k)}_{red}\) is a uniquely determined measure on \([\mathbb{R}^d \times (\mathbb{R}^d)^{k-1}]\) called the \textit{reduced \(k\)-th factorial moment measure}.

Moreover, if \(\alpha^{(k)}\) is absolutely continuous w.r.t. the Lebesgue measure on \((\mathbb{R}^d)^k\) one can define the \textit{k-th-order product density} \(p^{(k)}\) on \((\mathbb{R}^d)^{k-1}\) by

\[
\lambda \alpha^{(k)}_{red}(B) = \int_B p^{(k)}(x) \, dx \quad \text{for all bounded} \quad B \in (\mathbb{B}^d)^{(k-1)} .
\]

The \textit{reduced \(k\)-th-order Palm distribution} \(P^k_{x_1, \ldots, x_k}(Y)\), \(Y \in \mathcal{N}\), of a simple point process \(\Psi \sim P\) is the unique regular conditional probability distribution on \([\mathcal{N}, \mathcal{N}]\) (up to \(\alpha^{(k)}\)-null sets) satisfying the \textit{refined Campbell formula}

\[
\int_{\mathcal{N}} \sum_{x \in \psi}^k f(x_1, \ldots, x_k, \psi - \sum_{i=1}^k \delta_{x_i}) P(d\psi)
= \int_{\mathbb{R}^k} \sum_{x \in \psi}^k f(x_1, \ldots, x_k, \psi) P^k_{x_1, \ldots, x_k}(d\psi) \alpha^{(k)}(d(x_1, \ldots, x_k))
\]

for any bounded \(\mathcal{B}^k \otimes \mathcal{N}\)-measurable function \(f : \mathbb{R}^k \times \mathcal{N} \rightarrow \mathbb{R}^1\).

\(P^k_{x_1, \ldots, x_k}(Y)\) is interpreted as the conditional probability of the event \(\{\Psi - \sum_{i=1}^k \delta_{x_i} \in Y\}\) given the null event \(\{\prod_{i=1}^k \Psi(\{x_i\}) = 0\}\), see Kallenberg (1983) for details.

Note that a stationary Poisson process \(\Psi \sim P\) with intensity \(\lambda\) is characterized by the relations

\[
P^k_{x_1, \ldots, x_k}(Y) = P(Y) \quad \text{and} \quad \alpha^{(k)}(B_1 \times \cdots \times B_k) = \lambda^k |B_1| \cdots |B_k|
\]

for any \(B_1, \ldots, B_k \in \mathcal{B}^d\) and \(Y \in \mathcal{N}\), see Kallenberg (1983) or Hanisch (1982). Corresponding formulae for Poisson cluster processes are given in Heinrich (1988).

A (stationary) VT \(V(\psi) = \{C_i(\psi), i \in \mathbb{N}\}\) generated by a (stationary) simple point process \(\Psi = \sum_{i \in \mathbb{N}} \delta_{X_i}\) in \(\mathbb{R}^d\) is defined realizationwise. For any \(\psi = \sum_{i \in \mathbb{N}} \delta_{x_i} \in \mathcal{N}\) the cell \(C_i(\psi)\) of the non-random VT \(V(\psi)\) is defined to be the set of all points in \(\mathbb{R}^d\) which are as close or closer to the nucleus \(x_i\) than to any other \(x_j, j \neq i\), i.e.

\[
C_i(\psi) = \{x \in \mathbb{R}^d : ||x - x_i|| \leq ||x - x_j||, j \neq i\} = \bigcap_{j \neq i} H_{ij} ,
\]

where the point sets \(H_{ij} = \{x \in \mathbb{R}^d : ||x - x_i|| \leq ||x - x_j||\}, j \neq i\), are half-spaces containing \(x_i\). Obviously, the cells \(C_i(\psi)\) are convex \(d\)-polytopes and the property of local finiteness is \(\mathbb{P}\)-a.s. satisfied.

In addition, in this paper it is basically assumed that \(V(\psi)\) is a \textit{normal} tessellation (see e.g. Möller (1989)), that is,

\textbf{Condition (N)} For \(s \in \{0, 1, \ldots, d - 1\}\) each \(s\)-face of the \(C_i(\psi)\)’s lies in the boundary of \(d - s + 1\) adjacent cells,

is satisfied. For \(s = 0\) this condition says that exactly \(d + 1\) edges emanate from each node in a \(d\)-dimensional normal VT \(V(\psi)\). Clearly, Condition (N) imposes a restriction on the generating point process \(\Psi\). In Lemma 1.1
we formulate conditions in terms of the point process $\Psi$ implying that $V(\Psi)$ is normal. For this, we need the following two subsets of $(\mathbb{R}^d)^d$ and $(\mathbb{R}^d)^d+1$, respectively:

$$L_d = \{ (x_1, \ldots, x_d) \in (\mathbb{R}^d)^d : \det [x_1, \ldots, x_d] = 0 \} \quad \text{and}$$

$$S_{d+1} = \{ (x_1, \ldots, x_{d+1}) \in (\mathbb{R}^d)^{d+1} : \partial b(c(x_1, \ldots, x_{d+1}), r(x_1, \ldots, x_{d+1})) \}. $$

**Lemma 1.1** Let $\Psi \sim P$ be a stationary simple point process on $\mathbb{R}^d$ with intensity $\lambda > 0$. Then Condition (N) is satisfied if the sets

$$L_d \triangleq \left\{ \psi \in \mathcal{N} : \sum_{x_0, x_1, \ldots, x_d \in \psi} \mathbf{1}_{L_d}(x_1 - x_0, \ldots, x_d - x_0) \geq 1 \right\} \quad \text{and}$$

$$S_{d+1} \triangleq \left\{ \psi \in \mathcal{N} : \sum_{x_0, x_1, \ldots, x_{d+1} \in \psi} \mathbf{1}_{S_{d+1}}(x_1 - x_0, \ldots, x_{d+1} - x_0) \geq 1 \right\}$$

have $P$-measure zero. In particular, this holds if the $(d+2)nd$-order product density $p^{(d+2)}$ of $\Psi$ exists or if at least $\alpha_{\text{red}}^{(d+1)}(L_d) = \alpha_{\text{red}}^{(d+2)}(S_{d+1}) = 0$.

**Proof.** In analogy to the proof of Proposition 7.1 in Möller (1989) which states that a stationary PVT is normal, it suffices to show that, for $P$-almost every $\psi \in \mathcal{N}$ and any fixed $t \in \{1, \ldots, d\}$, each $(t+1)$-tuple of pairwise distinct atoms $x_0, x_1, \ldots, x_t \in \psi$ defines a $t$-dimensional affine subspace ($t$-flat) $F_t = x_0 + \text{span} \{x_1 - x_0, \ldots, x_t - x_0\}$ in $\mathbb{R}^d$ such that $x_0, x_1, \ldots, x_t$ but no further atom of $\psi$ belong to the relative boundary of a unique $t$-ball in $F_t$. But this is equivalent to the linear independence of the polynomials $x_1 - x_0, \ldots, x_t - x_0$ and $\text{pol}(c(x_0, \ldots, x_d), r(x_0, \ldots, x_d))$ for all $(d+1)$-tuples of pairwise distinct atoms $x_0, \ldots, x_d \in \psi$, that is, $\psi \in \mathcal{N} \setminus (L_d \cup S_{d+1})$. This proves the first part of Lemma 1.1.

Let $B_n \uparrow \mathbb{R}^d$ be an increasing sequence of compact sets so that

$$\mathcal{L}^{(n)}_d = \left\{ \psi \in \mathcal{N} : \sum_{x_0, x_1, \ldots, x_d \in \psi} \prod_{i=0}^d \mathbf{1}_{B_n}(x_i) \mathbf{1}_{L_d}(x_1 - x_0, \ldots, x_d - x_0) \geq 1 \right\} \subseteq \mathcal{N}$$

is an increasing sequence of sets with $\mathcal{L}_d = \bigcup_{n \in \mathbb{N}} \mathcal{L}^{(n)}_d$.

Using the above definition of factorial and reduced factorial moment measures of $\Psi$ we may proceed as follows:

$$P(\mathcal{L}^{(n)}_d) \leq \int_{\mathcal{N}} \sum_{x_0, x_1, \ldots, x_d \in \psi} \prod_{i=0}^d \mathbf{1}_{B_n}(x_i) \mathbf{1}_{L_d}(x_1 - x_0, \ldots, x_d - x_0) P(d\psi)$$

$$\leq \lambda \int_{B_n} \int_{(\mathbb{R}^d)^d} \prod_{i=1}^d \mathbf{1}_{B_n}(x_i) \mathbf{1}_{L_d}(x_1 - x_0, \ldots, x_d - x_0) \prod_{i=1}^d \alpha^{(d+1)}_{\text{red}}(d(x_1, \ldots, x_d)) \, dx_0$$

$$\leq \lambda \left| B_n \right| \alpha^{(d+1)}_{\text{red}}(L_d).$$

Thus, $\alpha^{(d+1)}_{\text{red}}(L_d) = 0$ implies $P(\mathcal{L}^{(n)}_d) = 0$ for every $n \in \mathbb{N}$ so that $P(\mathcal{L}_d) = 0$. In the same way one can verify that $\alpha^{(d+2)}_{\text{red}}(S_{d+1}) = 0$ implies $P(S_{d+1}) = 0$. Since $L_d$ resp. $S_{d+1}$ has Lebesgue measure zero in $(\mathbb{R}^d)^d$ resp. in $(\mathbb{R}^d)^{d+1}$ and the existence of $p^{(d+2)}$ implies the existence of all lower-order product densities $p^{(k)}$, $2 \leq k \leq d + 1$, Lemma 1.1 is completely proved.

Let $\Psi_V \sim P_V$ denote the (stationary) point process of nodes associated with a normal (stationary) VT $V(\Psi)$. For each realization $\psi \in \mathcal{N}$ (up to a $P$-null set) generating a normal VT $V(\psi)$, the simple counting measure of
nodes is given by

\[ \psi_V \triangleq \frac{1}{(d+1)!} \sum_{x_0, x_1, \ldots, x_d \in \Psi} \delta(x_0, x_1, \ldots, x_d) \mathbf{1}_{Y(x_0, x_1, \ldots, x_d)} (\psi - \delta_{x_0} - \delta_{x_1} \cdots - \delta_{x_d}). \]  

The typical cell of a stationary VT \( V(\psi) \) is that cell whose nucleus is the typical atom of the point process \( \Psi \sim P \). The interpretation of the reduced Palm distribution as conditional distribution \( P^o_\psi (\cdot) = P(\Psi - \delta_0 \in \cdot | \Psi(\{o\}) > 0) \) reveals that the typical cell of \( V(\psi) \) coincides with the origin cell of the VT \( V(\psi_0 + \delta_0) \), where \( \psi_0 \sim P^o_\psi \) denotes the reduced Palm point process. Therefore

\[ N_o \triangleq \frac{1}{d!} \sum_{x_1, \ldots, x_d \in \Psi_0} \mathbf{1}_{Y(o, x_1, \ldots, x_d)} (\psi_0 - \delta_{x_1} \cdots - \delta_{x_d}) \]  

is equal to random number of vertices of the typical cell.

