KERNEL ESTIMATION OF THE DIAMETER DISTRIBUTION IN BOOLEAN MODELS WITH SPHERICAL GRAINS

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We investigate asymptotic properties including MSE of kernel estimators of the second-order product density $\rho_d(x)$ of the point process of 'exposed tangent points' (for given direction $u \in S^{d-1}$) associated with a stationary $d$-dimensional Boolean model with convex compact grains. Under minimal conditions on the typical grain we prove that the square root of the kernel estimators is asymptotically normally distributed with constant variance which only depends on the chosen kernel function. In the particular case of spherical grains, as first shown by Molchanov and Stoyan (1994), the diameter distribution function $F(t)$ is just equal to the product of $\rho_d(u)$ and some function of $t \geq 0$ which can be estimated by standard methods. Using this fact we are able to derive a multivariate CLT for a suitably defined empirical diameter distribution function. Owing to this result we suggest a $\chi^2$-goodness-of-fit-test for testing a hypothetical diameter distribution.

Keywords: Boolean model with convex compact grains; point process of exposed tangent points; kernel product density estimator; mean squared error; asymptotic normality; spherical typical grain; empirical diameter distribution; $\chi^2$-goodness-of-fit-test

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1. INTRODUCTION

The Boolean model (BM) is the best studied and mostly used mathematical model to describe irregularly shaped sets in a Euclidean space $\mathbb{R}^d$, $d \geq 1$, consisting of randomly distributed clumps, see

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Matheron [14]; Serra [20]; Hall [6] and Stoyan et al. [21]. In most cases statistical analysis of a (stationary) BM is based on a single observation of the union set of clumps in some sampling window $W_n$ which is assumed to expand unboundedly in all directions. In this paper we are concerned with non-parametric statistics for some class of stationary BM's. In avoiding technical problems, let $W_n$ be the $d$-dimensional cube $[0,n]^d$. To be definite in describing our problem, we first give a rigorous definition of a stationary BM as to be the union set

$$\Xi(\lambda, Q) = \bigcup_{i \geq 1} (\Xi_i + X_i)$$

of independent copies $\Xi_1, \Xi_2, \ldots$ (grains) of a random compact set $\Xi_0$ (typical grain) having the distribution $Q$, where the grains are independently shifted by the atoms $X_1, X_2, \ldots$ (germ points) of a stationary Poisson process $\Pi = \sum_{i \geq 1} \delta_{X_i}$ with intensity $\lambda (= \text{mean number of germ points in the unit cube } [0,1]^d)$.

Here and throughout in this paper all random $\xi$ elements are defined on a common probability space $[\Omega, \mathcal{F}, P]$ and $E$, $\text{Var}$ resp. $\text{Cov}$ denote the expectation, variance resp. covariance w.r.t. $P$. In particular, $\Xi_0$ is a measurable mapping from $[\Omega, \mathcal{F}, P]$ into the Polish space of non-void compact subsets $\mathcal{K}$ of $\mathbb{R}^d$ equipped with the Hausdorff metric and $Q$ coincides with the image measure $P \circ \Xi_0^{-1}$ acting on the corresponding Borel subsets of $\mathcal{K}$, see Matheron [14]. Note that $\Xi(\lambda, Q)$ is a closed set ($P$-a.s.) if $E|\Xi_0 \oplus b(o, r)| < \infty$ for $r > 0$. Here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$, $\oplus$ stands for the Minkowski addition and $b(o, r) = \{y \in \mathbb{R}^d : ||y|| \leq r\}$ is the closed ball with radius $r$ centered at the origin $o = (0, \ldots, 0)$. For $x \neq o$ put $b(x, r) = b(o, r) + x$. For any convex $K \in \mathcal{K}$, the well-known Steiner formula provides the expansion $|K \oplus b(o, r)| = \sum_{k=0}^{d} \binom{d}{k} W_k^{(d)}(K)r^k$, where $W_k^{(d)}(K)$ denotes the $k$-th Minkowski functional ('Quermassintegral') of $K$, $k = 0, 1, \ldots, d$, see Stoyan et al. [21]. Note that $W_0^{(d)}(K) = |b(o, 1)| = \text{const.}, W_0^{(d)}(K) = |K|$ and in the planar case $W_1^{(2)}(K) = \text{(perimeter of } K)/2$. The basic assumption throughout this paper is

**Condition (B)** Let the typical grain $\Xi_0$ be $P$-a.s. convex and compact such that

$$E W_k^{(d)}(\Xi_0) < \infty \quad \text{for } k = 0, 1, \ldots, d - 1.$$
Basically, statistics of the BM (1.1) is aimed at estimating the unknown intensity $\lambda$ of the driving Poisson process and the distribution $Q$ of the typical grain $\Xi_0$ or at least some of its numerical characteristics, see Stoyan et al. [21]; Molchanov [18] and Cressie [3]. The effect of overlapping prevents to gain information from the individual grains directly, see Figure 2 below. On the other hand, since most of the 'naive' characteristics of the random closed set (1.1) such as volume fraction $p = P(o \in \Xi(\lambda, Q)) = E|\Xi \cap [0, 1]^d|$, the covariance $C(x) = P(o \in \Xi(\lambda, Q), x \in \Xi(\lambda, Q))$ and various types of contact distributions can be calculated explicitly in terms of $\lambda$ and mean Minkowski functionals of the typical grain $\Xi_0$ and since, moreover, their empirical counterparts $\hat{p}_n, \hat{C}_n(x)$ etc. (see (2.10), (2.11)) can be determined comparatively easily, various methods were developed to obtain estimates of the vector $\lambda, E W_k^{(d)}(\Xi_0), k = 0, 1, \ldots, d - 1$. For example, the \textit{method of intensities} or \textit{Weil's method}, see Weil [22], [18] (pp. 82/83), [21] and the \textit{minimum contrast method}, see Diggle [4]; Heinrich [9], [18] (pp. 80/81). This methods are also applicable to estimate those parameters of the distribution $Q$ which can be expressed as (continuous)
functions of the mean Minkowski functionals of $\Xi_0$. If only the unknown intensity $\lambda$ is of interest, the *method of Schmitt*, see [19], or *Laslett’s transform*, see [3], provide suitable estimates.

