Limit Distributions of Some Stereological Estimators in Wicksell’s Corpuscle Problem

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Abstract. Suppose that a homogeneous system of spherical particles (d-spheres) with independent identically distributed radii is contained in some opaque d-dimensional body, and one is interested to estimate the common radius distribution. The only information one can get is by making a cross-section of that body with an s-flat (1 ≤ s ≦ d − 1) and measuring the radii of the s-spheres and their midpoints. The analytical solution of (the hyper-stereological version of) Wicksell’s corpuscle problem is used to construct an empirical radius distribution of the d-spheres. In this paper we study the asymptotic behaviour of this empirical radius distribution for s = d − 1 and s = d − 2 under the assumption that the intersection volume becomes unboundedly large and the point process of the midpoints of the d-spheres is Brillinger-mixing. Among others we generalize and extend some results obtained in [1] and [2] under the Poisson assumption for the special case d = 3, s = 2.

AMS 2000 Subject Classification: Primary 60 D 05, 62 G 20; Secondary 60 F 05, 60 G 55

Keywords: asymptotic normality, Brillinger-mixing point processes, shot-noise process, α-stable distribution functions.

This article contains detailed proofs of the Theorems 1 - 4 stated with concised proofs in a paper that appeared under the same title in the journal “Image Analysis & Stereology” 26, No. 2, 63-71 (2007). An earlier draft without Theorems 5 and 6 and Theorems 1 - 4 under assumptions slightly different from those in the present version (and partly not complete) has been published in [6].

1. Introduction

Let $\Psi_d = \{[X_i, R_i] : i \geq 1\}$ be a stationary, independently marked point process in $\mathbb{R}^d$ with generic non-negative mark $R_0$ having the distribution function (briefly dF) $F_d$. The intensity measure $\Lambda_d(\cdot)$ of $\Psi_d$ is then given by $\Lambda_d(B \times (0, r]) = \lambda_d \nu_d(B) F_d(\tau)$, where $\nu_d$ denotes the d-volume and $\lambda_d = \mathbb{E}\{\Psi_d^* \cap [0, 1]^d\}$ is the intensity of the corresponding stationary non-marked point process $\Psi_d^* = \{X_i : i \geq 1\}$, see Stoyan et al. [10] for details. To formulate appropriate mixing conditions on $\Psi_d^*$ we need the higher-order cumulant measures $\gamma_k(\cdot)$ for any $k \geq 2$ defined on the Borel $\sigma$-field $B(\mathbb{R}^{dk})$, see e.g. [3] for a precise definition. The stationarity of $\Psi_d^*$ enables us to define an associated (signed) measure - the reduced kth-order cumulant measure - $\gamma_k^{(red)}(\cdot)$ on $B(\mathbb{R}^{d(k-1)})$ by disintegration w.r.t. $\nu_d$, i.e.

$$
\gamma_k(B_i) = \lambda_d \int_{B_k} \gamma_k^{(red)}(\times_{i=1}^{k-1} (B_i - x)) \nu_d(dx).
$$

Further, let $B_d(x, r)$ denote the closed sphere in $\mathbb{R}^d$ with radius $r > 0$ centered at $x$ and $\omega_d$ stands for the d-volume of the unit sphere $B_d(0, 1)$, i.e. $\omega_d = \pi^{d/2}/\Gamma(d/2 + 1)$. 


Wicksell’s corpuscle problem in its hyper-stereological version can be described as follows: The system of $d$-spheres $\Xi_d = \{ B_d(X_i, R_i) : i \geq 1 \}$ is intersected by the $s$-flat $H_s = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_{s+1} = \cdots = x_d = 0 \}$ (which can be identified with $\mathbb{R}^s$).

We assume that the collection of non-empty $s$-spheres $\Xi_s := \Xi_d \cap H_s = \{ B_s(\overline{X}_i, \overline{R}_i) : i \geq 1 \}$ in the linear subspace $H_s$ can be observed (all radii and midpoints are visible, without considering overlappings and edge-effects) in an expanding sampling window $W_n^{(s)} := n W^{(s)}$, where $W^{(s)}$ is a fixed convex set in $\mathbb{R}^s$ with unit $s$-volume, i.e. $\nu_s(W^{(s)}) = 1$, and $n$ runs through $\mathbb{N} = \{ 1, 2, \ldots \}$. Note that $B_s(\overline{X}_i, \overline{R}_i) \neq \emptyset$ if $\overline{R}_i := (R_i - \| \overline{X}_i \|_{d-s})^{1/2} > 0$. Here and in what follows, write $\overline{\pi}$ (resp. $\underline{\pi}$) to indicate the projection of $x \in \mathbb{R}^d$ onto $H_s$ (resp. onto the orthogonal complement of $H_s$); $\| \cdot \|_{d-s}$ denotes the Euclidean norm in $\mathbb{R}^{d-s}$. The system of non-empty $s$-spheres $B_s(\overline{X}_i, \overline{R}_i)$ is completely described by the stationary marked point process $\overline{\Psi}_s = \{ (\overline{X}_i, \overline{R}_i) : i \geq 1 \}$ in $\mathbb{R}^s$ with intensity measure $\overline{\lambda}_s(A \times (0, r]) = \overline{\lambda}_s(A) \overline{F}_s(r)$, where $\overline{F}_s$ denotes the df of the typical radius $\overline{R}_0$.

In the next section we restate the well-known explicit expressions of the df $\overline{F}_s$ and the intensity $\overline{\lambda}_s$ in terms of $F_d$ and $\lambda_d$ together with the corresponding inversion formulae. After that we present our results on the asymptotic behaviour (as $n \rightarrow \infty$) of appropriate empirical counterparts of the radius df $F_d$ which are obtained from a single observation of all $s$-spheres whose centers lie in $W_n^{(s)}$. In particular, we state asymptotic normality (Theorem 1) and weak consistency (Theorem 4) in the cases $s = d - 1$ and $s = d - 2$, respectively. Using the terminology of the limit theory for sums of independent identically distributed random variables we are in the situation of a non-normal domain of attraction of the Gaussian and the degenerate law, respectively, see Ibragimov and Linnik [7]. By $\implies$ and $\xrightarrow{P}$ we designate weak convergence and convergence in probability $\mathbb{P}$, respectively.

The Poisson framework as presupposed in [1] and [2] is replaced in the present paper by imposing a mixing condition on the point process $\Psi_d^s$. This special type of weak dependence between separated parts of the point field $\{ X_i : i \geq 1 \}$ requires the existence of moment measures of any order. It should be mentioned that similar asymptotic results under milder moment assumptions can be obtained for an absolutely regular point process $\Psi_d$, see [5], as well as for a Poisson cluster process $\Psi_d$, see [4].

However, it seems that the Poisson assumption can hardly be dropped in our Theorems 5 and 6 to derive $\alpha$-stable limits (with $\alpha = 2/(d - s)$) for the fluctuation of the corresponding empirical df’s of $F_d$ when $d - s \geq 2$. In the final section we put together the essential steps of the proofs of our results.

2. RELATIONSHIPS BETWEEN THE RADIUS DF’S

By means of the Campbell theorem and the relation $\overline{R}_i^2 = R_i^2 - \| \overline{X}_i \|_{d-s}^2 > 0$ the intensity measures $\overline{\lambda}_s$ and $\lambda_d$ are connected by the identity

$$\overline{\lambda}_s(A \times (a, b)) = \int_{\mathbb{R}^d \times (0, \infty)} 1\{ \sqrt{\max\{0, \rho^2 - \| \overline{X}_i \|_{d-s}^2\}} \in (A \times (a, b)) \} \Lambda_d(d(x, \rho))$$

for any $A \in \mathcal{B}(\mathbb{R}^s)$ and $0 \leq a < b \leq \infty$ which leads (after putting $A = [0, 1]^s$ and $a = r$, $b = \infty$) to the following Abel-type integral equation:
\[
\overline{X}_s (1 - F_s (r)) = \lambda_d \omega_{d-s} \int_r^\infty (\varrho^2 - r^2)^{(d-s)/2} dF_d(\varrho)
\]

\[
= \lambda_d (d - s) \omega_{d-s} \int_0^\infty (1 - F_d(\sqrt{r^2 + \varrho^2})) \varrho^{d-s-1} d\varrho.
\]