**Lemma 1.2** Let \( \Psi \sim P \) be a simple stationary (and isotropic) point process in \( \mathbb{R}^d \) with intensity \( \lambda \) satisfying \( \mathbb{E} \psi_{d+1}(\cdot) < \infty \) and Condition (N). Then \( \Psi_V \sim P_V \) is simple, stationary (and isotropic) and possesses the intensity

\[ \lambda_V = \frac{\lambda}{(d+1)!} \int_{\mathbb{R}^d} \sum_{x_0, x_1, \ldots, x_d \in \Psi} \mathbf{1}_B(c(x_0, x_1, \ldots, x_d)) \mathbf{1}_{Y(x_0, x_1, \ldots, x_d)} (\psi - \sum_{i=0}^d \delta_{x_i}) P(d\psi) \]

and

\[ \lambda \mathbb{E} N_o = (d+1) \lambda V. \]  

**Proof.** Using the definition of the \((d+1)\)st-order Palm distribution and the reduced \((d+1)\)st factorial moment measure we may write for any \( B \in \mathcal{B}_o^d \):

\[ \mathbb{E} \Psi_V (B) = \frac{1}{(d+1)!} \int_{\mathbb{R}^d} \sum_{x_0, x_1, \ldots, x_d \in \Psi} \mathbf{1}_B(c(x_0, x_1, x_2, \ldots, x_d)) \mathbf{1}_{Y(x_0, x_1, \ldots, x_d)} (\psi - \sum_{i=0}^d \delta_{x_i}) P(d\psi) \]

\[ = \frac{\lambda}{(d+1)!} \int_{\mathbb{R}^d} \mathbf{1}_B(c(x_0, x_1, x_2, \ldots, x_d + x_0)) \times P^o_{x_0, x_1, x_2, \ldots, x_d} (Y(x_0, x_1, x_2, \ldots, x_d + x_0)) o_{\psi}^{(d+1)} (d(x_1, \ldots, x_d)) dx_0. \]

Since \( c(x_0, x_1, x_2, \ldots, x_d + x_0) = c(o, x_1, \ldots, x_d) + x_0 \) and, by stationarity of \( \Psi \sim P \),

\[ P^o_{x_0, x_1, x_2, \ldots, x_d} (Y(x_0, x_1, x_2, \ldots, x_d + x_0)) = P^o_{0, x_1, \ldots, x_d} (Y(o, x_1, \ldots, x_d)), \]

we obtain by interchanging the integration that

\[ \mathbb{E} \Psi_V (B) = \lambda_V |B| \]

with \( \lambda_V \) as given in (1.5).

The relation (1.6) follows from well-known mean-value relationships being valid for more general stationary tessellations, see Möller (1989) and references therein. Its proof here is based on higher-order Palm theory. By virtue of the recursive relation \( (P^o_\psi)^{1}_{x_1, \ldots, x_d} (\cdot) = P^o_{0, x_1, \ldots, x_d} (\cdot) \) (see Hanisch (1982)) we may apply the refined Campbell formula (1.1) w.r.t. \( P^o_\psi \) proving the identity

\[ \int_{\mathbb{R}^d} \sum_{x_0, x_1, \ldots, x_d \in \Psi} \mathbf{1}_{Y(o, x_1, \ldots, x_d)} (\psi) P^o_\psi (d\psi) \]

\[ = \int_{\mathbb{R}^d} P^o_{0, x_1, \ldots, x_d} (Y(o, x_1, \ldots, x_d)) o_{\psi}^{(d+1)} (d(x_1, \ldots, x_d)). \]

Thus, combining (1.4) and (1.5) yields (1.6).
We close the introductory section with a formula for the intensity of nodes \( \lambda_V = E\Psi_V([0,1]^d) \) in a \( d \)-dimensional stationary PVT.

**Lemma 1.3** (See Møller (1994).) Let \( \Psi \sim P \) be a stationary Poisson process in \( \mathbb{R}^d \) with intensity \( \lambda \). Then

\[
\lambda_V = \kappa_d \lambda \quad \text{with} \quad \kappa_d = \frac{2d}{(d+1)} \frac{\pi^{d-1}}{\omega_d} \left( \frac{\omega_{d-1}}{\omega_d} \right)^d.
\]

(1.7)

In particular, \( \kappa_1 = 1 \), \( \kappa_2 = 2 \) and \( \kappa_3 = \frac{24}{35} \pi^2 \).

**Remark 1.4** For any normal planar stationary VT we have \( \lambda_V = 2\lambda \) and \( E_N = 6 \), whereas for \( d \geq 3 \) the ratio \( \lambda_V/\lambda \) is different from \( \kappa_d \) in general, see e.g. Møller (1994).

## 2 Second moment characteristics of the point process of nodes in a stationary Voronoi tessellation

We first consider the second factorial moment measure \( \alpha_V^{(2)} \) of \( \Psi_V \sim P_V \) defined by

\[
\alpha_V^{(2)}(A \times B) \triangleq \int_{\mathbb{R}^d} \sum_{x,y \in \psi}^* 1_A(x) 1_B(y) P_V(d\psi) \quad \text{for} \quad A,B \in \mathcal{B}_d.
\]

In the first step we derive an explicit representation of \( \alpha_V^{(2)}(A \times B) = E\Psi_V(A)\Psi_V(B) \) for \( A \cap B = \emptyset \) in terms of higher-order Palm distributions and factorial moment measures of \( \Psi \).

By a little combinatorics it is quickly checked that, for any two nonnegative symmetric functions \( f_1, f_2 \) on \( (\mathbb{R}^d)^{d+1} \) with bounded support,

\[
\sum_{x_0,\ldots,x_{d+1} \in \psi} f_1(x_0,\ldots,x_d) f_2(x_{d+1},\ldots,x_{2d+1}) = \sum_{j=0}^{d+1} j! \binom{d+1}{j} \binom{d+1}{j} \sum_{x_0,\ldots,x_{d-j+1} \in \psi} f_1(x_0,\ldots,x_d) f_2(x_{d-j+1},\ldots,x_{2d-j+1}).
\]

Furthermore,

\[
1_{Y_{d+1}}(\psi - \sum_{i=0}^d \delta_{x_i}, 1_{Y_j}(\psi - \sum_{i=d-j+1}^{2d-j+1} \delta_{x_i}) = \prod_{i=0}^{d-j} 1_{\psi(b(c_j,r_j))} (x_{d-j+1},\ldots,x_{2d-j+1}) 1_{Y_{d+1} \cap Y_j}(\psi - \sum_{i=0}^{2d-j+1} \delta_{x_i}),
\]

where, for notational simplicity, we have put

\[
c_j = c(x_{d-j+1},\ldots,x_{2d-j+1}),
\]

\[
r_j = r(x_{d-j+1},\ldots,x_{2d-j+1}),
\]

\[
Y_j = \{ \psi \in N : \psi(b(c_j,r_j)) = 0 \},
\]

for \( j = 0, 1, \ldots, d, d+1 \).

Combining the latter two identities with (1.3) we can express the product \( \psi_V(A)\psi_V(B) \) in terms of \( \psi \) so that, for disjoint sets \( A, B \in \mathcal{B}_d \),

\[
\alpha_V^{(2)}(A \times B) = \sum_{j=0}^d \frac{1}{j!(d-j+1)!} \beta_{j,d}(A \times B),
\]

(2.1)
where

\[ \beta_{j,d}(A \times B) = \int_{N_{x_0,\ldots,x_{2d-j+1}}} \sum_{x} 1_A(c_{d+1}) 1_B(c_j) \prod_{i=0}^{d-j} 1_{B_r(c_{d+1},r_{d+1})}(x_{d+i+1}) 1_{B_r(c_j,r_j)}(x_j) \times 1_{Y_{d+1} \cap Y_j}(\psi - \sum_{i=0}^{2d-j+1} \delta_{x_i}) P(d\psi). \]

By definition, \( \beta_{j,d}(A \times B)/[d(d-j+1)!]^2 \) is nothing else but the mean number of ordered pairs of simplices \( S(x_0, \ldots, x_d) \) and \( S(x_{d-j+1}, \ldots, x_{2d-j+1}) \) which have exactly one common \((j-1)\)-facet \( S(x_{d-j+1}, \ldots, x_d) \) (for \( j = 1, \ldots, d \)) such that their circumcentres \( c_{d+1} \) and \( c_j \) belong to \( A \) and \( B \), respectively, and no further atom of \( \Psi \) lies in \( b(c_{d+1}, r_{d+1}) \cup b(c_j, r_j) \). Applying the definition of the higher-order Palm distribution of \( \Psi \sim P \) we arrive at

**Theorem 2.1** Let \( \Psi \sim P \) be a simple stationary point process in \( \mathbb{R}^d \) satisfying Condition (N) and

\[ \mathbb{E} \Psi^{2d+2}([0,1]^d) < \infty. \]

Then, for any \( A, B \in \mathcal{B}_0^d \) with \( \emptyset \cap B = \emptyset \), the second factorial moment measure \( \alpha^{(2)}(A \times B) \) is given by (2.1), where

\[ \beta_{j,d}(A \times B) = \int_{[\mathbb{R}^d]^{2d-j+2}} 1_A(c_{d+1}) 1_B(c_j) \prod_{i=0}^{d-j} 1_{B_r(c_{d+1},r_{d+1})}(x_{d+i+1}) 1_{B_r(c_j,r_j)}(x_j) \times P_{x_0,\ldots,x_{2d-j+1}}(\{\psi(b(c_{d+1}, r_{d+1}) \cup b(c_j, r_j)) = 0\}) \alpha^{(2d-j+2)}(d(x_0, \ldots, x_{2d-j+1})). \]

In the particular case of a stationary Poisson process \( \Psi \sim P \) with intensity \( \lambda \) we have

\[ \beta_{j,d}(A \times B) = \lambda^{2d-j+2} \int_{[\mathbb{R}^d]^{2d-j+2}} 1_A(c_{d+1}) 1_B(c_j) \prod_{i=0}^{d-j} 1_{B_r(c_{d+1},r_{d+1})}(x_{d+i+1}) 1_{B_r(c_j,r_j)}(x_j) \times \exp\{-\lambda|b(c_{d+1}, r_{d+1}) \cup b(c_j, r_j)|\} \, dx_0, \ldots, x_{2d-j+1}. \]

The set function \( \beta_{j,d} \) can be extended to a locally finite measure on the Borel \( \sigma \)-field \( \mathcal{B}_0^d \otimes \mathcal{B}_0^d \) provided \( \alpha^{(2d-j+2)} \) exists, \( j = 0, 1, \ldots, d \). The stationarity of \( \Psi \sim P \) entails the invariance of \( \beta_{j,d} \) under diagonal shifts, i.e., \( \beta_{j,d}((A + x) \times (B + x)) = \beta_{j,d}(A \times B) \) for all \( x \in \mathbb{R}^d \). This enables the disintegration of the measures \( \beta_{j,d} \) w.r.t. the Lebesgue measure (see Daley and Vere-Jones (1988)), that is, there exist reduced measures \( \beta_{j,d}^{(\text{red})} \) on \([\mathbb{R}^d, \mathcal{B}_0^d]\) such that

\[ \beta_{j,d}(A \times B) = \int_A \beta_{j,d}^{(\text{red})}(B - x) \, dx. \]

The measures \( \beta_{j,d}^{(\text{red})} \) are obtained by disintegration of the factorial moment measures \( \alpha^{(k+1)} \) and using that 

\[ P'_{x_0+y,\ldots,x_{2d+j+y}}(\{\psi: \psi((.) + y) = 0\}) \text{ does not depend on } y \in \mathbb{R}^d \text{ for } k = d + 1, \ldots, 2d+1. \]

Disintegrating w.r.t. \( x_0 \) on the rhs of (2.2) yields

\[
\beta_{j,d}^{(\text{red})}(B) = \lambda \int_{[\mathbb{R}^d]^{2d-j+1}} 1_B(c_j - c_{d+1}) \prod_{i=0}^{d-j} 1_{B_r(c_{d+1},r_{d+1})}(x_{d+i+1}) 1_{B_r(c_j,r_j)}(x_j) \times P_{x_0,\ldots,x_{2d-j+1}}(\{\psi(b(c_{d+1}, r_{d+1}) \cup b(c_j, r_j)) = 0\}) \alpha^{(2d-j+2)}(d(x_1, \ldots, x_{2d-j+1}))
\]

for \( j = 0, 1, \ldots, d \) with \( c_{d+1}' = c(o, x_1, \ldots, x_d) \) and \( r_{d+1}' = r(o, x_1, \ldots, x_d) \).

For \( j = 1, \ldots, d \) we may also disintegrate the rhs of (2.2) w.r.t. the coordinate \( x_d \) (which belongs to

\[ \partial b(c_{d+1}, r_{d+1}) \cap \partial b(c_j, r_j). \]
This leads to an expression slightly different from the previous one:

\[
\beta_{j,d}^{(\text{red})}(B) = \lambda \int_{[e_{d-1},e_{d-1}+1)} 1_B(e_{d-1} + \epsilon_{d-1}, \epsilon_{d-1} + 1) \prod_{i=1}^{d-j+1} 1_{\psi}(\epsilon_{d-1} + r_{d+1}^{(i)})(x_{d+1}^{(i)}) \ 1_{\psi}(\epsilon_{d-1} + r_{d+1}^{(i)})(x_{d+1}^{(i)}) \ d(x_{d+1}^{(i)}) \ \text{for } j = 1, \ldots, d.
\]  

where \( e_{d-1} = c(0, x_d, x_d, \ldots, x_{2d-j-1}) \) for \( j = 1, \ldots, d \).