In the special case of a spherical grain $\Xi_0 = b(o, R_0)$ there exist several approaches to estimate the parameters of a known distribution function (DF) of $R_0$, for example by the above-mentioned minimum contrast method.

There are also attempts at non-parametric estimating (at least partially) of the radius or diameter DF from a single observation of (1.1) in $W_n$, see Chapt. 5.6 in Hall [6]; Molchanov [15] and Bortnik [2]. Using the so-called *method of tangent points*, Molchanov and Stoyan [16] succeeded in obtaining a kernel estimator for the diameter DF. The statistical and asymptotic properties of all these estimation techniques have been only rarely studied so far.

The goal of the present paper is to establish consistency properties and asymptotic normality of the empirical diameter DF suggested in [16].
The main result (Theorem 2.1) is formulated in Section 2 and proved in Section 5. In Section 3 the behaviour of expectation and variance of a kernel estimator of the second-order product density of the family of tangent point processes associated with the BM (1.1) is studied even in the case of an arbitrary convex typical grain. For this estimator a multivariate central limit theorem (CLT) is shown in Section 4.

Throughout, \( \Rightarrow \) denotes the weak convergence of random vectors and \( \int \) without limits means integration over the whole space \( (\mathbb{R}^d)^k \) for some \( k \geq 1 \). For the necessary facts from point process and random set theory the reader is referred to Stoyan et al. [21].

2. PRELIMINARIES AND MAIN RESULT

First, we introduce the family of tangent point processes associated with the BM (1.1). Let \( K \in \mathcal{K} \) be a convex set and \( u \in S^{d-1} = \partial b(o,1) \). Then we define the tangent point \( l(u, K) \) of \( K \) in direction \( u \) by

\[
l(u, K) := \text{lexmin}\{x \in \partial K|(-u, x) = h(-u, K)\},
\]

where 'lexmin' denotes the lexicographical minimum of a point set, \( \langle u, x \rangle \) the inner product of \( u \) and \( x \) and \( h(u, K) \) the support function of \( K \) in direction \( u \) defined by \( h(u, K) := \sup_{x \in K} \langle u, x \rangle \). To simplify the writing we put \( K^u := K - l(u, K) \).

For fixed \( u \in S^{d-1} \) we determine the unique tangent point \( l(u, \Xi_i) + X_i \) of each shifted grain \( \Xi_i + X_i \) and all those tangent points not covered by any other shifted grain yields the support of the point process \( \Psi_u \) of exposed tangent points associated with the BM (1.1):

\[
\Psi_u = \sum_{i \geq 1} \delta_{l(u, \Xi_i) + X_i} \prod_{j \neq i} (1 - 1_{\Xi_j + X_j}(l(u, \Xi_i) + X_i)). \tag{2.1}
\]

This point process turns out to be stationary (but not necessarily isotropic even if \( \Xi_0 \) is rotation-invariant under \( Q \)) with intensity \( \lambda_u = \lambda(1 - p) \). The second-order product density \( \varrho_u^{(2)} \) of \( \Psi_u \) exists and can be
calculated by the limit

$$\lambda_u \varrho_u^{(2)}(x) = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{E_\mu(b(o, \varepsilon_1)) \Psi_\mu(b(x, \varepsilon_2))}{|b(o, \varepsilon_1)||b(o, \varepsilon_2)|}$$

for \( x \neq o \),

which leads to, see Molchanov [7], (p. 38):

$$\lambda_u \varrho_u^{(2)}(x) = \lambda^2 \exp\{-\lambda E|\Xi_0 \cup (\Xi_0 - x)|\} f_u(x) \tag{2.2}$$

where

$$f_u(x) = \mathbb{P}(-x \notin \Xi_0^u) \mathbb{P}(x \notin \Xi_0^u) \quad \text{with} \quad \Xi_0^u = \Xi_0 - l(u, \Xi_0). \tag{2.3}$$

Using the abbreviation \( \varrho_u(x) := \lambda_u \varrho_u^{(2)}(x) \) and the well-known formulae.

\[
p = 1 - \exp\{-\lambda E|\Xi_0|\} \quad \text{and} \quad C(x) = 2p - 1 + \exp\{-\lambda E|\Xi_0 \cup (\Xi_0 - x)|\} \tag{2.4}
\]

for the volume fraction and covariance (as defined in Section 1) of the stationary BM (1.1), see Stoyan et al. [21], we obtain

$$\varrho_u(x) = \frac{\lambda_u^2(1 - 2p + C(x))}{(1 - p)^2} f_u(x). \tag{2.5}$$

In the particular case of spherical typical grain \( \Xi_0 = b(o, R_0) \) the function \( f_u(x) \) can be expressed by the DF \( F \) of the diameter \( D_0 = 2R_0 \).

