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Lothar Heinrich \(^a\), Zbyněk Pawlas \(^b\)

\(^a\) University of Augsburg, Institute of Mathematics, Universitätstr. 14, Augsburg, Germany

\(^b\) Dept. of Probability and Math. Statistics, Charles University of Prague,

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Weak and strong convergence of empirical distribution functions from germ-grain processes

LOTHAR HEINRICH*† and ZBYNĚK PAWLAS‡

†University of Augsburg, Institute of Mathematics, Universitätstr. 14, 86135 Augsburg, Germany
‡Charles University of Prague, Dept. of Probability and Math. Statistics, Sokolovská 83, 186 75 Praha 8, Czech Republic

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We observe randomly placed random compact sets (called grains or particles) in a bounded, convex sampling window \( W_n \) of the \( d \)-dimensional Euclidean space which is assumed to expand unboundedly in all directions as \( n \to \infty \). In addition, we suppose that the grains are independent copies of a so-called typical grain \( \Xi_0 \), which are shifted by the atoms of a homogeneous point process \( \Psi \) in such a way that each individual grain lying within \( W_n \) can be observed. We define an appropriate estimator \( \hat{F}_n(t) \) for the distribution function \( F(t) \) of some \( m \)-dimensional vector \( f(\Xi_0) = (f_1(\Xi_0), \ldots, f_m(\Xi_0)) \) (describing shape and size of \( \Xi_0 \)) on the basis of the corresponding data vectors of those grains which are completely observable in \( W_n \). As main results, we prove a Glivenko-type theorem for \( \hat{F}_n(t) \) and the weak convergence of the multivariate empirical processes \( \sqrt{\Psi(W_n)}(\hat{F}_n(t) - F(t)) \) to an \( m \)-dimensional Brownian bridge process as \( n \to \infty \).

Keywords: Marked point process; Germ-grain process; Glivenko’s theorem; Typical grain; Weak convergence; Skorohod space \( D(\mathbb{R}^m) \); Multivariate empirical distribution; Kolmogorov–Smirnov test

AMS Subject Classification: Primary: 60D05, 62M30; Secondary: 60G55, 62G30

1. Introduction and preliminaries

We consider a stationary \( d \)-dimensional ‘germ-grain process’

\[
\Xi = \{ \Xi_i + X_i, i \geq 1 \},
\]

which consists of two independent random components defined on a common probability space \([\Omega, \mathcal{A}, P]\) – a (weakly) stationary point process \( \Psi = \sum_{i \geq 1} \delta_{X_i} \) on \( \mathbb{R}^d \) with intensity \( \lambda = \mathbb{E}(\Psi([0, 1]^d)) \) and a sequence \( \{ \Xi_i, i \geq 1 \} \) of independent copies of a random compact set \( \Xi_0 \subset \mathbb{R}^d \) (called ‘typical grain’). To identify the points \( X_i \) in (1), we require additionally that \( P(c(\Xi_0) = o) = 1 \), where \( c(\mathcal{K}) \in \mathcal{K} \) is a canonical point assigned to each \( \mathcal{K} \in \mathcal{K} \) (=family of non-empty compact sets in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \)). For more details and

*Corresponding author. Email: heinrich@math.uni-augsburg.de
further background of point process theory and germ-grain (or particle) processes, we refer
the reader to [1–5].

Throughout we assume that only a single observation of \( \Xi \) in a ‘sampling window’ \( W_n \subset \mathbb{R}^d \)
is given, where the sequence of convex compact sets \( (W_n) \) grows unboundedly in all directions such that \( H_{d-1}(\partial W_n)/|W_n| \to 0 \) as \( n \to \infty \). Here and in what follows \( H_k(\cdot) \) designates the \( k \)-dimensional Hausdorff measure and \( |\cdot| = H_d(\cdot) \) the Lebesgue measure on \( \mathbb{R}^d \).

Let \( f(\Xi_0) = (f_1(\Xi_0), \ldots, f_m(\Xi_0)) \) be an \( m \)-dimensional random vector describing various shape and size parameters of \( \Xi_0 \), e.g. geometric functionals, direction of normal unit vectors at fixed points on the surface \( \partial \Xi_0 \) etc. For example, if \( \Xi_0 \) is a random segment or more generally a random rectifiable curve in \( \mathbb{R}^2 \), then non-parametric testing of the (joint) distribution function (df) of length \( H_1(\Xi_0) \) and angle between the tangent at \( c(\Xi_0) \) and the \( x \)-axis turns out to be a non-trivial statistical issue.

In order to estimate the \( m \)-variate df

\[
F(t) := P(f(\Xi_0) \leq t) = P(f_1(\Xi_0) \leq t_1, \ldots, f_m(\Xi_0) \leq t_m)
\]

for \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m \), we suppose that the \( m \)-dimensional data vectors \( f(\Xi_i) \) of those (shifted) grains \( \Xi_i + X_i \) lying completely within \( W_n \) are available. In other words, either (1) consists of non-overlapping grains or the set of points in \( W_n \) covered by more than one grain is negligible so that an exact measurement of each data vector \( f(\Xi_i) \) is possible; see Figure 1. Fibre, surface and manifold processes (see [4], [5, Chapter 9]) are typical examples of such germ-grain processes. In these examples, \( \Xi_0 \) is a random \( k \)-dimensional compact manifold with \( P(0 < H_k(\Xi_0) < \infty) = 1 \) for some \( k \in \{1, \ldots, d-1\} \). In case of \( P(|\Xi_0| > 0) > 0 \), the above restriction means that the grains \( \Xi_i + X_i \) in (1) are pairwise disjoint, e.g. if \( \Psi \) is a hard-core point process with hard-core distance \( h > 0 \) and \( \|\Xi_0\| := \sup\{\|x\| : x \in \Xi_0 \} \) is bounded and less than \( h/2 \); see Figure 4.

The key question we address in this paper is: How to define a suitable empirical df \( \hat{F}_n(t) \) for (2) by using only data vectors \( f(\Xi_i) \) of those grains \( \Xi_i + X_i \) lying completely within \( W_n \) such that the limit distribution of the maximum discrepancy \( \sup_{t \in \mathbb{R}^m} |\hat{F}_n(t) - F(t)| \) after blowing up with \( \sqrt{\Psi(W_n)} \) can be shown to exist quite similar to the classical situation of i.i.d. random vectors in \( \mathbb{R}^m \), see [6, 7] (and [8] for \( m = 1 \)).

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Figure 1. Realization of a fibre process as special germ-grain process.
Our sampling procedure (called ‘minus-sampling’) leads necessarily to weighted estimators of the form

\[ \hat{M}_n(g) := \sum_{i \geq 1} \text{1}_{\Xi_i + X_i \subseteq W_n} \frac{g(\Xi_i)}{|W_n \ominus \hat{\Xi}_i|} \]

being unbiased for the mean value \( \lambda \mathbb{E}_g(\Xi_0) \) for any \( Q \)-integrable functional \( g : \mathcal{K} \mapsto \mathbb{R}^1 \), see [5, Chapter 6]. Here \( Q \) denotes the distribution induced by \( X_0 \) on \( \mathcal{K} \), \( 1_\mathcal{B} \) stands for the indicator function of a set or event \( \mathcal{B} \) and we have used the set operations \( \hat{\mathcal{B}} = \{-x : x \in \mathcal{B}\} \) (reflection) and \( A \ominus B := \cup_{y \in B}(A + y) = \{x : x + \hat{\mathcal{B}} \subseteq A\} \) (Minkowski-subtraction). Further, let \( A \oplus B := \bigcup_{y \in B}(A + y) \) (Minkowski-addition) and \( b(x, r) \) denotes the ball with radius \( r \) centred at \( x \in \mathbb{R}^d \).

Note that the proof of the unbiasedness of (3) is a straightforward application of Campbell’s theorem for a stationary marked point process \( \Psi_\text{mark} = \sum_{i \geq 1} \delta_{(X_i, \Xi_i)} \) (which defines (1)) with intensity \( \lambda \) and mark distribution \( Q \), see [1, Chapter 10.5] or [5]:

\[ \mathbb{E}\hat{M}_n(g) = \lambda \int_{\mathcal{K}} \int_{\mathbb{R}^d} \text{1}_{W_n \ominus \hat{K}}(x) \frac{g(K)}{|W_n \ominus \hat{K}|} \, dx \, Q(dK) = \lambda \int_{\mathcal{K}} g(K) \, Q(dK). \]

In Section 2 we will define an empirical df \( \hat{F}_n(t), t \in \mathbb{R}^m \), and state an analogue to Glivenko’s theorem (Theorem 1) even for the more general case of germ-grain processes (1) governed by a stationary ergodic marked point process \( \Psi_\text{mark} \).