By the very definition of the reduced second factorial moment measure \( \alpha_{V,\text{red}}^{(2)} \), it follows from (2.1) that

\[
\alpha_{V,\text{red}}^{(2)}(B) = \int_B \psi(B)(P_{\psi})_{\theta}(d\psi) = \frac{1}{\lambda V} \sum_{j=0}^{d} \frac{\beta_{j,d}^{(\text{red})}(B)}{j![(d-j+1)!]^2}, \quad B \in B_\theta^d.
\]  

Next in this section we shall derive expressions of the second factorial moment of the number \( N_o \) and the asymptotic variance \( \sigma_{\psi}^2 \) which is defined by

\[
\sigma_{\psi}^2 = \lim_{n \to \infty} \frac{\mathbb{E}((\Psi_A(N_o) - \lambda V|A_n)|^2)}{|A_n|} = \lim_{n \to \infty} \frac{|A_n|}{\mathbb{E}((\lambda V_n - \lambda V)|A_n)|^2},
\]

where \( \lambda V_n \triangleq \Psi_A(N_o)/|A_n| \) is an unbiased estimator for the intensity \( \lambda V \) and \( A_n, n \in \mathbb{N} \), stands for a convex averaging sequence of sets in \( \mathbb{R}_d^d \), see Daley and Vere–Jones (1988), that is, the sets \( A_n \) are compact convex, increasing, and contain a ball with unboundedly growing radius, e.g. \( A_n = [0, n]^d \).

**Theorem 2.2** Let the assumptions of Theorem 2.1 be satisfied. Further, let the measures \( \beta_{j,d}^{(\text{red})}(\cdot) \), \( j = 1, \ldots, d \), be finite and the signed measure \( \beta_{j,d}^{(\text{red})}(\cdot) - ((d+1)! \lambda V)^2 \cdot | \cdot | \) be of bounded total variation.

Then the following formulae hold:

\[
\mathbb{E} N_o(N_o - 1) = \frac{1}{\lambda} \sum_{j=1}^{d} \frac{\beta_{j,d}^{(\text{red})}(\mathbb{R}_d^d)}{(j-1)!(d-j+1)!},
\]

\[
\sigma_{\psi}^2 = \lambda V + \sum_{j=1}^{d} \frac{\beta_{j,d}^{(\text{red})}(\mathbb{R}_d)}{j![(d-j+1)!]^2} + \lim_{n \to \infty} \frac{\beta_{j,d}^{(\text{red})}(A_n)}{((d+1)!|^2)} - \lambda V
\]

**Proof.** In accordance with the above notation we put

\[
Y_j = Y_1 Y_2 \cdots Y_d \cdots Y_d \cdots Y_d \cdots Y_{j-1} \cdots Y_{j-1} \cdots Y_{j-1} \cdots Y_{j-1} \cdots Y_{j-1} \cdots Y_{j-1} = \{ \psi \in N : \psi(b_{j,d}^{(i)})(x_{j,d}^{(i)}) = 0 \} \quad \text{for } j = 1, \ldots, d, d+1.
\]

Starting from (1.4) and repeating with obvious changes the steps leading to (2.2) it is easily seen that \( \mathbb{E} N_o(N_o - 1) \) equals

\[
\frac{1}{(d+1)!} \sum_{j=0}^{d-1} \sum_{i=1}^{d-j} \prod_{i=1}^{d-j} 1_{\psi}(\epsilon_{d-1} + r_{d+1}^{(i)})(x_{d+1}^{(i)}) \ 1_{\psi}(\epsilon_{d-1} + r_{d+1}^{(i)})(x_{d+1}^{(i)}) \ d(x_{d+1}^{(i)}) \ \text{for } j = 1, \ldots, d.
\]

Applying the refined Campbell formula (1.3) w.r.t. the Palm distribution \( P_{\psi} \) we are able to express the latter integral in terms of the void probability \( P_{\psi}^{(j,d)} \) and \( P_{\psi}^{(j,d)}(Y_{d+1} \cap Y_{j+1}^{d+1}) \) and the reduced factorial moment measures \( \alpha_{V,\text{red}}^{(2d-j+1)} \). Finally, comparing this new integral with (2.3) reveals that

\[
\mathbb{E} N_o(N_o - 1) = \frac{1}{\lambda} \sum_{j=0}^{d-1} \frac{\beta_{j+1,d}^{(\text{red})}(\mathbb{R}_d)}{j![(d-j)!]^2}.
\]
To verify the second formula (2.7) we first note that, by means of (2.5), the variance of the number of nodes in a set $A_n$ can be rewritten as

$$
\lambda_V |A_n| + \lambda_V \int_{A_n} \left( \alpha_{V,red}^{(2)}(A_n - x) - \lambda_V |A_n| \right) \, dx
$$

$$
= \lambda_V |A_n| + \sum_{j=0}^{d} \frac{1}{j! \| (d-j+1) \|^2} \int_{\mathbb{R}^d} |A_n \cap (A_n - x)| \beta_{j,d}^{(red)}(dx) - \lambda_V^2 \int_{\mathbb{R}^d} |A_n \cap (A_n - x)| \, dx.
$$

Finally, in view of the definition of $\sigma_V^2$ and the assumptions imposed on the measures $\beta_{j,d}^{(red)}$ and the sets $A_n$ the validity of (2.7) follows by Lebesgue’s dominated convergence theorem.

3 Pair correlation function of the nodes in a $d$-dimensional Poisson Voronoi tessellation

If $\Psi \sim P$ is additionally isotropic then $\lambda_V$ and the second moment function $K_V : [0, \infty) \mapsto [0, \infty)$, defined by

$$
\lambda_V K_V(r) \triangleq \alpha_{V,red}^{(2)}(b(o, r)) = \frac{1}{\lambda_V^2} \sum_{j=0}^{d} \beta_{j,d}^{(red)}(b(o, r)) \| (d-j+1) \|^2, \quad r > 0,
$$
determine the first and second moment properties of the motion-invariant point process of nodes $\Psi_V \sim P_V$ completely.

Moreover, if the second product density $p_V^{(2)}$ of $\Psi_V \sim P_V$ exists we may describe its second moment properties by the PCF $g_V : [0, \infty) \mapsto [0, \infty]$ which is defined by

$$
\lambda_V^2 g_V(r) \triangleq \mu^{(2)}(x) \quad \text{for} \quad x \in \mathbb{R}^d \quad \text{with} \quad \|x\| = r,
$$
or equivalently, $K_V(r) = d \omega_d \int_0^r g_V(s) \, s^{d-1} \, ds$ for $r > 0$.

By using the infinitesimal conditional probability interpretation of the Palm distribution $(P_V)_{\mathbb{R}^d}(\cdot)$, see Daley and Vere-Jones (1988), we may alternatively define $g_V(r)$ by the following limit:

$$
g_V(r) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{\alpha^{(2)}(b(o, \varepsilon) \times (b(o, r + \delta) \setminus b(o, r)))}{\lambda_V^2 \| b(o, \varepsilon) \| \| b(o, r + \delta) \setminus b(o, r) \|}.
$$

In this way we can interpret $g_V(r)$ as “measure of frequency” of pairs of nodes in $V(\Psi)$ having distance $r > 0$. The estimation of the second moment or pair correlation function is a central point of the statistical second-order analysis of motion-invariant point processes in $\mathbb{R}^d$, $d \geq 2$. For a detailed discussion of this topic we refer to Stoyan et al. (1995). In order to apply these methods to the statistical analysis of (Voronoi) tessellations we need an explicit or at least numerically tractable representation of $K_V$ and $g_V$, respectively.

Henceforth, the PCF of the nodes in a stationary PVT with cell intensity $\lambda > 0$ is denoted by $g_{V,\lambda}$. By the scale transformation

$$
g_{V,\lambda}(r) = g_{V,1}(\lambda^{1/d} r) \quad \text{for} \quad r > 0
$$

we may confine ourselves to the standard case $\lambda = 1$. For the sake of completeness, we mention here without proof that the PCF of the nodes in a one-dimensional PVT (which consists of the midpoints between consecutive Poisson points on the real line) takes the form:

$$
g_{V,\lambda}(r) = 1 + e^{-2\lambda r} (2\lambda r - 1), \quad r \geq 0.
$$

In the remaining part of this section we show the existence of the PCF $g_{V,\lambda}(r)$ for stationary planar and higher-dimensional PVT’s by direct calculation.

For this purpose we introduce $d$-dimensional spherical coordinates and a family of orthogonal $d \times d$-matrices with positive determinant and give a formula of the $d$-dimensional volume of two intersecting balls in $\mathbb{R}^d$. 

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Heinrich and Mache: Nodes in a Voronoi tessellation
3.1 Spherical coordinates on $S^{d-1}$

For $d \geq 2$, the mapping $\Theta_{d-1} \ni \theta = (\vartheta_1, \ldots, \vartheta_{d-1}) \mapsto s(\theta) = (s_1(\theta), \ldots, s_d(\theta))' \in S^{d-1}$ given by

\[
\begin{align*}
    s_1(\theta) &= \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-2} \sin \vartheta_{d-1}, \\
    s_2(\theta) &= \cos \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-2} \sin \vartheta_{d-1}, \\
    s_3(\theta) &= \cos \vartheta_2 \cdots \sin \vartheta_{d-2} \sin \vartheta_{d-1}, \\
    &\vdots \\
    s_{d-1}(\theta) &= \cos \vartheta_{d-2} \cdot \sin \vartheta_{d-1}, \\
    s_d(\theta) &= \cos \vartheta_{d-1},
\end{align*}
\]

for $-\pi \leq \vartheta_1 \leq \pi$ and $0 \leq \vartheta_2, \ldots, \vartheta_{d-1} \leq \pi$ establishes a unique representation of (Lebesgue-almost) every point on the $d$-dimensional unit sphere by a $(d-1)$-dimensional vector of spherical coordinates.

For describing an orientation-preserving rotation of the orthonormal basis of $\mathbb{R}^d$ which shifts $e_d = (0, \ldots, 0, 1)'$ to $s(\theta)$, we define an orthogonal $d \times d$-matrix $R_d(\theta)$, $\theta \in \Theta_{d-1}$, with $\det R_d(\theta) = 1$ as follows:

\[ R_d(\theta) \, e_i = s(\theta)_i \quad \text{for} \quad i = 1, \ldots, d, \]

where $e_i = (0, \ldots, 1, \ldots, 0)'$ designates the $i$-th unit (column) vector in $\mathbb{R}^d$, $\theta_d = \theta$ and

\[ \theta_i = \left( \vartheta_1, \ldots, \vartheta_{i-1}, \vartheta_i + \frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2} \right) \quad \text{for} \quad i = 1, \ldots, d-1. \]

In other words, $s(\theta_j)$ is the $j$-th column of the orthogonal matrix $R_d(\theta) = (s_j(\theta_i))_{i,j=1}^d$, where $s_i(\theta_j) = 0$ for $i = j + 2, \ldots, d$ and $j = 1, \ldots, d - 2$.