**Proposition 2.1** For \( \Xi_0 = b(o, R_0) \) with \( F(t) = \mathbb{P}(D_0 < t) \) we have

$$f_u(x) = \begin{cases} 
0 & \text{if } x = o \\
1 & \text{if } x \neq o \text{ and } \langle u, x \rangle = 0 \\
F(|x|^2/|\langle u, x \rangle|) & \text{if } \langle u, x \rangle \neq 0
\end{cases} \tag{2.6}$$

In particular,

$$F(t) = f_u(tu) \quad \text{for all } t > 0, u \in S^{d-1}. \tag{2.7}$$
Proof of Proposition 2.1 Let \( \langle u, x \rangle > 0 \). Since \( \Xi_0^u = b(R_0 u, R_0) \) it is immediately clear that \( P(-x \notin \Xi_0^u) = 1 \) and
\[
 f_u(x) = P(x \notin \Xi_0^u) = P(\|x - R_0 u\|^2 > R_0^2) = P(\|x\|^2 - 2R_0 \langle u, x \rangle > 0)
 = F(\|x\|^2/\langle u, x \rangle).
\]
Likewise, for \( \langle u, x \rangle < 0 \),
\[
 f_u(x) = P(-x \notin \Xi_0^u) = F(-\|x\|^2/\langle u, x \rangle).
\]
Finally, it is clear that \( \{o \notin \Xi_0^u\} = \{x \in \Xi_0^u\} = \emptyset \) for \( \langle u, x \rangle = 0 \).

From (2.5) and (2.7) we immediately get
\[
 F(t) = \frac{(1 - p)^2 \varrho_u(tu)}{\lambda_0^2(1 - 2p + C(tu))}.
\]
To estimate the density \( \varrho_u(x) \) we introduce the kernel-type estimator \( \hat{\varrho}_{u,n}(x) \) defined by
\[
 \hat{\varrho}_{u,n}(x) = \frac{1}{b_n |W_n|} \sum_{x_1, x_2 \in \Psi_u} w_n(x_1) k\left( \frac{x_2 - x_1 - x}{b_n} \right) \quad \text{for } x \in \mathbb{R}^d.
\]
Here the sum \( \sum^* \) stretches over all pairs \( x_1, x_2 \) of distinct points of the support \( s(\Psi_u) \) of the point process \( \Psi_u \) and \( k : \mathbb{R}^d \rightarrow \mathbb{R} \) denotes a kernel function associated with a sequence \( (b_n)_{n \geq 1} \) of bandwidths satisfying the following:

Condition (K) There exists some \( R > 0 \) such that
\[
 k(x) = 0 \quad \text{for } \|x\| > R, \quad M := \sup_{\|x\| \leq R} |k(x)| < \infty, \quad \int k(x) dx = 1 \quad \text{and}
 b_n > 0 \quad \text{for } n \geq 1, \quad \lim_{n \to \infty} b_n = 0, \quad \lim_{n \to \infty} nb_n = \infty.
\]
The characteristics \( p, C(x) \) and \( \lambda_u \) in (2.8) are replaced by their well-known ‘naive’ empirical counterparts
\[
 \hat{p}_n = \frac{|W_n \cap \Xi(\lambda, Q)|}{|W_n|}, \quad \hat{C}_n(x) = \frac{|W_n \cap \Xi(\lambda, Q) \cap (\Xi(\lambda, Q) - x)|}{|W_n|}, \quad x \in \mathbb{R}^d,
\]
These estimators turn out to be unbiased and strongly consistent as \( n \to \infty \) (since the BM is ergodic) and results on asymptotic normality were obtained by Mase [3]; Heinrich [9]; Molchanov and Stoyan [6]; Heinrich and Molchanov [11] and Garsia-Soidán [5].

Plugging in the above estimators (2.9) and (2.10), (2.11), (2.12) into (2.8) gives the following empirical DF \( \hat{F}_{n}(t) \) for \( F(t) \):

\[
\hat{F}_{u,n}(t) = \frac{(1 - \hat{p}_{n})^{2} \hat{g}_{u,n}(tu)}{\hat{\lambda}_{u,n}^{2}(1 - 2\hat{p}_{n} + \hat{C}_{n}(tu))}.
\]

The strong consistency of the estimators (2.10), (2.11) and (2.12) implies the following:

**Proposition 2.2** For any \( t > 0 \) and \( u \in S^{d-1} \)

\[
\hat{F}_{u,n}(t) \xrightarrow{n \to \infty} F(t) \text{ in probability P (resp. P - a.s.) iff } \\
\hat{g}_{u,n}(tu) \xrightarrow{n \to \infty} g_{u}(tu) \text{ in probability P (resp. P - a.s.).}
\]

Conditions ensuring pointwise as well as uniform P - a.s. convergence of kernel estimators of second-order product densities for a wide class of point processes were given in Heinrich and Liebscher [10]. The convergence in probability of \( \hat{g}_{u,n}(tu) \) will be discussed in Section 3.