Our principal asymptotic result formulated (Theorem 2) and proved in section 3 states the weak convergence of the \( m \)-parameter empirical processes

\[ Y_n(t) := \sqrt{\Psi(W_n)} \left( \hat{F}_n(t) - F(t) \right), \quad t \in \mathbb{R}^m, \quad n \geq 1. \]

It turns out that the weak limit of (4) (as \( n \to \infty \)) in the Skorohod space \( D(\mathbb{R}^m) \) can be identified with a mean zero Gaussian process \( Y(t), t \in \mathbb{R}^m \), having the covariance function \( \mathbb{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t) \), where \( s \wedge t = (\min(s_1, t_1), \ldots, \min(s_m, t_m)) \). Consequently, for \( m = 1 \) a Kolmogorov–Smirnov test can be established just as for i.i.d. samples, provided the df \( F(\cdot) \) is continuous.

To conclude Section 1, we mention the problem of how to use the information available in \( W_n \) of those shifted grains which cross the boundary \( \partial W_n \). Statisticians familiar with estimation from incomplete observations or censored data would suggest a Kaplan–Meier type estimator of the df \( F(\cdot) \). However, it seems that this idea might be successful only in the case of geometrically simple grains such as segments or circles. For edge-corrected Kaplan–Meier type and minus-sampling type estimators of the empty-space function (and other distance functions) of stationary point processes, we refer to [9, 10], respectively. A comparison study for both types of estimators can be found in [11].

2. Empirical distribution functions and Glivenko’s theorem

A quite natural empirical counterpart of the df (2) is given by

\[ \hat{F}_n(t) = \frac{1}{N_n} \sum_{i \geq 1} \text{1}_{\Xi_i + X_i \subseteq W_n} \text{1}_{(-\infty, t]}(f(\Xi_i)), \quad t = (t_1, \ldots, t_m) \in \mathbb{R}^m, \]

where \( (-\infty, t] = \prod_{j=1}^m (-\infty, t_j] \) and \( N_n = \sum_{i \geq 1} \text{1}_{\Xi_i + X_i \subseteq W_n} \) gives the number of grains which are completely observable in \( W_n \). Obviously, \( \hat{F}_n(t) \) is a discrete \( m \)-variate df which can
be shown to converge $\mathbb{P}$-a.s. (as $n \to \infty$) to $F(t)$ uniformly in $t \in \mathbb{R}^m$, provided the stationary marked point process $\Psi_{\text{mark}}$ is ergodic. However, the relation
\[
\frac{1}{\sqrt{|W_n|}} \mathbb{E} N_n \left( \tilde{F}_n(t) - F(t) \right) = \frac{\lambda}{\sqrt{|W_n|}} \int_K \left| W_n \ominus \tilde{K} \right| \left( \mathbf{1}_{(-\infty,t]}(f(K)) - F(t) \right) Q(dK)
\]
reveals that for $d \geq 2$, a zero mean weak limit of $\sqrt{|W_n|} \left( \tilde{F}_n(t) - F(t) \right)$ cannot exist. In other words, the empirical value $\tilde{F}_n(t)$ is not close enough to $F(t)$. To remove this shortcoming, we suggest the following empirical df which is based on the weighted estimator (3):
\[
\hat{F}_n(t) = \frac{1}{\hat{\lambda}_n} \sum_{i \geq 1} \frac{\mathbf{1}_{[X_i + \partial B(0, r) \subseteq W_n]} \mathbf{1}_{(-\infty,t]}(f(\Xi_i))}{|W_n \ominus \hat{\Xi}_i|}, \tag{6}
\]
where
\[
\hat{\lambda}_n = \sum_{i \geq 1} \frac{\mathbf{1}_{[X_i + \partial B(0, r) \subseteq W_n]} \mathbf{1}_{(-\infty,t]}(f(\Xi_i))}{|W_n \ominus \hat{\Xi}_i|} \tag{7}
\]
is an unbiased estimator for the intensity $\lambda$ implying that $\mathbb{E}(\hat{\lambda}_n(\hat{F}_n(t) - F(t))) = 0$ for any $t \in \mathbb{R}^m$ and $n \geq 1$. Here, $\hat{F}_n(\cdot)$ is again a discrete $m$-variate df with random jumps depending on the size of the grains $\Xi_i$. In the next section, we will see under which circumstances $\hat{F}_n(\cdot)$ can be substituted by the more convenient unbiased estimator $\hat{\lambda}_n := \Psi(W_n)/|W_n|$ for $\lambda$ which considers all points $X_i$ belonging to $W_n$.

Next, we formulate Glivenko’s theorem for $\hat{F}_n(t)$ even in the more general situation of stationary ergodic germ-grain processes with not necessarily independent grains. To avoid too large weights $|W_n \ominus \hat{\Xi}_i|^{-1}$ in (6) and (7), we put an additional moment condition on the diameter $\|\Xi_0\|$.

**Theorem 1** Let the germ-grain process (1) be defined by a stationary ergodic marked point process $\Psi_{\text{mark}} = \sum_{i \geq 1} \delta_{[X_i, \Xi_i]}$ with positive and finite intensity $\lambda$ and typical mark (grain) $\Xi_0$ satisfying $\mathbb{E}\|\Xi_0\|^q < \infty$ for some $q \geq d$. Further, let $(W_n)$ be an increasing sequence (i.e. $W_n \subseteq W_{n+1}$ for $n \geq 1$) of convex, bounded sets in $\mathbb{R}^d$ such that for $n \geq 1$,
\[
\frac{H_{d-1}(\partial W_n)}{|W_n|^{1-1/q}} \leq c_0 < \infty \quad \text{and} \quad \rho(W_n) := \sup\{r > 0 : b(x, r) \subseteq W_n\} \to \infty. \tag{8}
\]

Then
\[
\sup_{t \in \mathbb{R}^m} |\hat{F}_n(t) - F(t)| \to 0 \quad \text{P–a.s., where} \quad F(t) = \mathbb{P}(f(\Xi_0) \leq t). \tag{9}
\]

**Proof** The conditions put on the sequence $(W_n)$ are sufficient for the spatial individual ergodic theorem, see [1, p. 333] or [12], which, applied to the stationary ergodic marked point process $\Psi_{\text{mark}}$, implies that
\[
\frac{1}{|W_n|} \sum_{i \geq 1} \mathbf{1}_{W_n}(X_i) \mathbf{1}_{(-\infty,t]}(f(\Xi_i)) \to F(t) \quad \text{P–a.s.} \tag{10}
\]
for any $t \in \mathbb{R}^m$. Since the function $\rho \mapsto V(\rho) := |W_n \ominus b(o, \rho)|$ is differentiable for $0 \leq \rho < \rho(W_n)$ with continuous derivative $V'(\rho) = H_{d-1}(\partial(W_n \ominus b(o, \rho)))$, see [13, p. 207], we get
the identity

\[ V(0) - V(r) = |W_n \setminus (W_n \ominus b(o, r))| = \int_0^r H_{d-1}(\partial(W_n \ominus b(o, \rho))) \, d\rho \]

for \( 0 \leq r \leq \rho(W_n) \), which leads to the inequality

\[ |W_n \setminus (W_n \ominus b(o, r))| = |\{ x \in W_n : b(x, r) \cap W_n^c \neq \emptyset \}| \leq r \, H_{d-1}(\partial W_n) \]

(11) for any \( r > 0 \). Thus, setting \( r^n_\delta := \delta |W_n| / H_{d-1}(\partial W_n)(\to \infty) \) and \( W^n_\delta := W_n \ominus b(o, r^n_\delta) \), we immediately deduce that

\[ \frac{|W^n_\delta|}{|W_n|} \leq \frac{1}{1 - \delta} \]

respectively \( \frac{|W^n_\delta|}{|W_n|} \geq 1 - \delta \) for \( 0 \leq \delta < 1 \).

(12)

Further, it is easily seen that for any \( n \geq 1 \) and \( \delta \in [0, 1) \), \( W^n_\delta \) is a non-empty, convex subset of \( W_n \) with \( \rho(W^n_\delta) \to \infty \). Hence, according to the definition given in [12], \( (W^n_\delta) \) is a ‘regular generalized sequence’ of convex sets in \( \mathbb{R}^d \) to which the spatial individual ergodic theorem, see [12, p. 143], can be extended. This means that in (10) the sequence \( (W_n) \) can be replaced by \( (W^n_\delta) \) for any \( 0 \leq \delta < 1 \).