Letting \( r \to 0 \), the previous formula yields

\[
\overline{X}_s = \lambda_d \omega_{d-s} E R_0^{d-s}
\]

provided that \( E R_0^{d-s} < \infty \), whence it follows that

\[
1 - F_s (r) = \frac{1}{E R_0^{d-s}} \int_r^\infty (\varrho^2 - r^2)^{(d-s)/2} dF_d(\varrho)
\]

and the probability density function \( f_s \) of \( R_0 \) (which always exists!) takes the form

\[
f_s (r) = \frac{r (d - s)}{E R_0^{d-s}} \int_r^\infty (\varrho^2 - r^2)^{(d-s-2)/2} dF_d(\varrho).
\]

Here and throughout, the integral \( \int_r^\infty \) stretches over the interval \( (r, \infty) \). To express the radius \( F_d \) in terms of the radius \( F_s \), for any \( s \in \{1, \ldots, d-1\} \) one has to solve the above Abel-type integral equation by unfolding. For doing this we distinguish between the cases \( d-s \) is even and \( d-s \) is odd, respectively. Put \( q = [(d-s-1)/2] \), where \( [x] \) denotes the largest integer smaller than or equal to \( x \) and \( n!! = n(n-2) \cdots 4 \cdot 2 \) or \( 3 \cdot 1 \). Then the \( df \) \( F_d \) can be expressed in terms of the probability density function \( f_s \) by the following formulae:

\[
1 - F_d (r) = (-1)^q \frac{E R_0^{d-s}}{(d-s)!!} \left\{ \begin{array}{ll}
\frac{1}{q} g_s (r) & \text{, } d-s \text{ even} \\
\frac{2}{\pi} \int_r^\infty g_s (\varrho) (\varrho^2 - r^2)^{-1/2} d\varrho & \text{, } d-s \text{ odd}
\end{array} \right.
\]

with

\[
g_s (r) = \left\{ \begin{array}{ll}
\overline{f}_s (r) & \text{if } d-s = 1, 2 \\
\left( \frac{1}{r} \cdots \left( \frac{\overline{f}_s (r)}{r} \right)' \cdots \right)' & \text{if } d-s \geq 3
\end{array} \right.
\]

where in the last line \( q \) derivatives occur.

However, the statistical solution of the above integral equation leads to an inverse estimation problem which is rather unstable from both the computational and statistical viewpoint, see e.g. [1], [2], [8], [10], [11] for further details.

In the most important case \( s = d-1 \) it is rapidly verified by a straightforward application of Campbell’s theorem, see e.g. Stoyan et al. [10], that

\[
\hat{U}_n (r) = \frac{1}{\pi n^{d-1}} \sum_{i \geq 1} 1 (X_i \in W_n^{(d-1)}) \frac{1 (\overline{R}_i > r)}{\sqrt{\overline{R}_i^2 - r^2}}
\]
is an unbiased estimation of \( \lambda_d (1 - F_d(r)) \). On the other hand, this calculation reveals that the variance of \( \hat{U}_n(r) \) does not exist (which has been first noticed in [1]).

We refer to the fact that, for any fixed \( n \in \mathbb{N} \), the empirical process \( \hat{U}_n(r) \) regarded as a random function in \( r \geq 0 \) is by no means monotonically decreasing. It possesses downward jumps at the random points \( r = \mathbb{N}_n \), however, between two such jumps \( \hat{U}_n(r) \) is strictly increasing. Such strange behaviour of this stereological estimator of \( \lambda_d (1 - F_d(r)) \) gave rise to consider several modified and smoothed versions of \( \hat{U}_n(r) \), see e.g. [2] for an isotonic estimation and its asymptotic analysis.

3. ASYMPTOTIC RESULTS

3.1 The Case \( s = d - 1 \)

We first put together some mixing-type conditions for the point process \( \Psi_d^* = \{ X_i : i \geq 1 \} \) of the midpoints of the \( d \)-spheres.

**Condition 1** Assume that \( \Psi_d^* \) is Brillinger-mixing, i.e.,

\[
\int_{(\mathbb{R}^d)^{k-1}} |\gamma_{k}^{(red)}(d(x_1, \ldots, x_{k-1}))| < \infty \quad \text{for} \quad k \geq 2.
\]

**Condition 2** Assume that the reduced second-order cumulant measure \( \gamma_2^{(red)}(\cdot) \) satisfies

\[
\int_{\mathbb{R}^{d-1} \times A} |\gamma_2^{(red)}(dx)| \leq \text{const} \nu_1(A)
\]

for any bounded Borel set \( A \subset \mathbb{R}^1 \).

**Condition 3** Assume that the reduced second-order cumulant measure \( \gamma_2^{(red)}(\cdot) \) has finite total variation, i.e.,

\[
\int_{\mathbb{R}^d} |\gamma_2^{(red)}(dx)| < \infty.
\]

Sufficient conditions for some classes of point processes to be Brillinger-mixing are discussed in [4]. For example, Poisson cluster processes are Brillinger-mixing iff the number of points in the typical cluster has moments of any order. Also, several types of dependently thinned Poisson processes such as Matérn’s hard-core point processes possess this mixing property. If \( \Psi_d^* \) is additionally isotropic with pair correlation function \( g(r) \), see Stoyan et al. [10], then Condition 2 is satisfied if

\[
\sup_{a \geq 0} \int_0^\infty |g(\sqrt{r^2 + a}) - 1| r^{d-2} \, dr < \infty.
\]

This as well as Condition 3 are rather mild restrictions on the point process \( \Psi_d^* \).
Theorem 1 Let the Conditions 1 and 2 be satisfied. If

\[
\sigma^2(r) := \lambda_d \int_r^\infty (q^2 - r^2)^{-1/2} dF_d(q) < \infty
\]

(1)

for some fixed \( r \geq 0 \) and \( \mathbb{E} R_0 < \infty \), then

\[
\sqrt{\frac{\pi^2 n^{d-1}}{\log n^{d-1}}} \left( \hat{U}_n(r) - \lambda_d (1 - F_d(r)) \right) \xrightarrow{\mathbb{P}} N(0, \sigma^2(r)),
\]

where \( N(0, \sigma^2) \) denotes a zero mean Gaussian random variable with variance \( \sigma^2 \). Furthermore, the relation \( \lambda_d \mathbb{E} R_0 \tilde{f}_{d-1}(r) = r \sigma^2(r) \) shows that, for \( r > 0 \), condition (1) is equivalent to \( \tilde{f}_{d-1}(r) < \infty \).

Remark 1 Provided that \( F_d(0) = 0 \), Theorem 1 (for \( r = 0 \)) yields a central limit theorem for the unbiased estimator \( \hat{U}_n(0) \) of the intensity \( \lambda_d \).

Note that, without assuming Brillinger mixing - merely under Condition 3 - \( \hat{U}_n(r) \) turns out to be weakly consistent for \( \lambda_d (1 - F_d(r)) \). Hence, we get that

\[
\frac{\hat{U}_n(r)}{\hat{U}_n(0)} \xrightarrow{\mathbb{P}} 1 - F_d(r) \quad \text{for any} \quad r \geq 0.
\]

It should be noted that, in case \( \Psi^*_d \) is a stationary ergodic point process, the latter relation holds \( \mathbb{P}-\text{a.s.} \).

Remark 2 For \( r > 0 \) the assumption (1) is satisfied if the df \( F_d \) is \( \alpha \)-Hölder continuous for some \( \alpha > 1/2 \) in \( [r, r + \delta] \), i.e.,

\[
F_d(q) - F_d(r) \leq H_{\alpha,\delta} (q - r)^\alpha
\]

for \( r \leq q \leq r + \delta \) and some \( \delta > 0 \).

The multivariate extension of Theorem 1 (by employing the well-known method of Cramér-Wold) shows that the finite-dimensional distributions of the sequence of standardized empirical processes in Theorem 1 tend to those of a Gaussian ‘white noise’ process as \( n \to \infty \).

Theorem 2 Let the Conditions 1 and 2 and (1) for \( r \in \{r_1, \ldots, r_k\} \), \( 0 \leq r_1 < \cdots < r_k \leq \infty \), be satisfied. Then

\[
\sqrt{\frac{\pi^2 n^{d-1}}{\log n^{d-1}}} \left( \frac{\hat{U}_n(r_j) - \lambda_d (1 - F_d(r_j))}{\sigma^2(r_j)} \right)_{j=1}^k \xrightarrow{n \to \infty} N_k(\mathbf{0}, I_k)
\]

where \( N_k(\mathbf{0}, I_k) \) denotes a \( k \)-dimensional Gaussian random vector having zero mean components and a covariance matrix being equal to the unit matrix \( I_k \).