To conclude this section we formulate our main result concerning the joint asymptotic normality and the asymptotic independence of the components of the discrepancy vector

\[
\sqrt{\frac{b^{d}|W_{n}|}{n}}(\hat{F}_{u,n}(t_{i}) - F(t_{i}))_{i=1}^{r} \quad \text{as } n \to \infty.
\]

**Theorem 2.1** Let \( \Xi(\lambda, F) \) be a stationary BM in \( \mathbb{R}^{d} \) with spherical typical grain \( \Xi_{0} = b(o, R_{0}) \) having a diameter \( DF F \) satisfying \( \int_{0}^{\infty} r^{2d} dF(r) < \infty \) and the Lipschitz condition

\[
|F(t) - F(s)| \leq L|t - s| \quad \text{for } s, t \in (a,b), \ 0 \leq a < b \leq \infty. \]
Further, let the kernel function \( k \) satisfy Condition (K) and the bandwidths \( (b_n)_{n \geq 1} \) be chosen such that \( nb_n^{1+\gamma} \to \infty \) for some \( \gamma > 0 \) and \( nb_n^{1+2/d} \to 0 \) as \( n \to \infty \).

Then, for any finite collection of distinct points \( \{t_1, \ldots, t_s\} \subset (a, b) \), we have

\[
\sqrt{b_n d_n W_n} \left( \sqrt{\hat{F}_{u,n}(t_i)} - \sqrt{F(t_i)} \right)_{i=1}^s \xrightarrow{n \to \infty} \mathcal{N}(0, \sum_s).
\]

Here \( \mathcal{N}(0, \sum_s) \) denotes an \( s \)-dimensional Gaussian vector with mean zero and covariance matrix \( \sum_s = (\sigma_{ij})_{i,j=1}^s \), where

\[
\sigma_{ii} = \int k^2(x) dx / 4\lambda^2 (1 - 2p + C(t_iu)) \quad \text{and} \quad \sigma_{ij} = 0 \quad \text{if} \ i \neq j.
\]

Therefore, by the continuous mapping theorem,

\[
b_n d_n W_n \sum_{i=1}^s \frac{1}{\sigma_{ii}} \left( \sqrt{\hat{F}_{u,n}(t_i)} - \sqrt{F(t_i)} \right)^2 \xrightarrow{n \to \infty} \chi^2_s,
\]

where the random variable \( \chi^2_s \) is \( \chi^2 \)-distributed with \( s \) degrees of freedom. Moreover, (2.16) remains valid if the variances \( \sigma_{ii} \) are replaced by the empirical variances

\[
\hat{\sigma}_{ii}^2 = (1 - \hat{p}_n)^2 \int k^2(x) dx / 4\hat{\lambda}_{u,n}(1 - 2\hat{p}_n + \hat{C}_n(t_iu)).
\]

Remark 1 The second assertion of Theorem 2.1 suggests an asymptotic \( \chi^2 \)-goodness-of-fit-test to check a hypothetical diameter DF.

3. ASYMPTOTIC MEAN SQUARED ERROR OF THE KERNEL PRODUCT DENSITY ESTIMATOR

In this section we will derive bounds for the MSE (\( = \) mean squared error) of the kernel estimator \( \hat{p}_{u,n}(x) \) which allow to select an optimal (asymptotic) bandwidth minimizing the MSE. For doing this we first study the Lipschitz-continuity of the function \( \theta_u(x) = \lambda_u \theta_u^{(2)}(x) \) and then we determine the asymptotic behaviour of \( E_{\theta_{u,n}(x)} \) and
\( \text{Var} \tilde{\varrho}_{u,n}(x) \). The latter is based on the knowledge of the third- and fourth-order product densities \( \varrho_u^{(k)}(x_2, \ldots, x_k) \), \( k = 3, 4 \), of the point process (2.1) which are defined for pairwise distinct \( x_2, \ldots, x_k \neq o \) by

\[
\lambda_u \varrho_u^{(k)}(x_2, \ldots, x_k) := \lim_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \to 0} \frac{E \Psi_u(b(o, \varepsilon_1)) \Psi_u(b(x_2, \varepsilon_2)) \cdots \Psi_u(b(x_k, \varepsilon_k))}{|b(o, \varepsilon_1)| |b(o, \varepsilon_2)| \cdots |b(o, \varepsilon_k)|}, \quad k \geq 2.
\]

These densities exist and can be calculated explicitly, see Theorem 4.1 in Molchanov and Stoyan [16]. In a slightly changed form they are as follows:

\[
\lambda_u \varrho_u^{(3)}(x_2, x_3) = \lambda^3 \exp\{-\lambda \mathbb{E} |\Xi_0 \cup (\Xi_0 - x_2) \cup (\Xi_0 - x_3)|\}
\times \int_{\mathcal{K}} \int_{\mathcal{K}} \int_{\mathcal{K}} (1 - 1_{K_1^*(x_2)})(1 - 1_{K_1^*(-x_2)})(1 - 1_{K_1^*(-x_3)})
\times (1 - 1_{K_1^*(x_3)})(1 - 1_{K_1^*(x_3 - x_2)})
\times (1 - 1_{K_1^*(x_2 - x_3)}) Q(dK_1) Q(dK_2) Q(dK_3).
\]

(3.1)

and

\[
\lambda_u \varrho_u^{(4)}(x_2, x_3, x_4)
= \lambda^4 \exp\{-\lambda \mathbb{E} |\Xi_0 \cup (\Xi_0 - x_2) \cup (\Xi_0 - x_3) \cup (\Xi_0 - x_4)|\}
\times \int_{\mathcal{K}} \cdots \int_{\mathcal{K}} (1 - 1_{K_1^*(x_2)})(1 - 1_{K_1^*(-x_2)})
\times (1 - 1_{K_1^*(-x_3)})(1 - 1_{K_1^*(x_3)})(1 - 1_{K_1^*(-x_4)})
\times (1 - 1_{K_1^*(x_4)})(1 - 1_{K_1^*(x_3 - x_2)})
\times (1 - 1_{K_1^*(x_4 - x_2)})(1 - 1_{K_1^*(x_2 - x_3)})(1 - 1_{K_1^*(x_4 - x_3)})
\times (1 - 1_{K_1^*(x_2 - x_4)})(1 - 1_{K_1^*(x_3 - x_4)}) Q(dK_1) \cdots Q(dK_4).
\]

(3.2)

The following Lemma 3.1 shows how the (Lipschitz-) continuity of \( \varrho_u(x) \) depends on the (Lipschitz-) continuity of \( f_u(x) \).

