THERMODYNAMIC LIMIT AND LARGE DEVIATIONS
OF THE EMPIRICAL VOLUME FRACTION FOR
STATIONARY POISSON GRAIN MODELS

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Abstract. We investigate the (thermodynamic) limit of the scaled cumulant-generating function $L_n(z) = \frac{1}{|W_n|} \log \mathbb{E} \exp \{ z |\Xi \cap W_n| \}$ of the empirical volume fraction $\hat{p}_n = |\Xi \cap W_n|/|W_n|$ (where $|.|$ denotes the $d$-dimensional Lebesgue measure) in case of a $d$-dimensional Poisson grain model (Boolean model) $\Xi = \bigcup_{i \geq 1} (\Xi_i + X_i)$ defined by a stationary Poisson process $\Pi_\lambda = \sum_{i \geq 1} \delta_{X_i}$ with intensity $\lambda > 0$ and a sequence of independent copies $\Xi_1, \Xi_2, \ldots$ of a random compact set $\Xi_0$. For an increasing family of compact convex sets $\{W_n, n \geq 1\}$ which expand unboundedly in all directions we prove the existence and analyticity of the limit $\lim_{n \to \infty} L_n(z)$ on some disk in the complex plane whenever $\mathbb{E} \exp \{ a |\Xi_0| \} < \infty$ for some $a > 0$. Moreover, closely connected with this result, we obtain exponential inequalities and the exact asymptotics for the large deviation probabilities of $\hat{p}_n$ in the sense of Cramér and Chernoff.

Keywords: Poisson grain model with compact grains, volume fraction, Cox process, thermodynamic limit, correlation measures, cumulants, large deviations, Berry - Esseen bound, Chernoff-type theorem

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1. Introduction and Main Results

The Poisson grain model (PGM) (also known as Boolean model) is the best studied and mostly used random set model to describe systems of randomly distributed and irregularly shaped clumps in a Euclidean space $\mathbb{R}^d, d \geq 1$, see Matheron [14], Hall [8], Stoyan et al. [22]. It is the
basic model in stereology and stochastic geometry, see [22]. Statistical analysis of a stationary PGM is mostly based on a single realization of the union set of clumps in some region $W$ which is assumed to expand unboundedly in all directions, see Heinrich and Molchanov [11] and Molchanov [16]. To be definite in describing our problem, we first give a rigorous definition of a stationary PGM as to be the union set

$$(1.1) \quad \Xi := \bigcup_{i \geq 1} \left( \Xi_i + X_i \right)$$

of independent copies $\Xi_1, \Xi_2, \ldots$ (grains) of a random compact set $\Xi_0$ (typical grain) having the distribution $Q$, where the grains are independently shifted by the atoms $X_1, X_2, \ldots$ (germ points) of a stationary Poisson process $\Pi_\lambda = \sum_{i \geq 1} \delta_{X_i}$ with intensity $\lambda$ (i.e., mean number of germ points in the unit cube $[0,1]^d$). Throughout in this paper all random elements are defined on a common probability space $[\Omega, \mathcal{F}, \mathbb{P}]$ and $\mathbb{E}$ denotes the expectation w.r.t. $\mathbb{P}$. In particular, $\Xi_0$ is a measurable mapping from $[\Omega, \mathcal{F}, \mathbb{P}]$ into the space of non-void compact subsets $\mathcal{K}$ of $\mathbb{R}^d$ equipped with the Hausdorff metric and $Q$ coincides with the image measure $\mathbb{P} \circ \Xi_0^{-1}$ acting on the corresponding Borel $\sigma-$field $\mathcal{B}(\mathcal{K})$, see [14]. Note that $\Xi$ is a closed set ($\mathbb{P}$-a.s.) if $\mathbb{E}[\Xi_0 + B_r(o)] < \infty$ for $r > 0$, where $B_r(x)$ denotes the closed ball with radius $r > 0$ centered at $x \in \mathbb{R}^d$ and $|.|$ the Lebesgue measure in $\mathbb{R}^d$, see Heinrich [10].

The main aim of this paper is to prove the existence and analyticity of the limit (as $n \to \infty$) of

$$(1.2) \quad L_n(z) := \frac{1}{|W_n|} \log \mathbb{E} \exp \{ z |\Xi \cap W_n| \} \quad \text{on} \quad D_\Delta := \{ z \in C^1 : |z| < 1/\Delta \}$$

for some $0 < \Delta < \infty$ provided that an exponential moment of the volume $|\Xi_0|$ exists, i.e.

$$(1.3) \quad M(a) := \mathbb{E} \exp \{ a |\Xi_0| \} < \infty \quad \text{for some} \quad a > 0,$$

and $\{W_n, n \geq 1\}$ is a convex averaging sequence of sets in $\mathbb{R}^d$, i.e. $W_n \in \mathcal{K}$ is convex, $W_n \subseteq W_{n+1}$ for $n \geq 1$, and $\sup \{ r > 0 : B_r(x) \subseteq W_n \} \to \infty$, see Daley and Vere-Jones [2]. Because of the conspicuous analogy to similar problems in statistical physics, see Ruelle [19], we will call $L(z) = \lim_{n \to \infty} L_n(z)$ the thermodynamic limit of (the thermodynamic function) $L_n(z)$. The second aim closely connected with the first one is to derive inequalities and asymptotic relations (in the sense of H. Cramér and H. Chernoff) for probabilities of large deviations of the empirical volume fraction $\hat{p}_n := |\Xi \cap W_n| / |W_n|$ from its mean $p := \mathbb{E}\hat{p}_n = \mathbb{E}|\Xi \cap [0,1]^d| = \mathbb{P}(o \in \Xi)$. 

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In the special case of a bounded typical grain, that is $\Xi_0 \subseteq B_R(o)$ for some $0 < R < \infty$, both problems were solved satisfactorily in Götze et al. [6] by using the device of $m-$dependent random fields with block representation. The proving technique in the present paper is completely different from that in [6] and does not require any mixing properties of the PGM (1.1) as one would expect. In general, (1.3) does not imply any specific mixing rates, see Mase [13] and [11]. However, in case of a spherical typical grain, (1.3) induces an exponentially decaying $\beta-$mixing coefficient, see [11]. For another result under this condition, see Blaszczyszyn et al. [1].

Note that (1.3) does even not imply the closedness of $\Xi$ in general, see Appendix. For a positive random variable $X$ with infinite mean, the typical grain $\Xi_0 = [0, X] \times [0, 1/X]$ exhibits such an example for $d = 2$.

For this reason we choose the probability space $[\Omega, \mathcal{A}, \mathbb{P}]$ (for its existence, see Appendix) in such a way that the mapping $\mathbb{R}^d \times \Omega \ni (x, \omega) \mapsto 1_{\Xi(\omega)}(x)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A}-$measurable. This property allows to apply Fubini's theorem to the $0-1-$valued random field $\xi(x) = 1_{\Xi}(x)$, $x \in \mathbb{R}^d$, and implies that the function

$$p^{(k)}_\Xi(x_1, \ldots, x_k) := \mathbb{E}\xi(x_1) \cdots \xi(x_k) = \mathbb{P}(x_1 \in \Xi, \ldots, x_k \in \Xi)$$

is $\mathcal{B}(\mathbb{R}^d)$-measurable for any $k \geq 1$ and $|\Xi \cap W| = \int_W \xi(x) \, dx$ is a random variable over $[\Omega, \mathcal{A}, \mathbb{P}]$ for any bounded $W \in \mathcal{B}(\mathbb{R}^d)$. The functions (1.4) are expressible (and vice versa) by the corresponding probabilities for the complement set $\Xi^c$

$$p^{(k)}_{\Xi^c}(x_1, \ldots, x_k) := \mathbb{E}(1 - \xi(x_1)) \cdots (1 - \xi(x_k)) = \mathbb{P}(\Xi \cap \{x_1, \ldots, x_k\} = \emptyset) .$$

Since $(\Xi_i + X_i) \cap \{x_1, \ldots, x_k\} = \emptyset$ iff $X_i \notin (-\Xi_i) + \{x_1, \ldots, x_k\}$, the shape of the probability generating functional (5.2) (of a stationary independently marked Poisson process $\Pi_{\lambda, Q}$) for $v(x, K) = 1 - 1_{(-K)+\{x_1, \ldots, x_k\}}(x)$ yields

$$p^{(k)}_{\Xi^c}(x_1, \ldots, x_k) = \mathbb{E} \prod_{i \geq 1} \left( 1 - 1_{(-\Xi_i)+\{x_1, \ldots, x_k\}}(X_i) \right) = \exp \left\{ -\lambda E \left| \bigcup_{i=1}^k (-\Xi_0 - x_i) \right| \right\}$$

It should be noted that $p^{(k)}_{\Xi^c}(x_1, \ldots, x_k) = 1 - T_{\Xi^c}(\{x_1, \ldots, x_k\})$ for an arbitrary random closed set $\Xi$ with capacity functional $T_{\Xi}$, see [14]. The study of the sequence (1.2) is closely related with the behaviour of the higher-order mixed cumulants

$$c^{(k)}_{\Xi}(x_1, \ldots, x_k) := \Gamma(\xi(x_1), \ldots, \xi(x_k)) \quad \text{for} \quad k \geq 1$$
of the random field \( \{ \xi(x), x \in \mathbb{R}^{d} \} \), where the \textit{mixed cumulant} (semiinvariant) of any random variables \( Y_1, \ldots, Y_k \) (having a finite \( k \)th moment) is defined by

\[
\Gamma(Y_1, \ldots, Y_k) := i^{-k} \frac{\partial^k}{\partial s_1 \cdots \partial s_k} \log \mathbb{E} \left\{ \exp \left[ i \sum_{j=1}^{k} s_j Y_j \right] \right\} \bigg|_{s_1=\ldots=s_k=0}
\]

and \( \Gamma_k(Y) := \Gamma(Y, \ldots, Y) \) (obtained by putting \( Y = Y_1 = \ldots = Y_k \) in (1.8)) denotes the \( k \)-th \textit{cumulant} of \( Y \). Directly from (1.8) it is seen that, for \( k \geq 2 \),

\[
e^{(k)}(x_1, \ldots, x_k) := \Gamma(1 - \xi(x_1), \ldots, 1 - \xi(x_k)) = (-1)^k e^{(k)}(x_1, \ldots, x_k).
\]

We are now in a position to formulate our main result.

**Theorem 1.** Let \( \Xi \) be the PGM (1.1) with compact typical grain \( \Xi_0 \) satisfying (1.3) and let \( \{ W_n, n \geq 1 \} \) be a convex averaging sequence in \( \mathbb{R}^{d} \). Then, for any \( k \geq 2 \),

\[
\int_{[R^{d}]^{k-1}} \left| e^{(k)}(a, x_2, \ldots, x_k) \right| \, d(x_2, \ldots, x_k) \leq (k - 1)! H(a) \Delta(a)^{k-2},
\]

where \( H(a) := 8 \lambda M(a) (1 + \exp(\lambda E[\Xi_0]))/a^2 \) and \( \Delta(a) := 8 \lambda M(a) (1 + \exp(\lambda E[\Xi_0]))/a^2 \). Furthermore, the limit \( L(z) = \lim_{n \to \infty} L_n(z) \) exists and is analytic on the open disk \( D_{\Delta(a)} \).

The next result states Cramér’s large deviation relations for the random sequence \( |\Xi \cap W_n| \) and an optimal Berry-Esseen bound of the distance between \( F_n(x) := \mathbb{P}(\sqrt{|W_n|} (\hat{p}_n - p) \leq x\sigma_n) \) and the standard normal distribution function \( \Phi(x) = \int_{-\infty}^{x} e^{-t^2/2} \, dt/\sqrt{2\pi} \), where

\[
\sigma_n^2 := \frac{\text{Var}(|\Xi \cap W_n|)}{|W_n|} = \int_{R^{d}} \frac{|W_n \cap (W_n - x)|}{|W_n|} \left( e^{-\lambda E[\Xi_0]\min(\Xi_0 - x)} - e^{-2\lambda E[\Xi_0]} \right) \, dx.
\]

The following Theorem 2 is derived from (1.10) combined with a well-known lemma on large deviations for a single random variable due to V.A. Statulevičius [21], see also [20], Lemma 2.3.

**Theorem 2.** Let the assumptions of Theorem 1 be satisfied and in addition let \( \mathbb{E}[\Xi_0] > 0 \). Then there exists some \( n_0 \geq 1 \) such that

\[
0 < \inf_{n \geq n_0} \sigma_n^2 \leq \lim_{n \to \infty} \sigma_n^2 = \int_{R^{d}} e^{(2)}(a, x) \, dx.
\]

and for \( 0 \leq x \leq \sigma_n \sqrt{|W_n|}/2 \Delta(a)(1 + 4H_n) \) with \( H_n = H(a)/2\sigma_n^2 \), the following asymptotic relations hold:

\[
\frac{1 - F_n(x)}{1 - \Phi(x)} = \exp \left\{ \frac{x^3}{\sigma_n \sqrt{|W_n|}} \sum_{k=0}^{\infty} \mu_k \left( \frac{x}{\sigma_n \sqrt{|W_n|}} \right)^k \right\} \left( 1 + O \left( \frac{1 + x}{\sqrt{|W_n|}} \right) \right)
\]
and

\[ \frac{F_n(-x)}{\Phi(-x)} = \exp \left\{ -\frac{x^3}{\sigma_n \sqrt{|W_n|}} \sum_{k=0}^{\infty} \mu_k^{(n)} \left( \frac{-x}{\sigma_n \sqrt{|W_n|}} \right)^k \right\} \left( 1 + O \left( \frac{1 + x}{\sqrt{|W_n|}} \right) \right) \]

as \( n \to \infty \), where the coefficients

\[ \mu_k^{(n)} = \frac{1}{(k + 2)(k + 3)} \sum_{l=1}^{k+1} (-1)^{l-1} \binom{k + l + 1}{l} \sum_{k_1 + \ldots + k_l = k+1} \prod_{i=1}^l \frac{\Gamma_{k_i+2}(|\Xi \cap W_n|)}{\sigma_n^2 |W_n| (k_i + 1)!} \]

satisfy the estimate \( |\mu_k^{(n)}| \leq 4H_n \Delta(a) (2\Delta(a)(1 + 4H_n))^{k/(k+2)}(k+3) \) for \( k \geq 0 \).