As a simple application of Theorem 2 for \( k = 2 \), \( r_1 = 0 \), \( r_2 = r \) (using the asymptotic independence of the components) and Slutski’s theorem we obtain

Corollary 1 Let the Conditions 1 and 2, \( F_d(0) = 0 \) and (1) for \( r = 0 \) and some \( r > 0 \) be satisfied. Then
\[
\sqrt{\frac{\pi^2 n^{d-1}}{\log n^{d-1}}} \left( \frac{\hat{U}_n(r)}{U_n(0)} - (1 - F_d(r)) \right) \xrightarrow{n \to \infty} N(0, s^2(r)) ,
\]
where \( s^2(r) := (\sigma^2(r) + \sigma^2(0) (1 - F_d(r))^2) / \lambda_d^2. \)

There exists indeed a weakly consistent estimator of the asymptotic variance \( \sigma^2(r) \) (although its expectation does not exist) which is given by the following ‘ornormed’ random sum

\[
\hat{\sigma}_n^2(r) := \frac{1}{n^{d-1} \log n^{d-1}} \sum_{i \geq 1} 1(\bar{X}_i \in W_i^{(d-1)}) \frac{1(R_i > r)}{R_i^2 - r^2} .
\]

**Theorem 3** Under Condition 3 and \( \mathbb{E}R_0 < \infty \) it holds

\[
\hat{\sigma}_n^2(r) \xrightarrow{P} \sigma^2(r) \quad \text{for each} \quad r \geq 0 \quad \text{satisfying (1)} .
\]

Combining Theorem 1 with Theorem 3 together with Slutski’s theorem provides

**Corollary 2** Let the Conditions 1 and 2, \( \mathbb{E}R_0 < \infty \) and (1) for some fixed \( r \geq 0 \) be satisfied. Then

\[
\sqrt{\frac{\pi^2 n^{d-1}}{\sigma_n^2(r) \log n^{d-1}}} \left( \frac{\hat{U}_n(r)}{U_n(0)} - \lambda_d (1 - F_d(r)) \right) \xrightarrow{n \to \infty} N(0, 1) .
\]

**Remark 3** By means of Corollary 2 (applied to \( r = 0 \) provided \( F_d(0) = 0 \)) we are able to construct an asymptotically exact confidence interval for the unknown intensity \( \lambda_d \) of the midpoints of \( d \)-spheres.

In order to find an asymptotic confidence interval for \( 1 - F_d(r) \) we combine Corollary 1, Theorem 2 and Slutski’s theorem and obtain

**Corollary 3** Assume that the Conditions 1 and 2, \( \mathbb{E}R_0 < \infty \), \( F_d(0) = 0 \) and (1) for \( r = 0 \) and some fixed \( r > 0 \) are satisfied. Then

\[
\sqrt{\frac{\pi^2 n^{d-1}}{\hat{\sigma}_n^2(r) \log n^{d-1}}} \left( \frac{\hat{U}_n(0) - \lambda_d (1 - F_d(r))}{U_n(0)} \right) \xrightarrow{n \to \infty} \chi^2(0, 1) ,
\]

where \( \hat{\sigma}_n^2(r) := (\hat{\sigma}_n^2(r) \hat{U}_n^2(0) + \hat{\sigma}_n^2(0) \hat{U}_n^2(r)) / \hat{U}_n^2(0). \)

An immediate consequence of Theorem 2 and Slutski’s theorem is

**Corollary 4** Let the assumptions of Theorem 2 and \( \mathbb{E}R_0 < \infty \) be satisfied. Then

\[
\frac{\pi^2 n^{d-1}}{\log n^{d-1}} \sum_{j=1}^{k} \left( \frac{\hat{U}_n(r_j) - \lambda_d (1 - F_d(r_j))}{\hat{\sigma}_n^2(r_j)} \right)^2 \xrightarrow{n \to \infty} \chi^2_k ,
\]

where the random variable \( \chi^2_k \) is \( \chi^2 \)-distributed with \( k \) degrees of freedom.

The latter result can be used to test the goodness-of-fit of certain hypothesized radius df \( F_d \) (if \( \lambda_d \) is known).
3.2 The Case $s = d - 2$

Define the empirical process

$$
\hat{V}_n(r) = \frac{1}{\pi n^{d-2} \log n^{d-2}} \sum_{i \geq 1} 1(X_i \in W_n^{(d-2)}) \frac{1(\tilde{R}_i > r)}{\tilde{R}_i^2 - r^2}
$$

which has an infinite mean for any $r \geq 0$. Nevertheless, $\hat{V}_n(r)$ is weakly consistent for $\lambda_d (1 - F_d(r))$ under slight additional assumptions.

**Theorem 4** Under Condition 3 and $\mathbb{E} R_0^3 < \infty$ it holds

$$
\hat{V}_n(r) \xrightarrow{P_{n \to \infty}} \lambda_d (1 - F_d(r)) \quad \text{for any } r \geq 0,
$$

and therefore, together with $F_d(0) = 0$,

$$
\frac{\hat{V}_n(r)}{V_n(0)} \xrightarrow{P_{n \to \infty}} 1 - F_d(r) \quad \text{for any } r \geq 0.
$$

**Theorem 5** Let $\Psi_d = \{X_i : i \geq 1\}$ be a stationary Poisson process with intensity $\lambda_d$. If, in addition,

$$
\int_r^\infty |\log(q^2 - r^2)| dF_d(q) < \infty \quad (2)
$$

for some fixed $r \geq 0$ with $F_d(r) < 1$, then

$$
\log n^{d-2} \left( \frac{\hat{V}_n(r)}{\lambda_d (1 - F_d(r))} - 1 \right) - \log \left( \pi \lambda_d (1 - F_d(r)) \right)
\hspace{1cm}
- \int_r^\infty \log(q^2 - r^2) dF_d(q) - 1 + \gamma \xrightarrow{n \to \infty} S_1.
$$

where $\gamma := \lim_{n \to \infty} (1 + 1/2 + \cdots + 1/n - \log n) \simeq 0.5772$ denotes the Euler-Mascheroni constant and the random variable $S_1$ possesses a stable df with characteristic exponent $\alpha = 1$ and skewness parameter $\beta = 1$ having the characteristic function

$$
\mathbb{E} \exp\{it S_1\} = \exp \left\{ - \frac{\pi}{2} |t| - it \log |t| \right\} \quad \text{for } t \in \mathbb{R}^1.
$$

**Remark 4** Nolan [9] provides tables and numerical procedures for calculating the density of $S_1$ (and other stable densities). This gives at least in principle the possibility for testing the null hypothesis $H_0 : F_d = F_d^{(0)}$, $\lambda_d = \lambda_d^{(0)}$. 

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3.3 The Case $d - s > 2$

Of course, the previous cases are of particular interest in stereological practice for $d = 3, s = 2$, $d = 2, s = 1$ and $d = 3, s = 1$. To be complete we also investigate the asymptotic behaviour of a simple generalization of $\hat{U}_n(r)$ resp. $\hat{V}_n(r)$ to the case $d - s > 2$. The below result seems to be of interest for its own right (from the view point of pure asymptotics) and it gives insight how the instability increases when $d - s$ becomes greater than two.

Let $p := d - s$ and define

$$\hat{Y}_n^{(p)}(r) = \frac{1}{n^{sp/2}} \sum_{i \geq 1} 1(X_i \in W_n^{(s)}) \frac{1(\hat{R}_i > r)}{(\hat{R}_i^2 - r^2)^{p/2}}$$

**Theorem 6** Let $\Psi_d = \{X_i : i \geq 1\}$ be a stationary Poisson process with intensity $\lambda_d$ and $\mathbb{E}R_0^{p-2} < \infty$. Then, for any fixed $r \geq 0$ with $F_d(r) < 1$, it holds

$$\frac{\hat{Y}_n^{(p)}(r)}{\left(c_p \lambda_d \int r (p^2 - r^2)(p-2)/2 \, dF_d(r)\right)^{p/2}} \xrightarrow{n \to \infty} S_{2/p},$$

where $c_p = \omega_p \frac{p}{2} \Gamma(1 - \frac{2}{p}) \cos\left(\frac{\pi}{p}\right)$ and the random variable $S_{2/p}$ possesses a stable df with characteristic exponent $\alpha = 2/p \in (0, 1)$ and skewness parameter $\beta = 1$ having the characteristic function

$$\mathbb{E}\exp\{it S_{2/p}\} = \exp\left\{ - |t|^{2/p}\left(1 - i \text{sgn}(t) \tan\left(\frac{\pi}{p}\right)\right)\right\} \quad \text{for} \quad t \in \mathbb{R}.$$ 

4. PROOFS OF THE THEOREMS

First observe that the empirical processes $\hat{U}_n(r), \hat{V}_n(r), \hat{\sigma}_n^2(r)$, and $\hat{Y}_n^{(p)}(r)$ can be regarded as so-called shot-noise processes $\sum_{i \geq 1} f(X_i, \bar{X}_i, R_i)$ with different ‘response functions’ $f(\bar{X}, R)$ $\mathbb{R}^s \times \mathbb{R}^{d-s} \times \mathbb{R}$, see [3] and references therein. However, only $\hat{U}_n(r)$ has a finite first moment. In fact, applying Campbell’s theorem gives $\mathbb{E}\hat{U}_n(r) = \lambda_d (1 - F_d(r))$ and further $\mathbb{E}(\hat{U}_n(r))^m < \infty$ for $1 < m < 2$, but $\mathbb{E}(\hat{U}_n(r))^2 = \infty$. In order to prove Theorem 1 we have to replace the terms $(\hat{R}_i^2 - r^2)^{-1/2}$ (which are responsible for the large fluctuations of the sum) by truncated terms.