**Lemma 3.1** Let the stationary BM (1.1) satisfy Condition (B). Then

\[
|\varrho_u(x) - \varrho_u(y)| \leq \lambda^2 (1 - p)(|f_u(x) - f_u(y)|
+ 2\lambda \mathbb{E} |\Xi_0 \oplus b(o, 1)||x - y||).
\]
Proof of Lemma 3.1 From (2.4) and (2.5) it follows immediately that

\[ |g_u(x) - g_u(y)| \leq \lambda^2 \left( |f_u(x) - f_u(y)| \exp\left\{ -\lambda E|\Xi_0 \cup (\Xi_0 - x)| \right\} \\
+ f_u(y)| \exp\left\{ -\lambda E|\Xi_0 \cup (\Xi_0 - x)| \right\} \\
- \exp\left\{ -\lambda E|\Xi_0 \cup (\Xi_0 - y)| \right\} \right) \\
\leq \lambda^2 (1 - p) |f_u(x) - f_u(y)| \\
+ \lambda|E|\Xi_0 \cup (\Xi_0 - x)| - E|\Xi_0 \cup (\Xi_0 - y)|. \]

If \( \|x - y\| \geq 1 \), then

\[ |E|\Xi_0 \cup (\Xi_0 - x)| - E|\Xi_0 \cup (\Xi_0 - y)| | \leq 2|\Xi_0| \]
\[ \leq 2\|x - y\||E|\Xi_0 \oplus b(o, 1)|. \]

Otherwise, for \( \|x - y\| \leq 1 \), it is easily seen that

\[ |E(|\Xi_0 \cup (\Xi_0 - x)| - |\Xi_0 \cup (\Xi_0 - y)|)| \leq E|\Xi_0 \cap [(\Xi_0 - x) \Delta (\Xi_0 - y)]| \]
\[ \leq E|\Xi_0 \Delta (\Xi_0 + (x - y))|, \]

where \( A \Delta B = (A \setminus B) \cup (B \setminus A) \) denotes the symmetric difference of the point sets \( A \) and \( B \). By elementary considerations.

\[ \Xi_0 \Delta (\Xi_0 + x - y) \subseteq (\Xi_0 \oplus b(o, \|x - y\|)) \setminus (\Xi_0 \cap (\Xi_0 + x - y)), \]

whence it follows from Condition (B) by using Steiner's formula that

\[ E|\Xi_0 \Delta (\Xi_0 + (x - y))| \]
\[ \leq E(|\Xi_0 \oplus b(o, \|x - y\|))| - |\Xi_0 \cap (\Xi_0 + (x - y))|) \]
\[ = E(|\Xi_0 \oplus b(o, \|x - y\|))| - 2|\Xi_0| + |\Xi_0 \cup (\Xi_0 + (x - y))|) \]
\[ \leq 2E(|\Xi_0 \oplus b(o, \|x - y\|)| - |\Xi_0|) \]
\[ = 2 \sum_{j=1}^d \|x - y\|^{d-1} \binom{d}{j} E\mathbb{W}^{(d)}_j(\Xi_0) \]
\[ \leq 2\|x - y\||E|\Xi_0 \oplus b(o, 1)| - E|\Xi_0|. \]
Summarizing the obtained inequalities completes the proof of Lemma 3.1.

The following theorem presents the asymptotic behavior of the expectation and variance of the kernel estimator (2.9).

**Theorem 3.1** Let the stationary BM (1.1) satisfy Condition (B) and $E|\Xi_0|^2 < \infty$. Further, let the kernel function $k(\cdot)$ satisfy Condition (K).

1. If the function (2.3) is continuous at $x \neq o$, then

   \[
   \mathbb{E}\hat{\theta}_{u,n}(x) \xrightarrow{n \to \infty} \theta_u(x) \quad \text{and} \quad \text{Var} \hat{\theta}_{u,n}(x) \xrightarrow{n \to \infty} \theta_u(x) \int k^2(y)dy
   \]

   implying $\hat{\theta}_{u,n}(x) \xrightarrow{n \to \infty} \theta_u(x)$ in probability $P$.

2. Let $x \neq o$ and $\varepsilon > 0$. If there exists a constant $L_0 \geq 0$ such that

   \[|f_u(x) - f_u(z)| \leq L_0||x - z|| \quad \text{for all } z \in b(x, \varepsilon),\]

   then, for $n \geq 1$ with $b_n < \varepsilon/R$,

   \[
   |\mathbb{E}\hat{\theta}_{u,n}(x) - \theta_u(x)| \leq c_1 b_n \int \|y\|k(y)dy,
   \]

   for $n \geq 1$ with $b_n < \min\{\|x\|, \varepsilon\}/R$,

   \[
   b_n^d |W_n| \text{Var} \hat{\theta}_{u,n}(x) - \theta_u(x) \int k^2(y)dy \leq c_1 b_n \int \|y\|k^2(y)dy + c_2 b_n^d,
   \]

   and, for $n \geq 1$ with $b_n < \min\{\|x - y\|, \|x + y\|\}/2R$,

   \[
   |W_n| \text{Cov} (\hat{\theta}_{u,n}(x), \hat{\theta}_{u,n}(y)) \leq c_2 \quad \text{for } x, y \neq o, \ x \neq y, \ x \neq -y,
   \]

   where $c_1 = \lambda^2(1-p)(L_0 + 2\lambda E|\Xi_0| \oplus b(o, 1))$ and

   \[
   c_2 = 4(1-p)\lambda^3(1 + 2\lambda E|\Xi_0| + \lambda^2 E|\Xi_0|^2) \left( \int |k(z)|dz \right)^2.
   \]