Furthermore, there exists some constant \( c > 0 \) (depending on \( a \), \( \lambda \), \( M(a) \) and \( \sigma_n^2 \)) such that

\[ \sup_{x \in \mathbb{R}^d} |F_n(x) - \Phi(x)| \leq \frac{c}{\sqrt{|W_n|}} \quad \text{for} \quad n \geq n_0 \, . \]

The rest of this paper is organized as follows: In Section 2, we investigate (1.2) for a quite general random set model (Lemma 1 and 2) and put together the required tools from point process theory presented in a rather general setting (Lemma 3). In Section 3 we are concerned with the proof of Theorem 1 which is divided into several steps (Lemma 4 - 7), whereas the proof of Theorem 2 is deferred to Section 4. Furthermore, in Section 4, we derive some inequalities and a Chernoff-type theorem for large deviation (Theorem 3 and 4) of the empirical volume fraction \( \tilde{p}_n \). The Appendix contains among others the construction of a measurable random field \( \xi(x) = 1_{\Xi(x)}, x \in \mathbb{R}^d \), and a criterion for (non-)closedness of the PGM \( \Xi \) given by (1.1).

2. Preliminary Results and Relations to Point Processes

We first investigate the behaviour of the cumulants \( \Gamma_k(|\Xi \cap W_n|) \) and give a condition which guarantees the existence and analyticity of the limit of (1.2) for the support set \( \Xi = \text{supp}(\xi) \) of an arbitrary \( (\mathcal{B}(\mathbb{R}^d) \otimes \mathbb{R}^d) \) measurable, \( 0 - 1 \) valued, stationary random field \( \{\xi(x), x \in \mathbb{R}^d\} \).

We use the same notation as in Sect. 1. Lemma 2 states that this condition can be expressed by the total variation of the reduced cumulant measures of the Cox process

\[ \Pi^{(z)}_\Xi := \sum_{i \geq 1} (1 - 1_{\Xi(Y_i)}) \delta_{Y_i} \, , \]
which is directed by the random measure \( z \int_{\mathbb{R}^d} 1_{(.)}(x) (1 - \xi(x)) \, dx = z|\Xi^c \cap (.)| \), where \( \Pi_z = \sum_{i \geq 1} \delta_{Y_i} \) is a stationary Poisson process with intensity \( z > 0 \) being independent of \( \Xi \). In the second part of this section we introduce a family of correlation measures for arbitrary stationary point processes and derive (Lemma 3) a recurrence relation for the corresponding Lebesgue density functions provided they exist. Lemma 3 is the key to prove Theorem 1 and it seems to be of interest for its own.

**Lemma 1.** Let \( \{\xi(x), x \in \mathbb{R}^d\} \) be a measurable, \( 0 - 1 \)-valued, stationary random field on \( \mathbb{R}^d \) with support set \( \Xi := \{x \in \mathbb{R}^d : \xi(x) = 1\} \). Then, for any bounded \( W \in \mathfrak{A}(\mathbb{R}^d) \) and \( k \geq 2 \), we have

\[
|\Gamma_k(\Xi \cap W)| \leq |W| G_k(\Xi) \quad \text{with} \quad G_k(\Xi) := \int_{(\mathbb{R}^d)^{k-1}} |c^{(k)}_\Xi (o, x_2, \ldots, x_k)| \, d(x_2, \ldots, x_k).
\]

Further, let \( (W_n)_{n \geq 1} \) be a convex averaging sequence. If \( G_k(\Xi) < \infty \) for some \( k \geq 2 \), then

\[
\lim_{n \to \infty} \frac{\Gamma_k(\Xi \cap W_n)}{|W_n|} = \int_{(\mathbb{R}^d)^{k-1}} c^{(k)}_\Xi (o, x_2, \ldots, x_k) \, d(x_2, \ldots, x_k)
\]

and, if \( G_k(\Xi) \leq k! \, H \, \Delta^{k-2} \) for some \( H, \Delta > 0 \) and any \( k \geq 2 \), then the limit function \( L(z) = \lim_{n \to \infty} L_n(z) \) - the thermodynamic limit of \( L_n(z) \) - exist and is analytic on the open disk \( D_\Delta \), where \( L_n(z) \) is given by (1.2) with \( \Xi := \{x \in \mathbb{R}^d : \xi(x) = 1\} \) (instead of (1.1)). For \( z \in D_\Delta \), the thermodynamic limit \( L(z) \) admits the power series expansion

\[
L(z) = z^2 + \frac{z^2}{2} \int_{\mathbb{R}^d} (C(x) - p^2) \, dx + \sum_{k \geq 3} \frac{z^k}{k!} \int_{(\mathbb{R}^d)^{k-1}} c^{(k)}_\Xi (o, x_2, \ldots, x_k) \, d(x_2, \ldots, x_k),
\]

where \( p := p^{(1)}_\Xi (o) \) (‘volume fraction’ of \( \Xi \)) and \( C(x) := p^{(2)}_\Xi (o, x) \) (‘covariance’ of \( \Xi \)).

**Remark 1.** The absolute integrability of \( c^{(k)}_\Xi (o, x_2, \ldots, x_k) \) alone does not imply any rate of convergence in (2.2). Such rates can be derived if the random field \( \{\xi(x), x \in \mathbb{R}^d\} \) satisfies certain strong mixing condition.

**Proof of Lemma 1.** Using Fubini’s theorem and the definition (1.4) we may write

\[
\mathbb{E} \prod_{i=1}^k |\Xi \cap B_i|^k = \mathbb{E} \prod_{i=1}^k \int_{B_i} \xi(x_i) \, dx_i = \int_{B_1 \times \cdots \times B_k} p^{(k)}_\Xi (x_1, \ldots, x_k) \, dx_1 \ldots dx_k.
\]

A direct calculation of the logarithmic derivatives in (1.8) leads to

\[
\Gamma(Y_1, \ldots, Y_k) = \sum_{j=1}^k (-1)^{j-1} (j - 1)! \sum_{K_1 \cup \cdots \cup K_j = K} \prod_{i=1}^j \mathbb{E} \left( \prod_{k \in K_i} Y_{k_i} \right)
\]
see e.g. Saulis and Statulevičius [20], where the inner sum is taken over all decompositions of $K = \{1, ..., k\}$ into $j$ disjoint nonempty subsets $K_1, ..., K_j$. From (2.4) and (2.3) and by repeated application of Fubini’s theorem, we see that the integral $\int_{B_1 \times \cdots \times B_k} \Gamma(\xi(x_1), ..., \xi(x_k)) \, d(x_1, ..., x_k)$ coincide with $\Gamma((\Xi \cap B_1), ..., (\Xi \cap B_k))$. This means, setting $B_1 = ... = B_k = W$ and using (1.7), that

\[
(2.5) \quad \Gamma_k(\Xi \cap W) = \int_{W^k} c_x^{(k)}(x_1, x_2, ..., x_k) \, d(x_1, x_2, ..., x_k).
\]

The stationarity of the random field $\{\xi(x), x \in \mathbb{R}^d\}$ implies the invariance of the mixed cumulants (1.7) under diagonal shifts, i.e.

\[
(2.6) \quad c_x^{(k)}(x_1, x_2, ..., x_k) = c_{(x_2 - x_1)}^{(k)}(o, x_2 - x_1, ..., x_k - x_1)
\]

whence, by substituting $y_j = x_j - x_1$, $j = 2, ..., k$, it follows that

\[
\Gamma_k(\Xi \cap W) = \int_{(\mathbb{R}^d)^{k-1}} \int_{\mathbb{R}^d} c_x^{(k)}(o, y_2, ..., y_k) \, 1_W(x_1) \prod_{j=2}^{k} 1_W(y_j - x_1) \, dx_1 \, dy_2, ..., y_k
\]

\[
= \int_{(\mathbb{R}^d)^{k-1}} c_x^{(k)}(o, x_2, ..., x_k) \, |W \cap (W - x_2) \cap \cdots \cap (W - x_k)| \, dx_2, ..., x_k
\]

proving the first part of Lemma 1. The limit (2.2) is an immediate consequence of Lebesgue’s dominated convergence theorem and the fact that, in view of the geometric properties of the $W_n$’s, see Fritz [4],

\[
\lim_{n \to \infty} \frac{|W_n \cap (W_n - x_1) \cap \cdots \cap (W_n - x_{k-1})|}{|W_n|} = 1 \quad \text{for any fixed } x_1, ..., x_{k-1} \in \mathbb{R}^d.
\]

The power series expansion of (1.2) is as follows

\[
L_n(z) = pz + \sum_{k=2} \frac{\Gamma_k(\Xi \cap W)}{|W_n|} \frac{z^k}{k!}
\]

and hence, by our assumptions,

\[
|L_n(z) - pz| \leq \sum_{k=2} G_k(\Xi) \frac{|z|^k}{k!} \leq |z|^2 H \sum_{k=2} \frac{(|z| \Delta)^{k-2}}{1 - |z| \Delta} = \frac{|z|^2 H}{1 - |z| \Delta} \quad \text{for } |z| < 1/\Delta.
\]

Thus, for any $n \geq 1$, $L_n(z)$ is analytic on the open disk $D_\Delta$ and, by (2.2), $L_n(z)$ converges to $L(z)$ uniformly in any closed subset of $D_\Delta$ proving the analyticity of $L(z)$ on $D_\Delta$. \[Q\]

In order to obtain estimates of the form $G_n(\Xi) \leq n! H \Delta^{n-2}$ in case of the PGM (1.1), we first show that $(-z)^n c_{\Xi}^{(n)}$ coincide with the $n$th-order cumulant density of the Cox process (2.1).
In the second step we introduce a family of correlation measures \( \gamma^{(m,n)}_{\Psi} \) and their Lebesgue densities and study them for \( \Psi = \Pi^{[1]}_{\Xi} \). In the next Section 3 we perform a somewhat involved and rather lengthy inductive estimation technique to derive bounds of the total variation of these correlation measures in terms of moments of \( \Xi_0 \) when \( \Xi \) is given by (1.1). The basic idea of this method goes back to D. Ruelle [18] and [19], Chapt. 4.4, who developed it (without using the terminology of point processes) to prove the existence of thermodynamic limits for grand canonical Gibbs ensembles with pair interactions. An extension to ensembles with higher-order interactions has been tried by Greenberg [7], but it fails in our situation.

To begin with, we briefly recall the definition of the \( n \)-th order factorial moment (and cumulant) measure \( \alpha_{\Psi}^{(n)} \) (and \( \gamma_{\Psi}^{(n)} \)) of a point process \( \Psi = \sum_{i \geq 1} \delta_{Z_i} \) satisfying \( \mathbb{E} \Psi^n(K) < \infty \) for \( K \in \mathcal{K} \) by means of its probability generating functional \( G_{\Psi}[w] = \mathbb{E} \left( \prod_{i \geq 1} w(Z_i) \right) \), where \( w : \mathbb{R}^d \rightarrow [0,1] \) is Borel measurable such that \( 1 - w \) has bounded support, see e.g. Daley and Vere-Jones [2]. Setting \( w_{B_1,\ldots,B_n}(x) = 1 + \sum_{j=1}^n (v_j - 1) \mathbf{1}_{B_j}(x) \) for \( 1 - \frac{1}{n} \leq v_j \leq 1 \), \( i = 1, \ldots, n \) and bounded \( B_1, \ldots, B_n \in \mathfrak{B}(\mathbb{R}^d) \), we define

\[
\alpha_{\Psi}^{(n)}(x_{j=1}^n B_j) := \lim_{v_1,\ldots,v_n \uparrow 1} \frac{\partial^n}{\partial v_1 \cdots \partial v_n} G_{\Psi} \left[ w_{B_1,\ldots,B_n} \right]
\]

and

\[
\gamma_{\Psi}^{(n)}(x_{j=1}^n B_j) := \lim_{v_1,\ldots,v_n \uparrow 1} \frac{\partial^n}{\partial v_1 \cdots \partial v_n} \log G_{\Psi} \left[ w_{B_1,\ldots,B_n} \right] .
\]

If \( \alpha_{\Psi}^{(n)} \) resp. \( \gamma_{\Psi}^{(n)} \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^d \), then we denote the corresponding (factorial) moment resp. cumulant density by \( p_{\Psi}^{(n)} \) resp. \( \kappa_{\Psi}^{(n)} \). In the sequel we will often write \( p_{\Psi}^{(n)}(X_n) \) instead \( p_{\Psi}^{(n)}(x_1,\ldots,x_n) \), where \( X_n \) stands for the (unordered) point set \( \{x_1,\ldots(2.7)\)

\[
\gamma_{\Psi}^{(n)}(x_{j=1}^n B_j) = \int_{B_1} \gamma_{\Psi,\text{red}}^{(n)}(x_{j=2}(B_j - x)) \, dx \quad \text{for any bounded} \quad B_1,\ldots,B_n \in \mathfrak{B}(\mathbb{R}^d).
\]

Finally, a stationary point process \( \Psi \) is said to be Brillinger-mixing, see e.g. Ivanoff [12] or Heinrich and Schmidt [9], if \( \mathbb{E} \Psi^n([0,1]^d) < \infty \) and the total variation \( \text{var}(\gamma_{\Psi,\text{red}}^{(n)}) \) on \( \mathbb{R}^{d(n-1)} \) is finite for all \( n \geq 2 \).

**Lemma 2.** Let \( \Xi \) be the support set of a measurable, \( 0 - 1 \)-valued, stationary random field \( \{\xi(x), x \in \mathbb{R}^d\} \). Then the \( n \)-th order reduced cumulant measure \( \gamma_{\Pi^{[1]}_{\Xi},\text{red}}^{(n)} \) of the Cox process (2.1)
exists for any \( n \geq 2 \) and its total variation (if it exists) takes the form
\[
\text{var}(\gamma^{(n)}_{\Pi^z_\Xi}, \text{red}) = z^n G_n(\Xi) \quad \text{with} \quad \gamma^{(n)}_{\Pi^z_\Xi}(B) = (-z)^n \int_B c^{(n)}_{\Xi}(o, x_2, ..., x_n) \, d(x_2, ..., x_n)
\]
for \( B \in \mathcal{B}(\mathbb{R}^{d-n-1}) \). Consequently, \( \Pi^z_\Xi \) is Brillinger-mixing iff \( G_n(\Xi) < \infty \) for all \( n \geq 2 \).