More precisely, for any $\varepsilon > 0$, we introduce the ‘truncated’ shot-noise process

$$\tilde{U}_{n, \varepsilon}(r) = \frac{1}{\pi^{d/2}} \sum_{i \geq 1} \frac{1(X_i \in W_n^{(d-1)})}{\sqrt{\hat{R}_i^2 - r^2}} 1(\hat{R}_i^2 - r^2 > \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}})$$

and the nonnegative random integer

$$N_{n, \varepsilon}(r) = \sum_{i \geq 1} 1(X_i \in W_n^{(d-1)}) \frac{1(0 < \hat{R}_i^2 - r^2 \leq \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}})}{\frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}}.$$
First step. For any Borel set \( B \subseteq \mathbb{R}^1 \) we have the identity \( \{ \hat{U}_{n,e}(r) \in B \} \cap \{ N_{n,e}(r) = 0 \} = \{ \hat{U}_n(r) \in B \} \cap \{ N_{n,e}(r) = 0 \} \) and this in turn implies the estimate
\[
\left| P(\hat{U}_{n,e}(r) \in B) - P(\hat{U}_n(r) \in B) \right| \leq P(N_{n,e}(r) \geq 1) \leq E N_{n,e}(r)
\]  
(3)

For brevity put \( \alpha_n = (\varepsilon n^{d-1} \log n^{d-1})^{-1} \). By applying Campbell’s theorem and using that the assumptions (1) and \( ER_0 < \infty \) are satisfied we may write
\[
E N_{n,e}(r) = E \sum_{i \geq 1} 1(\mathbf{x}_i \in W_n^{(d-1)}) 1\left( 0 < R_i^2 - r^2 \leq \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}} \right)
\]
\[
= \lambda_d n^{d-1} \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \int_{-\sqrt{\varrho^2 - r^2}}^{\sqrt{\varrho^2 - r^2}} 1\left( \varrho^2 - x^2 - r^2 \leq \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}} \right) dx dF_d(\varrho)
\]
\[
= 2 \lambda_d n^{d-1} \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \left( \sqrt{\varrho^2 - r^2} - \sqrt{(\varrho^2 - r^2)(1 - \alpha_n/\varepsilon)} \right) dF_d(\varrho)
\]
\[
+ 2 \lambda_d n^{d-1} \int_{r}^{\sqrt{r^2 + \varepsilon}} \left( \sqrt{\varrho^2 - r^2} - \sqrt{\max\{0, \varrho^2 - r^2 - \alpha_n\}} \right) dF_d(\varrho)
\]
\[
\leq 2 \lambda_d n^{d-1} \alpha_n \left( \frac{1}{\varepsilon} \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \frac{\sqrt{\varrho^2 - r^2}}{\sqrt{\varrho^2 - r^2} - x^2} dF_d(\varrho) + \int_{r}^{\sqrt{r^2 + \varepsilon}} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2} - x^2} \right)
\]

Since, by our assumptions, both integrals in the last line exist, it follows that
\[
EN_{n,e}(r) \xrightarrow{n \to \infty} 0 \quad \text{for any} \quad \varepsilon > 0 .
\]  
(4)

Second step: Once more using assumption (1) we are able to verify that
\[
\sqrt{\frac{\pi^2 n^{d-1}}{\log n^{d-1}} \left( \hat{E}U_{n,e}(r) - \lambda_d (1 - F_d(r)) \right)} \xrightarrow{n \to \infty} 0 .
\]  
(5)

For this we again employ Campbell’s theorem combined with \( \int_0^1 (1 - w^2)^{-1/2} dw = \pi/2 \) which leads to
\[
\hat{E}U_{n,e}(r) = \frac{2 \lambda_d}{\pi} \left( \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \frac{\sqrt{\varrho^2 - r^2}(1 - \alpha_n/\varepsilon)}{\sqrt{\varrho^2 - r^2} - x^2} \frac{dx}{\sqrt{\varrho^2 - r^2 - x^2}} + \int_0^{\sqrt{r^2 + \varepsilon}} \frac{\sqrt{\varrho^2 - r^2 - \alpha_n}}{\sqrt{\varrho^2 - r^2 - x^2}} \frac{dx}{\sqrt{\varrho^2 - r^2 - x^2}} \right)
\]
\[
= \lambda_d (1 - F_d(r)) - \lambda_d \left( F_d(\sqrt{r^2 + \alpha_n}) - F_d(r) \right) - \frac{2 \lambda_d}{\pi} \int_{\sqrt{r^2 + \alpha_n}}^{1} \frac{1}{\sqrt{1 - \frac{x^2}{\varrho^2 - r^2}}} \frac{dx}{\sqrt{\varrho^2 - r^2 - x^2}} .
\]
By obvious rearrangements we get that

\[ F_d(\sqrt{r+\alpha_n}) - F_d(r) \leq \sqrt{\alpha_n} \int_r \frac{dF_d(q)}{\sqrt{q^2 - r^2}} \]

and

\[ \int_{\sqrt{1-\frac{\alpha_n}{\pi^2}}-1}^1 \frac{dx}{\sqrt{1-x^2}} \leq \int_0^1 \frac{dx}{\sqrt{1-x^2}} = 2 \sqrt{1 - \frac{\alpha_n}{\pi^2}} \leq 2 \sqrt{\frac{\alpha_n}{\pi^2}}. \]

Hence,

\[ |E\hat{U}_{n,\varepsilon}(r) - \lambda_d (1 - F_d(r))| \leq \lambda_d \left(1 + \frac{4}{\pi}\right) \sqrt{\frac{\alpha_n}{\pi^2}} \int_r \frac{dF_d(q)}{\sqrt{q^2 - r^2}}, \]

which, together with assumption (1), implies (5). Combining (4) and (5) shows that the sequences \( \sqrt{n^{d-1}/\log n^{d-1}} (\hat{U}_{n,\varepsilon}(r) - E\hat{U}_{n,\varepsilon}(r)) \) and \( \sqrt{n^{d-1}/\log n^{d-1}} (U_n(r) - \lambda_d (1 - F_d(r))) \) possess the same limit distribution with mean zero. We shall verify that this limit distribution is Gaussian with variance \( \sigma^2(r)/\pi^2 \).