In particular, for $d = 2$ and $d = 3$,

\[ R_2(\theta) = \begin{pmatrix} \cos \vartheta_1 & \sin \vartheta_1 \\ -\sin \vartheta_1 & \cos \vartheta_1 \end{pmatrix} \quad \text{and} \quad R_3(\theta) = \begin{pmatrix} \cos \vartheta_1 & \sin \vartheta_1 \cos \vartheta_2 & \sin \vartheta_1 \sin \vartheta_2 \\ -\sin \vartheta_1 & \cos \vartheta_1 \cos \vartheta_2 & \cos \vartheta_1 \sin \vartheta_2 \\ 0 & -\sin \vartheta_2 & \cos \vartheta_2 \end{pmatrix}. \]

3.2 Lebesgue measure of the union of two intersecting balls

Assume that $b(c_1, r_1)$ and $b(c_2, r_2)$ possess a common point set with positive Lebesgue measure which is uniquely determined by the distance of their centers $r = \|c_1 - c_2\|$ and their radii $r_1$ and $r_2$. We prefer to express $r_1$ and $r_2$ by more suitable parameters $\rho_1$ and $\rho_2$ given by

\[ \rho_1 = (r_1 - r_2)/r \quad \text{and} \quad \rho_2 = (r_1 + r_2)/r \]

so that with the abbreviation

\[ \nu_d(\rho_1, \rho_2) \triangleq \nu \left( a, \frac{\rho_1 + \rho_2}{2} \right) \cup \nu \left( e_d, \frac{\rho_2 - \rho_1}{2} \right), \]

we may write

\[ |b(c_1, r_1) \cup b(c_2, r_2)| = r^d \nu_d(\rho_1, \rho_2) = \left| b \left( a, \frac{r}{2}(\rho_1 + \rho_2) \right) \cup b \left( e_d, \frac{r}{2}(\rho_2 - \rho_1) \right) \right|, \]

whence it follows that

\[ \lim_{r \to 0} r^d \nu_d \left( \rho_1, \frac{\rho_2}{r} \right) = \left| b(a, \rho_2/2) \right| = \omega_d \left( \frac{\rho_2}{2} \right)^d \quad \text{for all} \quad \rho_1, \rho_2 \geq 0. \]

Further, it is easily verified that

\[
\begin{align*}
    b(c_1, r_1) \cap b(c_2, r_2) &\neq \emptyset \quad \text{iff} \quad \rho_2 \geq 1, \\
    b(c_1, r_1) \subseteq b(c_2, r_2) &\quad \text{iff} \quad \rho_2 \geq 1 \quad \text{and} \quad \rho_1 \leq -1, \\
    b(c_1, r_1) \supseteq b(c_2, r_2) &\quad \text{iff} \quad \rho_2 \geq 1 \quad \text{and} \quad \rho_1 \geq 1.
\end{align*}
\]

We are mainly interested in the non-trivial intersection of the balls, that is, when $|\rho_1| \leq 1$ and $\rho_2 \geq 1$. Since, for

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reasons of symmetry, \( \nu_d(\rho_1, \rho_2) = \nu_d(-\rho_1, \rho_2) \), we only need to consider the case \( 0 \leq \rho_1 \leq 1 \), i.e. \( 0 \leq \alpha \leq \frac{\pi}{2} \), see Fig. 1.

![Fig. 1](image)

**Fig. 1** The two cases of penetration of \( b(c_1, r_1) \) and \( b(c_2, r_2) \) (projected into the \((e_1, e_d)\)–plane) for \( r_1 \geq r_2 \) (i.e. \( 0 \leq \rho_1 \leq 1 \) or \( 0 \leq \alpha \leq \frac{\pi}{2} \)) with \( P_{1, 2} = \left( \frac{\pi}{2}, \sqrt{(1 - \rho_1^2)(\rho_2^2 - 1)}, 0, \ldots, 0, \frac{\pi}{2}(1 + \rho_1 \rho_2) \right) \).

Case (i): \( r_1^2 - r_2^2 \leq r^2 \)

i.e. \( \rho_1 \rho_2 \leq 1 \) or \( \frac{\pi}{2} \leq \beta \leq \pi \)

Case (ii): \( r_1^2 - r_2^2 \geq r^2 \)

i.e. \( \rho_1 \rho_2 \geq 1 \) or \( 0 \leq \beta \leq \frac{\pi}{2} \)

For any \( d \geq 1 \), we have

\[
\nu_d(\rho_1, \rho_2) = \frac{\omega_{d-1}}{2} \left( (\rho_1 + \rho_2)^d c_d(\rho_1, \rho_2) + (\rho_2 - \rho_1)^d c_d(-\rho_1, \rho_2) \right), \quad (3.4)
\]

where

\[
c_d(\rho_1, \rho_2) = \int_\alpha^\pi \sin^d x \, dx = \begin{cases} \displaystyle \frac{\omega_d}{2 \omega_{d-1}} \left( 1 + \cos \alpha + \frac{\cos \alpha}{\pi} \sum_{i=1}^{(d-1)/2} \frac{\omega_{2i-1}}{\omega_{2i}} \sin^{2i} \alpha \right) & \text{if } d \geq 1 \text{ odd} \\ \displaystyle \frac{\omega_d}{\pi \omega_{d-1}} \left( \pi - \alpha + \frac{\cos \alpha}{2} \sum_{i=1}^{d/2} \frac{\omega_{2i-1}}{\omega_{2i-2}} \sin^{2i-1} \alpha \right) & \text{if } d \geq 2 \text{ even} \end{cases}
\]

with \( 0 \leq |\rho_1| \leq 1 \), \( \rho_2 \geq 1 \) and \( \alpha = \alpha(\rho_1, \rho_2) \in [0, \pi] \), see also Muche (2005) for an alternative representation of this integral in terms of \( \Gamma \)–functions.

The one-to-one correspondence between \( \alpha = \alpha(\rho_1, \rho_2) \) resp. \( \beta = \beta(\rho_1, \rho_2) \) and the pair \( \rho_1, \rho_2 \) can be obtained by elementary manipulations with trigonometric functions:

\[
\sin \alpha = \frac{\sqrt{(1 - \rho_1^2)(\rho_2^2 - 1)}}{\rho_1 + \rho_2} \quad \text{and} \quad \cos \alpha = \frac{1 + \rho_1 \rho_2}{\rho_1 + \rho_2}, \quad (3.5)
\]

\[
\sin \beta = \frac{\sqrt{(1 - \rho_1^2)(\rho_2^2 - 1)}}{\rho_2 - \rho_1} \quad \text{and} \quad \cos \beta = \frac{1 - \rho_1 \rho_2}{\rho_2 - \rho_1}.
\]

In particular, for \( d = 2 \),

\[
(\rho_1 + \rho_2)^2 \ c_2(\rho_1, \rho_2) = \frac{1}{2} \left( (1 + \rho_1 \rho_2) \sqrt{(1 - \rho_1^2)(\rho_2^2 - 1)} + (\rho_1 + \rho_2)^2 \left( \pi - \arccos \frac{1 + \rho_1 \rho_2}{\rho_1 + \rho_2} \right) \right)
\]
and, for $d = 3$,

$$(\rho_1 + \rho_2)^3 c_0(\rho_1, \rho_2) = \frac{1}{3} (2(\rho_1 + \rho_2)^3 + 3(1 + \rho_1 \rho_2)(\rho_1 + \rho_2)^2 - (1 + \rho_1 \rho_2)^3).$$

We are now ready to replace the $d \times (2d - j + 2)$ coordinates $x_0, x_1, \ldots, x_{2d-j+1} \in \mathbb{R}^d$ in the integrand of $\beta_{j,d}(b(\alpha, \varepsilon) \times (b(\alpha, \varepsilon + \delta) \setminus b(\alpha, \varepsilon)))$ by $r, \rho_1, \rho_2, c = c_{d+1} = (c_1, \ldots, c_d)'$ and the spherical coordinates $\theta, \phi_0, \ldots, \phi_{2d-j+1}, \varphi_0, \ldots, \varphi_{d-j}, \varphi_{d+1}, \ldots, \varphi_{2d-j+1}$. The center of the circumball $b(c_j, r_j)$ is then given by

$$c_j = c + r R_2(\theta) e_d = c + r s(\theta)$$

for $\theta \in \Theta_{d-1},$

whereas the spherical coordinates $(\phi_0, \varphi_0), \ldots, (\phi_{d-j}, \varphi_{d-j}) \in \Theta_{d-2} \times [0, \pi]$ resp. $(\phi_{d-j+1}, \varphi_{d-j+1}) \in \Theta_{d-2} \times [0, \beta]$ describe the positions of the points $x_{d-j+1}, \ldots, x_{d-1}$ resp. $x_{d-j+1}, \ldots, x_{2d-j+1}$ on the three spheres

(i) $\partial b(c, \frac{\tau}{2}(\rho_1 + \rho_2)) \setminus b(c_j, \frac{\tau}{2}(\rho_2 - \rho_1))$ resp.

(ii) $\partial b(c, \frac{\tau}{2}(\rho_1 + \rho_2)) \cap \partial b(c_j, \frac{\tau}{2}(\rho_2 - \rho_1))$ ($(d - 1)$-dimensional sphere) resp.

(iii) $\partial b(c, \frac{\tau}{2}(\rho_2 - \rho_1)) \setminus b(c, \frac{\tau}{2}(\rho_1 + \rho_2))$

for $j = 1, \ldots, d$. In case of $j = 0$ (that is, none of the points $x_0, \ldots, x_{2d+1}$ lies in the sphere (ii)) we distinguish the cases

(a) $b(c, \frac{\tau}{2}(\rho_1 + \rho_2)) \cap b(c_j, \frac{\tau}{2}(\rho_2 - \rho_1)) \neq \emptyset$, i.e. $\rho_2 \geq 1$

and

(b) $b(c, \frac{\tau}{2}(\rho_1 + \rho_2)) \cap b(c_j, \frac{\tau}{2}(\rho_2 - \rho_1)) = \emptyset$, i.e. $0 \leq \rho_2 < 1$.

We consider the cases $d = 2$ and $d \geq 3$ separately.

For $d = 3$ and $j = 0, 1, \ldots, d$, we introduce the coordinate transformations

$$T_{3,d} : \mathbb{R}^d \times [-1, 1] \times [0, \infty) \times [0, \pi]^{2d-j+1} \times \Theta_{d-2}^{2d-j+2} \longrightarrow (\mathbb{R}^d)^{2d-j+2} :$$

$$x_i = c + r(R(\theta) R_2(\varepsilon))_{i,d} = c + r(R_2(\theta) + r s(\theta))$$

for $\phi_1 = (\varphi_{i,d-2}) \in \Theta_{d-2}$ and $\varphi_1 \in [0, \pi]$ for $i = 0, \ldots, d - j$,

$x_i = c + r(R(\theta) R_2(\varepsilon))_{i,d} = c + r(R_2(\theta) + r s(\theta))$

for $\phi_1 = (\varphi_{i,d-2}) \in \Theta_{d-2}$ and $\varphi_1 \in [0, \beta]$ for $i = d - j + 1, \ldots, d$.

For $d = 2$ and $j = 0, 1, 2$, we use the coordinate transformations

$$T_{j,2} : \mathbb{R}^2 \times [-1, 1] \times [0, \infty) \times [-\pi, \pi] \times [0, \infty) \times [-\pi, \pi]^{6-j-2} \longrightarrow (\mathbb{R}^2)^{6-j} :$$

$$j = 2 :$$

$$x_0 = \frac{(c_1)}{(c_2)} + \frac{r}{2}(\rho_1 + \rho_2) \left( \frac{\sin(\theta + \varphi_0)}{\cos(\theta + \varphi_0)} \right), \quad \varphi_0 \in [-\pi, -\alpha] \cup [\alpha, \pi],$$

$$x_1 = \frac{(c_1)}{(c_2)} + \frac{r}{2}(\rho_1 + \rho_2) \left( \frac{\sin(\theta \pm \delta)}{\cos(\theta \pm \delta)} \right),$$

$$x_2 = \frac{(c_1)}{(c_2)} + \frac{r}{2}(\rho_1 + \rho_2) \left( \frac{\sin(\theta \pm \alpha)}{\cos(\theta \pm \alpha)} \right),$$

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\[ x_3 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + r \begin{pmatrix} \sin \vartheta \\ \cos \vartheta \end{pmatrix} + \frac{r}{2}(\rho_2 - \rho_1) \begin{pmatrix} \sin(\vartheta + \varphi_3) \\ \cos(\vartheta + \varphi_3) \end{pmatrix}, \quad \varphi_3 \in [-\beta, \beta], \]

\[ j = 1: \]

\[ x_i = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{r}{2}(\rho_1 + \rho_2) \begin{pmatrix} \sin(\vartheta + \varphi_i) \\ \cos(\vartheta + \varphi_i) \end{pmatrix}, \quad \varphi_i \in [-\pi, -\alpha] \cup [\alpha, \pi], \quad i = 0, 1, \]

\[ x_2 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{r}{2}(\rho_1 + \rho_2) \begin{pmatrix} \sin(\vartheta + \varphi_i) \\ \cos(\vartheta + \varphi_i) \end{pmatrix}, \quad \varphi_i \in [-\beta, \beta], \quad i = 3, 4, \]

\[ j = 0: \]

\[ x_i = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{r}{2}(\rho_1 - \rho_2) \begin{pmatrix} \sin(\vartheta + \varphi_i) \\ \cos(\vartheta + \varphi_i) \end{pmatrix}, \quad \frac{r}{2}(\rho_1 - \rho_2) \begin{pmatrix} \sin(\vartheta + \varphi_i) \\ \cos(\vartheta + \varphi_i) \end{pmatrix}, \quad \varphi_i \in [-\beta, \beta], \quad i = 3, 4, 5, \]

\[ \text{for } c_1, c_2 \in \mathbb{R}^1, -\pi \leq \vartheta \leq \pi, -1 \leq \rho_1 \leq 1, \rho_2 \geq 1 \text{ (resp. } |\rho_1| \leq \rho_2, 0 \leq \rho_2 \leq 1), r \geq 0. \]

In order to apply the integral transformation formula we need the Jacobian \( J_{j,d} \) of the coordinate transformation \( T_{j,d} \) which is formally defined to be a \( d(2d - j + 2) \times d(2d - j + 2) \) functional determinant consisting of the partial derivatives of the coordinates of the vector \( (x_0, \ldots, x_{2d-j+1}) \in \mathbb{R}^{2d-j+1} \) w.r.t. the coordinates of the vector \( (c, \rho_1, \rho_2, \vartheta, \varphi_0, \ldots, \varphi_{d-j}, \varphi_{d-j+1}, \ldots, \varphi_{2d-j+1}, \phi_{d-j+1}, \ldots, \phi_{2d-j+1}) \).