**Proof of Lemma 2.** From the shape of the probability generating functional of a Cox process directed by an arbitrary random measure, see [2], p. 262, we deduce that
\[
G_{\Pi^z_\Xi}[w] = \mathbb{E} \exp \left\{ z \int_{\mathbb{R}^d} (w(x) - 1) 1_{\Xi^c}(x) \, dx \right\}
\]
which in turn, using the above definitions of moment and cumulant measures, provides
\[
a^{(n)}_{\Pi^z_\Xi}(\chi^{(n)}_{j=1} B_j) = z^n \mathbb{E} \prod_{j=1}^n [\Xi^c \cap B_j] \quad \text{and} \quad \gamma^{(n)}_{\Pi^z_\Xi}(\chi^{(n)}_{j=1} B_j) = z^n \Gamma([\Xi^c \cap B_1, ..., \Xi^c \cap B_n]) \nonumber
\]
Hence, repeating the steps in the proof of Lemma 1 leading to (2.3) and (2.5) (with \( \Xi^c \) instead of \( \Xi \)), we recognize that, for \( B \in \mathcal{B}(\mathbb{R}^d) \),
\[
a^{(n)}_{\Pi^z_\Xi}(B) = z^n \int_B p^{(n)}_{\Xi^c}(X_n) \, dX_n \quad \text{and} \quad \gamma^{(n)}_{\Pi^z_\Xi}(B) = z^n \int_B c^{(n)}_{\Xi^c}(X_n) \, dX_n,
\]
where \( X_n = \{x_1, ..., x_n\} \) and \( dX_n = d(x_1, ..., x_n) \). Thus, the \( n \)th-order cumulant density of \( \Pi^z_\Xi \) equals \( z^n c^{(n)}_{\Xi^c} \). The proof is completed by appealing to (2.6), (2.7) and the very definition of total variation. □

We now introduce a further family of (signed) measures \( \gamma^{(m,n)}_{\Psi} \) on \( \mathcal{B}(\mathbb{R}^{m+n}) \) for \( n, m \geq 0 \) associated with the point process \( \Psi \) which is assumed to admit moment measures of order \( m+n \). For bounded \( A_1, ..., A_m, B_1, ..., B_n \in \mathcal{B}(\mathbb{R}^d) \) define
\[
\gamma^{(m,n)}_{\Psi}(\chi^{m}_{i=1} A_i \times \chi^{n}_{j=1} B_j) = \lim_{\substack{u_1, ..., u_{m+n} \to 0 \\
u_1, ..., u_{m+n} \uparrow 1}} \frac{\partial^{n+m}}{\partial \nu_1 \cdots \partial \nu_n \partial u_1 \cdots \partial u_m} \left[ \frac{G_{\Psi}[w^{A_1, ..., A_m, B_1, ..., B_n}]}{G_{\Psi}[w^{B_1, ..., B_n}]} \right].
\]
For the sake of distinction, let us call \( \gamma^{(m,n)}_{\Psi} \) the (factorial) correlation measure of order \((m,n)\).

In case the moment density \( p^{(m+n)}_{\Psi} \) exists, \( \gamma^{(m,n)}_{\Psi} \) has a Lebesgue density \( c^{(m,n)}_{\Psi} \) which we call correlation density of order \((m,n)\). Note that D. Ruelle [18] was apparently the first who introduced the densities \( c^{(m,n)}_{\Psi} \) via an algebraic method in order to study cluster properties of the correlation functions of classical gases.

It is evident that \( \gamma^{(m,n)}_{\Psi} \) is symmetric in the first \( m \) as well as in the second \( n \) components,
but not completely symmetric. By logarithmic differentiation with respect to \( v_1 \) in the above definition of \( \gamma_\Psi^{(n)} \) we see that \( \gamma_\Psi^{(n)} \) coincides with \( \gamma_\Psi^{(1,n-1)} \) for \( n \geq 1 \) and, thus, \( c_\Psi^{(n)} = c_\Psi^{(1,n-1)} \) for \( n \geq 1 \) provided the densities exist. Moreover, for fixed \( m \geq 1 \) and any \( n \geq 1 \), the following relation between factorial moment and correlation measures holds:

\[
(2.8) \quad \alpha_\Psi^{(m+n)}(x_{i=1}^n A_i \times x_{j=1}^n B_j) = \sum_{\sum_j \leq n \leq N-\{1,\ldots,n\}} \gamma_\Psi^{(m-|J|)}(x_{i=1}^m A_i \times x_{j \in J} B_j) \alpha_\Psi^{(n-|J|)}(x_{j \in N \setminus J} B_j),
\]

where the summation extends over all subsets \( J \) of \( N \) with \( |J| \) elements. For reasons of consistency, put \( \gamma_\Psi^{(m,0)}(x_{i=1}^m A_i) = \alpha_\Psi^{(m)}(x_{i=1}^m A_i) \) ( = 1 for \( m = 0 \)) and \( \gamma_\Psi^{(0,n)}(x_{j=1}^n B_j) = 0 \) for \( n \geq 1 \).

In order to verify (2.8), let us briefly write \( D_u \) for \( \frac{\partial}{\partial u_1 \ldots \partial u_N} \) and \( D_v \) for \( \frac{\partial}{\partial v_1 \ldots \partial v_N} \) and put \( f(u,v) = \omega_{B_1 \ldots B_N} \omega_{A_1 \ldots A_N} \) and \( g(v) = \omega_{B_1 \ldots B_N} \). Then (2.8) is obtained by applying Leibniz’s rule for higher-order derivatives of products of functions to the right-hand side of the identity

\[
D_v D_u (f(u,v)) = D_v (g(v) \cdot h(v)) \quad \text{with} \quad h(v) = \frac{D_u(f(u,v))}{g(v)}.
\]

We conclude this section with a recursive representation of the correlation density \( c_\Psi^{(m,n)} \) in terms of the densities \( c_\Psi^{(m-1+j,n-j)} \), \( j = 1,\ldots,n \). For notational convenience, we omit the superscripts (if confusion is excluded) and write \( c_\Psi(X_m, Y_n) \) instead \( c_\Psi^{(m,n)}(x_1,\ldots,x_m,y_1,\ldots,y_n) \), where \( X_m = \{x_1,\ldots,x_m\} \) and \( Y_n = \{y_1,\ldots,y_n\} \) are two disjoint sets of distinct points in \( \mathbb{R}^d \). Further, put \( X_{m-1} = X_m \setminus \{x_1\} \) and let \( |Y| \) denote the cardinality of a finite point set \( Y \subset \mathbb{R}^d \).

**Lemma 3.** Let \( \Psi \) be a point process on \( \mathbb{R}^d \) with strictly positive factorial moment densities \( p_\Psi^{(k)} \) for \( k = 1,\ldots,m+n\geq 1 \). Then we have

\[
(2.9) \quad c_\Psi(X_m,Y_n) = \sum_{\emptyset \subseteq V \subseteq \mathbb{Y}^n} (-1)^{|V|} K_\Psi(X_m,Y) c_\Psi(Y \cup X_{m-1}, Y_n \setminus Y),
\]

where

\[
(2.10) \quad K_\Psi(X_m,Y) := \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \frac{p_\Psi(V \cup X_m)}{p_\Psi(V \cup X_{m-1})} \quad \text{for} \quad m,|Y| \geq 1 , \ Y \subseteq \mathbb{Y}_n
\]

and \( K_\Psi(X_m,\emptyset) = p_\Psi(X_m)/p_\Psi(X_{m-1}) \) for \( m \geq 1 \) and \( K_\Psi(\emptyset,Y_n) = 0 \) for \( n \geq 1 \).

**Proof of Lemma 3.** The relation (2.8) reads in terms of densities as follows

\[
(2.11) \quad p_\Psi(X_m \cup Y_n) = \sum_{\emptyset \subseteq Y \subseteq \mathbb{Y}^n} c_\Psi(X_m,Y) \ p_\Psi(Y_n \setminus Y).
\]
Given the moment density functions \( p_\psi(Y), Y \subseteq Y_n \), with \( p_\psi(\emptyset) = 1 \), there exist unique symmetric functions \( p_\psi^*(Y), Y \subseteq Y_n \), with \( p_\psi^*(\emptyset) = 1 \) satisfying the equations

\[
(2.12) \quad \sum_{\emptyset \subseteq V \subseteq Y} p_\psi^*(V) p_\psi(Y \setminus V) = 0 \quad \text{for} \quad \emptyset \neq Y \subseteq Y_n.
\]

By means of the functions \( p_\psi^*(Y) \) we may invert the ‘convolution equation’ (2.11) by calculating the sum

\[
\sum_{\emptyset \subseteq V \subseteq Y} p_\psi^*(Y \setminus Y) p_\psi(X_m \cup Y) = \sum_{\emptyset \subseteq V \subseteq Y} p_\psi^*(Y \setminus V) \sum_{\emptyset \subseteq V \subseteq Y} c_\psi(X_m, V) p_\psi(Y \setminus V)
\]

\[
= \sum_{\emptyset \subseteq V \subseteq Y} c_\psi(X_m, V) \sum_{Y : V \subseteq Y \subseteq Y_n} p_\psi^*(Y \setminus Y) p_\psi(Y \setminus V)
\]

Since, by (2.12), the second sum in the last line vanishes for all proper subsets \( V \subset Y_n \), the whole last line is equal to \( c_\psi(X_m, Y_n) \). Using this identity and the relation

\[
\sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} K_\psi(X_m, V) = \frac{p_\psi(Y \cup X_m)}{p_\psi(Y \cup X_{m-1})} \quad \text{for} \quad \emptyset \subseteq Y \subseteq Y_n
\]

obtained from (2.10) by using the Möbius inversion formula, see Rota [17], we may proceed with

\[
c_\psi(X_m, Y_n) = \sum_{\emptyset \subseteq V \subseteq Y_n} p_\psi^*(Y_n \setminus Y) p_\psi(Y \cup X_{m-1}) \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} K_\psi(X_m, V)
\]

\[
= \sum_{\emptyset \subseteq V \subseteq Y_n} (-1)^{|V|} K_\psi(X_m, V) \sum_{Y : V \subseteq Y \subseteq Y_n} p_\psi^*(Y_n \setminus Y) p_\psi(Y \cup X_{m-1})
\]

\[
= \sum_{\emptyset \subseteq V \subseteq Y_n} (-1)^{|V|} K_\psi(X_m, V) \sum_{\emptyset \subseteq U \subseteq Y_n \setminus Y} p_\psi^*(Y_n \setminus V \setminus U) p_\psi(U \cup V \cup X_{m-1}).
\]

Applying again the above derived identity, we see that the second sum in the last line equals \( c_\psi(V \cup X_{m-1}, Y_n \setminus V) \) proving the asserted relation (2.9). \( \Box \)

3. Absolute Integrability of the Correlation Densities of the Cox Process \( \Pi_1^{(1)} \) and Proof of Theorem 1

Throughout this section we consider the factorial moment and correlation densities \( p_\psi \) and \( c_\psi \) merely with respect to Cox process \( \Pi_1^{(1)} \) defined by (2.1) for the PGM (1.1). For notational
ease, we indicate this by omitting the subscript $\Psi$ at $p_\Psi$, $c_\Psi$ and $K_\Psi$. Our aim consists in obtaining bounds of the integrals $\int_{R^n} |c(o,Y_n)| \, dY_n$ ( = $G_n(\Xi)$ by Lemma 2 ) under suitable moment conditions on $|\Xi_0|$. For doing this, however, our inductive proving technique requires to estimate the integrals $\int_{R^n} |c(X_m,Y_n)| \, dY_n$ uniformly in $X_m \in R^{dm}$ for any $m \geq 1$.

Let $X_m, X_{m-1}$ and $Y_n$ be the finite point sets introduced at the end of Section 2. Further, for any finite subset $Y \subset R^d$, put $\Xi_0(Y) := \bigcup_{y \in Y} (\Xi_0 - y)$ (and thus $\Xi_0^c(Y) = \bigcap_{y \in Y} (\Xi_0^c - y)$).

For any $n \geq 1$ and $Y \subseteq Y_n$ define

$$S(X_m,Y) := \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \exp \{ E(x_1,X_{m-1}^l,V) \},$$

where

$$E(x;U,V) := \lambda E[(\Xi_0 - x) \cap \Xi_0^c(\Xi_0 - x) \cap \Xi_0^c(V) | x \notin U]$$

and, for any $Y \subseteq Y_{n-1} := Y_n \setminus \{y_n\}$,

$$T(y_n,X_m,Y) := \sum_{\emptyset \subseteq V \subseteq Y} (-1)^{|V|} \exp \{ -E(x_1,y_n;X_{m-1}^l,V) \} ,$$

where

$$E(x,y;U,V) := \lambda E[(\Xi_0 - x) \cap (\Xi_0 - y) \cap \Xi_0^c(U) \cap \Xi_0^c(V) | x \neq y, x \notin U, y \notin V] .$$

Note that $E(x,y;U,\emptyset) = E(x;U,\emptyset) = 0$ implying $S(X_m,\emptyset) = T(y_n,X_m,\emptyset) = 1$.