Third step: By virtue of Condition 2 we may verify that

\[ \lim_{n \to \infty} \frac{\pi^2 n^{d-1}}{\log n^{d-1}} \text{Var}(\hat{U}_{n,\varepsilon}(r) - \sigma^2(r)) \leq \lambda_d \int_r \frac{dF_d(q)}{\sqrt{q^2 - r^2}} \text{ for any } \varepsilon > 0. \quad (6) \]

To derive this estimate we rewrite the variance of \( \hat{U}_{n,\varepsilon}(r) \) by applying Campbell’s formula and remembering the definition of the second-order reduced factorial cumulant measure \( \gamma_2^{(red)}(\cdot) \)

\[
\text{Var}(\hat{U}_{n,\varepsilon}(r)) = \frac{2\lambda_d}{\pi^2 n^{d-1}} \left( \int_{\sqrt{r^2+\varepsilon}}^{\infty} \sqrt{(q^2-r^2)(1-\alpha_n/\varepsilon)} \frac{dx}{\sqrt{q^2 - r^2 - x^2}} \frac{dF_d(q)}{\sqrt{q^2 - r^2 - x^2}} \right) + \frac{\lambda_d}{\pi^2 n^{2(d-1)}} \int_0^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{1}(\|\varepsilon\| \leq \|\tau\| - (y+z)^2 > \max\{\varepsilon, \tau^2 - r^2\}/\varepsilon^2 n^{d-1} \log n^{d-1}) \times \frac{\text{1}(\|\tau + \varepsilon\| \leq \|\bar{\tau}\| - (y+z)^2 > \max\{\tau, \tau^2 - r^2\}/\varepsilon^2 n^{d-1} \log n^{d-1})}{\sqrt{r^2 + \varepsilon - (y+z)^2}} \right. \]

\[
\times \gamma_2^{(red)}(d(\bar{y}, y)) d(\bar{\tau}, \varepsilon) dF_d(\tau) = T_1^{(n)} + T_2^{(n)} + T_3^{(n)}. \]
The first term is easy to treat and yields the following limit:

\[
\frac{\pi^2 n^{d-1}}{\log n^{d-1}} T_1^{(n)} = \frac{2 \lambda_d}{\log n^{d-1}} \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}} \int_0^\infty \frac{dx}{1 - x^2} \sqrt{1 - \alpha_n / \varepsilon}
\]

\[
= \frac{\lambda_d}{\log n^{d-1}} \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}} \left( - \log(1 - \sqrt{1 - \alpha_n / \varepsilon}) + \log(1 + \sqrt{1 - \alpha_n / \varepsilon}) \right)
\]

\[
= \frac{\lambda_d}{\log n^{d-1}} \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}} \left( \log \left(c^2 n^{d-1} \log n^{d-1}\right) + 2 \log \left(1 + \sqrt{1 - \alpha_n / \varepsilon}\right) \right)
\]

\[
\xrightarrow{n \to \infty} \lambda_d \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}} .
\]

Analogously,

\[
\frac{\pi^2 n^{d-1}}{\log n^{d-1}} T_2^{(n)} = \frac{2 \lambda_d}{\log n^{d-1}} \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 + \alpha_n}} \int_0^1 \frac{dx}{1 - x^2} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}}
\]

\[
= \frac{\lambda_d}{\log n^{d-1}} \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 + \alpha_n}} \left( \log \left(c^2 \alpha_n / \varrho^2\right) + 2 \log \left(1 + \sqrt{1 - \alpha_n / \varrho^2}\right) \right) \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}}
\]

leading to

\[
\lim_{n \to \infty} \frac{\pi^2 n^{d-1}}{\log n^{d-1}} T_2^{(n)} \leq \lambda_d \int_{r^d + \varepsilon}^{\infty} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}} \quad \text{for any} \quad \varepsilon > 0 .
\]

Condition 2 and \( \int_0^1 (1 - w^2)^{-1/2} dw = \pi / 2 \) guarantee that the third summand can be bounded by

\[
\frac{\pi^2 n^{d-1}}{\log n^{d-1}} T_3^{(n)} \leq \frac{\lambda_d \log n^{d-1}}{\log n^{d-1}} \left( 2 \int_0^\infty \int_0^\infty \frac{1}{\sqrt{\varrho^2 - r^2}} \left(1 - \alpha_n / \varrho^2\right) \frac{1}{\sqrt{\varrho^2 - r^2}} \frac{dF_d(\varrho)}{\varrho^2 - r^2} \right)^2
\]

\[
\leq \frac{\lambda_d \pi^2 \log n^{d-1}}{\log n^{d-1}} ,
\]

which together with the above relations immediately confirm (6).
In the final step we make use of Condition 1 and derive bounds of the cumulants of order $m \geq 3$ (abbreviated by the symbol $\text{Cum}_m$) of $(\pi^2 n^{d-1}/\log n^{d-1})^{1/2} \hat{U}_{n,\varepsilon}(r)$, which are uniformly bounded in $n$ and tend to zero as $\varepsilon$ does so.

More precisely, using the representation formula for cumulants of general shot-noise processes obtained in Heinrich and Schmidt [3] we get

$$\text{Cum}_m\{\hat{U}_{n,\varepsilon}(r)\} =$$

$$= \sum_{p=1}^{m} \frac{1}{p!} \sum_{m_1 + \ldots + m_p = m \atop m_j \geq 1, j = 1, \ldots, p} \frac{m!}{m_1! \cdots m_p!} \int_{\mathbb{R}^d} \prod_{j=1}^{p} \int_{0}^{\infty} \left( f_{n,\varepsilon}(x_j, \varrho) \right)^{m_j} dF(\varrho) \gamma_p(d(x_1, \ldots, x_p))$$

$$= \lambda_d \int_{\mathbb{R}^d} \int_{0}^{\infty} \left( f_{n,\varepsilon}(x, \varrho) \right)^m dF(\varrho) \, dx + \lambda_d \sum_{p=2}^{m} \frac{1}{p!} \sum_{m_1 + \ldots + m_p = m \atop j = \min\{p, m_j \geq 2\}} \frac{m!}{m_1! \cdots m_p!}$$

$$\times \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{p-1}} \prod_{j=1}^{p} \int_{0}^{\infty} \left( f_{n,\varepsilon}(x_j + x_j', \varrho) \right)^{m_j} dF(\varrho) \gamma_p^{(red)}(d(x_j; j \neq J))$$

$$\times \int_{0}^{\infty} \left( f_{n,\varepsilon}(x_j, \varrho) \right)^{m_j} dF(\varrho) \, dx_J ,$$

where the ‘response function’ $f_{n,\varepsilon} | \mathbb{R}^{d-1} \times \mathbb{R}^1 \times [0, \infty) \mapsto \mathbb{R}^1$ of the truncated shot-noise process $\hat{U}_{n,\varepsilon}(r)$ is given by

$$f_{n,\varepsilon}(x, \varrho) = \frac{1}{\pi n^{d-1}} \frac{1(\varpi \in W_{n}^{(d-1)})}{\sqrt{\varrho^2 - r^2 - \varrho^2}} \frac{1}{1(\varrho^2 - r^2 - \varrho^2 > \frac{\max\{\varepsilon, \varrho^2 \varepsilon^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}})} \text{ for } x = (\varpi, \varrho).$$

Since $0 \leq f_{n,\varepsilon}(x, \varrho) \leq \sqrt{\varepsilon \log n^{d-1}/\pi^2 n^{d-1}}$ uniformly in $x \in \mathbb{R}^d$ and $\varrho \in (0, \infty)$, we arrive at the estimate

$$|\text{Cum}_m\{\hat{U}_{n,\varepsilon}(r)\}| \leq \lambda_d C_m \left( \frac{\varepsilon \log n^{d-1}}{\pi^2 n^{d-1}} \right)^{(m-2)/2} \int_{\mathbb{R}^d} \int_{0}^{\infty} f_{n,\varepsilon}^2(x, \varrho) dF(\varrho) \, dx \text{ for } m \geq 3 ,$$

where the constant $C_m$ depends on the total variations of the signed measures $\gamma_p^{(red)}(\cdot)$ for $p = 2, \ldots, m$ in the following way

$$C_m = 1 + \sum_{p=2}^{m} \frac{1}{p!} \sum_{m_1 + \ldots + m_p = m \atop m_j \geq 1, j = 1, \ldots, p} \frac{m!}{m_1! \cdots m_p!} \int_{(\mathbb{R}^d)^{p-1}} |\gamma_p^{(red)}(d(x_1, \ldots, x_{p-1})| .$$
Making use of the abbreviation $\alpha_n$ introduced at the beginning of the proof we find that

\[
\int_{\mathbb{R}^d} \int_{0}^{\infty} f_{n,\varepsilon}(x, g) \, dF(g) \, dx \leq \frac{2}{\pi^2 n^{d-1}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{1(\varrho^2 - r^2 - y^2 > \alpha_n (\varrho^2 - r^2)/\varepsilon)}{\varrho^2 - r^2 - y^2} \, dF(g) \, dy
\]

\[
= \frac{2}{\pi^2 n^{d-1}} \int_{\varepsilon}^{\infty} \frac{dF(g)}{\sqrt{\varrho^2 - r^2}} \int_{0}^{\sqrt{1 - \alpha_n/\varepsilon}} \frac{dy}{1 - y^2} \]

\[
= \frac{1}{\pi^2 n^{d-1}} \log \left( \frac{\varepsilon}{\alpha_n} \right) + 2 \log (1 + \sqrt{1 - \alpha_n/\varepsilon}) \int_{\varepsilon}^{\infty} \frac{dF(g)}{\sqrt{\varrho^2 - r^2}}
\]

\[
\leq \frac{\log n^{d-1}}{\pi^2 n^{d-1}} \left( 1 + \frac{\log (4 \varepsilon^2 \log n^{d-1})}{\log n^{d-1}} \right) \int_{\varepsilon}^{\infty} \frac{dF(g)}{\sqrt{\varrho^2 - r^2}}
\]

Thus, summarizing the above steps yields the estimate

\[
\lim_{n \to \infty} \left( \frac{\pi^2 n^{d-1}}{\log n^{d-1}} \right)^{m/2} \left| \text{Cum}_m \{ \hat{U}_{n,\varepsilon}(r) \} \right| \leq \varepsilon^{(m-2)/2} C_m \sigma^2(r) \quad \text{for any } \varepsilon > 0 \text{ and } m \geq 3.
\]

This last step confirms the asymptotic normality of the truncated shot-noise process $\hat{U}_{n,\varepsilon}(r)$ by applying the classical ‘method of moments’.