The following lemma shows that \( J_{j,d} \) can be expressed as product of some elementary functions and two \((d + 1) \times (d + 1)\)-determinants with a comparatively simple structure. Note that \( J_{j,d} \) does not depend on \( \mathbf{c} = (c_1, \ldots, c_d)^T \).

**Lemma 3.1** For \( d \geq 3 \) and \( j = 0, 1, \ldots, d \), the Jacobian \( J_{j,d} \) is (up to the sign) equal to

\[
2^{d-j} \left( \begin{array}{c} \frac{r}{2} \end{array} \right)^{d(2d-j+1)-1} (\rho_1 + \rho_2)^{(d-1)(d+1)} (\rho_2 - \rho_1)^{(d-1)(d-j+1)} \left( \frac{\sin^{d-2} \alpha}{\sin \beta} \right)^j \\
\times \prod_{i=0}^{2d-j+1} (\sin \varphi_{1,2} \sin^2 \varphi_{i,3} \ldots \sin^{d-3} \varphi_{i,d-2}) \prod_{i=0}^{d-j} \sin^{d-2} \varphi_i \prod_{i=d+1}^{2d-j+1} \sin^{d-2} \varphi_i \\
\times \sin \vartheta \sin^2 \vartheta \ldots \sin^{d-2} \vartheta \sin^{d-2} \vartheta \sin^{d-2} \vartheta \\
\text{and} \]

\[
D_{j,d}^{(1)} = D_{j,d}^{(2)}(\rho_1, \rho_2, \varphi_0, \ldots, \varphi_{d-j}, \phi_0, \ldots, \phi_d) \\
\text{take the form} \]

\[
D_{j,d}^{(1)} = \det \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \alpha \\
\end{pmatrix} \quad \text{and} \quad D_{j,d}^{(2)} = \det \begin{pmatrix} 1 \\ \vdots \\ 1 \\ \phi_{d-j+1} \\
\end{pmatrix}.
\]
For \( d = 2 \) and \( j = 1, 2 \), the rhs of (3.6) must be multiplied by the factor \( 2 \). This results from the two possibilities of choosing the sign of \( \alpha \) in the coordinate transformations \( T_{1,2} \) and \( T_{2,2} \).

The proof of Lemma 3.1 makes use of the well-known transformation rules for determinants. However, the combination of all these steps is rather involved and requires a lot of manipulations. A detailed proof is given in the report Heinrich and Muche (1994), copies of which can be obtained from the corresponding author and the pdf-file of this report is available on http://www.math.uni-augsburg.de/stochastik/heinrich/papers/pcfvoro.pdf.

**Remark 3.2** For the special cases \( j = 0 \) and \( j = d \) we have:

\[
D_{0,d}^{(1)} = \det \begin{pmatrix} 1 & s(\phi_0, \varphi_0)' \\ \vdots & \vdots \\ 1 & s(\phi_d, \varphi_d)' \end{pmatrix}, \quad D_{0,d}^{(2)} = \det \begin{pmatrix} 1 & s(\phi_{d+1}, \varphi_{d+1})' \\ \vdots & \vdots \\ 1 & s(\phi_{2d+1}, \varphi_{2d+1})' \end{pmatrix} \tag{3.7}
\]

and, after a further reduction of the order of \( D_{d,d}^{(1)} \) and \( D_{d,d}^{(2)} \),

\[
D_{d,d}^{(1)} = \sin^{d-1} \alpha (\cos \varphi_0 - \cos \alpha) D_{d-1},
\]

\[
D_{d,d}^{(2)} = \sin^{d-1} \beta (\cos \varphi_{d+1} - \cos \beta) D_{d-1} \quad \text{with} \quad D_{d-1} = \det \begin{pmatrix} 1 & s(\phi_1)'
\vdots & \vdots \\ 1 & s(\phi_d)' \end{pmatrix}.
\]

Here we should mention the well-known fact that \( |D_{0,d}^{(1)}/d!| \) is equal to the \( d \)-dimensional Lebesgue measure of the \( d \)-simplex spanned by the unit vectors \( s(\phi_0, \varphi_0), \ldots, s(\phi_d, \varphi_d) \in S^{d-1} \). Likewise,

\[
|D_{0,d}^{(2)}/d!| = |S(s(\phi_{d+1}, \varphi_{d+1}), \ldots, s(\phi_{2d+1}, \varphi_{2d+1}))|.
\]

Now, introducing the new coordinates it follows from Theorem 2.1 and Lemma 3.1 that

\[
\beta_{j,d}(b(o, \varepsilon) \times (b(o, r + \delta) \setminus b(o, r))) = \lambda^2 \int_{b(o,c)} \int_0^\infty \int_{-1}^1 \exp \left\{ - \lambda \rho^p \nu_d(\rho_1, \rho_2) \right\} \lambda \rho^{2d-j} \rho^{d-1}
\]

\[
\times \int_{\theta_{d-1}} \mathbf{1}_{b(o,r+\delta)\setminus b(o,r)}(\mathbf{c} + \rho \mathbf{s}(\theta)) I_{j,d}(\rho_1, \rho_2, \alpha, \beta) \sin \vartheta_2 \sin^2 \vartheta_3 \ldots \sin^{d-2} \vartheta_{d-1} d\vartheta d\rho_1 d\rho_2 d\rho d\mathbf{c}
\]

for \( j = 1, \ldots, d \), where

\[
I_{j,d}(\rho_1, \rho_2, \alpha, \beta) \triangleq \left( \frac{\rho_2^2 - \rho_1^2}{2^{d-1}(2d-j+2)+1} \right)^{d-1}(\rho_2 - \rho_1)^{d-1} \sin^d \beta \]

\[
\times \prod_{i=0}^{d-j+1} \sin^{d-2} \varphi_i \sin^2 \varphi_i, \beta \ldots \sin^{d-3} \varphi_i, \beta \ldots \sin \varphi_{i-1} \sin \varphi_{i+1} d(\varphi_0, \ldots, \varphi_{d-j-1})
\]

\[
\times \left( \frac{\sin^{d-2} \alpha}{(\rho_2 - \rho_1)^{d-1} \sin^2 \beta} \right)^j \Delta_{j,d}(\rho_1, \rho_2).
\]

with

\[
\Delta_{j,d}(\rho_1, \rho_2) \triangleq \int_{\varepsilon_{d-j}}^{\rho_{d-j}} \int_{0, \varepsilon_{d-j}}^{\varphi_{d-j}} \int_{\varphi_{d-j+1}}^{\varphi_{d-j+1}} \left| \frac{D_{j,d}^{(1)}}{D_{j,d}^{(2)}} \right| d(\varphi_0, \ldots, \varphi_{2d-j-1})
\]

\[
\times \prod_{i=0}^{d-j} \sin^{d-2} \varphi_i d(\varphi_0, \ldots, \varphi_{d-j}) \prod_{i=d-j+1}^{2d-j+1} \sin^{d-2} \varphi_i d(\varphi_{d-j+1}, \ldots, \varphi_{2d-j+1}).
\]

To find the corresponding representation for \( \beta_{d,d}(b(o, \varepsilon) \times (b(o, r + \delta) \setminus b(o, r))) \) we have additionally to consider the case \( b(o, 1/2(\rho_1 + \rho_2)) \cap b(e_1, 1/2(\rho_2 - \rho_1)) = \emptyset \) (and hence \( \alpha = 0, \beta = \pi \)) which is equivalent to
Next we will simplify the parameter integrals the abbreviation expression
\[
\begin{align*}
\lambda^2 \int_0^\infty \int_0^\infty \int_{-\rho_2}^{\rho_2} \exp \left\{ -\frac{r^d}{\lambda^2} \omega_d \left[ (\rho_1 + \rho_2)^d + (\rho_2 - \rho_1)^d \right] \right\} (\lambda \rho^d)^{2d} \rho^{d-1} \\
\times \int_{\Theta_{d-1}} 1_{b(o,r+a)}(c) \frac{\rho s(\theta)}{\lambda^2} \left( \sin \psi_2 \sin^2 \psi_3 \cdots \sin^{d-2} \psi_{d-1} \right) d\theta \, d\rho_1 \, d\rho_2 \, d\rho \, dc,
\end{align*}
\]
where, in view of (3.7), \( I_{0,d}(\rho_1, \rho_2, 0, \pi) = V_d^2 (\rho_2 - \rho_1)^{(d-1)(d+1)} / 2^{2(d-1)(d+1)} \) with
\[
V_d \triangleq \int_{\Theta_{d-1}} D^{(1)}_{0,d} \prod_{i=0}^{d-1} (\sin \varphi_{i,2} \sin^2 \varphi_{i,3} \cdots \sin^{d-2} \varphi_{i,d-2} \sin^{d-2} \varphi_i) d(\phi_0, \varphi_0, \ldots, \phi_d, \varphi_d).
\]
Taking into account the geometrical interpretation of the determinants (3.7), the expected \( d \)-dimensional volume of a \( d \)-simplex whose vertices \( X_0, \ldots, X_d \) are uniformly distributed on the unit sphere \( S^{d-1} \) is equal to
\[
E[S(X_0, \ldots, X_d)] = \frac{V_d}{d! (d-\omega_d)^{d+1}}.
\]
On the other hand, the computation of the intensity \( \lambda_V \) for a PVT (see Møller (1994) or Miles (1971)) yields that
\[
\kappa_d = E[S(X_0, \ldots, X_d)] \frac{d! (d-1) \omega_d}{d+1} = \frac{V_d}{d^2 (d+1) \omega_d}.
\]
The above integral representations reveal that the following limits exist:
\[
\lim_{\delta \to 0} \beta_{j,d} \frac{\beta_{j,d}(b(o, \rho) \times (b(o, r+\delta) \setminus b(o, r)))}{|b(o, \rho)| |b(o, r+\delta) \setminus b(o, r)|} \leq \begin{cases} 
  g_{j,d}(\lambda^{1/d} \rho) & \text{for } j = 1, \ldots, d, \\
  g_{0,d}(\lambda^{1/d} \rho) + g_{1,d}(\lambda^{1/d} \rho) & \text{for } j = 0,
\end{cases}
\]
where
\[
g_{0,d}(\rho) = \frac{4V_d}{\kappa_d^2} \left( \frac{r^d}{2^d} \int_0^\infty \int_0^{\rho_2} \exp \left\{ -\frac{r^d}{\lambda^2} \omega_d \left[ (\rho_1 + \rho_2)^d + (\rho_2 - \rho_1)^d \right] \right\} (\rho_2^d - \rho_1^d)^{d-1} \, d\rho_1 \, d\rho_2,
\]
\[
g_{j,d}(\rho) = \frac{2^{2-j}}{\kappa_d^2} \left( \frac{r^d}{2^d} \int_1^\infty \int_0^\infty \exp \left\{ -r^d \nu_d(\rho_1, \rho_2) \right\} \times (\rho_2^d - \rho_1^d)^{(d-1)(d-1)+j} \left[ (1 - \rho_1^d) (\rho_2^d - 1)^{(d-3)} - 2 \right] d\rho_1 \, d\rho_2 \right)
\]
for \( j = 0, 1, \ldots, d \), \( d \geq 3 \). Note that for \( d = 2 \) and \( j = 1, 2 \) the right-hand side of (3.9) must be doubled, see Lemma 3.1.