From (1.6) it is clear that

$$\frac{p(V \cup X_m)}{p(V \cup X_{m-1})} = \exp \{ -\lambda E[(\Xi_0 - x_1) \cap \Xi_0^c(V \cup X_{m-1})] \}
= \exp \{ -\lambda E[(\Xi_0 - x_1) \cap \Xi_0^c(X_{m-1}^l)] \} \exp \{ E(x_1;X_{m-1}^l,V) \} .$$

so that, by (2.10) and (3.1),

$$(3.3) \quad K(X_m,Y) = \exp \{ -\lambda E[(\Xi_0 - x_1) \cap \Xi_0^c(X_{m-1}^l)] \} S(X_m,Y) .$$

Next we establish a recursive representation of $S(X_m,Y_n)$ with respect to $Y_n$ in combination with the non-negative terms $T(y_n,X_m,Y)$ for $Y \subseteq Y_{n-1}$. It turns out, see Lemma 5 below, that the integrals $\int_{R^n} T(y_n,X_m,Y_{n-1}) \, dY_n$ can be represented as functionals of certain PGM (3.6) which enables us to derive upper bounds of them under reasonable moment conditions on the volume of the typical grain $\Xi_0$. By means of these bounds and the following Lemma 4 we find corresponding bounds of $\int_{R^n} |S(X_m,Y_n)| \, dY_n$ which in turn, using (2.9) with (3.3), enable us to establish the desired bounds of $\int_{R^n} |c(X_m,Y_n)| \, dY_n$, see Lemma 7 below.
Lemma 4. We have

\[
S(X_m, Y_n) = S(X_m, Y_{n-1}) \left(1 - \exp\{E(x_1; X_{m-1}^t, \{y_n\})\}\right) - \exp\{E(x_1; X_{m-1}^t, \{y_n\})\} \\
\times \sum_{\emptyset \subseteq Y \subseteq Y_{n-1}} T(y_n, X_m, Y) \exp\{E(x_1; X_{m-1}^t, Y)\} S(X_m \cup Y, Y_{n-1} \setminus Y).
\]

In the special cases \(n = 1, 2\) we get the following expressions:

\[
S(X_m, \{y_1\}) = 1 - \exp\{E(x_1; X_{m-1}^t, \{y_1\})\}
\]

\[
S(X_m, \{y_1, y_2\}) = S(X_m, \{y_1\}) \left(1 - \exp\{E(x_1; X_{m-1}^t, \{y_2\})\}\right) - \exp\{E(x_1; X_{m-1}^t, \{y_1\}) + E(x_1; X_{m-1}^t, \{y_2\})\} T(y_2, X_m, \{y_1\}).
\]

Proof of Lemma 4. By the definition of the terms \(E(x_1; U, V)\) and \(E(x, y; U, V)\) and the relation 
\(|A| + |B| - |A \cap B| = |A \cup B|\) for bounded \(A, B \in \mathcal{B}(\mathbb{R}^d)\), we get

\[
E(x_1; X_{m-1}^t, Y \cup \{y_n\}) = E(x_1; X_{m-1}^t, \{y_n\}) + E(x_1; X_{m-1}^t, Y) - E(x_1, y_n; X_{m-1}^t, Y)
\]

for any \(Y \subseteq Y_{n-1}\). Further, we may rewrite the sum \(S(X_m, Y_n)\) as follows:

\[
\sum_{\emptyset \subseteq Y \subseteq Y_{n-1}} (-1)^{|Y|} \left(\exp\{E(x_1; X_{m-1}^t, Y)\} - \exp\{E(x_1; X_{m-1}^t, Y \cup \{y_n\})\}\right).
\]

This combined with the foregoing relation leads to

\[
S(X_m, Y_n) = S(X_m, Y_{n-1}) - \exp\{E(x_1; X_{m-1}^t, \{y_n\})\} \\
\times \sum_{\emptyset \subseteq Y \subseteq Y_{n-1}} (-1)^{|Y|} \exp\{E(x_1; X_{m-1}^t, Y) - E(x_1, y_n; X_{m-1}^t, Y)\}.
\]

A simple application of the Möbius inversion formula, see [17], to the terms (3.2) yields

\[
\exp\{-E(x_1, y_n; X_{m-1}^t, Y)\} = 1 + \sum_{\emptyset \subset U \subseteq Y} (-1)^{|U|} T(y_n, X_m, U), \quad Y \subseteq Y_{n-1}.
\]

Inserting this identity on the right-hand-side of the previous equality we arrive at

\[
S(X_m, Y_n) = S(X_m, Y_{n-1}) \left(1 - \exp\{E(x_1; X_{m-1}^t, \{y_n\})\}\right) - \exp\{E(x_1; X_{m-1}^t, \{y_n\})\} \\
\times \sum_{\emptyset \subseteq Y \subseteq Y_{n-1}} \sum_{\emptyset \subset U \subseteq Y} (-1)^{|Y| - |U|} T(y_n, X_m, U) \exp\{E(x_1; X_{m-1}^t, Y)\}.
\]

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By interchanging the sums and substituting \( V = Y \setminus U \) we obtain that

\[
\sum_{0 \leq Y \subseteq Y_{n-1}} \sum_{0 \leq U \subseteq Y} (-1)^{|Y|-|U|} T(y_n, X_m, U) \exp\{E(x_1; X_{m-1}^t, Y)\} = \sum_{0 \leq U \subseteq Y_{n-1}} \sum_{Y \cup U \subseteq Y_{n-1}} (-1)^{|Y|-|U|} T(y_n, X_m, U) \exp\{E(x_1; X_{m-1}^t, (Y \cup U))\}.
\]

Since \(|A \cup B| = |B| + |A \cap B^c|\) for any bounded \( A, B \in \mathcal{B}(R^d)\),

\[
E(x_1; X_{m-1}^t, V \cup U) = E(x_1; X_{m-1}^t, U) + E(x_1; U \cup X_{m-1}^t, V),
\]

whence, by definition (3.1), it follows that

\[
\sum_{0 \leq U \subseteq Y_{n-1} \setminus V} (-1)^{|V|} \exp\{E(x_1; X_{m-1}^t, V \cup U)\} = \exp\{E(x_1; X_{m-1}^t, U)\} S(X_m \cup U, Y_{n-1} \setminus U).
\]

Finally, assembling all above identities we obtain the assertion of Lemma 4. □

**Lemma 5.** Let \( \Xi \) be the PGM (1.1) with compact typical grain \( \Xi_0 \) satisfying \( E|\Xi_0|^{|n+1|} < \infty \) for some fixed \( n \geq 2 \). Then, for any \( m \geq 1 \),

\[
\sup_{X_m} \int_{R^d} T(y_n, X_m, Y_{n-1}) \, dY_n \leq (n-1)! \sum_{k=1}^{n-1} \frac{\lambda^k}{k!} \frac{E|\Xi_0|^{|n+1|+2}}{n_1!} \frac{E|\Xi_0|^{|n+2|+1}}{n_2!} \ldots \frac{E|\Xi_0|^{|n+k|+1}}{n_k!}.
\]

If condition (1.3) is satisfied, then the estimate

\[
\sup_{X_m} \int_{R^d} T(y_n, X_m, Y_{n-1}) \, dY_n \leq n! \left(\frac{2}{a}\right)^n \frac{\lambda M(a)}{a} \left(1 + \frac{\lambda M(a)}{a}\right)^{n-2}
\]

holds for all \( n \geq 2 \) and \( m \geq 1 \).

**Proof of Lemma 5.** According to the definition (3.2),

\[
T(y_n, X_m, Y_{n-1}) = \sum_{0 \leq Y \subseteq Y_{n-1}} (-1)^{|Y|} \exp\{-E(x_1, y_n; X_{m-1}^t, Y)\}.
\]

Now, for any non-empty \( Y \subseteq Y_{n-1} \), we introduce a new PGM \( \Xi(x_1, y_n; X_{m-1}^t, Y) \) governed by

\[
\Pi_\lambda = \sum_{i \geq 1} \delta_{X_i} \quad \text{and the typical grain} \quad (\Xi_0 - x_1) \cap (\Xi_0 - y_n) \cap \Xi_0(X_{m-1}^t) \cap \Xi_0(Y),
\]

that is

\[
\Xi(x_1, y_n; X_{m-1}^t, Y) := \bigcup_{i \geq 1} \left( (\Xi_i - x_1) \cap (\Xi_i - y_n) \cap \Xi_i(X_{m-1}^t) \cap \Xi_i(Y) + X_i \right),
\]

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where $\Xi_t(Y) = \bigcup_{y \in Y} (\Xi_t - y)$ and $\Xi_t(X_{m-1}') = \bigcap_{j=2}^m (\Xi_t - x_j)$.

Obviously, for each realization of the PGM (3.6), we have

$$\Xi(x_1, y_n; X_{m-1}', Y) = \bigcup_{y \in Y} \Xi(x_1, y_n; X_{m-1}', \{y\}) \quad \text{for} \quad Y \subseteq Y_{n-1}. $$

Applying the well-known formula $P(o \in \Xi) = 1 - \exp\{-\lambda E[\Xi_0]\}$ (being valid for the PGM (1.1)) to the stationary PGM (3.6) we see that

$$\exp\{-E(x_1, y_n; X_{m-1}', Y)\} = P(o \notin \Xi(x_1, y_n; X_{m-1}', Y)) $$

$$= 1 - P\left( \bigcup_{y \in Y} \{o \in \Xi(x_1, y_n; X_{m-1}', \{y\})\} \right).$$

Since $\sum_{\emptyset \subseteq Y \subseteq Y_{n-1}} (-1)^{|Y|} = 0$, it follows from the inclusion-exclusion principle that

$$T(y_n, X_m, Y_{n-1}) = -\sum_{\emptyset \subseteq Y \subseteq Y_{n-1}} (-1)^{|Y|} P\left( \bigcup_{y \in Y} \{o \in \Xi(x_1, y_n; X_{m-1}', \{y\})\} \right) $$

$$= P\left( \prod_{i=1}^{n-1} \{o \in \Xi(x_1, y_n; X_{m-1}', \{y_i\})\} \right) = E\left( \prod_{i=1}^{n-1} \mathbb{1}_{\Xi(x_1, y_n; X_{m-1}', \{y_i\})}(o) \right).$$

Thus, by Fubini’s theorem,

$$\int_{R^{dn}} T(y_n, X_m, Y_{n-1}) \, dy_n = \int_{R^d} E\left( \int_{R^d} \mathbb{1}_{\Xi(x_1, y_n; X_{m-1}', \{y\})}(o) \, dy \right)^{n-1} \, dy_n$$

and, for each realization of (3.6),

$$\int_{R^d} \mathbb{1}_{\Xi(x_1, y_n; X_{m-1}', \{y\})}(o) \, dy \leq \int_{R^d} \sum_{i \geq 1} \mathbb{1}_{(\Xi_t - x_1) \cap (\Xi_t - y_n) \cap (\Xi_t - y)}(X_i) \, dy $$

$$\leq \sum_{i \geq 1} |\Xi_t| \mathbb{1}_{(\Xi_t - x_1) \cap (\Xi_t - y_n)}(X_i),$$

whence, by applying the polynomial formula and using Fubini’s theorem again, we get that

$$E\left( \int_{R^d} \mathbb{1}_{\Xi(x_1, y_n; X_{m-1}', \{y\})}(o) \, dy \right)^{n-1}$$

$$\leq \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{n_1, \ldots, n_k = n-1} \frac{(n-1)!}{n_1! \cdots n_k!} \prod_{j=1}^k \mathbb{1}_{(\Xi_{i_j} - x_1) \cap (\Xi_{i_j} - y_n) \cap (\Xi_{i_j} | \Xi_{i_{j+1}})}(X_j) \mathbb{1}_{\Xi_{i_j} | \Xi_{i_{j+1}}}(o_j) $$

$$= (n-1)! \sum_{k=1}^{n-1} \frac{\lambda^k}{k!} \sum_{n_1, \ldots, n_k = n-1} \prod_{j=1}^k \frac{E[(\Xi_0 - x_1) \cap (\Xi_0 - y_n) \cap (\Xi_0 | \Xi_{i_j})]}{n_j!}.$$
Here the sum \( \sum^* \) stretches over \( k \)-tuples of pairwise distinct indices and the last equality is obtained by applying the Campbell-type formula (5.3) for \( f_j(x, K) = 1_{(K-x_1) \cap (K-y_n)}(x) |K|^n \).

Together with the obvious relation

\[(3.7) \quad \int_{R^d} |(\Xi_0 - x_1) \cap (\Xi_0 - y_n)| \, dy_n = |\Xi_0|^2 \]

we finally arrive at the desired estimate (3.4).

The existence of the exponential moment \( M(a) \) of \( |\Xi_0| \) implies \( E|\Xi_0|^k \leq k! M(a) a^{-k} \) for all \( k \geq 1 \). Inserting this moment bounds in the right-hand side of (3.4) and taking into account

\[
\sum_{n_1 + \cdots + n_k = n-1} (n_1 + 2) \prod_{i=1}^k (n_i + 1) \leq \left( \frac{n-2}{k-1} \right) 2^n \quad \text{,}
\]

we obtain that

\[
\int_{R^d} E \left( \int_{R^d} 1_{\Xi(x_1, y_n; x_m \setminus \{x_1\}, \{y\})} (o) \, dy \right)^{n-1} \, dy_n \leq \frac{n!}{a^n} \sum_{k=1}^{n-1} \frac{(\lambda M(a))^k}{a^k k!} \left( \frac{n-2}{k-1} \right) 2^n \\
\leq n! \left( \frac{2}{a} \right)^n \frac{\lambda M(a)}{a} \left( 1 + \frac{\lambda M(a)}{a} \right)^{n-2}.
\]

But this is exactly the desired estimate (3.5). Thus, Lemma 5 is completely proved. \( \square \)

**Lemma 6.** Let \( \Xi \) be the PGM (1.1) with compact typical grain \( \Xi_0 \) satisfying \( E|\Xi_0|^{n+1} < \infty \) for some fixed \( n \geq 1 \). Then, for any \( m \geq 1 \),

\[(3.8) \quad \sup_{x_m} \int_{R^n} |S(X_m, Y_n)| \, dY_n \leq c_n(\lambda) < \infty \quad ,
\]

where the constant \( c_n(\lambda) \) depends on \( \lambda \) and the first \( n+1 \) moments of \( |\Xi_0| \).

Moreover, if condition (1.3) is satisfied, then (3.8) holds with

\[(3.9) \quad c_n(\lambda) = n! A^n \quad B (1 + B)^{n-1}
\]

for all \( n \geq 1 \) and \( m \geq 1 \) with \( A = \frac{2}{\alpha} \left( 1 + \exp\{\lambda E|\Xi_0|\} \right) = \frac{2(2-p)}{\alpha \|1-p\|} \) and \( B = \frac{\lambda M(a)}{a} \).