In order to prove (9) we first show that for all fixed \( t \in \mathbb{R}^m \),

\[
\hat{\lambda}_n \hat{F}_n(t) = \sum_{i \geq 1} \frac{1_{\{X_i + \mathbb{Z} \subseteq W_n\}}}{|W_n \ominus \mathbb{Z}_i|} 1_{(-\infty, t]}(f(\mathbb{Z}_i)) \to \lambda \, F(t) \quad \text{P-a.s.}
\]

(13)

or equivalently, by using the sequence of events \( A_n(\varepsilon) := \{ |\hat{\lambda}_n \hat{F}_n(t) - \lambda \, F(t)| \geq \varepsilon \} \),

\[ \mathbb{P}
\left( \bigcup_{k \geq n} A_k(\varepsilon) \right) \to 0 \quad \text{for any} \quad \varepsilon > 0.
\]

Define a further sequence of events \( B_n(\delta) := \bigcap_{i : X_i \in W_n} \{ \| \mathbb{Z}_i \| \leq r^n_\delta \} \) with \( \delta \in [0, 1) \). Then

\[ \mathbb{P}
\left( \bigcup_{k \geq n} A_k(\varepsilon) \right) \leq \mathbb{P}
\left( \bigcup_{k \geq n} A_k(\varepsilon) \cap B_k(\delta) \right) + \mathbb{P}
\left( \bigcup_{k \geq n} B_k^c(\delta) \right).
\]

The first part of (8) yields the estimate \( r^n_k \geq \delta |W_k|^{1/q}/c_0 =: \rho^n_k \) for \( k \geq 1 \). By exploiting the monotonicity of the sequence \( (\rho^n_k) \) and applying Campbell’s theorem, we obtain that

\[
\mathbb{P}
\left( \bigcup_{k \geq n} B_k^c(\delta) \right) \leq \mathbb{P}
\left( \bigcup_{k \geq n} \bigcup_{i : X_i \in W_k} \{ \| \mathbb{Z}_i \| \geq \rho_k^\delta \} \right)
\]

\[
= \mathbb{P}
\left( \bigcup_{i : X_i \in W_n} \{ \| \mathbb{Z}_i \| \geq \rho_k^\delta \} \bigcup \bigcup_{k \geq n} \{ \| \mathbb{Z}_i \| \geq \rho_k^\delta \} \right)
\]

\[
\leq \mathbb{E}
\sum_{i \geq 1} 1_{W_n}(X_i) 1_{[\rho^\delta_n, \infty)}(\| \mathbb{Z}_i \|) + \sum_{k \geq n} \mathbb{E}
\sum_{i \geq 1} 1_{W_{k+1} \setminus W_k}(X_i) 1_{[\rho^\delta_{k+1}, \infty)}(\| \mathbb{Z}_i \|)
\]

\[
= \lambda \sum_{k \geq n} |W_k| \mathbb{P}(\rho^\delta_k \leq \| \mathbb{Z}_0 \| < \rho^\delta_{k+1})
\]

\[
\leq \frac{\lambda c_0^q}{d q} \, \mathbb{E}(\| \mathbb{Z}_0 \|^q \, 1_{[\rho^\delta_0, \infty)}(\| \mathbb{Z}_0 \|) \downarrow 0 \quad \text{as} \quad n \to \infty.
\]

(14)
Obviously, if $\|\Xi_i\| \leq r^\delta_n$ then $W^\delta_n \subseteq W_n \supseteq \Xi_i \subseteq W_n$. Hence, using the abbreviation $X(G, t) := |G|^{-1} \sum_{i \geq 1} 1_{G(X_i)} 1_{(-\infty, x]}(f(\Xi_i))$ for bounded Borel sets $G \subset \mathbb{R}^d$ and choosing $\delta = \varepsilon/(2\varepsilon + 2\lambda)$ in (12) we find that for any $k \geq 1$,

\[
A_k(\varepsilon) \cap B_k(\delta) = \left\{ \lambda_k \hat{F}_k(t) - \lambda F(t) \geq \varepsilon \right\} \cap \bigcap_{i : X_i \subseteq W_k} \{ \|\Xi_i\| \leq r^\delta_k \}
\]

\[
\subseteq \left\{ \frac{|W_k|}{|W^\delta_k|} X(W_k, t) \geq \lambda F(t) + \varepsilon \right\} \cup \left\{ \frac{|W^\delta_k|}{|W_k|} X(W_k, t) \leq \lambda F(t) - \varepsilon \right\}
\]

\[
\subseteq \left\{ X(W_k, t) - \lambda F(t) \geq \lambda \left( \frac{|W^\delta_k|}{|W_k|} - 1 \right) + \varepsilon \frac{|W^\delta_k|}{|W_k|} \right\}
\]

\[
\cup \left\{ X(W_k, t) - \lambda F(t) \leq \lambda \left( \frac{|W^\delta_k|}{|W_k|} - 1 \right) - \varepsilon \frac{|W^\delta_k|}{|W_k|} \right\}
\]

\[
\subseteq \left\{ X(W_k, t) - \lambda F(t) \geq \frac{\varepsilon}{2} \right\} \cup \left\{ X(W_k, t) - \lambda F(t) \leq -\frac{\varepsilon}{2} \right\}.
\]

As stated above, we have $X(W^\delta_n, t) \underset{n \to \infty}{\longrightarrow} \lambda F(t)$ $P$-a.s. for any $\delta \in [0, 1)$. Thus, from the previous relation it follows that $P\left( \bigcup_{k \geq n} A_k(\varepsilon) \cap B_k(\delta) \right) \underset{n \to \infty}{\longrightarrow} 0$ for $\delta := \varepsilon/(2\varepsilon + 2\lambda)$ and this together with $P\left( \bigcup_{k \geq n} B^\varepsilon_k(\delta) \right) \underset{n \to \infty}{\longrightarrow} 0$ proves (13). In the same way, we get $\hat{\lambda}_n \underset{n \to \infty}{\longrightarrow} \lambda$ $P$-a.s., which in turn implies that $\hat{F}_n(t) \underset{n \to \infty}{\longrightarrow} F(t)$ $P$-a.s. for any fixed $t \in \mathbb{R}^m$.

The proof of the uniform $P$-a.s. convergence in (9) consists in applying a standard technique relying on the boundedness and monotonicity of $\hat{F}_n(t)$ (in each component of $t = (t_1, \ldots, t_m)$). For details the reader is referred to [14], where a similar case is treated. This completes the proof of Theorem 1.

**Remark 1** If $\Psi_{\text{mark}}$ is independently marked with an i.i.d. sequence of random compact sets $(\Xi_i)_{i \geq 1}$, then it suffices to assume that the unmarked point process $\Psi = \sum_{i \geq 1} \delta_{X_i}$ is stationary and ergodic; see [2].

**Remark 2** If $\|\Xi_0\| \leq c_1 < \infty$ $P$-a.s. then the first part of (8) can be dropped. Note that $\rho(W_n) \underset{n \to \infty}{\longrightarrow} \infty$ is equivalent to $|W_n|/H_{d-1}(\partial W_n) \underset{n \to \infty}{\longrightarrow} \infty$, since for any convex compact set $W_n \subset \mathbb{R}^d$ with $|W_n| > 0$ the inclusion

\[
\frac{\rho(W_n)}{d} \leq \frac{|W_n|}{H_{d-1}(\partial W_n)} \leq \rho(W_n)
\]

holds. (The first inequality is a direct consequence of a result proved by Wills [15] and the second one follows from (11) with $r = \rho(W_n)$ and the obvious fact that $W_n \supseteq b(o, \rho(W_n)) = 0$).

### 3. Weak convergence of empirical distribution functions

In this section, we will prove the announced weak convergence of the centred and normalized sequence $Y_n(t)$ (see (4)) of random processes on $\mathbb{R}^m$, where the empirical df $\hat{F}_n(t)$ is defined by (6). This result can be formulated for the germ-grain process (1) with all the independence assumptions made at the beginning of section 1. On the other hand, in contrast to Theorem 1 the assumptions of strict stationarity and ergodicity can be considerably weakened in proving the weak convergence results stated subsequently.
A point process $\Psi = \sum_{i \geq 1} \delta_{X_i}$ on $\mathbb{R}^d$ is called ‘weakly’ or (‘second-order’) stationary with intensity $\lambda > 0$, if $\mathbb{E} \Psi^2([0, 1]^d) < \infty$, $\mathbb{E} \Psi(\cdot) = \lambda |\cdot|$ and the second-order factorial moment measure $\alpha^{(2)}(A \times B) = \mathbb{E} \sum_{i,j \geq 1} 1_A(X_i) 1_B(X_j)$ (where $A, B \subseteq \mathbb{R}^d$ are bounded Borel sets and the sum $\sum_x$ runs over pairwise distinct indices) is invariant against diagonal shifts, i.e. $\alpha^{(2)}(\{(A + x) \times (B + x)\}) = \alpha^{(2)}(A \times B)$ for any $x \in \mathbb{R}^d$.