Here we have used the relations (3.5), \( \lambda_V = \kappa_d \lambda \) (see Lemma 1.3), and the integral
\[
\int_{\Theta_{d-1}} \sin \psi_2 \sin^2 \psi_3 \cdots \sin^{d-2} \psi_{d-1} d(\psi_1, \psi_2, \ldots, \psi_{d-1}) = d \omega_d.
\]
Next we will simplify the parameter integrals \( g_{0,d}^* \) and \( g_{j,d} \). First, the term \( \Delta_{j,d}(\rho_1, \rho_2) \) can be rewritten by using the abbreviation
\[
P_{j,d}(\rho_1, \rho_2, \phi_d, \phi_d - 1, \ldots, \phi_d) \triangleq \int_{\Theta_{d-1}} 1_{b(o,r+a)}(c) \frac{\rho s(\theta)}{\lambda^2} \left( \sin \psi_2 \sin^2 \psi_3 \cdots \sin^{d-2} \psi_{d-2} \sin^{d-2} \psi_d \right) d(\phi_0, \varphi_0, \ldots, \phi_{d-j})
\]
\[
\times \prod_{i=0}^{d-j} (\sin \varphi_{i,2} \sin^2 \varphi_{i,3} \cdots \sin^{d-3} \varphi_{i,d-2} \sin^{d-2} \varphi_i) d(\phi_0, \varphi_0, \ldots, \phi_{d-j}).
\]
Note that, in view of the volume interpretation of the number $|D_{j,d}|$, the term $P_{j,d}(\rho_1, \rho_2, \phi_{d-j+1}, \ldots, \phi_d)$ can be expressed by

$$
d! \left( (d-1)\omega_{d-1} \int_0^\pi \sin^{d-2} \varphi \, d\varphi \right)^{d-j+1} E[|S(X_0, \ldots, X_{d-j}, s(\phi_{d-j+1}, \alpha), \ldots, s(\phi_d, \alpha))|],$$

where the (conditional) expectation $E$ is taken w.r.t. the independent and uniformly distributed random points $X_0, \ldots, X_{d-j} \in \{\phi(\varphi, \phi) : \phi \in \Omega_{d-2}, \varphi \in [\alpha, \pi] \} \subset S^{d-1}$.

Now, substituting $\varphi_i$ by $-\varphi_i - \pi$ for $i = d + 1, \ldots, 2d - j + 1$ and making use of the relation

$$\beta(\rho_1, \rho_2) = \arccos \frac{1 - \rho_1 \rho_2}{\rho_1 - \rho_2} = \pi - \alpha(-\rho_1, \rho_2)$$

shows that the integral

$$\int_{\Theta_{d-2}^{d-j+1}} \int_{[0,\pi]^d} |D_{j,d}(\rho_1, \rho_2, \phi_{d-j+1}, \ldots, \phi_d)| \prod_{i=d+1}^{2d-j+1} \sin^{d-2} \varphi_i \times d(\varphi_{i+1}, \ldots, \varphi_{2d-j+1}) \prod_{i=d+1}^{2d-j+1} \sin \varphi_{i-2} \sin \varphi_{i-3} \cdots \sin \varphi_{i-3} \varphi_{i,d-2}) \, d(\phi_{d+1}, \ldots, \phi_{2d-j+1})$$

coincides with $P_{j,d}(-\rho_1, \rho_2, \phi_{d-j+1}, \ldots, \phi_d)$.

Therefore,

$$\Delta_{j,d}(\rho_1, \rho_2) = \int_{\Theta_{d-2}^{d-j+1}} P_{j,d}(\rho_1, \rho_2, \phi_{d-j+1}, \ldots, \phi_d) P_{j,d}(-\rho_1, \rho_2, \phi_{d-j+1}, \ldots, \phi_d) \times \prod_{i=d+1}^{d+j+1} \left( \sin \varphi_{i-2} \sin \varphi_{i-3} \cdots \sin \varphi_{i-3} \varphi_{i,d-2} \right) \, d(\phi_{d+1}, \ldots, \phi_d). \quad (3.10)$$

By the coordinate transformation

$$r_1 = \frac{r}{2} \omega_d^{1/d} (\rho_1 + \rho_2) \quad \text{and} \quad r_2 = \frac{r}{2} \omega_d^{1/d} (\rho_2 - \rho_1)$$

the function $g_{0,d}$ becomes a considerably simpler parameter integral:

$$g_{0,d}(r) = \frac{V_d^2}{\kappa_d \omega_d} \int_0^{1/d} \int_0^{1/d} r_1^{d-1} r_2^{d-1} \, dr_1 \, dr_2 \times \left( \frac{V_d}{\kappa_d \omega_d} \right)^2 (d-1)! \int_0^{\omega_d r^d} f_d \left( \frac{\omega_d^{1/d} r - \rho^{1/d}}{d} \right)^d e^{-r^d} \, dr$$

where

$$f_d(x) = \frac{1}{(d-1)!} \int_0^x e^{-\rho \rho^{d-1}} \, d\rho = 1 - e^{x(1 + x + \cdots + x^{d-1}/(d-1)!)}.$$  

From (3.8) it is rapidly seen that

$$\frac{g_{0,d}(r)}{(d+1)!^2} = \frac{1}{(d-1)!} \int_0^{r^d} f_d \left( \frac{\omega_d^{1/d} r - \rho^{1/d}}{d} \right)^d e^{-\rho \rho^{d-1}} \, d\rho. \quad (3.11)$$

Splitting up the latter integral into two integrals over $[0, \omega_d r^d/2d]$ and $[\omega_d r^d/2d, \omega_d r^d]$ we get

$$1 - \frac{g_{0,d}(r)}{(d+1)!^2} \leq 2 \left( 1 - f_d(\omega_d r^d/2d) \right), \quad r \geq 0. \quad (3.12)$$
Hence, the integral
\[ \Gamma_{0,d}^{**} = d \omega_d \int_0^\infty \left(1 - \frac{g_{0,d}^*(r)}{((d+1)!)^2}\right) r^{d-1} \, dr \]
exists. It turns out after somewhat lengthy calculations (which are omitted here) that this integral can be calculated explicitly in a closed form:
\[ \Gamma_{0,d}^{**} = 2d + \pi \sum_{i=1}^{d-1} \frac{l}{\sin(\pi d / l)} \prod_{j=1}^{d-1} \left(i + \frac{l}{d} \right) \left(1 + \sum_{k=1}^{d-1} \frac{1}{k!} \prod_{j=1}^{k} \left(j - \frac{l}{d}\right)\right). \quad (3.13) \]

Finally, combining (2.1), (3.1) and the above definition of the functions \( g_{0,d}, \ldots, g_{d,d} \) and \( g_{0,d}^* \) we arrive at the following

**Theorem 3.3** The PCF \( g_{V,\lambda} \) of the point process of nodes \( \Psi_V \sim P_V \) by a \( d \)-dimensional stationary PVT \( V(\Psi) \) \((d \geq 2)\) with cell intensity \( \lambda > 0 \) exists and takes the form:
\[ g_{V,\lambda}(r) = \frac{g_{0,d}(\lambda^{1/d} r)}{[(d+1)!]^2} + \sum_{j=0}^{d} \frac{g_{j,d}(\lambda^{1/d} r)}{[(d - j + 1)!]^2} \quad \text{for} \quad r > 0, \quad (3.14) \]
where the functions \( g_{0,d}, \ldots, g_{d,d} \) and \( g_{0,d}^* \) are given by (3.9), (3.10) and (3.11), respectively.

**Remark 3.4** Besides PVT’s Poisson hyperplane tessellations form a further well-studied class of random subdivisions of \( \mathbb{R}^d \), see e.g. Schneider and Weil (2000) or Stoyan et al. (1995). In comparison with \( g_{V,\lambda}(r) \) the PCF \( g_{H,\lambda}(r) \) of the point process \( \Psi_H \sim P_H \) of intersection points (nodes) generated by a stationary and isotropic \( d \)-dimensional Poisson hyperplane process with intensity \( \lambda \) (mean total \((d-1)\)-volume of all hyperplanes in the unit cube \([0,1]^d\)) can be expressed simply by a weighted sum of power functions with negative exponents:
\[ g_{H,\lambda}(r) = 1 + \sum_{k=1}^{d-1} \binom{d-1}{k} \left(\frac{\omega_{d-k}}{\omega_d}\right)^2 \frac{d \omega_d}{\omega_{d-1}} \cdot \frac{1}{(kr)^k} \quad \text{for} \quad r > 0. \quad (3.15) \]

In distinction to the PCF \( g_{V,\lambda}(r) \) the function \( g_{H,\lambda}(r) - 1 \) is positive, strictly decreasing and approaches zero very slowly with order \( O(r^{-1}) \) as \( r \to \infty \). The latter fact reflects long-range dependences within Poisson hyperplane processes. In the special cases \( d = 2 \) and \( d = 3 \) formula (3.15) is seen from the shape of the corresponding \( K \)-function which is obtained by applying Slivnyak’s theorem to the Poisson line and plane process, see Arns et al. (2005). In Heinrich et al. (2006) a closed-form expression for the variance \( \text{Var} \Psi_H(b(o, r)) \) is polynomial of degree \( 2d - 1 \) in \( r \) and the relationship
\[ \text{Var} \Psi_H(b(o, r)) = \lambda_H \omega_d^d + d \omega_d \lambda_H^2 \int_0^{2r} |b(o, r) \cap b((u, 0, \ldots, 0), r)| \, g_{H,\lambda}(u) - 1 \, du, \]
where \( \lambda_H = \omega_d \omega_{d-1} / (d \omega_d)^d \lambda^d \) denotes the intensity of nodes, are used to derive (3.15) simply by comparison of coefficients.

In the planar case a more explicit form of \( g_{V,\lambda} \) is derived in the next section.


4 Pair correlation function of the nodes in a planar Poisson Voronoi tessellation

Putting \( d = 2 \) in (3.14) yields

\[
g_{V,2}(r) = \frac{g_{0,2}(\sqrt{\lambda r})}{36} + \frac{g_{0,2}(\sqrt{\lambda r})}{36} + \frac{g_{1,2}(\sqrt{\lambda r})}{4} + \frac{g_{2,2}(\sqrt{\lambda r})}{2}. \tag{4.1}
\]

In this special case we will find explicit expressions for the functions \( \Delta_{j,d}(p_1, p_2) \), see (3.10).