**Proof of Lemma 6.** In view of the obvious inequalities

\[ E(x_1; X'_{m-1}, \{y_1\}) \leq E |(\Xi_0 - x_1) \cap (\Xi_0 - y_1)| \leq \lambda E|\Xi_0| \]

\[ E(x_1; X'_{m-1}, \{y_1\}) \leq E |(\Xi_0 - x_1) \cap (\Xi_0 - y_1)| \leq \lambda E|\Xi_0| \]

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and $e^x - 1 \leq x e^x$ for $x \geq 0$ together with (3.7) we see that

$$\int_{R^d} |S(X_m, \{y_1\})| \, dy_1 = \int_{R^d} \left( \exp\{E(x_1; X_{m-1}', \{y_1\})\} - 1 \right) \, dy_1$$

$$\leq \lambda \exp\{\lambda E[\Xi_0]\} \int_{R^d} E[(\Xi_0 - x_1) \cap (\Xi_0 - y_1)] \, dy_1 = \frac{\lambda}{1 - p} E[\Xi_0]^2.$$

Define

$$A_{m,n} := \sup_{X_m} \int_{R^{d \times n}} |S(X_m, Y_n)| \, dY_n \quad \text{and} \quad B_{m,n} := \sup_{X_m} \int_{R^{d \times n}} T(y_n, X_m, Y_{n-1}) \, dY_n$$

for $m, n \geq 1$. From Lemma 4 and $E(x_1; X_{m-1}', Y) \leq \lambda E[\Xi_0]$ for $Y \subseteq Y_{m-1}$ we get the inequality

$$(3.10) \quad A_{m,n} \leq A_{m,n-1} \frac{\lambda}{1 - p} E[\Xi_0]^2 + \frac{1}{(1 - p)^2} \sum_{k=1}^{n-1} \binom{n-1}{k} B_{m,k+1} A_{m+k,n-k-1}$$

with $A_{m,0} = 1$ and $A_{m,1} \leq \lambda E[\Xi_0]^2/(1 - p)$ for any $m \geq 1$. Since, by Lemma 5, $B_{m,k+1} \leq k! C_k$ for $k \geq 1$, where $C_k$ depends on $\lambda$ and the first $k+2$ moments of $|\Xi_0|$ but not on $m$, we recognize by induction on $n$, that $A_{m,n} \leq n! D_n$, where $D_0 = 1$, $D_1 = \lambda E[\Xi_0]^2/(1 - p)$ and

$$D_n = \frac{1}{(1 - p)^n} \left( D_{n-1} C_0 + \frac{1}{1 - p} \sum_{k=1}^{n-1} C_k D_{n-k-1} \right)$$

for $n \geq 2$ with $C_0 := \lambda E[\Xi_0]^2$. Therefore, $A_{m,n}$ does not depend on $m$ and is bounded by terms involving merely $\lambda$ and $E[\Xi_0]^k, k = 1, 2, \ldots, n + 1$. This proves the first part of Lemma 6.

We will also prove (3.9) by induction on $n$. From (1.3) we get $E[\Xi_0]^2 \leq 2 M(a)/a^2$ implying

$$A_{m,1} \leq D_1 \leq \lambda \exp\{\lambda E[\Xi_0]\} E[\Xi_0]^2 \leq \frac{2 \lambda M(a)}{a^2 (1 - p)}$$

which is even slightly stronger than (3.9) for $n = 1$. Assume now the validity of (3.9) for $n = 1, \ldots, N - 1$. Taking into account the estimates $B_{m,k+1} \leq k! C_k$ with $C_k = (k + 1) B\left(\frac{2}{a}\right)^{k+1} (1 + B)^{k-1}$ for $k \geq 1$ as stated in Lemma 5 together with $C_0 \leq 2 B/a$, we may write

$$D_N \leq \frac{1}{N (1 - p)} \left( A^{N-1} B \left(1 + B\right)^{N-2} \frac{2 B}{a} + \right.$$

$$+ \frac{1}{1 - p} \sum_{k=1}^{N-1} (k + 1) B \left(\frac{2}{a}\right)^{k+1} (1 + B)^{k-1} A^{N-k-1} \left(1 + B\right)^{N-k-1})$$

After a short calculation using that $\sum_{k \geq 1} (k + 1) \left(\frac{2}{a} \right)^k = (1 - p)(3 - p)$, we arrive at

$$D_N \leq A^N B \left(1 + B\right)^{N-1} \quad \text{for} \quad N \geq 2.$$
Thus, the second part of Lemma 6 proved. □

**Lemma 7.** Let \( \Xi \) be the PGM (1.1) with compact typical grain \( \Xi_0 \) satisfying \( \mathbb{E}|\Xi_0|^{n+1} < \infty \) for some fixed \( n \geq 1 \). Then, for any \( m \geq 1 \),

\[
(3.11) \quad \sup_{X_m} \int_{R^{d_m}} |c(X_m, Y_n)| \, dY_n \leq c_{m,n}(\lambda) < \infty,
\]

where the constant \( c_{m,n}(\lambda) \) depends on \( m \), \( \lambda \) and the first \( n+1 \) moments of \( |\Xi_0| \).

If condition (1.3) is satisfied, then (3.11) holds with

\[
(3.12) \quad c_{m,n}(\lambda) = n! \, 2^{m+1} \, A \, B \left( 4A \, (1+B) \right)^{n-1}
\]

for all \( n \geq 1 \) and \( m \geq 1 \) with \( A \) and \( B \) as in Lemma 6.

**Proof of Lemma 7.** Replacing of \( K(X_m, Y) \) in (2.9) by (1.6) leads to

\[
c(X_m, Y_n) = \exp\{-\lambda \mathbb{E}[\Xi_0 - x_1] \cap \Xi_0(X_{m-1}^t)\} \times \sum_{0 \subseteq Y \subseteq Y_n} (-1)^{|Y|} S(X_m, Y) \, c(Y \cup X_{m-1}^t, Y_n \setminus Y).
\]

Since \( c(X_m, \emptyset) = \exp\{-\lambda |\Xi_0(X_m)|\} \leq 1 \), by (1.6), and \( c(\emptyset, Y_n) = 0 \) for \( n \geq 1 \), by definition, and \( S(X_m, \emptyset) = 1 \) for \( m \geq 1 \) and since both \( S(X, Y) \) and \( c(X, Y) \) are symmetric in \( Y \subseteq Y_n \) for fixed \( X \), we deduce from the latter recurrence relation the following inequality:

\[
(3.13) \quad \int_{R^{d_m}} |c(X_m, Y_n)| \, dY_n \leq \int_{R^{d_m}} |c(X_{m-1}^t, Y_n)| \, dY_n + \int_{R^{d_m}} |S(X_m, Y_n)| \, dY_n
\]

\[
+ \sum_{k=1}^{n-1} \binom{n}{k} \int_{R^{d_k}} |S(X_m, Y_k)| \, dY_k \sup_{Y_k} \int_{R^{d(n-k)}} |c(Y_k \cup X_{m-1}^t, Y_n \setminus Y_k)| \, d(Y_n \setminus Y_k).
\]

For any \( m \geq 1 \) we have

\[
\int_{R^d} |c(X_m, \{y\})| \, dy = \int_{R^d} e^{-\lambda \mathbb{E} |\Xi_0(X_n) \cap \Xi_0 - y|} \left( 1 - e^{-\lambda \mathbb{E} |\Xi_0(X_n) \cap \Xi_0 - y|} \right) \, dy 
\]

\[
\leq \lambda (1 - p) \, \mathbb{E} |\Xi_0(X_n)| \mathbb{E} |\Xi_0|^2 \leq \lambda \, m \, \mathbb{E} |\Xi_0|^2
\]

Using the estimate (3.8) of Lemma 6 and applying (3.13) successively to the remaining integrals on the right-hand side of (3.13) we obtain a bound of the left-hand side of (3.13) in terms of \( c_k(\lambda), k = 1, \ldots, n \), and \( \sup_{Y_n} \int_{R^d} |c(X \cup Y, \{y_n\})| \, dy_n, Y \subseteq Y_{n-1}, X \subseteq X_{m-1}^t \). This combined with the foregoing inequality proves (3.11).
We now assume (1.3) giving \( \mathbb{E}|\Xi_0|^2 \leq 2M(a)/a^2 \) so that together with \( m \leq 2^{m-1} \),
\[
\int_{\mathbb{R}^d} |c(X_m, \{y\})| \, dy \leq 2^{m-1} \, A \, B
\]
which implies (3.12) for \( n = 1 \) and \( m \geq 1 \). Let now (3.12) hold for all \( m, n \geq 1 \) satisfying \( m + n \leq M + N \). Then, making use of estimate (3.9) of Lemma 6, it follows from (3.13) that
\[
\int_{\mathbb{R}^{m+n}} |c(X_M, Y_N)| \, dY_N \leq N! 2^{M-2} (4A)^N B (1 + B)^{N-1} + N! A^N B (1 + B)^{N-1}
\]
\[
+ N! \sum_{k=1}^{N-1} A^k B (1 + B)^{k-1} \, 2^{M+k-2} (4A (1 + B))^{N-k}
\]
\[
= N! 2^{M-1} (4A)^N B (1 + B)^{N-1} \left[ \frac{1}{2} + \frac{1}{2^{2N+M-1}} + \sum_{k=1}^{N-1} \frac{1}{2^{k+1}} \right].
\]
Thus, the validity of (3.12) for \( m + n = M + N \) follows because the sum in brackets does not exceed one for \( M + N \geq 2 \). This completes the proof of Lemma 7. \( \square \)

**Proof of Theorem 1.** As an immediate consequence of (3.12) for \( m = 1 \) and Lemma 2 applied to the stationary PGM (1.1), we obtain that
\[
G_{n+1}(\Xi) = \int_{\mathbb{R}^{m+n}} |c(\{v\}, Y_n)| \, dY_n \leq c_{1,n}(\lambda) \leq n! 4A^B (4A (1 + B))^{n-1}
\]
for all \( n \geq 1 \). Thus, by the definition of \( A \) and \( B \) in Lemma 5, we get (1.10) with \( H(a) = 4AB \) and \( \Delta(a) = 4A(1 + B) \). Finally, the existence and analyticity of the thermodynamic limit \( L(z) \) of the function (1.2) on the disk \( D_{\Delta(a)} \) follows from the second part of Lemma 1. \( \square \)

### 4. Proof of Theorem 2 and Further Results on Large Deviation

**Proof of Theorem 2.** From Lemma 1 and (3.14) we get the estimate
\[
|\Gamma_k(\Xi)\cap W| \leq |W| (k - 1)! \, H(a) \, \Delta(a)^{k-2} \quad \text{for} \quad k \geq 2
\]
and any \( W \in \mathcal{B}(\mathbb{R}^d) \). For the standardized random variable \( \xi_n := (|\Xi| \cap W_n) - p \, |W_n|)/\sigma_n \sqrt{|W_n|} \) (with \( n \geq n_0 \)) (4.1) implies that
\[
|\Gamma_k(\xi_n)| \leq (k - 1)! \, \frac{H(a)}{\sigma_n^2 \Delta_n^{k-2}} \leq \frac{k! \, H_n}{\Delta_n^{k-2}} \quad \text{for} \quad k \geq 3
\]
with \( H_n = H(a)/2\sigma_n^2 \) and \( \Delta_n = \sigma_n \sqrt{|W_n|/\Delta(a)} \). Note that the asymptotic variance \( \lim_{n \to \infty} \sigma_n^2 = \int_{\mathbb{R}^d} c_{1,1}^2(a, x) \, dx \) in (1.11) is finite and strictly positive iff \( 0 < \mathbb{E}|\Xi|^2 < \infty \). In this case one can
find suitable upper and lower bounds of $c_{\xi}^{(2)}(\alpha, \mathbf{x}) = \exp\{-\lambda \mathbb{E}[\mathbb{E}_0 \cup (\Xi_0 - x)]\} - \exp\{-2\lambda \mathbb{E}[\mathbb{E}_0]\}$ leading to the inclusion
\[
\exp\{-2\lambda \mathbb{E}[\mathbb{E}_0]\} \left(1 - \exp\{-\lambda \mathbb{E}[\mathbb{E}_0]\}\right) \frac{\mathbb{E}[\mathbb{E}_0]}{\mathbb{E}[\mathbb{E}_0]} \leq \int_{\mathbb{R}^d} c_{\xi}^{(2)}(\alpha, \mathbf{x}) \, d\mathbf{x} \leq \lambda \mathbb{E}[\mathbb{E}_0] \exp\{-\lambda \mathbb{E}[\mathbb{E}_0]\}.
\]
The estimate (4.2) enables us to apply a well-known lemma on large deviations of a single random variable proved in [21] to $\xi_n$ which immediately provides the asymptotic relations (1.12) and (1.13) as well as the Berry-Esseen bound (1.15) stated in Theorem 2. To be precise, according to the result in [21], the relations (1.12) and (1.13) are only valid in the narrower interval $0 \leq x \leq \delta^* \Delta_n$ for any $\delta^* < \delta_0 (1 + \delta_0)/2$, where $\delta_0 \in (0, 1)$ denotes the unique real root of $(1 - \delta)^3 = 6 H_n \delta$. Indeed, since $H_n \geq 1/2$, by (3.14) for $n = 1$, we have $\delta_0 (1 + \delta_0) < \delta_0/(1 - \delta)^3 = 1/6 H_n \leq 1/(1 + 4 H_n)$. Using again (4.1) and the inequality $(k + l + 1)^k \leq 2^{k+l}$ we can estimate the coefficients (1.14) as follows:
\[
|\mu_k^{[n]}| \leq \frac{1}{(k + 2)(k + 3)} \sum_{i=1}^{k+1} \sum_{k_i \geq 1} \left(\Delta(a) \frac{H(a)}{\sigma_n^2}\right) \prod_{l=1}^{k} \left(2 \frac{H(a)}{\sigma_n^2}\right)^{l-1} = \frac{2^k \Delta(a)^{k+1}}{(k + 2)(k + 3)} \sum_{i=1}^{k+1} \left(\frac{2 H(a)}{\sigma_n^2}\right)^{l-1} = \frac{4 H_n \Delta(a)}{(k + 2)(k + 3)} \left(2 \Delta(a) (1 + 4 H_n)^k\right)
\]
for $k \geq 0$. Therefore the series $\sum_{k \geq 0} |\mu_k^{[n]}| \sigma_n \sqrt{|W_n|}^{k}$ converges absolutely for $|x| \leq \Delta_n/2(1 + 4 H_n)$ and the $O-$ terms in (1.12) and (1.13) can be easily verified by evaluating the remainder terms given in Statulevičius [21]. Thus, (1.12) and (1.13) are valid for the whole interval $0 \leq x \leq \Delta_n/2(1 + 4 H_n)$ which completes the proof of Theorem 2. □

We turn now to study large deviation inequalities for the unbiased estimators
\[
\hat{p}_W := \frac{|\Xi \cap W|}{|W|} \quad \text{and} \quad \hat{C}_W(x) := \frac{|\Xi \cap (\Xi - x) \cap W|}{|W|}, \quad x \in \mathbb{R}^d.
\]
of the volume fraction $p = P(o \in \Xi)$ and the covariance $C(x) = P(o \in \Xi, x \in \Xi)$, respectively, in case of the PGM (1.1) observed on a sampling window $W \in \mathfrak{W}(\mathbb{R}^d)$, see [22]. In the below Theorem 3, relying on Lemma 7 and (3.14) resp. Lemma 1 and (1.10), we state probability inequalities for the deviation of the estimators (4.3) from their means $p$ and $C(x)$ under finite-order resp. exponential moment assumptions. Theorem 4 supplements Theorem 2 by determining a Chernoff rate function for the sequence $\hat{p}_n$ in terms of the thermodynamic limit $L(x)$.