Obviously, the (factorial) ‘covariance measure’ $\gamma^{(2)}(A \times B) := \alpha^{(2)}(A \times B) - \lambda^2 |A||B|$ has the same invariance property which, by disintegration (see [1, Chapter 10.4]), provides the existence of a unique signed measure $\gamma_{\text{red}}^{(2)}(\cdot)$ on $\mathbb{R}^d$ called ‘reduced covariance measure’ of $\Psi$—such that

$$\gamma^{(2)}(A \times B) = \lambda \int_A \gamma_{\text{red}}^{(2)}(B - x) \, dx.$$ 

Further, let $|\gamma_{\text{red}}^{(2)}(B)|$ be the total variation of $\gamma_{\text{red}}^{(2)}(\cdot)$ over the Borel set $B \subseteq \mathbb{R}^d$; see [1, Appendix A1.3].

In order to prepare the proof of Theorem 2 below, we first study the weak consistency of the estimators $\hat{\lambda}_n$, see (7), and $\lambda_n^* = \Psi(W_n)/|W_n|$ to the intensity $\lambda$. In what follows, let $\overset{P}{\longrightarrow}$ indicate convergence in probability.

Weak stationarity of $\Psi$ implies immediately the unbiasedness of $\lambda_n^*$ and that the variance of $\Psi(W_n)$ can be expressed by $\gamma_{\text{red}}^{(2)}(\cdot)$ in the following way:

$$\text{Var} \, (\Psi(W_n)) = \lambda |W_n| + \lambda \int_{\mathbb{R}^d} |W_n \cap (W_n - x)| \, \gamma_{\text{red}}^{(2)}(dx).$$

Hence, by $W_n \oplus \tilde{W}_n = \{x \in \mathbb{R}^d : W_n \cap (W_n - x) \neq \emptyset\}$, we obtain that

$$\mathbb{E} \left( \lambda_n^* - \lambda \right)^2 = \frac{\text{Var} \, (\Psi(W_n))}{|W_n|^2} \leq \frac{\lambda}{|W_n|} \left( 1 + |\gamma_{\text{red}}^{(2)}(W_n \oplus \tilde{W}_n)| \right) \overset{n \to \infty}{\longrightarrow} 0$$

if the additional assumption

$$\frac{|\gamma_{\text{red}}^{(2)}(W_n \oplus \tilde{W}_n)|}{|W_n|} \overset{n \to \infty}{\longrightarrow} 0 \quad \text{as} \quad |W_n| \overset{n \to \infty}{\longrightarrow} \infty \quad (\text{e.g. if} \quad |\gamma_{\text{red}}^{(2)}(\mathbb{R}^d)| < \infty)$$

is satisfied. Thus, $\lambda_n^*$ is mean-square consistent for $\lambda$ and this in turn gives $\overset{P}{\longrightarrow} \lambda$ as $|W_n| \overset{n \to \infty}{\longrightarrow} \infty$.

The corresponding result for $\hat{\lambda}_n$ is formulated in the following lemma.

**Lemma 1** Let the conditions of Theorem 1 be fulfilled, where ergodicity of $\Psi_{\text{mark}}$ can be replaced by the assumption that the (strictly) stationary unmarked point process $\Psi(\cdot) = \Psi_{\text{mark}}((\cdot) \times \mathcal{K})$ has second moments and its reduced covariance measure $\gamma_{\text{red}}^{(2)}(\cdot)$ satisfies (16). Then $\hat{\lambda}_n$ is an unbiased, weakly consistent estimator for $\lambda$, i.e.

$$\hat{\lambda}_n \overset{P}{\longrightarrow} \lambda.$$ (17)
Proof  To verify (17), we again apply the truncation technique used in the proof of Theorem 1. With the notation used there we may write for any \( \varepsilon, \delta \in (0, 1) \), that
\[
\mathbb{P}(|\hat{\lambda}_n - \lambda| \geq \varepsilon) = \mathbb{P}(|[\hat{\lambda}_n - \lambda] \geq \varepsilon) \cap B_n(\delta)) + \mathbb{P}(|[\hat{\lambda}_n - \lambda] \geq \varepsilon) \cap B_n^c(\delta)) \\
\leq \mathbb{P}(\lambda_n^* - \lambda \geq \varepsilon/2) + \mathbb{P}(|[\hat{\lambda}_n - \lambda_n^*] \geq \varepsilon/2) \cap B_n(\delta)) + \mathbb{P}(B_n^c(\delta)). \quad (18)
\]

In view of the decomposition
\[
\hat{\lambda}_n - \lambda_n^* = \frac{1}{|W_n|} \sum_{i \geq 1} 1_{u_i}(X_i) \frac{|W_n \setminus (W_n \ominus \mathcal{Z}_i)|}{|W_n \ominus \mathcal{Z}_i|} - \frac{1}{|W_n\setminus(W_n \ominus \mathcal{Z}_i)|} \\
\text{and since } \|\mathcal{Z}_i\| \leq r_n^\delta \text{ implies the inclusion } W_n^\delta := W_n \ominus b(o, r_n^\delta) \subseteq W_n \ominus \mathcal{Z}_i, \text{ we arrive at}
\]
\[
\left\{ |\hat{\lambda}_n - \lambda_n^*| \geq \frac{\varepsilon}{2} \right\} \cap B_n(\delta) \subseteq \left\{ \Psi(W_n)|W_n \setminus W_n^\delta| \geq \frac{\varepsilon}{4} \right\} \cup \left\{ \frac{|W_n \setminus W_n^\delta|}{|W_n^\delta|} \geq \frac{\varepsilon}{4} \right\}.
\]

By Markov’s inequality and (12),
\[
\mathbb{P}\left(\left\{ |\hat{\lambda}_n - \lambda_n^*| \geq \frac{\varepsilon}{2} \right\} \cap B_n(\delta)\right) \leq \frac{8 \lambda |W_n \setminus W_n^\delta|}{\varepsilon |W_n^\delta|} \leq \frac{8 \lambda \delta}{\varepsilon (1 - \delta)}.
\]

Finally, (15) and Chebyshev’s inequality yield \( \mathbb{P}(|\lambda_n^* - \lambda| \geq \varepsilon/2) \xrightarrow{n \to \infty} 0 \) so that (18) and (14) with \( \delta = \varepsilon^2/2 \) imply that
\[
\lim_{n \to \infty} \mathbb{P}(|\hat{\lambda}_n - \lambda| \geq \varepsilon) \leq 8 \lambda \varepsilon \quad \text{for any } 0 < \varepsilon < 1.
\]

This proves the assertion (17) of Lemma 1. ■

The previous proof reveals the following.

Remark 3  If in Lemma 1 \( \Psi_{\text{mark}} \) is independently marked, then it suffices to require weak stationarity of \( \Psi(\cdot) = \Psi_{\text{mark}}(\cdot \times \kappa) \).

Next we summarize the essential facts concerning weak convergence of sequences of \( m \)-parameter random processes \( X_n(t), t \in \mathbb{R}^m \), living in \( D(\mathbb{R}^m) \), the set of real functions on \( \mathbb{R}^m \) which are right continuous with finite left limits existing everywhere (for a precise definition of the limits, see [16] or [17])—briefly the ‘càdlàg-functions’ on \( \mathbb{R}^m \). We first recall that, following [6, 16], the set \( D[s, t] \) of càdlàg-functions defined on the closed hyper-rectangle \( [s, t] := \prod_{j=1}^m [s_j, t_j] \) (for \( s = (s_1, \ldots, s_m) \in \mathbb{R}^m \) and \( t = (t_1, \ldots, t_m) \in \mathbb{R}^m \) with \( s < t \), i.e. \( s_j < t_j \) for \( j = 1, \ldots, m \)) can be equipped with a metric \( \rho_{s,t} \), making \( D[s, t] \) to a Polish space which generalizes in a natural way the one-dimensional Skorohod-space \( D[a, b] \) for \( -\infty < a < b < \infty \); see [8, Chapter 3] for details. Weak convergence \( X_n(\cdot) \xrightarrow{n \to \infty} X(\cdot) \) of random elements \( X_n(\cdot) \) in \( D[s, t] \) is then defined in the usual way and criteria for the convergence in terms of mixed moments of increments of \( X_n(\cdot) \) over neighbouring hyper-rectangles in \( [s, t] \) are given in [6].