**Remark 2** The estimate (3.5) holds without any continuity assumptions.
Since the mean squared error $\text{MSE}(\hat{\theta}_{u,n}(x)) := \mathbb{E}(\hat{\theta}_{u,n}(x) - \theta_u(x))^2$ is simply decomposed into variance and squared bias
\[
\text{MSE}(\hat{\theta}_{u,n}(x)) = \text{Var}(\hat{\theta}_{u,n}(x)) + (\mathbb{E}(\hat{\theta}_{u,n}(x) - \theta_u(x))^2,
\]
the above Theorem 3.1 provides a bound for $\text{MSE}(\hat{\theta}_{u,n}(x))$ from which an asymptotic MSE-optimal bandwidth $b_n^*$ up to a constant can be derived.

**Corollary 3.1** We have
\[
\text{MSE}(\hat{\theta}_{u,n}(x)) \leq \frac{\theta_u(x)}{|W_n| b_n^d} \int k^2(y)dy + \frac{c_1}{|W_n| b_n^{d-1}} \int ||y||^2k^2(y)dy \]
\[
+ \frac{c_2}{|W_n|} + c_1^2 b_n^2 \left( \int ||y||\|k(y)\|dy \right)^2.
\]

By equating the first and fourth term on the r.h.s. we find the asymptotic MSE-optimal bandwidth to be
\[
b_n^* = cn^{-d/(d+2)} \quad \text{for some constant } c > 0.
\]

**Proof of Theorem 3.1** The calculation of expectation and variance of $\hat{\theta}_{u,n}$ is based on the following identity which is a consequence of Campbell’s theorem from point process theory, see e.g., Stoyan et al. [21]:
\[
\mathbb{E} \sum_{x_1, \ldots, x_k \in \Psi_u} * f(x_1, \ldots, x_k)
\]
\[
= \lambda_u \int f(x_1, \ldots, x_k) \theta_u^{(k)}(x_k - x_1, \ldots, x_2 - x_1) d(x_1, \ldots, x_k)
\]
for any absolutely integrable function $f: (\mathbb{R}^d)^k \mapsto \mathbb{R}^1$.

Applying the preceding formula for $k = 2$ leads to
\[
\mathbb{E}\hat{\theta}_{u,n}(x) = \frac{1}{|W_n| b_n^d} \int 1_{W_u}(x_1) k \left( \frac{x_2 - x_1 - x}{b_n} \right) \theta_u(x_2 - x_1) d(x_1, x_2)
\]
\[
= \theta_u(x) \int k(y)dy + \int (\theta(x + b_ny) - \theta_u(x))k(y)dy.
\]

Since, by Lemma 3.1, the continuity of $\theta_u(\cdot)$ at $x \neq o$ implies that of $\theta_u(\cdot)$ at the same point, the first part of the statement (I) follows immediately from Condition (K). In the same way, by using Lemma 3.1
and the supposed Lipschitz-continuity of \( f_u(\cdot) \) in some neighbourhood of \( x \neq 0 \),

\[
|\mathbb{E}\hat{\theta}_{u,n}(x) - \varrho_u(x)| \leq \int |\varrho_u(x + b_nz) - \varrho_u(x)||k(z)|dz
\leq c_1 b_n \int \|z\||k(z)|dz
\]

provided that \( b_n R \leq \varepsilon \).

To prove (3.4) and (3.5) we first express the covariance \( \text{Cov}(\hat{\theta}_{u,n}(x), \hat{\theta}_{u,n}(y)) \) in terms of the product densities \( \varrho_u^{(k)}(x_2, \ldots, x_k) \), \( k = 2, 3, 4 \). For this we make use of the identity given at the beginning of the proof and a general decomposition formula derived in Heinrich [8], p. 97. We also refer the reader to a detailed computation of \( \text{Cov}(\hat{\theta}_{u,n}(x), \hat{\theta}_{u,n}(x)) \) for \( u, v \in S^{d-1}, u \neq v \), in Heinrich and Werner [12].

In this way we obtain

\[
\mathbb{E}\hat{\theta}_{u,n}(x)\hat{\theta}_{u,n}(y) - \mathbb{E}\hat{\theta}_{u,n}(x)\mathbb{E}\hat{\theta}_{u,n}(y) = \frac{1}{|W_n|^2 b_n^{2d}} \sum_{i=1}^{7} A_i(x, y) \quad (3.6)
\]

where the summands on the r.h.s. of (3.6) take the following form:

\[
A_1(x, y) = \lambda_u \int 1_{W_n}(x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{x_2 - y}{b_n}\right) \varrho_u^{(2)}(x_2)d(x_1, x_2)
\]

\[
A_2(x, y) = \lambda_u \int 1_{W_n}(x_1) 1_{W_n}(x_2 + x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{-x_2 - y}{b_n}\right) \varrho_u^{(2)}(x_2)d(x_1, x_2)
\]