The proof of Theorem 3 is quite similar to that of Theorem 4. For this reason we present a detailed proof only in case of Theorem 4 and outline the essential proving steps.

Let $\delta > 0$ be arbitrarily small, but fixed and $\varepsilon > 0$ be chosen small enough (in fact, $\varepsilon = \varepsilon_n$ can be thought of as a positive sufficiently slowly decreasing null sequence). Define in analogy to $\hat{U}_{n,\varepsilon}(r)$ the truncated shot-noise process

\[
\hat{V}_{n,\varepsilon}(r) = \frac{1}{\pi n^{d-2} \log n^{d-2}} \sum_{i \geq 1} 1(X_i \in W_{n}^{(d-2)}) \frac{1(\mathbf{R}_i^2 - r^2 > \max \{\varepsilon, R_i^2 - r^2\})}{\varepsilon^2 n^{d-2} \log n^{d-2}}
\]

and let $M_{n,\varepsilon}(r)$ denote the above random integer $N_{n,\varepsilon}(r)$ with $d-2$ instead of $d-1$.

Using the analog to the ‘truncation inequality’ (3) and Chebychev’s inequality we get

\[
P(\left| \hat{V}_n(r) - \lambda_d(1 - F_d(r)) \right| \geq \delta) \leq P(M_{n,\varepsilon}(r) \geq 1) + P(\left| \hat{V}_{n,\varepsilon}(r) - \lambda_d(1 - F_d(r)) \right| \geq \delta)
\]

\[
\leq EM_{n,\varepsilon}(r) + \frac{\text{Var}(\hat{V}_{n,\varepsilon}(r))}{\delta^2} + \frac{(E\hat{V}_{n,\varepsilon}(r) - \lambda_d(1 - F_d(r)))^2}{\delta^2}.
\]

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The following relations can be proved for any \( \varepsilon > 0 \):

\[
\mathbb{E} M_{n,\varepsilon}(r) \xrightarrow{n \to \infty} 0 \quad \text{(since } \mathbb{E} R_0^2 < \infty \text{)},
\]

\[
\lim_{n \to \infty} |\mathbb{E} \hat{V}_{n,\varepsilon}(r) - \lambda_d (1 - F_d(r))| \leq \lambda_d \left( F_d(\sqrt{r^2 + \varepsilon}) - F_d(r) \right)
\]

and

\[
\lim_{n \to \infty} \text{Var}(\hat{V}_{n,\varepsilon}(r)) \leq \frac{\varepsilon \lambda_d}{\pi} \int_{\mathbb{R}^d} |\gamma^{(red)}_2(x)| \text{d}x.
\]

Next we present detailed proofs of these three relations. Using the abbreviation \( \beta_n = (\varepsilon n^{d-2} \log n^{d-2})^{-1} \) we may write the expectation \( \mathbb{E} M_{n,\varepsilon}(r) \) as follows:

\[
\mathbb{E} M_{n,\varepsilon}(r) = \mathbb{E} \sum_{i \geq 1} \mathbf{1}(X_i \in W_n^{(d-2)}) \mathbf{1} \left( 0 < R_i^2 - r^2 \leq \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon n^{d-2} \log n^{d-2}} \right)
\]

\[
= \lambda_d n^{d-2} \int_0^\infty \int_0^\infty \mathbf{1} \left( \sqrt{\frac{\varepsilon}{\beta_n}} - \frac{\sqrt{r^2 + \varepsilon}}{\beta_n} \leq \frac{\max\{\varepsilon, \sqrt{x^2 + y^2} - r^2\}}{\varepsilon n^{d-2} \log n^{d-2}} \right) \text{d}x \text{d}F_d(\theta)
\]

\[
= 2 \pi \lambda_d n^{d-2} \int_0^\infty \int_0^\infty \mathbf{1} \left( \sqrt{\frac{\varepsilon}{\beta_n}} - \frac{\sqrt{r^2 + \varepsilon}}{\beta_n} \leq \frac{\max\{\varepsilon, \sqrt{x^2 + y^2} - r^2\}}{\varepsilon n^{d-2} \log n^{d-2}} \right) \text{d}x \text{d}F_d(\theta)
\]

\[
= \pi \lambda_d n^{d-2} \left( \frac{\beta_n}{\varepsilon} \int_0^\infty \left( \sqrt{\frac{\varepsilon}{\beta_n}} - \frac{\sqrt{r^2 + \varepsilon}}{\beta_n} \right) \text{d}F_d(\theta) + \int_0^\sqrt{r^2 + \varepsilon} \min\{\sqrt{x^2 + y^2} - r^2, \beta_n\} \text{d}F_d(\theta) \right).
\]

Thus, by \( n^{d-2} \beta_n \xrightarrow{n \to \infty} 0 \), (7) is shown.

Furthermore, by introducing planar polar coordinates we are led to

\[
\mathbb{E} \hat{V}_{n,\varepsilon}(r) = \frac{\lambda_d}{\pi \log n^{d-2}} \int_0^\infty \int_0^{\sqrt{2 + \varepsilon}} \mathbf{1} \left( \sqrt{\frac{\varepsilon}{\beta_n}} - \frac{\sqrt{r^2 + \varepsilon}}{\beta_n} \leq \frac{\max\{\varepsilon, \sqrt{x^2 + y^2} - r^2\}}{\varepsilon n^{d-2} \log n^{d-2}} \right) \text{d}x \text{d}F_d(\theta)
\]

\[
= \frac{\lambda_d}{\log n^{d-2}} \left( \int_0^\sqrt{2 + \varepsilon} \frac{\text{d}x \text{d}F_d(\theta)}{\sqrt{r^2 + \varepsilon} - r^2 - x} + \int_0^\sqrt{2 + \varepsilon} \frac{\text{d}x \text{d}F_d(\theta)}{\sqrt{r^2 + \varepsilon} - r^2 - x} \right)
\]

\[
= \frac{\lambda_d}{\log n^{d-2}} \left( 1 - F_d(\sqrt{r^2 + \varepsilon}) \right) \log \left( \frac{\varepsilon}{\beta_n} \right) + \frac{\lambda_d}{\log n^{d-2}} \int_0^\sqrt{2 + \varepsilon} \log \left( \frac{\sqrt{x^2 + y^2} - r^2}{\beta_n} \right) \text{d}F_d(\theta),
\]

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whence, in view of \( \log(\varepsilon/\beta_n)/\log n^{d-2} \to 1 \) for all \( \varepsilon > 0 \), relation (8) follows. In like manner, we may express the variance of the truncated shot-noise process \( \hat{V}_{n,\varepsilon}(r) \):

\[
\text{Var}(\hat{V}_{n,\varepsilon}(r)) = \frac{\lambda_d \pi n^{d-2}}{(\pi n^{d-2} \log n^{d-2})^2} \times \left( \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \int_0^{(\varepsilon^2 - r^2)(1 - \beta_n/\varepsilon)} dx \frac{dF_d(q)}{(q^2 - r^2 - x)^2} + \int_{\sqrt{r^2 + \beta_n}}^{\infty} \int_0^{\varepsilon^2 - r^2 - \beta_n} dx \frac{dF_d(q)}{(q^2 - r^2 - x)^2} \right)
\]

\[+ \frac{\lambda_d}{(\pi n^{d-2} \log n^{d-2})^2} \int_0^{\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1(\exists \in W_n^{(d-2)}) \frac{dF_d(\tau)}{\tau^2 - r^2 - \|y + x\|^2} 1(\tau^2 - r^2 - \|y + x\|^2 > \max\{\varepsilon, \tau^2 - r^2\}) \frac{\max\{\varepsilon, \tau^2 - r^2\}}{\varepsilon^2 n^{d-2} \log n^{d-2}} \]

\[\times \ \gamma_{2}^{(red)}(d(\overline{y}, y)) d(\overline{x}, \overline{z}) dF_d(\tau) dF_d(\tau) = T_4^{(n)} + T_5^{(n)} + T_6^{(n)}. \]