For \( m = 1 \), the extension of weak convergence to random processes defined on an infinite interval goes back to papers of Stone, Lindvall and Whitt, see references in [18, Chapter 4.4.1] or [19, Chapter 12.9] and references therein. In the Billingsley’s monograph [8, Chapter 16, p. 191] the reader can find a criterion for the weak convergence in \( D(0, \infty) \) and \( D(\mathbb{R}) \), respectively, which applies almost verbatim to the higher-dimensional case. A detailed study...
of weak convergence in the space \( D([0, \infty)^m, E) \) of càdlàg-functions taking values in a Polish space \( E \) can be found in \([17]\). In our context we need the extension of Skorohod-space \( D[s, t] \) to the corresponding space \( D(\mathbb{R}^m) \). The crucial point is the introduction of a metric \( \varphi \) in \( D(\mathbb{R}^m) \); see \([8, \text{pp. 168–179, 191}] \) for \( m = 1 \) and \([17, \text{pp. 182–184}] \) for \( m \geq 2 \), which is defined in such way that for \( x(\cdot), x_n(\cdot) \in D(\mathbb{R}^m), n = 1, 2, \ldots, \)

\[
\varphi(x_n(\cdot), x(\cdot)) \rightarrow 0 \quad \text{iff} \quad \varphi_{s,t}(r_{s,t}x_n(\cdot), r_{s,t}x(\cdot)) \rightarrow 0 \quad \text{for all continuity points } s, t \in \mathbb{R}^m \text{ of } x(\cdot) \text{ satisfying } s < t. \]

Here \( r_{s,t} : D(\mathbb{R}^m) \rightarrow D[s, t] \) is defined by \( r_{s,t}(u) = u \) for \( u \in [s, t] \), and \([s, t] \subset \mathbb{R}^m \) will be called ‘continuity hyper-rectangle of \( x(\cdot) \)’ if the càdlàg-function \( x : \mathbb{R}^m \hookrightarrow \mathbb{R}^1 \) is continuous at both its ‘lower-left’ vertex \( s \) and its ‘upper-right’ vertex \( t \).

In this way weak convergence in \( D(\mathbb{R}^m) \) can be reduced to weak convergence in the more familiar space \( D[s, t] \); see \([6, 16]\). A criterion for weak convergence in \( D(\mathbb{R}^m) \) generalizing \([18, \text{Proposition 4.18}] \) for \( m = 1 \) is stated in the following.

**Proposition 1** If \( \{X_n(\cdot), n \geq 1\} \) and \( X(\cdot) \) are random elements of \( D(\mathbb{R}^m) \) then

\[
X_n(\cdot) \overset{n \rightarrow \infty}{\rightarrow} X(\cdot) \quad \text{in} \quad D(\mathbb{R}^m)
\]

iff for any continuity hyper-rectangle \([s, t] \) of \( X(\cdot) \) (i.e. \( P(X(\cdot) \text{ is continuous at } s \text{ and } t) = 1 \)), we have

\[
r_{s,t}X_n(\cdot) \overset{n \rightarrow \infty}{\rightarrow} r_{s,t}X(\cdot) \quad \text{in} \quad D[s, t].
\]

Note that the latter limiting relations are needed only for two sequences \( s^{(k)} = (s^{(k)}_1, \ldots, s^{(k)}_m) \) and \( t^{(k)} = (t^{(k)}_1, \ldots, t^{(k)}_m) \) of continuity points of \( X(\cdot) \) satisfying

\[
\max\{s^{(k)}_1, \ldots, s^{(k)}_m\} \rightarrow -\infty \quad \text{and} \quad \min\{t^{(k)}_1, \ldots, t^{(k)}_m\} \rightarrow \infty, \quad \text{which do always exist.}
\]

By the very definition of an \( m \)-variate df \( F(t) \) and its empirical counterpart \( \hat{F}_n(t) \), see \((6)\), we immediately get the following.

**Proposition 2** The random processes \( \{Y_n(\cdot), n \geq 1\} \) defined by \((4)\) as well as the mean zero Gaussian process \( Y(\cdot) \) with covariance function \( \mathbb{E}Y(s)Y(t) = F(s \land t) - F(s)F(t) \) belong \( P \)-a.s. to the subspace \( D_0(\mathbb{R}^m) \) containing those \( x \in D(\mathbb{R}^m) \) which have finite one-sided limits

\[
\lim_{t \rightarrow t^*}\, x(t) \quad \text{for any } t^* = (t^*_1, \ldots, t^*_m) \text{ with } t^*_i \in \{-\infty, +\infty\} \text{ for some } i \in \{1, \ldots, m\}; \text{ see } [17].
\]

Our next step towards the proof of \( Y_n(\cdot) \overset{n \rightarrow \infty}{\rightarrow} Y(\cdot) \) in \( D(\mathbb{R}^m) \) consists in showing that \( Y(\cdot) \) is the weak limit of the following sequence of \( m \)-parameter empirical processes:

\[
Z_n(t) = \frac{1}{\sqrt{\lambda |W_n|}} \sum_{i \geq 1} I_{W_n}(X_i) \, (1_{(-\infty,t]}(f(\Xi_i)) - F(t)), \quad t \in \mathbb{R}^m.
\]

**Lemma 2** Let the germ-grain process \((1)\) be defined by a weakly stationary point process \( \Psi = \sum_{i \geq 1} \delta_{X_i} \), with intensity \( \lambda > 0 \) and an independent sequence \( \{\Xi_i, i \geq 1\} \) of i.i.d. copies of the typical grain \( \Xi_0 \). If in addition, the reduced covariance measure \( \gamma_{\text{red}}^{(2)}(\cdot) \) of \( \Psi \) satisfies condition \((16)\), then

\[
Z_n(\cdot) \overset{n \rightarrow \infty}{\rightarrow} Y(\cdot) \quad \text{in} \quad D(\mathbb{R}^m)
\]

provided that \( |W_n| \overset{n \rightarrow \infty}{\rightarrow} \infty \), where \( Y(t), t \in \mathbb{R}^m \), denotes the Gaussian process of Proposition 2.
Proof A straightforward application of the classical CLT for i.i.d. random variables shows that

$$U_N(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( 1_{(-\infty,t]}(f(\Xi_i)) - F(t) \right) \Longrightarrow \mathcal{N}(0, F(t)(1 - F(t))) \quad \text{for any } t \in \mathbb{R}^m,$$

where $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian random variable with mean $\mu$ and variance $\sigma^2$. Since, after a short calculation, $\mathbb{E}U_N(s)U_N(t) = F(s \wedge t) - F(s)F(t)$ for any $N \geq 1$, we get in the same way

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{P} c_i U_N(t(i)) \Longrightarrow \mathcal{N}(0, F(t_1)(1 - F(t_1))\ldots F(t_p)(1 - F(t_p)))$$

for any $t(1), \ldots, t(p) \in \mathbb{R}^m$ and $c_1, \ldots, c_P \in \mathbb{R}$, where

$$\sigma^2(t(1), \ldots, t(p)) = \sum_{k,l=1}^{P} c_k c_l \left( F(t(k) \wedge t(l)) - F(t(k))F(t(l)) \right).$$

A well-known so-called ‘transfer theorem’ see [20, Chapter 4] or [21, Chapter 8.7], tells us that the previous CLT remains valid if the number of summands $N$ is replaced by the random number $\Psi(W_n)(\rightarrow \infty)$. Here we have used the fact that the random integers $\Psi(W_n)$ are independent of the i.i.d. sequence $f(\Xi_i), i \geq 1$. Further, by using Lemma 1 and Slutsky’s theorem, we find that

$$\frac{1}{\sqrt{\Psi(W_n)}} \sum_{k=1}^{P} c_k U_{\Psi(W_n)}(t(k)) \Longrightarrow \mathcal{N}(0, \sigma^2(t(1), \ldots, t(p)))$$

Hence, by applying the Cramér–Wold device,

$$(Z_n(t(1)), \ldots, Z_n(t(p))) \Longrightarrow (Y(t(1)), \ldots, Y(t(p)))$$

for any $t(1), \ldots, t(p) \in \mathbb{R}^m$. It remains to verify the tightness of the sequence $Z_n(\cdot), n \geq 1$, by estimating the mixed fourth moment $\mathbb{E}Z_n^2((u, v])Z_n^2((\tilde{u}, \tilde{v}])$ for each pair of neighbouring half-open rectangles $(u, v], (\tilde{u}, \tilde{v}]$ having a common $(m-1)$-dimensional face and lying in some continuity hyper-rectangle $[s,t]$ of $Y(\cdot)$. Note that the ‘increment’ of $Z_n(\cdot)$ around $(u, v], u < v$, is defined by

$$Z_n((u, v]) = \sum_{\varepsilon_1, \ldots, \varepsilon_m \in \{0,1\}} (-1)^{\varepsilon_1+\cdots+\varepsilon_m} Z_n(u_1 + \varepsilon_1(v_1 - u_1), \ldots, u_m + \varepsilon_m(v_m - u_m)).$$