\[
A_3(x, y) = \lambda_u \int 1_{W_n}(x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{x_3 - y}{b_n}\right) \varrho_u^{(3)}(x_2, x_3)d(x_1, x_2, x_3)
\]

\[
A_4(x, y) = \lambda_u \int 1_{W_n}(x_1) 1_{W_n}(x_3 + x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{-x_3 - y}{b_n}\right) \varrho_u^{(3)}(x_2, x_3)d(x_1, x_2, x_3)
\]

\[
A_5(x, y) = \lambda_u \int 1_{W_n}(x_1) 1_{W_n}(x_2 + x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{x_3 - x_2 - y}{b_n}\right) \varrho_u^{(3)}(x_2, x_3)d(x_1, x_2, x_3)
\]

\[
A_6(x, y) = \lambda_u \int 1_{W_n}(x_1) 1_{W_n}(x_3 + x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{x_3 - x_2 - y}{b_n}\right) \varrho_u^{(3)}(x_2, x_3)d(x_1, x_2, x_3)
\]

\[
A_7(x, y) = \lambda_u \int 1_{W_n}(x_1) 1_{W_n}(x_3 + x_1) k\left(\frac{x_2 - x}{b_n}\right) k\left(\frac{x_3 - x_2 - y}{b_n}\right) \varrho_u^{(3)}(x_2, x_3)d(x_1, x_2, x_3)
\]
\[ A_6(x, y) = \lambda_u \int 1_{w_n(x_1)} 1_{w_n(x_3 + x_1)} k\left( \frac{x_2 - x}{b_n} \right) k\left( \frac{x_2 - x_3 - y}{b_n} \right) \]
\[ \cdot \varrho_u^{(3)}(x_2, x_3)d(x_1, x_2, x_3) \]
\[ A_7(x, y) = \lambda_u \int 1_{w_n(x_1)} 1_{w_n(x_3 + x_1)} k\left( \frac{x_2 - x}{b_n} \right) k\left( \frac{x_4 - x_3 - y}{b_n} \right) \]
\[ \times (\varrho_u^{(4)}(x_2, x_3, x_4) - \lambda_u \varrho_u^{(2)}(x_2) \varrho_u^{(2)}(x_4 - x_3)) \]
\[ d(x_1, x_2, x_3, x_4) \]

To prove (3.4) we need to estimate the preceding terms for \( x = y \).
Since, after substituting \( z = (x_2 - x)/b_n \),
\[ A_1(x, x) = |W_n| b_n^d \varrho_u(x) \int k^2(z)dz \]
\[ + |W_n| b_n^d \int k^2(z)\varrho_u(x + b_nz) - \varrho_u(x))dz \]

we get in analogy to (3.3) that
\[ \left| A_1(x, x) - |W_n| b_n^d \varrho_u(x) \int k^2(z)dz \right| \leq |W_n| b_n^{d+1} c_1 \int ||z||k^2(z)dz. \] (3.8)

On the other hand, for \( x \neq y \),
\[ A_1(x, y) = \lambda_u |W_n| b_n^d \int k(z)k\left( z + \frac{x - y}{b_n} \right) \varrho_u(x + b_nz)dz. \]

Hence, in view of Condition (K),
\[ A_1(x, y) = 0 \quad \text{for } b_n < ||x - y||/2R. \]

In the same way, for any \( x, y \neq o \),
\[ A_2(x, y) = \lambda_u b_n^d \int |W_n \cap (W_n - x - b_nz)|k(z) k\left( -z - \frac{x + y}{b_n} \right) \]
\[ \varrho_u^{(2)}(x + b_nz)dz \]
entailing that
\[ A_2(x, y) = 0 \quad \text{for } b_n < ||x + y||/2R. \]
The crucial point to find upper bounds for $A_3(x, y), \ldots, A_6(x, y)$ is the uniform boundedness of the third-order product density (3.1). It is easily checked that $\theta^{(3)}_{u}(x_2, x_3) \leq \lambda^2$ for all $x_2, x_3 \in \mathbb{R}^d$. This bound combined with a suitable substitution of the integration variables in the kernel function (e.g., in $A_3(x, y)$ replace $x_2, x_3$ by $z_2 = (x_2 - x)/b_n$ and $z_3 = (x_3 - y)/b_n$) provides the uniform estimates

$$|A_i(x, y)| \leq (1 - p) \lambda^3 |W_n| b_n^{2d} \left( \int |k(z)|dz \right)^2 \text{ for } i = 3, 4, 5, 6. \quad (3.9)$$

Now we treat the term $A_7(x, y)$ which reflects the weak stochastic dependency between distant parts of a BM. Setting

$$\sigma(x_2, x_3, x_4) := \lambda u \theta^{(4)}_{u}(x_2, x_3, x_4) - \lambda^2 \theta^{(2)}_{u}(x_2) \theta^{(2)}_{u}(x_4 - x_3),$$

the above definition of $A_7(x, y)$ gives immediately

$$|A_7(x, y)| \leq |W_n| \int \left| k \left( \frac{x_2 - x}{b_n} \right) k \left( \frac{x_4 - x_3 - y}{b_n} \right) \sigma(x_2, x_3, x_4) \right| d(x_2, x_3, x_4).$$

$$\quad (3.10)$$

Since, by (2.2) and (2.3), the product $\lambda^2 \theta^{(2)}_{u}(x_2) \theta^{(2)}_{u}(x_4 - x_3)$ can be rewritten as