We know that

\[
\kappa_2 = 2, \quad V_2 = 24\pi^2 \quad \text{and} \quad \Delta_{j,2}(p_1, p_2) = P_{j,2}(p_1, p_2) P_{j,2}(-p_1, p_2),
\]

where the functions \( P_{j,2}(p_1, p_2) \), \( j = 0, 1, 2 \), depend only on \( p_1 \) and \( p_2 \). In particular,

\[
P_{2,2}(p_1, p_2) = \int_{[-\pi, \alpha] \cup [\alpha, \pi]} \int_{[-\pi, \alpha] \cup [\alpha, \pi]} \left| \begin{array}{ccc} \sin \varphi_0 & \cos \varphi_0 & 1 \\ \\ \sin \varphi_1 & \cos \varphi_1 & 1 \\ \\ \sin \varphi_2 & \cos \varphi_2 & 1 \\ \end{array} \right| d\varphi_0 d\varphi_1,
\]

\[
P_{1,2}(p_1, p_2) = \int_{[-\pi, \alpha] \cup [\alpha, \pi]} \int_{[-\pi, \alpha] \cup [\alpha, \pi]} \left| \begin{array}{ccc} \sin \varphi_0 & \cos \varphi_0 & 1 \\ \\ \sin \varphi_1 & \cos \varphi_1 & 1 \\ \sin \varphi_2 & \cos \varphi_2 & 1 \\ \end{array} \right| d\varphi_0 d\varphi_1 d\varphi_2.
\]

After some tedious calculations using the identity

\[
\left| \begin{array}{ccc} \sin \varphi_0 & \cos \varphi_0 & 1 \\ \\ \sin \varphi_1 & \cos \varphi_1 & 1 \\ \sin \varphi_2 & \cos \varphi_2 & 1 \\ \end{array} \right| = 4 \sin \frac{\varphi_1 - \varphi_0}{2} \sin \frac{\varphi_2 - \varphi_0}{2} \sin \frac{\varphi_2 - \varphi_1}{2}
\]

we arrive at

\[
P_{2,2}(p_1, p_2) = 4 \sin \alpha \left[ (\pi - \alpha) \cos \alpha + \sin \alpha \right] \tag{4.2}
\]

\[
P_{1,2}(p_1, p_2) = 4 \left[ (\pi - \alpha)(1 + 2 \cos^2 \alpha) + 3 \sin \alpha \cos \alpha \right] \tag{4.3}
\]

\[
P_{0,2}(p_1, p_2) = 24 \left[ (\pi - \alpha)^2 - (\pi - \alpha) \sin \alpha \cos \alpha - 2 \sin^2 \alpha \right]. \tag{4.4}
\]

Inserting these expressions in (3.9) and taking into account the remark concerning the validity of (3.9) for \( d = 2 \) we obtain that

\[
g_{2,2}(r) = \frac{r^4}{16} \int_1^\infty \exp \left\{ -r^2 \nu_2(p_1, p_2) \right\} F_2(p_1, p_2) F_2(-p_1, p_2) d\rho_1 d\rho_2,
\]

\[
g_{1,2}(r) = \frac{r^6}{4} \int_1^\infty \exp \left\{ -r^2 \nu_2(p_1, p_2) \right\} \det F_1(p_1, p_2) F_1(-p_1, p_2) d\rho_1 d\rho_2,
\]

\[
g_{0,2}(r) = \frac{g_{0,2}(\sqrt{\lambda r})}{4} \int_1^\infty \exp \left\{ -r^2 \nu_2(p_1, p_2) \right\} F_0(p_1, p_2) F_0(-p_1, p_2) d\rho_1 d\rho_2,
\]

where

\[
F_2(p_1, p_2) = (p_1 + p_2) \left[ (1 + p_1 p_2) \left( (p_1^2 + p_2^2)(1 + p_1^2 - 1) \right) \right],
\]

\[
F_1(p_1, p_2) = (p_1 + p_2) \left[ (1 + p_1 + p_2)(1 + p_1 p_2) \left( \pi - \arccos \frac{1 + p_1 p_2}{p_1 + p_2} \right) \right],
\]

\[
+ 3(1 + p_1 p_2) \sqrt{(1 - p_1^2)(p_2^2 - 1)}.
\]
\[ F_0(\rho_1, \rho_2) = (\rho_1 + \rho_2)^2 \left( (\pi - \arccos \frac{1 + \rho_1 \rho_2}{\rho_1 + \rho_2})^2 - (1 + \rho_1 \rho_2) \sqrt{(1 - \rho_1^2)(\rho_2^2 - 1)} \right) \]

\[ \times \left( \pi - \arccos \frac{1 + \rho_1 \rho_2}{\rho_1 + \rho_2} - 2(1 - \rho_1^2)(\rho_2^2 - 1) \right) \]

and

\[ \nu_2(\rho_1, \rho_2) = \frac{1}{2} \sqrt{(1 - \rho_1^2)(\rho_2^2 - 1) + \frac{\pi}{2}(\rho_1^2 + \rho_2^2)} \]

\[ - \left( \frac{\rho_1 + \rho_2}{2} \right)^2 \arccos \frac{1 + \rho_1 \rho_2}{\rho_1 + \rho_2} - \left( \frac{\rho_2 - \rho_1}{2} \right)^2 \arccos \frac{1 - \rho_1 \rho_2}{\rho_2 - \rho_1}. \]

Furthermore, the rhs of (3.11) can be rewritten for \( d = 2 \) as follows:

\[ \frac{g_{0,2}(r)}{36} = 1 - \left( 1 + \frac{\pi r^2}{16} + \frac{\pi^2 r^4}{16} \right) e^{-\pi r^2} - \frac{r \sqrt{\pi/2}}{8} (15 + 2\pi r^2 + \pi^2 r^4) e^{-\pi r^2/2} \int_0^r \sqrt{\pi/2} e^{-t^2} dt. \]

This function is easy to calculate and the above double integrals \( g_{j,2}(r), \; j = 0, 1, 2 \), are computed on a sufficiently dense equidistant grid of \( r \)-values by standard numerical integration procedures. The graphs of these functions together with \( g_{V,1} \) are shown in Fig. 2.

**Remark 4.1** The following particular points are of special interest for comparing the planar PVT with other planar tessellations:

![Fig. 2 Plot of the PCF \( g_{V,1}(r) \) and its additive components for \( d = 2 \)](image)
Extremal points of \( g_{V,λ} \):

\[
(\text{r}_{\text{min}}, g_{V,λ}(\text{r}_{\text{min}})) \simeq \left( 0.541, 0.76 \right), \quad (\text{r}_{\text{max}}, g_{V,λ}(\text{r}_{\text{max}})) \simeq \left( 1.107 \right)
\]

Zeros of \( g_{V,λ}(r) - 1 \):

\[
r_1 \simeq 2.291 \sqrt{\frac{λ}{g}} \quad \text{and} \quad r_2 \simeq 1.052 \sqrt{\frac{λ}{g}}.
\]

5 Some characteristic properties of the pair correlation function of the nodes

In this section we investigate the behaviour of the PCF \( g_{V,λ}(r) \) for small and large values of \( r > 0 \) and give explicit formulae for \( σ_V^2 \) and \( E N_0(N_0 - 1) \), see (2.6) and (2.7), in terms of the “moments”

\[
Γ_{j,d} = \frac{dω}{d} \int_0^{∞} g_{j,d}(r) r^{d-1} dr.
\]

First in this section we show the existence and positivity of the limits of \( r^{j-1} g_{j,d}(r) \), \( j = 0, 1, \ldots, d \), as \( r \to 0 \) and derive exponential bounds of \( g_{j,d}(r) \) for sufficiently large \( r \)-values.

**Theorem 5.1** There exist the limits

\[
γ_{j,d} = \lim_{r \to 0} r^{j-1} g_{j,d}(r) > 0 \quad \text{for} \quad j = 0, 1, \ldots, d,
\]

which imply the existence of the limit

\[
\lim_{r \to 0} r^{d-1} g_{V,λ}(r) = \frac{γ_{d,d}}{λ^{2d} / d!} = \frac{2(d - 1)(d^2 - 1)ω_{d-1}^2}{d^2 k_2^d(ω_dλ)^{2/d}} \left( \frac{ω_{d-1}}{ω_d} \right)^{d/d} \Gamma \left( \frac{d^2 - 1}{d} \right) I_d,
\]

where the number \( I_d \) is defined by (5.6) below. Therefore, \( g_{V,λ}(r) \) possesses a \((d - 1)\)-st-order pole at \( r = 0 \). Furthermore, there exists a constant \( C_d \) such that

\[
1 - g_{V,λ}(r) ≤ C_d(λ r^d)^{d/(d-1)}e^{-ω_d r^{d/(d-1)}} \quad \text{for} \quad r ≥ 2(ω_dλ)^{-1/d}, \quad j = 0, 1, \ldots, d.
\]

**Corollary 5.2** For the most important cases \( d = 2 \) and \( d = 3 \) we get \( I_2 = \frac{32}{7π} \) and \( I_3 = \frac{128}{7π} \) so that (5.3) gives

\[
g_{V,λ}(r) \simeq \frac{16}{9π^2 √π} \frac{0.18013}{r} \quad \text{as} \quad r \downarrow 0 \quad \text{for} \quad d = 2,
\]

\[
g_{V,λ}(r) \simeq \frac{350Γ(2/3)}{243π^2} \left( \frac{3}{4π} \right)^{2/3} \frac{1}{λ^{2/4π^2}} \simeq 0.07604 \quad \text{as} \quad r \downarrow 0 \quad \text{for} \quad d = 3.
\]

Moreover, for \( d = 2 \),

\[
γ_{1,2} = 6 - \frac{105}{2π^2} ≃ 0.6806, \quad γ_{0,2} = \frac{192}{25π^3} (3712 - 375π^2) ≃ 2.6994.
\]

**Proof.** Rewrite (3.9) upon substituting the variable \( ρ_2 \) by \( ρ_2/r \):

\[
\begin{align*}
\gamma_{j,d} = & \frac{1}{k_2^d (2d-j)!} \int_0^{∞} \int_0^{1} \exp \left\{ -r^d \nu_d (ρ_1, ρ_2/r) \right\} \Delta_{j,d} (ρ_1, ρ_2/r) \times (ρ_2^2 - r^2 ρ_1^2)^{(d-1)(d-j+1)} dρ_2 \left( (1 - ρ_1^2)(ρ_2^2 - r^2) \right)^{d(d-3)} dρ_1 dρ_2.
\end{align*}
\]

Passing to the limit \( r \downarrow 0 \) under the integral and using (3.3) and after that substituting \( τ = ω_d r^{d/(d-1)} \) leads to

\[
\gamma_{j,d} = \frac{1}{k_2^d (2d-j)!} \int_0^{∞} \exp \left\{ -ω_d τ^d \right\} τ^{d(d-1)(d-j+1)} dτ \left( d - 1 \right)^2 \left( d - j + 2 \right)^2 V_{j,d}
\]

\[
\approx \frac{2}{d k_2^d} \omega_d \sum_{i=0}^{d(j-1)+2} \frac{(d-1)(d-j+1)+1}{d} V_{j,d}.
\]

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where
\[
V_{j,d} = \int_0^1 (1 - \rho^2)^{d/2} \Delta_{j,d}^*(\rho, \infty) \, d\rho \quad \text{for} \quad j = 0, 1, \ldots, d.
\] (5.5)

Note that, since \(\cos \alpha(\rho, \infty) = \cos \beta(\rho, \infty) = \rho\), we obtain \(\Delta_{j,d}(\rho, \infty)\) from (3.10) by replacing the angles \(\alpha, \beta\) by \(\arccos \rho\). Thus, in view of (3.10) the numbers \(V_{j,d}\) are positive and finite so that (5.2) is shown. Further note that the numbers \(V_{1,2}\) and \(V_{2,2}\) must be doubled according to the remark after (3.9), see also Remark 3.1.

In the most important case \(j = d\) we use the special form of the determinants \(D_{d,d}^{(1)}\) and \(D_{d,d}^{(2)}\) given in Remark 3.2. After inserting these expressions into (3.9) we obtain that
\[
\Delta_{d,d}(\rho, \infty) = \omega_d^2 M_{d-1}(1 - \rho^2)^{d-1} I_d(\rho) I_d(-\rho),
\]
where \(I_d(\rho) = (d - 1) \int_0^\pi \arccos \rho (\rho - \cos \varphi) \sin^{d-2} \varphi \, d\varphi\) and
\[
M_{d-1} = \int_{\Theta_d} |D_{d-1}|^2 \prod_{i=1}^d (\sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \ldots \sin \varphi_{d-2}) \, d(\phi_0, \ldots, \phi_d).
\]

Thus, \(M_{d-1}/((d - 1)!)^2 (d - 1)^d\omega_{d-1}\) is the second moment of the \((d - 1)\)-dimensional Lebesgue measure of the \((d - 1)\)-simplex with vertices at \(d\) independent and uniformly distributed points on \(S^{d-2}\). From Theorem 2 in Miles (1971) (see also Lemma 4.2.1 in Møller (1994)) we find that
\[
M_{d-1} = (d - 1) d! \omega_{d-1}^2.
\]

After some manipulations we see that
\[
I_d(\rho) = I_d(-\rho) + \rho d \omega_d/\omega_{d-1} = (1 - \rho^2)^{d/2} + \rho(d - 1) \int_{\arccos \rho}^\pi \sin^{d-2} \varphi \, d\varphi.
\]

Inserting the preceding formulae into (5.5) for \(j = d\) gives
\[
V_{d,d} = (d - 1) d! \omega_{d-1}^2 I_d.
\]

where
\[
I_d = \int_0^1 (1 - \rho^2)^{(d-2)/(d+1)} I_d(\rho)(I_d(\rho) - \rho d \omega_d/\omega_{d-1}) \, d\rho \quad \text{for} \quad d \geq 3 \quad \text{and}
\]
\[
I_2 = 2 \int_0^1 \left(\sqrt{1 - \rho^2} + \rho(\pi - \arccos \rho)\right) \left(\sqrt{1 - \rho^2} - \rho \arccos \rho\right) \, d\rho = \frac{32}{27}.
\] (5.6)

This, together with \(\Gamma\left(\frac{d^2 - 1}{d}\right) = \frac{d^2 - 1}{d} \Gamma\left(\frac{d^2 - 1}{d}\right)\), proves (5.3).