**Theorem 3.** Let $\Xi$ be the PGM (1.1) with compact typical grain $\Xi_0$ satisfying $\mathbb{E}|\Xi_0|^s < \infty$ for some real $s \geq 2$. Further, let $W \subset \mathbb{R}^d$ be a bounded Borel set. Then there exists a positive
constants $c_s^{(1)}(\lambda)$ and $c_s^{(2)}(\lambda)$ (depending on $\lambda$ and the moments $E|\Xi_0|^k$, $k = 1, \ldots, [s], s$) such that, for $\varepsilon > 0$,

$$P( |\hat{p}_W - p| \geq \varepsilon ) \leq c_s^{(1)}(\lambda) \varepsilon^{-s} |W|^{-s/2} \quad \text{and} \quad P( |\hat{C}_W(x) - C(x)| \geq \varepsilon ) \leq c_s^{(2)}(\lambda) \varepsilon^{-s} |W|^{-s/2} \quad \text{for any} \quad x \in \mathbb{R}^d. \quad (4.4)$$

If $\Xi_0$ satisfies condition (1.3), then the following Bernstein-type inequality holds:

$$P( \hat{p}_W - p \geq \varepsilon ) \leq \begin{cases} \exp\left\{ -\frac{1 - \rho}{2 H(a)} \varepsilon^2 |W| \right\} & 0 \leq \varepsilon \leq \frac{H(a) \rho}{\Delta(a) (1 - \rho)} \\ \exp\left\{ -\frac{\rho}{\Delta(a)} \varepsilon |W| \right\} & \varepsilon \geq \frac{H(a) \rho}{\Delta(a) (1 - \rho)} \end{cases} \quad (4.6)$$

for any $0 < \rho < 1$ and $H(a), \Delta(a)$ from Theorem 1.

Proof of Theorem 3. For any integer $N \geq 2$, the $N$th moment of a random variable $Y$ can be expressed by its cumulants $\Gamma_k(Y), k = 1, \ldots, N$, (by inverting (2.4)) in the following way:

$$EY^N = \sum_{k=1}^{N} \frac{N!}{k!} \sum_{n_1 \geq 1, \ldots, n_k = N} \frac{\Gamma_{n_1}(Y)}{n_1!} \cdots \frac{\Gamma_{n_k}(Y)}{n_k!}. \quad (4.7)$$

Consider (4.7) for $Y = |\Xi \cap W| - p|W|$ with $p = E[\Xi \cap [0, 1)^d]$. Since $\Gamma_1(Y) = EY = 0$ and, by Lemma 1 combined with (3.14), $|\Gamma_n(Y)| \leq c_{1,n-1}(\lambda) |W|$ for $n = 2, \ldots, N$, we are led to

$$|EY^N| \leq N! \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{|W|^k}{k!} \sum_{n_1 \geq 2, \ldots, n_k \geq N} \frac{c_{1,n_1-1}(\lambda)}{n_1!} \cdots \frac{c_{1,n_k-1}(\lambda)}{n_k!} \leq c_N^{(1)}(\lambda) \max\{|W|, |W|^{N/2}\},$$

where $c_N^{(1)}(\lambda)$ depends on the first $N$ moments of $|\Xi_0|$. Hence, for an even integer $s \geq 2$, (4.4) follows from Chebyshev’s inequality. To prove (4.4) for any real $s \geq 2$ we next show

$$E\left| |\Xi \cap W| - p|W| \right|^s \leq c_s^{(1)}(\lambda) |W|^{s/2} \quad \text{for} \quad s \geq 2 \quad (4.8)$$

provided that $|W| \geq 1$. For this, we introduce the ‘truncated’ stationary PGM $\Xi_W := \bigcup_{t \geq 1} (\Xi_t^W + X_t)$ generated by $\Pi_\lambda = \sum_{t \geq 1} \delta_{X_t}$ and the typical grain

$$\Xi_0^W := \Xi_0 \cap B_R^W(\varepsilon),$$

where the random variable $R_0^W := \sup\{r > 0: |\Xi_0 \cap B_r(\varepsilon)| \leq |W|^\alpha \}$ takes the value $\infty$, if $|\Xi_0| < |W|^\alpha$. Here, we put $\alpha = N/2(N + 2 - s)$ if $N < s < N + 2$ for some even integer $N \geq 2$. 21
Define $Y_W := |\Xi_W \cap W| - |W| E[\Xi_W \cap [0,1]^d]$ and let $\tilde{\Xi}_W$ denote the PGM with typical grain $\Xi_0 \setminus \Xi_W^W$. Since $\tilde{\Xi}_W \subset \Xi$ and $\Xi \setminus \Xi_W \subset \tilde{\Xi}_W$ we have

$$|Y| \leq |Y_W| + \max\{|\tilde{\Xi}_W \cap W|, |W| E[\tilde{\Xi}_W \cap [0,1]^d]\}$$

which implies

$$(4.9) \quad E|Y|^s \leq 2^{s-1} E|Y_W|^s + 2^{s-1} E|\tilde{\Xi}_W \cap W|^s + 2^{s-1} \left(\lambda E[\Xi_0 \setminus \Xi_W^W ||W||^s]\right)^s.$$ 

By definition of $\Xi_W^W$, 

$$E[\Xi_0 \setminus \Xi_W^W]^k = E[\Xi_0 \cap B_R^W(0)]^k 1_{\{\Xi_0 \cap |W| \geq |\Xi_0|\}} \leq E[\Xi_0]^s |W|^{-\alpha s} \quad \text{for} \quad 0 < k \leq s$$.

Thus, $E[\Xi_0 \setminus \Xi_W^W ||W|| \leq E[\Xi_0]^s |W|^{1-\alpha s} \leq E[\Xi_0]^s \sqrt{|W|}$. Next, applying (5.3),

$$E[\tilde{\Xi}_W \cap W]^s \leq |W|^{s-N} E\left(\sum_{i \geq 1} \left((\Xi_0 \setminus \Xi_W^W) + X_i \right) \cap W\right)^N$$

$$\leq |W|^{s-N} N! \sum_{k=1}^{N} \frac{\lambda |W|^k}{k!} \sum_{n_1 + \ldots + n_k = N} \prod_{i=1}^{k} \frac{E[\Xi_0 \setminus \Xi_W^W]^{n_i}}{n_i!}$$

$$\leq c_1(N) |W|^{s-N} \max_{1 \leq k \leq N} \left\{\left(\lambda E[\Xi_0]^s\right)^k |W|^{k(1-\alpha s) + \alpha N}\right\}$$

$$\leq c_2(N, \lambda) |W|^{s/2}$$

Since, by Lyapunov’s inequality, $E|Y_W|^s \leq (E|Y_W|^{N+2})^{s/(N+2)}$, we only need to verify that $E|Y^W|^N \leq c_3(N, \lambda) |W|^{(N+2)/2}$ which in turn follows from (4.7) (with $Y_W$ and $N + 2$ instead of $Y$ and $N$) whenever $|\Gamma_{N+2}(|\Xi^W \cap W|)| \leq c_3(N, \lambda) |W|^{(N+2)/2}$. A thorough examination of the proofs of the Lemmas 5 - 7 reveals that the constant $c_{1,n}(\lambda)$ in (3.14) takes on the form

$$c_{1,n}(\lambda) = \lambda E[\Xi_0]^{n+1} + b_{1,n}^{(1)}(\lambda) E[\Xi_0]^n + b_{1,n}^{(2)}(\lambda),$$

where $b_{1,n}^{(1)}(\lambda)$ and $b_{1,n}^{(2)}(\lambda)$ are certain polynomials in $\exp\{\lambda E[\Xi_0]\}$ and the first $n-1$ moments of $|\Xi_0|$. Hence, by $E[\Xi_0]^{N+2} \leq |W|^{\alpha(N+2-s)} E[\Xi_0]^s$ and, we get the desired upper bound of $|\Gamma_{N+2}(|\Xi^W \cap W|)|$. Putting together the above estimates yields (4.8) and hence (4.4) is proved for any real $s \geq 2$.

To establish the second inequality (4.5) we use that $\Xi \cup (\Xi - x)$ is also a stationary PGM with typical grain $\Xi_0 \cup (\Xi_0 - x)$ and volume fraction $p(x) := P(o \in \Xi \cup (\Xi - x))$. In view of the obvious decomposition

$$\tilde{C}_W(x) - C(x) = \hat{N}_W - p + \frac{|(\Xi - x) \cap W|}{|W|} - p - \left(\frac{\left|(\Xi \cup (\Xi - x)) \cap W\right|}{|W|} - p(x)\right)$$
we obtain the inequality (4.5) by applying (4.4) to the three stationary PGM’s \( \Xi \), \( \Xi - x \) and \( \Xi \cup (\Xi - x) \). Finally, to prove the exponential inequality (4.6) we employ again a Chebyshev-type inequality. In this way we obtain, for \( \varepsilon \geq 0 \) and \( 0 \leq h \leq \rho/\Delta(a) \), that

\[
P(\hat{p}_W - p \geq \varepsilon) \leq \exp\{-h |W| (\varepsilon + p) + \log E e^{h |\Xi \cap W|}\}
\]

\[
\leq \exp\{-h |W| \varepsilon + \frac{h^2 H(a)}{2} |W| \sum_{k \geq 2} (h \Delta(a))^{-k} \}
\]

\[
\leq \exp\left\{-h |W| \varepsilon + \frac{h^2 H(a)}{2(1 - \rho)} |W| \right\}
\]

Taking \( h = \varepsilon (1 - \rho)/H(a) \) for \( 0 \leq \varepsilon \leq H(a)\rho/\Delta(a)(1 - \rho) \) proves the first part of (4.6), whereas the second part is obtained by setting \( h = \rho/\Delta(a) \) in the latter inequality. \( \Box \)

We conclude this section with a Chernoff-type theorem, see Dembo and Zeitouni [3] and references therein, for the empirical volume fraction \( \hat{p}_n \) which provides an extension and refinement of the relation (1.12) for the \( x \)-values \( x(\varepsilon) = \varepsilon \sqrt{|W_n|/\sigma_n} \) with \( \varepsilon \in (0, \varepsilon^*) \), where \( \varepsilon^* \) is determined by the slope of the function \( L(h) \) at \( h = 1/\Delta(a) \). This result touches the question whether \( \hat{p}_n \) satisfies the large deviation principle which seems to be unknown so far (to the best of the author’s knowledge). Without giving details we only mention that the limit \( \lim_{n \to \infty} L_n(h) \) exists on the negative real axis which can be shown by the methods of Chapt. 3.4 in [19]. For some related problems concerning large deviation principles for stationary independently marked Poisson processes, see Georgii and Zessin [5].

**Theorem 4.** Under the assumptions of Theorem 2 the following large deviation relation holds in the interval \( 0 \leq \varepsilon < \varepsilon^* := \lim_{h \to 1/\Delta(a)} L'(h) \) – \( p \):

\[
\lim_{n \to \infty} \frac{\log P(\hat{p}_n - p \geq \varepsilon)}{|W_n|} = \inf_{0 \leq h < 1/\Delta(a)} g(h) = g(h_0(\varepsilon)),
\]

where \( g(h) := L(h) - h (\varepsilon + p) \) and \( h_0(\varepsilon) \) is the unique solution of the equation \( L'(h) = \varepsilon + p \).

**Proof of Theorem 4.** As in the proof of Theorem 3, using (1.2) and the notation \( \hat{p}_n = \hat{p}_{W_n} \),

\[
P(\hat{p}_n - p \geq \varepsilon) \leq \exp\left\{|W_n| \left( L_n(h) - h (\varepsilon + p) \right)\right\}
\]

for any \( h \geq 0 \), whence, by virtue of Theorem 1, it follows that

\[
\limsup_{n \to \infty} \frac{1}{|W_n|} \log P(\hat{p}_n - p \geq \varepsilon) \leq g(h) \quad \text{for} \quad 0 \leq h < 1/\Delta(a).
\]
Thus, the limit on the left-hand side is bounded from above by
\[ \inf_{0 \leq h < 1/\Delta(a)} g(h). \]

Relation (4.10) is proved as soon as we show that
\[
\liminf_{n \to \infty} \frac{1}{|W_n|} \log \mathbb{P}(\hat{\rho}_n - p \geq \epsilon) \geq \inf_{0 \leq h < 1/\Delta(a)} g(h).
\]

For brevity, let \( \zeta_n(\epsilon) := |\Xi \cap W_n| - (\epsilon + p)|W_n| \). Then, for any \( \delta > 0 \) and \( h \geq 0 \),
\[
P(\hat{\rho}_n - p \geq \epsilon) \geq P(\hat{\rho}_n - p \in (\epsilon, \epsilon + \delta])
\]

(4.12)
\[
\geq \exp\left\{ |W_n| \left( L_n(h) - h(\epsilon + p + \delta) \right) \right\} \frac{\mathbb{E}\exp\{h \zeta_n(\epsilon)\} 1_{\{\zeta_n(\epsilon) \in [0, \delta|W_n|]\}}}{\mathbb{E}\exp\{h \zeta_n(\epsilon)\}}.
\]

Due to the properties of cumulant-generating functions, see e.g. [3], p. 27, the functions \( L_n \), \( n \geq 1 \), are convex on the whole real axis and \( L''_n(h) > 0 \) for every \( h \in \mathbb{R}^1 \) and \( n \geq n_0 \) with \( n_0 \) from Theorem 2. In view of Theorem 1, \( L'_n(h) \to h \) for \( h \to h_0 \) and both \( L'_n(h) \) (for \( n \geq n_0 \)) and \( L(h) \) are strictly increasing for \( 0 \leq h < 1/\Delta(a) \). Likewise, \( L''_n(h) \to L''(h) \) with \( L''(h) \geq 0 \) for \( 0 \leq h < 1/\Delta(a) \). Hence, for each \( \epsilon \in [0, \epsilon^*] \) and sufficiently large \( n \), there exists a unique \( h_n = h_n(\epsilon) \in [0, 1/\Delta(a)] \) satisfying the equation \( L'_n(h_n) = \epsilon + p \). Moreover, we have \( h_n \to h_0 \), where \( h_0 = h_0(\epsilon) \) is the unique solution of \( L'(h) = \epsilon + p \). Since \( g(h) \) is a convex function and \( g'(h_0) = 0 \), it follows that \( g(h) = \inf_{0 \leq h < 1/\Delta(a)} g(h) \). Consequently, putting \( h = h_n \) on the right-hand side of (4.12) and taking into account that \( L_n(h_n) \to L(h_0) \), we arrive at
\[
\liminf_{n \to \infty} \frac{\log \mathbb{P}(\hat{\rho}_n - p \geq \epsilon)}{|W_n|} \geq g(h_0) - h_0 \delta + \liminf_{n \to \infty} \frac{1}{|W_n|} \log \left( G_n(\delta|W_n|) - G_n(0) \right),
\]

where the distribution function \( G_n(x) = \mathbb{E}\exp\{h_n \zeta_n(\epsilon)\} 1_{\{\zeta_n(\epsilon) \leq x\}}/\mathbb{E}\exp\{h_n \zeta_n(\epsilon)\} \) possesses the Fourier-Stieltjes transform \( \hat{G}_n(t) = \mathbb{E}\exp\{(it + h_n) \zeta_n(\epsilon)\}/\mathbb{E}\exp\{h_n \zeta_n(\epsilon)\} \). Using (1.2) and \( L'_n(h_n) = \epsilon + p \) we may write
\[
\log \hat{G}_n(t) = |W_n| \left( -it(\epsilon + p) + L_n(it + h_n) - L_n(h_n) \right)
\]
\[
= - |W_n| t^2 \int_0^1 (1 - \vartheta) L''_n(i\vartheta t + h_n) \, d\vartheta, \quad \text{if} \quad |t| + h_n < 1/\Delta(a),
\]

where the last line is obtained by partial integration of \( L''_n(it\vartheta + h_n) \) with respect to \( \vartheta \in [0, 1] \).