Some obvious rearrangements show that

\[ T_4^{(n)} + T_5^{(n)} = \frac{\lambda_d}{\pi n^{d-2} (\log n^{d-2})^2} \left( \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \frac{dF_d(q)}{q^2 - r^2} \int_{\beta_n/\varepsilon}^{\infty} \frac{dx}{x^2} + \int_{\sqrt{r^2 + \beta_n}}^{\infty} \int_0^{\varepsilon^2 - r^2 - \beta_n} dx \frac{dF_d(q)}{(q^2 - r^2)^2} \right) \]

\[= \frac{\lambda_d}{\pi n^{d-2} (\log n^{d-2})^2} \left( \int_{\sqrt{r^2 + \varepsilon}}^{\infty} \frac{dF_d(q)}{q^2 - r^2} (\beta_n - 1) \right) \int_{\sqrt{r^2 + \beta_n}}^{\infty} \left( \frac{\varepsilon^2 - r^2}{\beta_n} - 1 \right) \frac{dF_d(q)}{(q^2 - r^2)^2} \]

\[\leq \frac{\lambda_d}{\pi n^{d-2} (\log n^{d-2})^2 \beta_n} \to 0 \]
and
\[
|T_6^{(n)}| \leq \frac{\lambda_d \varepsilon}{\pi \log n^{d-2}} \int_{\mathbb{R}^d} |\gamma_2^{(red)}(dy)| \int_0^\infty \int_0^\infty \frac{1}{g^2 - r^2 - x} \frac{1}{g^2 - r^2 - x} \frac{\max\{\varepsilon, g^2 - r^2\}}{\frac{\varepsilon}{\log n^{d-1}}} \, dx \, dF_d(g)
\]
\[
= \frac{\lambda_d \varepsilon}{\pi \log n^{d-2}} \int_{\mathbb{R}^d} |\gamma_2^{(red)}(dy)| \times \left(1 - F_d(\sqrt{r^2 + \varepsilon})\right) \log \left(\frac{\varepsilon}{\beta_n}\right) + \int \frac{\log \left(\frac{g^2 - r^2}{\beta_n}\right)}{\sqrt{r^2 + \beta_n}} \, dF_d(g)
\]
\[
\leq \frac{\lambda_d \varepsilon}{\pi \log n^{d-2}} \log \left(\frac{\varepsilon}{\beta_n}\right).
\]
These estimates imply immediately (9).

Combining the relations (7), (8), (9), Condition 3, and the right-continuity of the df $F_d$ completes the proof of Theorem 4.

As announced above we outline some calculations needed to prove Theorem 3. First we introduce a truncated version of the estimator $\hat{\sigma}_n^2(r)$ and calculate its mean by using Campbell’s formula. Let
\[
\hat{\sigma}_{n,\varepsilon}^2(r) := \frac{1}{n^{d-1} \log n^{d-1}} \sum_{i \geq 1} \frac{1}{\bar{R}_i^2 - r^2} \frac{1}{\max\{\varepsilon, \bar{R}_i^2 - r^2\}} \frac{\varepsilon}{\log n^{d-1}}
\]so that the expectation $\mathbb{E}\hat{\sigma}_{n,\varepsilon}^2(r)$ exists for any $\varepsilon > 0$. More precisely,
\[
\mathbb{E}\hat{\sigma}_{n,\varepsilon}^2(r) = \frac{2 \lambda_d}{\log n^{d-1}} \int_0^\infty \int_0^\infty \frac{1}{\sqrt{g^2 - r^2 - x^2}} \frac{1}{\sqrt{g^2 - r^2 - x^2}} \frac{\max\{\varepsilon, g^2 - r^2\}}{\frac{\varepsilon}{\log n^{d-1}}} \, dx \, dF_d(g)
\]
\[
= \frac{2 \lambda_d}{\log n^{d-1}} \left(\int_0^{\sqrt{g^2 - r^2 - x^2}} \frac{dx \, dF_d(g)}{\sqrt{g^2 - r^2 - x^2}} + \int_0^{\sqrt{g^2 - r^2 - x^2}} \frac{dx \, dF_d(g)}{\sqrt{g^2 - r^2 - x^2}}\right)
\]
\[
= \frac{\lambda_d}{\log n^{d-1}} \int_0^{\sqrt{g^2 - r^2 - x^2}} \frac{dx \, dF_d(g)}{\sqrt{g^2 - r^2 - x^2}} \left(\log \left(\frac{\varepsilon}{\alpha_n}\right) + 2 \log \left(1 + \sqrt{-\frac{\alpha_n}{\varepsilon}}\right)\right)
\]
\[
+ \frac{\lambda_d}{\log n^{d-1}} \int_0^{\sqrt{g^2 - r^2 - x^2}} \left(\log \left(\frac{g^2 - r^2}{\alpha_n}\right) + 2 \log \left(1 + \sqrt{-\frac{\alpha_n}{g^2 - r^2}}\right)\right) \frac{dx \, dF_d(g)}{\sqrt{g^2 - r^2}}.
\]
whence, together with $\log(\varepsilon/\alpha_n)/\log n^{d-1} \to 1$ and $\lim_{n \to \infty} \int_0^{\sqrt{g^2 - r^2 - x^2}} (g^2 - r^2)^{-1/2} \, dF_d(g) = 0$, it follows that
\[
\lim_{n \to \infty} \mathbb{E}\sigma_n^2(r) \leq \sigma^2(r) \quad \text{and} \quad \lim_{n \to \infty} \left| \mathbb{E}\sigma_n^2(r) - \sigma^2(r) \right| \leq \lambda_d \int_r^{\sqrt{r^2 + \varepsilon}} \frac{dF_d(\varrho)}{\sqrt{\varrho^2 - r^2}}
\]

(10)

for all \( \varepsilon > 0 \). It remains to show that

\[
\lim_{n \to \infty} \text{Var}(\tilde{\sigma}^2_{n, \varepsilon}(r)) \leq \varepsilon \sigma^2(r) \int_{\mathbb{R}^d} |\gamma_2^{(\text{red})}(d\varrho)|.
\]

(11)

In analogy to the computation of \( \text{Var}(\tilde{U}_{n, \varepsilon}(r)) \) and \( \text{Var}(\tilde{V}_{n, \varepsilon}(r)) \) we have

\[
\text{Var}(\tilde{\sigma}^2_{n, \varepsilon}(r)) = \frac{2\lambda_d n^{d-1}}{(n^{d-1} \log n)^{d-1}}
\]

\[
\times \left( \int_{\mathbb{R}^d} \frac{\sqrt{(\varrho^2 - r^2)(1 - \alpha_n/\varepsilon)}}{\varrho^2 - r^2 - x^2} \frac{dx}{\varepsilon^2 - r^2 + \alpha_n} \frac{dF_d(\varrho)}{(\varrho^2 - r^2 - x^2)^2} + \int_{\mathbb{R}^d} \frac{\sqrt{(\varrho^2 - r^2 + \alpha_n)}}{\varrho^2 - r^2 + \alpha_n} \frac{dx}{\varepsilon^2 - r^2 - x^2} \frac{dF_d(\varrho)}{(\varrho^2 - r^2 - x^2)^2} \right)
\]

\[
+ \frac{\lambda_d}{(n^{d-1} \log n)^{d-1}} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1(\varrho \in W_{n, \varepsilon}(d-1)}{\varrho^2 - r^2 - x^2} \frac{1(\varrho^2 - r^2 + \alpha_n)}{\varepsilon^2 - r^2 + \alpha_n} \frac{dx}{(\varrho^2 - r^2 - x^2)^2} \frac{dF_d(\varrho)}{(\varrho^2 - r^2 - x^2)^2} 
\]

\[
\times \gamma_2^{(\text{red})}(d(\gamma, y)) \frac{dF_d(y)}{(\gamma^2 - \alpha_n/\varepsilon)^{3/2}} \left( \frac{\sqrt{1 - \alpha_n/\varepsilon}}{\gamma^2 - \alpha_n/\varepsilon} \right) \frac{dF_d(\tau)}{(\gamma^2 - \beta_n/\varepsilon)^{3/2}}
\]

\[
\times \frac{2\lambda_d \varepsilon}{n^{d-1} \log n} \int_{\mathbb{R}^d} \gamma_2^{(\text{red})}(d\gamma) \int_{\mathbb{R}^d} \frac{1(\varrho^2 - r^2 + \alpha_n)}{\varrho^2 - r^2 - x^2} \frac{dx}{(\varrho^2 - r^2 - x^2)^2} \frac{dF_d(\varrho)}{(\varrho^2 - r^2 - x^2)^2} 
\]

\[
\leq \frac{4\sigma^2(r)}{n^{d-1} \log n} + \varepsilon \int_{\mathbb{R}^d} \gamma_2^{(\text{red})}(d\gamma) \mathbb{E}\tilde{\sigma}^2_{n, \varepsilon}(r).
\]