Since the functions \(\Delta_{j,d}(\rho_1, \rho_2), \ j = 0, 1, 2,\) can be expressed explicitly by (4.2)–(4.4), we arrive at the following formulæ:
\[
\gamma_{1,2} = \frac{2}{\pi^2} \int_0^\pi \left[(2 + \cos \varphi)(2\pi - \varphi) + 3 \sin \varphi\right] \left[(2 + \cos \varphi)\varphi - 3 \sin \varphi\right] \, d\varphi
\]

and
\[
\gamma_{0,2} = \frac{810}{\pi^6} \int_0^{\pi/2} \left[(\pi - \varphi)^2 - (\pi - \varphi) \sin \varphi \cos \varphi - 2 \sin^2 \varphi\right] \left[\varphi^2 + \varphi \sin \varphi \cos \varphi - 2 \sin^2 \varphi\right] \sin \varphi \, d\varphi.
\]

By direct calculation via MATHEMATICA we obtain the values given in Corollary 5.2.

To verify (5.4) we first consider the case \(d \geq 3\). Using the inequality \(\nu_d(\rho_1, \rho_2) \geq \omega_d \rho_2^2/2d\) and the fact that \(\Delta_{j,d}(\rho_1, \rho_2)\) is uniformly bounded by some constant \(C_{d,1}\), we immediately get from (3.9) that
\[
g_{j,d}(r) \leq \frac{4 C_{d,1}}{2r^{d-j}} \omega_d^{d-j} s^{d(2d-j)} \int_1^\infty e^{-s^2 \rho^2} \rho^{(d-1)(2d-j)+2} \, d\rho.
\]
with \( s = \omega_d^{1/d} r / 2 \). Finally, upon substituting \( y = s^d \rho^d \) and applying the inequality \( \int_0^\infty e^{r y} y^p / dy \leq C_{d,2} e^{-s^d} s^p \) for \( s \geq 1 \) and \( p = (d - 1)(2d - j + 1) \), we obtain

\[
g_{j,d}(r) \leq C_{d,3} r^{(2d-j-1)} e^{-\omega_d r^d / 2^d} \quad \text{for} \quad r \geq 2 / \omega_d^{1/d}, \quad j = 0, \ldots, d, \tag{5.7}
\]

where \( C_{d,2} \) and \( C_{d,3} \) are constants only depending on \( d \). The same upper bound can be shown for \( g_{j,d}(r) \), \( j = 0, 1, 2 \), by using the more explicit form of these functions derived in Sect. 4. Combining (5.7) and (3.12) yields (5.4). This completes the proof of Theorem 5.1.

In the remaining part of this section we specify the general formulae of Theorem 2.2 to stationary PVT’s and give the corresponding numerical results for \( d = 2 \) and \( d = 3 \). Having in mind the definition of the functions \( g_{j,d}(r) \), \( j = 0, 1, \ldots, d \), (3.9) and (5.1) we recognize that

\[
\beta_{0,d}^{(red)}(\mathbb{R}^d) = \lambda_d^d d \omega_d \int_0^\infty g_{j,d}(r \lambda_1^1 / d) r^{d-1} dr = \lambda \kappa_d^2 \Gamma_{j,d}
\]

for \( j = 0, 1, \ldots, d \) and, by the definitions of \( g_{0,d}^{(red)}(r) \) and \( \Gamma_{0,d}^* \),

\[
\frac{\beta_{0,d}^{(red)}(A_n)}{[\pi (d + 1)]^d} - \lambda_d^d |A_n| \xrightarrow{n \to \infty} \lambda \kappa_d^2 \left( \frac{\Gamma_{0,d}}{[\pi (d + 1)]^d} - \Gamma_{0,d}^* \right)
\]

Finally, the numbers (5.1) are obtained from (3.9) by changing the integration so that we arrive at

**Theorem 5.3** For the \( d \)-dimensional stationary PVT \( V(\Psi) \) with cell intensity \( \lambda \) the asymptotic variance (2.5) takes the form

\[
\sigma_V^2 = \lambda \kappa_d (1 + \kappa_d \sigma_d^2) \quad \text{with} \quad \sigma_d^2 = \frac{\sum_{j=0}^{d} \frac{\Gamma_{j,d}}{[(d-j+1)!]^2(j-1)!}}{\Gamma_{0,d}^*}, \tag{5.8}
\]

and the second factorial moment of the number of vertices of the typical Poisson Voronoi polytope can be expressed as follows:

\[
\mathbb{E}N_o(N_o-1) = \kappa_d^2 \sum_{j=1}^{d} \frac{\Gamma_{j,d}}{[(d-j+1)!]^2(j-1)!}, \tag{5.9}
\]

where the numbers \( \Gamma_{j,d}, j = 0, 1, \ldots, d \), (defined by (5.1)) are expressible as double integrals

\[
\Gamma_{j,d} = \frac{(2d-j)! \omega_d}{2^{d-j} \pi_d^d} \int_0^\infty \int_0^1 (\rho_2^2 - \rho_1^2)^{(d-1)(d-j+1)+j} \left( (1 - \rho_1^2) (\rho_2^2 - 1) \right)^{\frac{d}{2} - 1} \Delta_{j,d}(\rho_1, \rho_2) d\rho_1 d\rho_2
\]

and \( \kappa_d, \nu_d(\rho_1, \rho_2), \Delta_{j,d}(\rho_1, \rho_2) \) and \( \Gamma_{0,d}^* \) are given by (1.7), (3.4), (3.10) and (3.13), respectively. The rhs of \( \Gamma_{1,2} \) and \( \Gamma_{2,2} \) must additionally be multiplied by 2, see Lemma 3.1.

By means of (4.2), (4.3) and (4.4) the numbers \( \Gamma_{0,2}, \Gamma_{1,2} \) and \( \Gamma_{2,2} \) can be written as a double integral over elementary functions of \( \rho_1 \) and \( \rho_2 \). Standard numerical integration procedures (in MAPLE 8) yield

**Corollary 5.4** For the planar case we have \( \Gamma_{0,2}^* = 2 + \frac{3}{2} \pi \approx 7.5343 \),

\[
\Gamma_{0,2} \simeq 57.20719718, \quad \Gamma_{1,2} \simeq 19.78081170, \quad \Gamma_{2,2} = 3
\]

and, together with \( \kappa_2 = 2 \),

\[
\sigma_2^2 = 0.5, \quad \sigma_V^2 = 4 \lambda \quad \text{and} \quad \mathbb{E}N_o(N_o-1) = \Gamma_{1,2} + 4 \Gamma_{2,2} \simeq 31.78081170.
\]

To obtain the corresponding numerical values for \( d = 3 \) we need highly effective numerical integration techniques because the multiple parameter integrals \( \Delta_{j,3}(\rho_1, \rho_2), j = 0, 1, 2, 3 \), cannot be expressed in an explicit form. The details of the calculations which lead to the below Corollary 5.5 as well as to Table 2 and the plots in Fig. 3 are carried out in Heinrich et al. (1998).
Corollary 5.5 For the spatial case we have \( \Gamma_{0,3} = 6 + \frac{560}{81\sqrt{3}} \pi \simeq 18.5398 \),
\[ \Gamma_{1,3} \simeq 6232.6308, \quad \Gamma_{2,3} \simeq 353.3236, \quad \Gamma_{3,3} \simeq 19.1713, \quad \Gamma_{4,3} = \frac{35}{\pi^2} \simeq 3.5462, \]
and, together with \( \kappa_3 = \frac{24}{35} \pi^2 \),
\[ \sigma^2_\Psi = 5.084, \quad \sigma^2_{\Psi} \simeq 239.578 \lambda \quad \text{and} \quad EN_{\alpha}(N_\alpha - 1) \simeq 750.261. \]

It should be noted that the variance of \( N_\alpha \) was already calculated for \( d = 2 \) and \( d = 3 \) in two unpublished papers of Brakke (1985 b, c). The procedures used there are quite different from ours.

6 Asymptotic normality of the number of nodes

In the final section we study the asymptotic distribution of the number of nodes contained in a large piece of a \( d \)-dimensional stationary PVT. To be precise, let \( \Psi_V \sim P_V \) be the stationary point process of nodes in \( V(\Psi) \) generated by a stationary Poisson process with intensity \( \lambda > 0 \). Our aim is to construct an approximate (asymptotically exact) \( 100(1 - \alpha)\% \) confidence interval for the unknown cell intensity \( \lambda \). To derive a central limit theorem for the number \( \Psi_V(A_n) \) of nodes within a cube \( A_n = [0, n]^d \) as \( n \to \infty \). Making use of the \( \beta \)-mixing property of a stationary PVT expressed in terms of a suitable \( \beta \)-mixing coefficient having exponential decay (see Theorem 2.1 in Heinrich (1994)), we may deduce from Corollary 2.3 in Heinrich (1994) that the unbiased estimator \( \lambda_{V,n} = \Psi_V(A_n)/|A_n| \) for \( \lambda_V \) is asymptotically normally distributed as \( n \to \infty \) with mean \( \lambda_V = \kappa_d \lambda \) and the asymptotic variance \( \sigma^2_{\Psi} = \lambda \kappa_d (1 + \kappa_d \sigma^2_d) \), see (2.5) and (5.8). A corresponding result for the number of nodes induced by a \( d \)-dimensional Poisson hyperplane tessellation (which is not \( \beta \)-mixing) has been recently proved in Heinrich et al. (2006). For another approach to central limit theorems for random tessellations the reader is referred to Penrose and Yukich (2001).
Theorem 6.1 Under the assumptions of Theorem 5 we have

$$P \left( \frac{\Psi_V(A_n) - \kappa_d \lambda \rho |A_n|}{\sqrt{\lambda \kappa_d (1 + \kappa_d \sigma_d^2)}} \leq x \right) \xrightarrow{n \to \infty} \Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt$$  

(6.1)

for any $x \in \mathbb{R}^1$, whence it follows that $\lambda$ belongs to the interval

$$K_n(\alpha) = \frac{1}{\kappa_d |A_n|} \left[ \left( \frac{\sqrt{\Psi_V(A_n)} - \frac{a_0}{2} \sqrt{1 + \kappa_d \sigma_d^2}}{\sqrt{\lambda \kappa_d (1 + \kappa_d \sigma_d^2)}} \right)^2 ; \left( \sqrt{\Psi_V(A_n)} + \frac{a_0}{2} \sqrt{1 + \kappa_d \sigma_d^2} \right)^2 \right]$$

with probability $1 - \alpha$ as $n \to \infty$, where $z_{\alpha/2}$ satisfies the equation $\Phi(z_{\alpha/2}) = 1 - \frac{\alpha}{2}$. In particular, $\kappa_2 = 2$, $\sigma_d^2 = 0.5$ and $\kappa_3 \simeq 6.7677$, $\sigma_3^2 \simeq 5.084$.

Proof. It remains to prove the second assertion of Theorem 6.1. Since $\sqrt{\lambda V_n}$ converges in probability to $\sqrt{\lambda V}$ as $n \to \infty$ it follows from (6.1) and the well-known properties of weak convergence that

$$P \left( \frac{|A_n|}{\sqrt{\lambda V_n}} \leq x \right) \xrightarrow{n \to \infty} \Phi(x), \quad x \in \mathbb{R}^1.$$

Hence, it is easily seen that $K_n(\alpha)$ is an asymptotically exact $100(1 - \alpha)$% confidence interval for $\lambda$. \hfill \Box

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Table 1 Values of the PCF $g_{V,1}(r)$ for $d = 2$

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Table 2 Values of the PCF $g_{V,1}(r)$ for $d = 3$