An application of Theorem 1 shows that \( \log \hat{G}_n(t/\sqrt{|W_n|}) \to -t^2 L''(h_0)/2 \) for all \( t \in \mathbb{R}^1 \) which in turn implies \( G_n(x/\sqrt{|W_n|}) \to \Phi(x/\sqrt{L''(h_0)}) \) provided that \( L''(h_0) > 0 \). In this case, \( G_n(\delta|W_n|) - G_n(0) \to 1/2 \) proving (4.11) and, thus, the desired relation (4.10) holds. If
$L''(h_0(\varepsilon_0)) = 0$ for certain $\varepsilon_0 \in (0, \varepsilon^*)$, then there exists some $\eta > 0$ such that $L''(h_0(\varepsilon)) > 0$ for $\varepsilon \in [\varepsilon_0 - \eta, \varepsilon_0 + \eta] \setminus \{\varepsilon_0\}$. Since $\log P(\hat{p}_n - p \geq \varepsilon)$ is non-increasing in $\varepsilon$, it follows that
\[
g(h_0(\varepsilon_0 + \eta)) \leq \liminf_{n \to \infty} \frac{\log P(\hat{p}_n - p \geq \varepsilon_0)}{|W_n|} \leq \limsup_{n \to \infty} \frac{\log P(\hat{p}_n - p \geq \varepsilon_0)}{|W_n|} \leq \ g(h_0(\varepsilon_0 - \eta)) .
\]
Hence, having in mind the continuity of $h_0(\cdot)$, the proof of Theorem 4 is finished. \(\square\)

5. Appendix

For any random closed set $\Xi$ (defined as $(\mathfrak{A}, \sigma_f)$-measurable mapping $\Xi : [\Omega, \mathfrak{A}, P] \to [\mathcal{F}, \sigma_f]$, where $\mathcal{F}$ is the (metrizable) space of all closed set in $\mathbb{R}^d$ and $\sigma_f$ its Matheron $\sigma$-field), the mapping $(x, \omega) \mapsto 1_{\Xi(\omega)}(x)$ is $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{A}$-measurable since $(x, F) \mapsto 1_F(x)$ is $\mathfrak{B}(\mathbb{R}^d) \otimes \sigma_f$ measurable, see Chap.2 in [14]. As we will see below, the PGM (1.1) is no longer $\mathcal{F}$ a.s. closed in $\mathbb{R}^d$ if $E[\Xi_0] < \infty$ and $E[\Xi_0 + B_\varepsilon(o)] = \infty$ for any $\varepsilon > 0$.

In order to preserve the $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{A}$-measurability of the indicator function $1_{\Xi(\omega)}(x)$ (needed to apply Fubini’s theorem), we define the not necessarily closed PGM (1.1) as set-valued, measurable mapping of an independently marked Poisson process $\Pi_{\lambda, Q} := \sum i \in 1 \delta_{[X_i, \Xi_i]}$ with mark distribution $Q(\mathcal{L}) = P(\Xi_0 \in \mathcal{L}), \mathcal{L} \in \mathfrak{B}(\mathcal{K})$.

More precisely, let $M_{\mathcal{K}}$ denote the space of all integer-valued measures $\psi$ on $[\mathbb{R}^d \times \mathcal{K}, \mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{B}(\mathcal{K})]$ satisfying $\psi(B \times \mathcal{K}) < \infty$ for each bounded $B \in \mathfrak{B}(\mathbb{R}^d)$ and let $\mathfrak{M}_{\mathcal{K}}$ be the $\sigma$-field generated by the sets $\{\psi \in M_{\mathcal{K}} : \psi(B \times \mathcal{L}) = n\}$ for $n \geq 0$, bounded $B \in \mathfrak{B}(\mathbb{R}^d)$ and $\mathcal{L} \in \mathfrak{B}(\mathcal{K})$.

Each $\psi \in M_{\mathcal{K}}$ admits a representation $\psi = \sum_{i \geq 1} \delta_{[X_i(\psi), k_i(\psi)]}$ as sum of Dirac measures with respect to the at most countable set of atoms $[x_i(\psi), k_i(\psi)]$, $i \geq 1$, where each atom is counted according to its multiplicity. Note that the mappings $M_{\mathcal{K}} \ni \psi \mapsto [x_i(\psi), k_i(\psi)] \in \mathbb{R}^d \times \mathcal{K}$ are measurable, see [15]. Finally, define
\[
\Xi(\psi) := \bigcup_{i \geq 1} \left( x_i(\psi) + k_i(\psi) \right) .
\]

**Proposition 1.** The mapping $(x, \psi) \mapsto \xi(x, \psi) := 1_{\Xi(\psi)}(x)$ is $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{M}_{\mathcal{K}}$-measurable, that is $(x, \psi) \in \mathbb{R}^d \times M_{\mathcal{K}} : x \notin \Xi(\psi)$ is in $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{M}_{\mathcal{K}}$.

**Proof of Proposition 1.** Let $\mathcal{B}$ be the countable set of open balls in $\mathbb{R}^d$ having rational radii and midpoints with rational coordinates. For any sequence $\{K_n, n \geq 1\}$ in $\mathcal{K}$ satisfying $K_n \uparrow \mathbb{R}^d$ put
\[ \Xi_n(\psi) := \bigcup_{i \in \mathcal{K}} \{ x_i(\psi) + k_i(\psi) \}. \] Obviously, \( \Xi_n(\psi) \in \mathcal{K} \) and \( \Xi(\psi) = \bigcup_{n \geq 1} \Xi_n(\psi) \).

It is readily checked that the set \( \xi^{-1}(\{0\}) = \{ (x, \psi) \in R^d \times M_K : x \notin \Xi(\psi) \} \) coincide with \( \bigcap_{n \geq 1} \bigcup_{B \in B}(B \times \{ \psi \in M_K : \Xi_n(\psi) \cap B = \emptyset \}) \). But this set belongs to \( \mathfrak{B}(R^d) \otimes \mathfrak{M}_K \) if \( \{ \psi : \Xi_n(\psi) \cap B = \emptyset \} = \{ \psi : \sum_{i \geq 1} 1_{R^d \cap (K \times K)}(x_i(\psi), k_i(\psi)) = 0 \} \) in \( \mathfrak{M}_K \) for any \( n \geq 1 \) and \( B \in B \), where \( \mathcal{R}_B := \{ (x, K) \in R^d \times \mathcal{K} : (x + K) \cap B \neq \emptyset \} \). Since the mapping \( \psi \mapsto \sum_{i \geq 1} f(x_i(\psi), k_i(\psi)) \) is \( \mathfrak{M}_K \)-measurable whenever \( f : R^d \times \mathcal{K} \mapsto R^d \) is \( \mathfrak{B}(R^d) \otimes \mathfrak{B}(\mathcal{K}) \)-measurable, see [15], we only need to verify that \( \mathcal{R}_B \in \mathfrak{B}(R^d) \otimes \mathfrak{B}(\mathcal{K}) \) for \( B \in B \).

Since there exists a sequence of closed balls \( B_n \) such that \( B_n \uparrow B \) and thus \( \mathcal{R}_{B_n} \uparrow \mathcal{R}_B \), it suffices to show \( \mathcal{R}_C \in \mathfrak{B}(R^d) \otimes \mathfrak{B}(\mathcal{K}) \) for any closed ball \( C \). For this, remember that \( \mathcal{K}^D := \{ K \in \mathcal{K} : K \cap D = \emptyset \} \in \mathfrak{B}(\mathcal{K}) \) for any open ball \( D \), see [14]. Thus, the proof of Proposition 1 is completed by noting that

\[ \mathcal{R}_C = \{ (x, K) \in R^d \times \mathcal{K} : K \cap (C - x) = \emptyset \} = \bigcup_{B \in B} (B \times \mathcal{K}^{D(B,C)}) \]

where \( D(B,C) = \{ u - v : u \in C, v \in B \} \) is an open ball for \( B \in B \). □

Let there be given an unmarked point process \( \Psi = \sum_{i \geq 1} \delta_{X_i} \) on \( R^d \) and a random compact set \( \Xi_0 \) with distribution \( Q = P \circ \Xi_0^{-1} \) on \( \mathcal{K}, \mathfrak{B}(\mathcal{K}) \). The corresponding independently marked point process \( \Psi_Q = \sum_{i \geq 1} \delta_{[X_i, \Xi_i]} \) on \( R^d \) with mark distribution \( Q \) is then defined, see [2] or [22], to be a random element (over \( [\Omega, \mathfrak{A}, \mathbb{P}] \)) taking values in \( [M_K, \mathfrak{M}_K] \), the distribution of which is uniquely determined by its probability generating functional \( G_{\Psi_Q}[v] = E\left( \prod_{i \geq 1} v(X_i, \Xi_i) \right) = G_{\Psi}[v_Q] \) (see Sect. 2), where \( v_Q(.) := f_K(v(., K) Q(dK)) \) and the function \( v : R^d \times \mathcal{K} \mapsto [0, 1] \) is Borel measurable such that \( 1 - v(., K) \) has bounded support for all \( K \in \mathcal{K} \). In the special case of a stationary independently marked Poisson process \( \Pi_{\lambda, Q} = \sum_{i \geq 1} \delta_{[X_i, \Xi_i]} \) the shape of \( G_{\Pi_{\lambda, Q}}[\cdot] \), see e.g. [2] or [22], implies that

\[
G_{\Pi_{\lambda, Q}}[v] = \exp\left\{ \lambda \int_{R^d} \int_{\mathcal{K}} (v(x, K) - 1) Q(dK) \, dx \right\}
\]

and, furthermore, the following Campbell-type formula holds:

\[
E\left( \sum_{i_1, \ldots, i_k \geq 1} \prod_{j=1}^{k} f_j(X_{i_j}, \Xi_{i_j}) \right) = \lambda^k \prod_{j=1}^{k} \int_{R^d} \int_{\mathcal{K}} f_j(x, K) Q(dK) \, dx
\]

for any measurable functions \( f_1, \ldots, f_k : R^d \times \mathcal{K} \mapsto [0, \infty] \), where the sum \( \sum^* \) on the left-hand side of (5.3) stretches over \( k \)-tuples of pairwise distinct indices.
As announced we conclude the Appendix by showing that, under the assumption $E|\Xi_0| < \infty$, the condition $E|\Xi_0 + B_\varepsilon(o)| < \infty$ for some $\varepsilon > 0$ is not only sufficient as shown in [10], but even necessary for the closedness of the stationary PGM $\Xi = \Xi(\Pi_{\lambda,Q})$.

**Proposition 2.** Let $\Xi_0$ be a compact typical grain of the PGM (1.1) satisfying $E|\Xi_0| < \infty$ and $E|\Xi_0 + B_\varepsilon(o)| = \infty$ for any $\varepsilon > 0$. Then $P(\Xi$ is closed in $R^d) = 0$.

**Proof of Proposition 2.** Choose $K_n \in \mathcal{K}$, $n \geq 1$, such that $K_n \uparrow R^d$ as $n \to \infty$ and let $\Xi_n(\psi)$ be defined as in the proof of Proposition 1.

Obviously, $\{\psi \in M_\mathcal{K} : \Xi_n(\psi) \cap B_\varepsilon(o) = \emptyset\} \downarrow \{\psi \in M_\mathcal{K} : \Xi(\psi) \cap B_\varepsilon(o) = \emptyset\}$ as $n \to \infty$.

Further, since $(\Xi_i + X_i) \cap B_\varepsilon(o) \neq \emptyset$ iff $-X_i \in \Xi_i + B_\varepsilon(o)$, we find using (5.2) that

$$P(\Xi_n(\Pi_{\lambda,Q}) \cap B_\varepsilon(o) = \emptyset) = E \prod_{i \geq 1} \left(1 - 1_{K_n}(X_i) 1_{\Xi_i+B_\varepsilon(o)}(-X_i)\right)$$

$$= \exp\{-\lambda E[(-K_n) \cap (\Xi_0 + B_\varepsilon(o))]\}.$$

By the monotone convergence theorem and our assumptions,

$$\lim_{n \to \infty} E[(-K_n) \cap (\Xi_0 + B_\varepsilon(o))] = E[\Xi_0 + B_\varepsilon(o)] = \infty$$

which means for the stationary PGM $\Xi = \Xi(\Pi_{\lambda,Q})$ that

$$P(\Xi \cap B_\varepsilon(o) \neq \emptyset) = 1 - \lim_{n \to \infty} P(\Xi_n(\Pi_{\lambda,Q}) \cap B_\varepsilon(o) = \emptyset)) = 1 \text{ for any } \varepsilon > 0.$$

Thus,

$$P(\Xi \text{ is closed}) = P(\Xi \text{ is closed, } \cap_{m \geq 1} \{\Xi \cap B_1/m(o) \neq \emptyset\}) \leq P(o \in \Xi) = 1 - \exp\{-\lambda E[\Xi_0]\} < 1.$$

Similarly, $P(\Xi \text{ is closed}) \leq P(x_1 \in \Xi, \ldots, x_n \in \Xi)$ for any $x_1, \ldots, x_n \in R^d$. In view of (1.4), (1.5) and (1.6) the probability $p^{(n)}_{\Xi}(x_1, \ldots, x_n)$ is arbitrarily close to $(P(o \in \Xi))^n$ whenever the distances between the points $x_1, \ldots, x_n$ are sufficiently large. This proves the assertion of Proposition 2. $\square$. 