For brevity put $g_{(u,v]}(K) = 1_{[u,v]}(f(K)) - v_F((u, v])$ for $K \in \mathcal{K}$ and $u < v$, where $v_F(\cdot)$ designates the probability measure on the Borel sets of $\mathbb{R}^m$ generated by the df $F(\cdot)$. 
In view of the independence assumptions made in Lemma 2 and \( \mathbb{E} g_{(a,v)}(\Xi) = 0 \), we obtain
\[
\mathbb{E} Z^2_{n}((u,v)) Z^2_{n}((\tilde{u}, \tilde{v})) = \frac{1}{\lambda^2 |W_n|^2} \sum_{p \geq 1} \mathbb{P}(\Psi(W_n) = p) \sum_{i,j=1}^{p} \sum_{k,l=1}^{p} \mathbb{E} g_{(a,v)}(\Xi_i) g_{(a,v)}(\Xi_j) g_{(\tilde{a}, \tilde{v})}(\Xi_k) g_{(\tilde{a}, \tilde{v})}(\Xi_l)
\]
\[
= \frac{\mathbb{E}(\Psi(W_n))}{\lambda^2 |W_n|^2} \mathbb{E} g_{(a,v)}(\Xi_0) g_{(\tilde{a}, \tilde{v})}(\Xi_0) + \frac{\alpha(2)(W_n \times W_n)}{\lambda^2 |W_n|^2} \times \left( \mathbb{E} g_{(a,v)}(\Xi_0) \mathbb{E} g_{(\tilde{a}, \tilde{v})}(\Xi_0) + 2 \left( \mathbb{E} g_{(a,v)}(\Xi_0) g_{(\tilde{a}, \tilde{v})}(\Xi_0) \right)^2 \right) \leq \frac{\nu_F((u,v)) \nu_F((\tilde{u}, \tilde{v}))}{\lambda |W_n|} \left( 1 + 3 |\gamma^{(2)}_{\text{red}}|(W_n \oplus \tilde{W}_n) + 3 \lambda |W_n| \right) \leq c(\lambda) \nu_F((u,v)) \nu_F((\tilde{u}, \tilde{v}))
\]
for large enough \( n \), which proves the desired tightness condition; see [6].

Here we have used (15) and the relations \( \mathbb{E} g_{(a,v)}^2(\Xi_0) = \nu_F((u,v)) \) \( (1 - \nu_F((u,v))) \) and
\[
\mathbb{E} g_{(a,v)}^2(\Xi_0) g_{(\tilde{a}, \tilde{v})}(\Xi_0)
\]
\[
= \nu_F((u,v)) \nu_F((\tilde{u}, \tilde{v})) \left( \nu_F((u,v)) + \nu_F((\tilde{u}, \tilde{v})) - 3 \nu_F((u,v)) \nu_F((\tilde{u}, \tilde{v})) \right).
\]
Thus, by [6, Theorem 4] and some additional comments given there, we have shown that
\[
Z_n(\cdot) \xrightarrow{n \to \infty} Y(\cdot) \quad \text{in } D[s,t]
\]
for any continuity hyper-rectangle \([s,t]\) of \( Y(\cdot) \). This together with Proposition 1 completes the proof of (19).

\textit{Remark 4}  Note that the weak convergence of empirical dfs for i.i.d. samples taking values in the unit cube \([0,1]^m\) and having a not necessarily continuous df \( F(\cdot) \) has been already proven by Neuhaus [16]. Further refinements of this result can be found in [7]. Lemma 2 extends this result to a random number of i.i.d. observations in \( \mathbb{R}^m \) driven by a stationary point process.

\textit{Remark 5}  By applying a large deviations inequality for empirical dfs due to Kiefer and Wolfowitz [22], we obtain the estimate
\[
\mathbb{P}(\sup_{t \in \mathbb{R}^m} |U_N(t)| \geq r \sqrt{N}) \leq A e^{-ar^2 N} \quad \text{for all } N \geq 1 \quad \text{and all } r > 0,
\]
where \( a \) and \( A \) are positive constants. This result and the independence assumption of Lemma 2 yield
\[
\mathbb{P} \left( \sup_{t \in \mathbb{R}^m} |Z_n(t)| \geq r \lambda_n^m \sqrt{|W_n|} \right) \leq A e^{-a \lambda r^2 \Psi(W_n)} \left( = A \exp \left( -\lambda \left( 1 - e^{-a \lambda r^2} \right) |W_n| \right) \right),
\]
where the equality included in parenthesis holds if the point process \( \Psi(\cdot) \) is Poisson.

Now we are in a position to prove our final result:

\textit{Theorem 2}  Let the germ-grain process (1) be defined by a weakly stationary point process \( \psi = \sum_{i \geq 1} \delta_{X_i} \) with intensity \( \lambda > 0 \) and an independent sequence \( \{\Xi_i, i \geq 1\} \) of i.i.d. copies of the typical grain \( \Xi_0 \). Assume that \( |\gamma^{(2)}_{\text{red}}| (\mathbb{R}^d) < \infty \) and \( \mathbb{E} \| \Xi_0 \|_q < \infty \) for some \( q \geq d \) and let \( (W_n) \) be an increasing sequence of convex, bounded sets in \( \mathbb{R}^d \) satisfying (8).
Lemma 2 and \[8,\) Theorem 3.1] imply

\[ EY(s)Y(t) = F(s \land t) - F(s)F(t). \]

**Proof** First note that by applying Lemma 1 and Slutsky-type arguments, see e.g. \[8,\) Theorem 3.1], the weak limit (if it exists) of \(Y_n(t), t \in \mathbb{R}^m\), coincides with that of

\[
\hat{Z}_n(t) = \frac{\hat{Y}_n}{\sqrt{\lambda_n}} Y_n(t) = \sqrt{\frac{|W_n|}{\lambda_n}} \sum_{i \geq 1} \frac{1_{W_n \cap \Xi_i}(X_i)}{|W_n \cap \Xi_i|} \left( 1_{(-\infty,t]}(f(\Xi_i)) - F(t) \right).
\]

This means that Lemma 2 and \[8,\) Theorem 3.1] imply \(Y_n(\cdot) \xrightarrow{n \to \infty} Y(\cdot)\) in \(D(\mathbb{R}^m)\) whenever we can verify that \(\varrho_{s,t}(Z_n(\cdot), \hat{Z}_n(\cdot)) \xrightarrow{P} 0\) or slightly stronger that

\[
\sup_{u \in [s,t]} |Z_n(u) - \hat{Z}_n(u)| \xrightarrow{n \to \infty} 0 \quad \text{for any continuity hyper-rectangle } [s, t] \subseteq \mathbb{R}^m.
\]

For this, we consider the difference process \(\Delta_n(\cdot) := \sqrt{\lambda} \left( Z_n(\cdot) - \hat{Z}_n(\cdot) \right)\) on the event \(B_n(\delta) = \bigcap_{i:X_i \in \mathbb{W}_n} \{ \| \Xi_i \| \leq r_n^\delta \}\) with \(r_n^\delta = \delta |W_n|/H_{d-1}(W_n)\) and \(\delta \in (0, 1)\) (as in the proof of Theorem 1) which allows to replace the grains \(\Xi_i\) (with germs \(X_i \in W_n\)) by the truncated grains \(\Xi_i^\delta = \Xi_i \cap b(o, r_n^\delta)\) such that

\[
\Delta_n(t) 1_{B_n(\delta)} = \left( \Delta_n^\delta(t) - \Delta_n^{\delta,0}(t) \right) 1_{B_n(\delta)} \quad \text{for all } t \in \mathbb{R}^m,
\]

where

\[
\Delta_n^\delta(t) := \sqrt{|W_n|} \sum_{i \geq 1} \left( \frac{1_{W_n \cap \Xi_i}(X_i)}{|W_n \cap \Xi_i|} - \frac{1_{W_n \cap \Xi_i^\delta}(X_i)}{|W_n \cap \Xi_i^\delta|} \right) 1_{(-\infty,t]}(f(\Xi_i))
\]

and

\[
\Delta_n^{\delta,0} := \sqrt{|W_n|} \sum_{i \geq 1} \left( \frac{1_{W_n}(X_i)}{|W_n|} - \frac{1_{W_n \cap \Xi_i^\delta}(X_i)}{|W_n \cap \Xi_i^\delta|} \right).
\]