Thus, (11) is an immediate consequence of (10). To accomplish the proof of Theorem 4 we remember that, in analogy to the proof of Theorem 3 and in view of (3), we have

\[
P(|\tilde{\sigma}^2_n(r) - \sigma^2(r)| \geq \delta) \leq P(N_{n, \varepsilon}(r) \geq 1) + \frac{\text{Var}(\tilde{\sigma}^2_{n, \varepsilon}(r))}{\delta^2} + \frac{(\mathbb{E}\tilde{\sigma}^2_{n, \varepsilon}(r) - \sigma^2(r))^2}{\delta^2}
\]

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for arbitrarily small, but fixed $\delta > 0$. Due to (4), (10), and (11) the right-hand side of the latter inequality tends to zero as $\varepsilon = \varepsilon_n \downarrow 0$.

The proofs of the Theorems 5 and 6 rely on the exponential shape of the generating functional of the stationary, independently marked Poisson process $\Psi_d$, which is as follows:

$$
\mathbb{E} \prod_{i \geq 1} v(X_i, R_i) = \exp \left\{ \lambda_d \int_{\mathbb{R}^d} \int_0^\infty \left( v(x, \rho) - 1 \right) dF_d(\rho) \, dx \right\}
$$

for any Borel-measurable, complex-valued function $v(\cdot)$ on $\mathbb{R}^d \times [0, \infty)$ satisfying $\int_{\mathbb{R}^d} \int_0^\infty |v(x, \rho) - 1| dF_d(\rho) \, dx < \infty$, see e.g. Stoyan et al. [10].

Choosing

$$
v(x, \rho) = \exp \left\{ \frac{it}{\pi n^{d-2}} \left( \frac{\rho^2 - \|x\|^2}{\rho^2 - r^2} \right) \right\}
$$

yields the following expression for the logarithm of the characteristic function $\mathbb{E} \exp \{it \log n^{d-2} \hat{V}_n(r)\}$:

$$
\lambda_d \frac{n^{d-2}}{\pi} \int_{\mathbb{R}^d} \int_0^\infty \left( \exp \left\{ \frac{it}{\rho^2 - \|x\|^2} \right\} - 1 \right) dF_d(\rho) \, dx
$$

$$
\lambda_d \frac{n^{d-2}}{\pi} \int_0^\infty \int_r^\infty \left( \exp \left\{ \frac{it}{\rho^2 - r^2} \right\} - 1 \right) dy \, dF_d(\rho)
$$

$$
= \lambda_d \frac{n^{d-2}}{\pi} \int_r^\infty \int_0^\infty \frac{\exp \{itz\} - 1}{z^2} \, dz \, dF_d(\rho) .
$$

The inner integral in (12) can be approximated by elementary functions with explicit remainder term in the following way:

$$
\int_r^\infty \frac{\exp \{itz\} - 1}{z^2} \, dz = -\frac{\pi}{2} |t| - it \log |t| + it \left( 1 - \gamma - \log A \right) + \frac{At^2}{2} (1 + A |t|) \theta ,
$$

where $A = (a_n (\rho^2 - r^2))^{-1}$, $a_n = \pi n^{d-2}$, and $\theta$ denotes some complex number satisfying $|\theta| \leq 1$. Next, splitting the outer integral in (12) into two integrals over $(r_n(\varepsilon), \infty)$ and $(r, r_n(\varepsilon)]$ with $r_n(\varepsilon) = \sqrt{r^2 + (\varepsilon a_n)^{-1}}$, we arrive at

$$
\log \mathbb{E} \exp \{it \log n^{d-2} \hat{V}_n(r)\} = \lambda_d \left( 1 - F_d(r_n(\varepsilon)) \right) \left( -\frac{\pi}{2} |t| - it \log |t| + it \left( 1 - \gamma + \log a_n \right) \right)
$$

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\[ +it \lambda_d \int_{r_n(\varepsilon)}^\infty \log(\rho^2 - r^2) \, dF_d(\rho) + \frac{\varepsilon \lambda_d}{2} t^2 (1 + \varepsilon |t|) \theta + 2 \lambda_d \tilde{\theta} a_n \int_r^{r_n(\varepsilon)} (\rho^2 - r^2) \, dF_d(\rho) \]

with some complex number \( \tilde{\theta} \) satisfying \(|\tilde{\theta}| \leq 1 \). Since, in view of (2), the last term in the previous line vanishes as \( n \to \infty \) for any \( \varepsilon > 0 \) and also

\[
\log n^{d-2} \left( F_d(r_n(\varepsilon)) - F_d(r) \right) \\
\leq \frac{\log n^{d-2}}{\log(\varepsilon a_n)} \int_r^{r_n(\varepsilon)} | \log(\rho^2 - r^2) | \, dF_d(\rho) \xrightarrow{n \to \infty} 0 ,
\]

it follows from the foregoing equation (after replacing \( t \) by \( t/\lambda_d (1 - F_d(r)) \) and some further rearrangements) that

\[
\log \mathbb{E} \exp \left\{ it \log n^{d-2} \left( \frac{\tilde{V}_n(r)}{\lambda_d (1 - F_d(r))} - 1 \right) \right\} \\
\xrightarrow{n \to \infty} \log \mathbb{E} \exp \left\{ it S_1 \right\} + it \int_0^\infty \log(\rho^2 - r^2) \, dF_d(\rho) \left/ \frac{1 - F_d(r)}{1 - F_d(r)} \right. + it \log \left( \pi \lambda_d (1 - F_d(r)) \right) + it (1 - \gamma) 
\]

which is nothing else but the assertion of Theorem 5.

To prove Theorem 6 we make use of the subsequent representation of \( L_n^{(p)}(t) := \log \mathbb{E} \exp \{ it \tilde{Y}_n^{(p)}(r) \} \) which can be derived in analogy to (12) by using the generating functional of the Poisson process \( \Psi_n^d = \{ X_i : i \geq 1 \} : \)

\[
L_n^{(p)}(t) = \lambda_d \omega_p \int_r^{\infty} \int_0^\infty \frac{\exp \{ it z \} - 1}{z^{1+2/p}} \left( \rho^2 - r^2 - z^{-2/p} n^{-s} \right)^{-1+1/p} \, dz \, dF_d(\rho) .
\]

The following formula goes back to L. Euler and can be found in any ‘Table of Integrals’ for \( 0 < \alpha < 1 \):

\[
\int_0^\infty \frac{\exp \{ it z \} - 1}{z^{1+\alpha}} \, dz = \frac{\Gamma(1-\alpha)}{\alpha} \cos \left( \frac{\alpha \pi}{2} \right) \left| t \right|^\alpha \left( -1 + i \, \text{sgn}(t) \, \tan \left( \frac{\pi \alpha}{2} \right) \right) , \tag{13}
\]

where \( \Gamma(1-\alpha) = \int_0^\infty e^{-x} x^{-\alpha} \, dx \).

Therefore, applying (13) for \( \alpha = \frac{2}{p} \) we obtain after a simple calculation that

\[
L_n^{(p)}(t) \xrightarrow{n \to \infty} c_p \lambda_d I_p(r) \log \mathbb{E} \exp \left\{ it S_{2/p} \right\} = \log \mathbb{E} \exp \left\{ it \left( c_p \lambda_d I_p(r) \right)^{p/2} S_{2/p} \right\} ,
\]

where \( I_p(r) = \int_r^\infty (\rho^2 - r^2)^{(p-2)/2} \, dF_d(\rho) \) and \( c_p \) is as defined in Theorem 6. Thus, replacing \( t \) by \( t/(c_p \lambda_d I_p(r))^{p/2} \) completes the proof of Theorem 6.
References