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6. Supplements and Relations to Statistical Physics

1. Correlation measures and higher-order Palm moment measures

We start with an alternative derivation of relation (2.8). By the definition of the higher-order factorial moment measures \( \alpha_{k}^{[m]} \), \( k \geq 1 \) of a point process \( \Psi = \sum_{i \geq 1} \delta_{X_{i}} \) via probability generating functions, see e.g. [2], we obtain the expansion

\[
E \left( \prod_{i=1}^{m} \Psi(A_{i}) \prod_{j=1}^{k} z_{j}^{\Psi(B_{j})} \right) = \sum_{k} \left( \frac{z_{1} \ldots z_{k}-1}{\nu_{1} \ldots \nu_{k}} \right) \alpha_{k}^{[m]} \left( \sum_{i=1}^{m} A_{i} \times B_{1}^{\nu_{1}} \times \ldots \times B_{n}^{\nu_{k}} \right)
\]

for fixed \( k, m \geq 0 \), \( 0 \leq z_{1}, \ldots, z_{k} \leq 1 \) and pairwise disjoint, bounded \( A_{1}, \ldots, A_{m}, B_{1}, \ldots, B_{n} \in \mathfrak{M}(R^{d}) \). Here and in the next formula sum \( \sum_{k} \) runs over all \( k \)-tuples of non-negative integers \( \nu_{1}, \ldots, \nu_{k} \) with \( |\nu| := \nu_{1} + \ldots + \nu_{k} \). Analogously, by the definition of the correlation measures \( \gamma_{k}^{[m,n]} \) given in Section 2,

\[
E \left( \prod_{i=1}^{m} \Psi(A_{i}) \prod_{j=1}^{k} z_{j}^{\Psi(B_{j})} \right) = \sum_{k} \left( \frac{z_{1} \ldots z_{k}-1}{\nu_{1} \ldots \nu_{k}} \right) \gamma_{k}^{[m,n]} \left( \sum_{i=1}^{m} A_{i} \times B_{1}^{\nu_{1}} \times \ldots \times B_{n}^{\nu_{k}} \right),
\]

whence by multiplying the the last equality by \( E \left( \prod_{i=1}^{m} \Psi(B_{i}) \right) \) and comparing the coefficients, it follows that

\[
(6.1) \quad \alpha_{k}^{[m]} \left( A^{[m]} \times B^{\nu} \right) = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda! \mu!} \alpha_{k}^{[\lambda]} \left( B^{\lambda} \right) \gamma_{k}^{[m,\nu]} \left( A^{[m]} \times B^{\nu} \right)
\]

with the abbreviations \( A^{[m]} = \sum_{i=1}^{m} A_{i} \), \( B^{\nu} = \sum_{j=1}^{k} B_{j}^{\nu_{j}} \) and \( \nu! = \nu_{1} \ldots \nu_{k} \).

Let \( P_{X_{m}}^{r} \) denote the \( r \)-th order reduced Palm distribution (w.r.t. \( X_{m} = \{ x_{1}, \ldots, x_{m} \} \)) of \( \Psi \) acting on the measurable space \( [M, \mathfrak{M}] \) of locally finite counting measures \( \varphi \) on \( R^{d} \), for details see O. Kallenberg, Random measures, Academic Press, London (1986). Define

\[
\alpha_{k}^{[\nu]} \left( B^{\nu} | X_{m} \right) := \int_{M} \prod_{j=1}^{k} \varphi(B_{j})(\varphi(B_{j}) - 1) \cdots \varphi(B_{j})(\varphi(B_{j}) - \nu_{j} + 1) \cdot P_{X_{m}}^{\nu} \left( d\varphi \right)
\]

\[
= \lim_{\varepsilon_{1}, \ldots, \varepsilon_{m} \downarrow 0} \frac{\alpha_{k}^{[m+r]} \left( \sum_{i=1}^{m} B_{i} \left( x_{i} \right) \times B^{\nu} \right)}{\alpha_{k}^{[m]} \left( \sum_{i=1}^{m} B_{i} \left( x_{i} \right) \right)}
\]

where the limit in the last line should be understood in the sense of a Radon-Nikodym derivative for \( \alpha_{k}^{[m]} \) - almost every \( X_{m} \in R^{dm} \). In the same manner we may define \( \gamma_{k}^{[\nu]} \left( B^{\nu} | X_{m} \right) \) simply by substituting the numerator \( \alpha_{k}^{[m+r]} \) by \( \gamma_{k}^{[m+\nu]} \). Using (6.1) we are led to the recursive relation

\[
(6.2) \quad \gamma_{k}^{[\nu]} \left( B^{\nu} | X_{m} \right) = \alpha_{k}^{[\nu]} \left( B^{\nu} | X_{m} \right) - \sum_{\lambda+\mu=\nu, \mu \neq \nu} \frac{\nu!}{\lambda! \mu!} \alpha_{k}^{[\lambda]} \left( B^{\lambda} \right) \gamma_{k}^{[\nu]} \left( B^{\nu} | X_{m} \right).
\]

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2. Connections with Kirkwood-Salzburg equations

In view of (1.6) it is easily seen that

$$E \exp \{ z | \Xi^c \cap W_n | \} = \sum_{k \geq 0} \frac{z^k}{k!} E | \Xi^c \cap W_n |^k = 1 + \sum_{k \geq 1} \frac{z^k}{k!} \int_{W_n} e^{-\lambda U(x_1, \ldots, x_k)} d(x_1, \ldots, x_k),$$

where $U(x_1, \ldots, x_k) := E | \cup_{i=1}^k (\Xi_0 - x_k) | = \sum_{m=1}^k \sum_{1 \leq i_1 < \cdots < i_m \leq k} V_m(x_{i_1}, \ldots, x_{i_m})$. From the viewpoint of statistical physics the function $U(x_1, \ldots, x_k)$ may be interpreted as the potential energy of $k$ particles located at $x_1, \ldots, x_k$ which is determined by the $m$-body potentials $V_m(x_1, \ldots, x_m) = (-1)^{m-1} E \cap_{j=1}^m (\Xi_0 - x_j)$. In this context, $Z_n(\lambda, z) := E \exp \{ z | \Xi^c \cap W_n | \}$ for $z \in R^1$ is called grand canonical partition function of the continuous system of point particles.

Obviously, it is more convenient to investigate the limit of $L_n^c(z) := \log E \exp \{ z | \Xi^c \cap W_n | \} / |W_n|$ rather than to treat the sequence $L_n(z) = z + L_n^c(-z)$ directly. By logarithmic differentiation we obtain

$$(6.3) \quad \frac{d}{dz} L_n^c(z) = \frac{1}{|W_n|} \int_{W_n} \frac{1}{Z_n(\lambda, z)} \sum_{k \geq 0} \frac{z^k}{k!} \int_{W_n^k} e^{-\lambda U(x_1, y_1, \ldots, y_k)} d(y_1, \ldots, y_k) \, dx_1$$

$$= \frac{1}{|W_n|} \int_{W_n} z^{-1} \rho_n^{(x)}(x_1) \, dx_1,$$

where $\rho_n^{(x)}(x_1)$ is the first one of the functions $\rho_n^{(x)}(x_1, \ldots, x_m)$ defined by

$$\rho_n^{(x)}(X_m) = \prod_{i=1}^m \frac{1}{Z_n(\lambda, z)} \sum_{k \geq 0} \frac{z^{m+k}}{k!} \int_{W_n^k} e^{-\lambda U(X_m \cup Y_k)} dY_k, \quad X_m = \{ x_1, \ldots, x_m \}$$

for $m \geq 1$. $\rho_n^{(x)}(X_m)$ is called finite volume correlation function of order $m$ and $z^{-m} \rho_n^{(x)}(X_m)$ may be interpreted as the probability density of finding $m$ different particles at the positions $x_1, \ldots, x_m$. Paraphrasing the proof of Theorem 1.2 in Greenberg [7], we obtain the following system of integral equations - the so-called Kirkwood-Salzburg equations, see also [19] :

$$(6.4) \quad \rho_n^{(x)}(X_m) = \sum_{k \geq 0} \frac{z(-1)^k}{k!} \int_{W_n^k} \prod_{i=1}^m 1_{W_n}(x_i) K(X_m, Y_k) \rho_n^{(x)}(X_{m-1} \cup Y_k) \, dY_k$$

for $m \geq 1$ and $\rho_n^{(x)}(X_0) := 1$, where $X_0 = \emptyset$, $X_{m-1} = X_m \setminus \{ x_1 \}$ and $K(\cdot, \cdot)$ is given by (3.3).

Next define the linear operator $K_n$ by

$$(K_n g)(X_m) := \left\{ \begin{array}{ll} 0 & \text{if } m = 0 \\ \sum_{k \geq 0} \frac{(-1)^k}{k!} \int_{W_n^k} K(X_m, Y_k) g_{m-1+k}(X_{m-1} \cup Y_k) \, dY_k & \text{if } m \geq 1 \end{array} \right.$$
\[ \|g\|_\eta := \sup_{m \geq 0} \{ \eta^{-m} \text{ess sup}_{X_m} |g_m(X_m)| \} < \infty \text{ for some } \eta > 0. \]

By means of \( K_n \) the equations (6.4) can be shortly written as a single equation in \( E_\eta \):

\[ (6.5) \quad \rho_n^{(z)} = z K_n \rho_n^{(z)} + \alpha \text{ with } \rho_n^{(z)} = (\rho_n^{(z)}(X_m))_{m \geq 0}, \quad \alpha = (1, 0, 0, \ldots). \]

If (1.3) is satisfied and \( \eta < 4/\Delta(a) \), then, by Lemma 6, the linear operator \( K_n : E_\eta \to E_\eta \)

is bounded with norm \( \|K_n\|_{L(E_\eta)} \leq 4/\eta(4 - \eta \Delta(a)) \). Therefore, putting \( \eta = 2/\Delta(a) \), we get \( |z| \|K_n\|_{L(E_\eta)} < 1 \) on the disk \( D_{\Delta(a)} = \{ z \in \mathbb{C}^1 : |z| < 1/\Delta(a) \} \). Thus, solving (6.5)

yields the \( E_\eta \)-valued analytic function \( \rho_n^{(z)} = (I - z K_n)^{-1} \alpha = \sum_{k \geq 0} \frac{z^k}{k!} K_n^k \alpha \) on \( D_{\Delta(a)} \), each

component of which is uniformly bounded (for all \( n \geq 1 \)). The same holds true for the limit \( \rho^{(z)} := (\rho^{(z)}(X_m))_{m \geq 0} = \lim_{n \to \infty} \rho_n^{(z)} \) (called correlation functional) in D. Ruelle, J. Math.

Physics 6, 201 - 220). In particular, \( \rho^{(z)}(x_1) = \lim_{n \to \infty} \rho_n^{(z)}(x_1) \) is analytic on \( D_{\Delta(a)} \) and does not depend on \( x_1 \in \mathbb{R}^d \). It takes the form

\[ \rho^{(z)}(x_1) = \sum_{k \geq 0} \frac{z^{k+1}}{k!} \int_{\mathbb{R}^{d_k}} c^{(k+1)}(a, y_1, \ldots, y_k) \, d(y_1, \ldots, y_k) = \sum_{k \geq 0} \frac{z^{k+1}}{k!} \int_{\mathbb{R}^{d_k}} c(a, Y_k) \, dY_k \]

for all \( x_1 \in \mathbb{R}^d \). Moreover, for \( m \geq 1 \),

\[ \rho^{(z)}(X_m) = \sum_{k \geq 0} \frac{z^{k+m}}{k!} \int_{\mathbb{R}^{d_k}} c(X_m, Y_k) \, dY_k = z^m p^{(m)}(X_m) \sum_{k \geq 0} \frac{z^k}{k!} \gamma^{(k)}_{\Pi^{(1)}}(\mathbb{R}^{d_k}|X_m). \]

Therefore, the limit of (6.3) (as \( n \to \infty \)) exists and is analytic which immediately implies the existence and analyticity of the thermodynamic limit \( L^c(z) = \lim_{n \to \infty} L^c_n(z) \) for \( |z| < 1/\Delta(a) \).

3. The thermodynamic limit of \( L_n(z) \) for a scaled PGM

For any \( \kappa > 0 \) define a stationary PGM \( \Xi^{(\kappa)} \) (derived from (1.1)) generated by the scaled typical grain \( \Xi^{(\kappa)}_0 := \kappa^{-1/d} \Xi_0 \) and the intensity \( \lambda^{(\kappa)} = \kappa \lambda \). Since \( \lambda^{(\kappa)} E|\Xi^{(\kappa)}_0| = \lambda E|\Xi_0| \),

the volume fraction \( E|\Xi^{(\kappa)} \cap [0, 1]^d| \) is constant equal to \( p \). Furthermore, it rapidly seen that \( c^{(k)}_{\Xi^{(\kappa)}}(x_1, \ldots, x_k) = c^{(k)}_{\Xi}(\kappa^{1/d}x_1, \ldots, \kappa^{1/d}x_k) \) for \( k \geq 2 \) implying

\[ |\Gamma_k(|\Xi^{(\kappa)} \cap W_n|)| \leq \frac{|W_n|}{k-1} \int_{\mathbb{R}^{d(k-1)}} |c^{(k)}_{\Xi}(a, x_2, \ldots, x_k)| \, d(x_2, \ldots, x_k), \]

if \( E|\Xi_0|^k < \infty \) and, therefore,

\[ \lim_{n \to \infty} \frac{\log E \exp\{z |\Xi^{(\kappa)} \cap W_n|\}}{|W_n|} = \kappa L(z/\kappa) \quad \text{for} \quad |z| < \kappa/\Delta(a) \]

provided (1.3) is satisfied.