For any \(\varepsilon > 0\) and \(\delta \in (0, 1)\), we have

\[
P\left( \sup_{u \in [s,t]} |\Delta_n(u)| \geq \varepsilon \right) = P\left( \sup_{u \in [s,t]} |\Delta_n(u)| \geq \varepsilon, B_n(\delta) \right) + P\left( \sup_{u \in [s,t]} |\Delta_n(u)| \geq \varepsilon, B_n^c(\delta) \right)
\]

\[
\leq P\left( \sup_{u \in [s,t]} |\Delta_n^\delta(u)| \geq \frac{\varepsilon}{2} \right) + P\left( |\Delta_n^{\delta,0}| \geq \frac{\varepsilon}{2} \right) + P(B_n^c(\delta)).
\]

The moment assumption imposed on \(\| \Xi_0 \|\) combined with (8) implies (as an immediate consequence of (14)) that \(P(B_n^c(\delta_n)) \xrightarrow{n \to \infty} 0\) for certain sequence \(\delta_n > 0\) with \(\delta_n \downarrow 0\) as \(n \to \infty\). The second term in the latter line will be estimated using Chebyshev’s inequality. Having in mind \(E\Delta_n^{\delta,0} = 0\), we can express and estimate the second moment \(E(\Delta_n^{\delta,0})^2\) by means of the
The last inequality follows from
\[ \gamma(\delta n, \Delta_1) \leq \delta \]
Chebyshev's inequality yield
\[ \left( \frac{1}{\lambda W_n} \int_{\mathbb{R}^d} E\left( \frac{1_{W_n}(x) - \frac{1_{W_n \ominus \Sigma_0^d}(x)}{|W_n \ominus \Sigma_0^d|}}{|W_n|} \right)^2 dx + \lambda |W_n| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E\left( \frac{1_{W_n}(x) - \frac{1_{W_n \ominus \Sigma_0^d}(x)}{|W_n \ominus \Sigma_0^d|}}{|W_n|} \right)^2 dx \times E\left( \frac{1_{W_n}(y + x) - \frac{1_{W_n \ominus \Sigma_0^d}(y + x)}{|W_n \ominus \Sigma_0^d|}}{|W_n|} \right) \gamma^{(2)}(dy, dx) \]
\[ \leq \lambda \left( 1 + |\gamma^{(2)}(\mathbb{R}^d)| \right) \int_{\mathbb{R}^d} E\left( \frac{|W_n - (W_n \ominus \Sigma_0^d)|}{|W_n \ominus \Sigma_0^d|} \right)^2 dx \]
\[ = \lambda \left( 1 + |\gamma^{(2)}(\mathbb{R}^d)| \right) E\left( \frac{|W_n - (W_n \ominus \Sigma_0^d)|}{|W_n \ominus \Sigma_0^d|} \right) \leq \lambda \left( 1 + |\gamma^{(2)}(\mathbb{R}^d)| \right) \frac{\delta}{1 - \delta}. \]

The last inequality follows from (12). Hence,
\[ P \left( |\Delta_{n,0}^d| \geq \frac{\varepsilon}{2} \right) \rightarrow 0 \quad \text{for any } \varepsilon > 0. \]

In the same way as earlier, we get
\[ \mathbb{E}(\Delta_n^d(t))^2 \leq \lambda \left( 1 + |\gamma^{(2)}(\mathbb{R}^d)| \right) \frac{\delta}{1 - \delta}, \]
which implies \( \Delta_n^d(t) \xrightarrow{P} 0 \) for all \( t \in \mathbb{R}^m \).

In order to verify the uniform convergence sup\( u \in [s, t] |\Delta_n^d(u)| \xrightarrow{P} 0 \) for any continuity hyper-rectangle \([s, t]\), we rewrite \( \Delta_n^d(t) \) as sum \( \Delta_{n,1}^d(t) + \Delta_{n,2}^d(t) \), with
\[ \Delta_{n,1}^d(t) = \frac{1}{|W_n|} \sum_{i \geq 1} A_n^d(X_i, \Xi, (-\infty, t]) \quad \text{and} \quad \Delta_{n,2}^d(t) = \frac{1}{|W_n|} \sum_{i \geq 1} a_n^d(X_i, (-\infty, t]), \]
where the functions \( A_n^d(x, K, B) \) and \( a_n^d(x, B) \) are defined for any \( x \in \mathbb{R}^d, B \subseteq \mathbb{R}^m \) and \( K \subseteq K \) with \( c(K) = 0 \) as follows:
\[ A_n^d(x, K, B) = a_n^d(x, K, B) - a_n^d(x, B) \quad \text{with} \quad a_n^d(x, B) = \mathbb{E}a_n^d(x, \Xi, 0, B) \]
and
\[ a_n^d(x, K, B) = \left( 1_{W_n}(x) - \frac{|W_n|}{|W_n \ominus (\tilde{K} \cap b(o, r_n^d))|} 1_{W_n \ominus (\tilde{K} \cap b(o, r_n^d))}(x) \right) 1_B(f(K)). \]

From (12) we get
\[ |a_n^d(x, K, B)| \leq \max \left\{ 1, \frac{\delta}{1 - \delta} \right\} 1_{W_n}(x) 1_B(f(K)), \]
and hence for \( 0 < \delta \leq 1/2, \)
\[ |a_n^d(x, B)| \leq 1_{W_n}(x) \nu_F(B) \quad \text{and} \quad |A_n^d(x, K, B)| \leq 1_{W_n}(x) (1_B(f(K)) + \nu_F(B)). \] (21)

It is obvious that both random processes \( \Delta_{n,1}^d(\cdot) \) and \( \Delta_{n,2}^d(\cdot) \) belong to \( D(\mathbb{R}^m) \) for any \( \delta \in (0, 1) \). Further, calculating the second moment of \( \Delta_{n,1}^d(t) \) along the above lines and using Chebyshev's inequality yield \( \Delta_{n,1}^d(t) \xrightarrow{P} 0 \) for any fixed \( t \in \mathbb{R}^m \) and \( i = 1, 2 \).
The proof of the weak convergence \( \Delta_{n,i}^δ(\cdot) \xrightarrow{n \to \infty} 0 \) in \( D[s, t] \) for some continuity hyper-rectangle (implying \( \sup_{u \in [s, t]} |\Delta_{n,i}^δ(u)| \xrightarrow{n \to \infty} 0 \)) relies on the tightness of the sequences \( \{\Delta_{n,i}^δ(\cdot), n \geq 1\} \) in \( D[s, t] \) which will be verified now.

By virtue of the independence between \( \Psi = \sum_{i \geq 1} \delta_x^i \) and the i.i.d. sequence of grains \( \{\xi_i, i \geq 1\} \) together with \( \mathbb{E} A^δ(x, \xi_0, B) = 0 \) and (21) we obtain for any two disjoint (neighbouring) hyper-rectangles \((u, v)\) and \(\bar{(u, v)}\) and \(0 < \delta \leq 1/2\) that

\[
\begin{align*}
\mathbb{E} \left( \Delta_{n,1}^δ((u, v)) \right)^2 \left( \Delta_{n,1}^δ((\bar{u}, \bar{v})) \right)^2 &= \frac{\lambda}{|W_n|^2} \int_{\mathbb{R}^d} \mathbb{E} \left( A^δ_n(x, \xi_0, (u, v)) \right)^2 \left( A^δ_n(x, \xi_0, (\bar{u}, \bar{v})) \right)^2 \, dx \\
&+ \frac{1}{|W_n|^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E} \left( A^δ_n(x, \xi_0, (u, v)) \right)^2 \mathbb{E} \left( A^δ_n(y, \xi_0, (\bar{u}, \bar{v})) \right)^2 \alpha^{(2)}(dx, dy) \\
&+ \frac{2}{|W_n|^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E} \left( A^δ_n(x, \xi_0, (u, v)) A^δ_n(y, \xi_0, (\bar{u}, \bar{v})) \right)^2 \alpha^{(2)}(dx, dy) \\
&\leq 4 \lambda \nu_F((u, v)) \nu_F((\bar{u}, \bar{v})) + \frac{48 \nu^{(2)}(W_n \times W_n) \nu_F((u, v)) \nu_F((\bar{u}, \bar{v}))}{|W_n|^2}.
\end{align*}
\]

Similarly, by making use of \( \int_{\mathbb{R}^d} a^δ_n(x, (u, v)) \, dx = 0 \), we find that

\[
\begin{align*}
\mathbb{E} \left( \Delta_{n,2}^δ((u, v)) \right)^2 &= \frac{\lambda}{|W_n|} \int_{\mathbb{R}^d} \left( a^δ_n(x, (u, v)) \right)^2 \\
&+ \int_{\mathbb{R}^d} a^δ_n(x, (u, v)) a^δ_n(y + x, (u, v)) \gamma^{(2)}(dy) \, dx \\
&\leq \lambda \left( 1 + |\gamma^{(2)}(\mathbb{R}^d)| \right) \nu_F((u, v))^2.
\end{align*}
\]

This and the previous estimate together with \( \alpha^{(2)}(W_n \times W_n) \leq \lambda |W_n| |\lambda| W_n| + |\gamma^{(2)}(\mathbb{R}^d)| \), see (15), yield

\[
\mathbb{E} \left( \Delta_{n,1}^δ((u, v)) \right)^2 \left( \Delta_{n,1}^δ((\bar{u}, \bar{v})) \right)^2 \leq C \nu_F((u, v)) \nu_F((\bar{u}, \bar{v})) \text{ for } i = 1, 2
\]

proving the tightness of \( \Delta_{n,1}^δ(\cdot) \) and \( \Delta_{n,2}^δ(\cdot) \) in \( D[s, t] \). Summarizing the above steps and (20) completes the proof of Theorem 2.

In view of Proposition 2 and using the fact that the mapping \( x(\cdot) \mapsto \sup_{u \in \mathbb{R}^m} |x(u)| \) is continuous on the subspace \( D_0(\mathbb{R}^m) \) (which results essentially from its continuity on the spaces \( D[s, t] \) for \( s, t \in \mathbb{R}^m \)), see [7, 8, 16, 17], Theorem 2 and the continuous mapping theorem yield

\[
\sup_{t \in \mathbb{R}^m} |Y_n(t)| \xrightarrow{n \to \infty} \sup_{t \in \mathbb{R}^m} |Y(t)|,
\]

where the df of the limit depends on \( F(\cdot) \) for \( m \geq 2 \), see e.g. [22].

The case \( m = 1 \) is of special interest for testing the goodness-of-fit of some hypothetical distribution function \( F(t) = \mathbb{P}(f(\xi_0) \leq t) \) for \( t \in \mathbb{R}^1 \). We know that in this case the Gaussian limit process \( Y(\cdot) \) is stochastically equivalent to \( W^o(F(\cdot)) \), where \( W^o(\cdot) \) is the Brownian bridge – a zero mean Gaussian process on \([0, 1]\) with \( \mathbb{E} \, W^o(s)W^o(t) = s \wedge t - s \wedge t \), see e.g. [8].
**COROLLARY 1** Under the assumptions of Theorem 2 (for \( m = 1 \)) we have

\[
\sup_{t \in \mathbb{R}^1} |Y_n(t)| \Longrightarrow \sup_{t \in \mathbb{R}^1} |W^\circ(F(t))|.
\]

Furthermore, if \( F(\cdot) \) is continuous then the limit df \( P(\sup_{0 \leq t \leq 1} |W^\circ(t)| \leq x) \) coincides with well-known Kolmogorov df; see [8, p. 103].

### 4. Concluding remarks and examples

The independence assumptions put on the germ-grain process (1) in Theorem 2 are crucial in proving the existence of a weak limit and guarantee that the covariance function of the Gaussian limit process \( Y(\cdot) \) depends only on the df \( F(\cdot) \). Theorem 2 above suggests generalizations in various directions. One of them is to consider empirical marginal dfs in germ-grain processes with dependent grains and dependences between the sequence of grains and the point process of germs. Random tessellations seem to be tractable structures of this kind which give rise to future investigations. In any case these models have to satisfy certain (strong) mixing-type conditions as they have been shown and applied e.g. in [23] or [24], respectively. On the other hand, the covariance function of the expected Gaussian limit process will reflect the dependence structure of the model under consideration. There is an obvious similarity with the asymptotic behaviour of empirical processes related with strictly stationary sequences; see e.g. [25].

To conclude with, we demonstrate the applicability of the above results with three examples based on simulated germ-grain processes. Some further applications to real-life data, the reader can find in [26]. In Figure 2 the point process of germs is a Poisson process and the grains are unions of two connected arcs. The lengths of the arcs are independent and have uniform distribution on the interval \([0, a]\), \( a > 0 \).

In Figure 3 the underlying point process of germs forms a Matérn cluster process, that is, a Poisson cluster process with Poisson distributed cluster size and cluster members being uniformly distributed in a circle of fixed radius; see [5]. The grains are segments with exponentially distributed lengths (with mean \( 1/a \)) and uniformly distributed orientation.

The germ-grain process in Figure 4 is generated by a Matérn hard-core process II, see [5], and non-overlapping rectangular grains with edges parallel to the coordinate axes. Width and height of the typical rectangle \( \Xi_0 \) are independent random variables with dfs \( F_w(t) = (t/a)^2 \), \( t \in [0, a] \), and \( F_h(t) = (t/b)^2 \), \( t \in [0, b] \), respectively. The df of the area \( f(\Xi_0) = |\Xi_0| \) takes then the form \( F(t) = t^2/(a^2 b^2(1 + \log(a^2 b^2/t^2))) \) for \( 0 \leq t \leq ab \).

Figure 2. Realization of a Poisson fibre process (left) and df of fibre length \( F(\cdot) \) with empirical df \( \hat{F}_n(\cdot) \) (right).
Figure 3. Realization of a Matérn’s cluster segment process (left) and df of segment length $F(\cdot)$ with empirical df $\hat{F}_n(\cdot)$ (right).

Figure 4. Germ-grain process derived from a Matérn hard-core process II and non-overlapping rectangles (left) and df $F(\cdot)$ of rectangle area with empirical df $\hat{F}_n(\cdot)$ (right).

Table 1. Maximal deviations of empirical dfs $\hat{F}_n(\cdot)$ and $\tilde{F}_n(\cdot)$ from $F(\cdot)$.

<table>
<thead>
<tr>
<th></th>
<th>Figure 2</th>
<th>Figure 3</th>
<th>Figure 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(\Xi_0)$</td>
<td>$H_1(\Xi_0)$</td>
<td>$H_1(\Xi_0)$</td>
<td>$</td>
</tr>
<tr>
<td>$F(t)$</td>
<td>$\frac{t^2}{2a^2}, t \in [0, a]$</td>
<td>$1 - \exp\left(-at\right)$, $\frac{t^2}{a^2b^2}\left(1 - \log\frac{t^2}{a^2b^2}\right)$, $1 - \frac{(2a - t)^2}{2a^2}, t \in [a, 2a]$</td>
<td>$t \in [0, \infty)$, $t \in [0, ab]$</td>
</tr>
<tr>
<td>$\Psi(W_n)$</td>
<td>126</td>
<td>228</td>
<td>104</td>
</tr>
<tr>
<td>$N_n$</td>
<td>102</td>
<td>201</td>
<td>89</td>
</tr>
<tr>
<td>$\sup</td>
<td>\hat{F}_n(t) - F(t)</td>
<td>$</td>
<td>0.0543</td>
</tr>
<tr>
<td>$\sup</td>
<td>\tilde{F}_n(t) - F(t)</td>
<td>$</td>
<td>0.0725</td>
</tr>
<tr>
<td>$\sup</td>
<td>Y_n(t)</td>
<td>$</td>
<td>0.6093</td>
</tr>
<tr>
<td>Kolmogorov df</td>
<td>0.1483</td>
<td>0.0298</td>
<td>0.8888</td>
</tr>
</tbody>
</table>

The graphs of the empirical dfs $\hat{F}_n(\cdot)$ calculated according to (6) are displayed on the right-hand side of Figures 2–4. Among others, the maximal deviations of the empirical dfs $\hat{F}_n(\cdot)$ and $\tilde{F}_n(\cdot)$ from $F(\cdot)$ are summarized in Table 1. These values reveal that in each of the three cases the ‘natural’ empirical df $\hat{F}_n(\cdot)$ given by (5) deviates considerably more from $F(\cdot)$ than $\tilde{F}_n(\cdot)$ from $F(\cdot)$. The null hypothesis that the df of $f(\Xi_0)$ coincides with $F(\cdot)$ is not rejected at the 5%-level for each of the three models, since the values in the bottom line of Table 1 do not exceed 0.95, that is $\sup |Y_n(t)| \leq 1.358 = 95$%-quantile of the Kolmogorov df.
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