$$\lambda^4 \exp\{-\lambda(E|\Xi_0 \cup (\Xi_0 - x_2)| + E|\Xi_0 \cup (\Xi_0 - (x_4 - x_3))|)\}$$

$$\quad \int_{K} \cdots \int_{K} (1 - 1_{K^*_1}(x_2))$$

$$\quad \times (1 - 1_{K^*_2}(-x_2))(1 - 1_{K^*_3}(x_4 - x_3))(1 - 1_{K^*_4}(x_3 - x_4))$$

$$\quad Q(dK_1) \cdots Q(dK_4)$$

and together with the obvious decomposition

$$E|\Xi_0 \cup (\Xi_0 - x_2) \cup (\Xi_0 - x_3) \cup (\Xi_0 - x_4)|$$

$$= E|\Xi_0 \cup (\Xi_0 - x_2)|$$

$$+ E|\Xi_0 \cup (\Xi_0 - (x_4 - x_3))|$$

$$- E|(\Xi_0 \cup (\Xi_0 - x_2)) \cap ((\Xi_0 - x_3) \cup (\Xi_0 - x_4))|$$
we obtain from (3.2) the following bound for $|\sigma(x_2, x_3, x_4)|$:

\[
(1-p)\lambda^d \int_{\mathcal{K}} \cdots \int_{\mathcal{K}} \left| (1 - 1_{K_1^*}(-x_3))(1 - 1_{K_2^*}(x_3))(1 - 1_{K_4^*}(-x_4)) \right|
\]

\[
(1 - 1_{K_1^*}(x_4))(1 - 1_{K_2^*}(x_3 - x_2))(1 - 1_{K_3^*}(x_2 - x_4))
\]

\[
- \exp\{-\lambda E|\Xi_0 \cup (\Xi_0 - x_2)\} \cap ((\Xi_0 - x_3) \cup (\Xi_0 - x_4)) | Q(dK_1) \cdots Q(dK_4)
\]

\[
\leq (1-p)\lambda^d (P(x_3 \in \Xi_0^u) + P(-x_3 \in \Xi_0^u))
\]

\[
+ P(x_4 \in \Xi_0^u) + P(-x_4 \in \Xi_0^u)
\]

\[
+ P(x_3 - x_2 \in \Xi_0^u) + P(x_2 - x_3 \in \Xi_0^u) + P(x_4 - x_2 \in \Xi_0^u)
\]

\[
+ P(x_2 - x_4 \in \Xi_0^u)
\]

\[
+ \lambda E|\Xi_0 \cup (\Xi_0 - x_2)\} \cap ((\Xi_0 - x_3) \cup (\Xi_0 - x_4))
\]

The latter bound results from the elementary inequalities $1 - \Pi_{i=1}^m (1 - x_i) \leq \sum_{i=1}^m x_i$ for $0 \leq x_1, \ldots, x_m \leq 1$ and $1 - e^{-x} \leq x$ for $x \geq 0$. Next, we replace $|\sigma(x_2, x_3, x_4)|$ in the integral on the r.h.s. of (3.10) by the obtained upper bound of $|\sigma(x_2, x_3, x_4)|$. Finally, performing the appropriate variable transformations in (3.10) and taking into account the identities

\[
\int P(x \in \Xi_0^u)dx = E|\Xi_0| \quad ('\text{Robbins' theorem}', \text{ see } [18], \text{ p. } 13)
\]

\[
\int E|\Xi_0| \cap (\Xi_0 + x)|dx = E|\Xi_0|^2,
\]

we arrive at

\[
|A_7(x, y)| \leq 4(1 - p)\lambda^d b_z^{2d} |W_n| (2E|\Xi_0| + \lambda E|\Xi_0|^2) \left( \int |k(z)|dz \right)^2.
\]

Finally, collecting together the estimates (3.8), (3.9) and (3.10) for $x = y$ resp. $x \neq y$, we obtain from (3.6) the desired inequalities (3.4)
resp. (3.5). Note that the continuity assumptions put on \( f_u(\cdot) \) are only relevant to determine the asymptotic behaviour of \( A_1(x, x) \). Therefore, the second limiting relation of part (I) follows directly from (3.7) combined with (3.9) and (3.10) for \( x = y \). This completes the proof of Theorem 3.1.

4. ASYMPTOTIC NORMALITY OF THE KERNEL PRODUCT DENSITY ESTIMATOR

The proof of Theorem 2.1 is essentially based on the multivariate central limit theorem (CLT) for the kernel product density estimator (2.9) which is formulated and proved in this section. The proving idea of the below Theorem 4.1 consists in approximating the sequence \( \hat{\theta}_{u,n}(x) \) by a triangular array of \( m \)-dependent random fields which in turn is a consequence of the Poisson property of the germ process and the assumed mutual independence between different grains and between grains and germs. The corresponding Lindeberg-type CLT for \( m \)-dependent random fields has been proved in Heinrich [8].

**Theorem 4.1** Let the stationary BM (1.1) satisfy Condition (B) and \( \mathbb{E}|\Xi_0|^2 < \infty \) and let the kernel function \( k(\cdot) \) satisfy Condition (K). Further, let \( x_1, \ldots, x_s \in \mathbb{R}^d \setminus \{0\} \) be pairwise distinct points of continuity of the function \( f_u(\cdot) \) and let \( nb_n^{\gamma+\gamma} \to \infty \) for some \( \gamma > 0 \). Then

\[
\sqrt{b_n^d W_n} \left( \hat{\theta}_{u,n}(x_i) - \mathbb{E}\hat{\theta}_{u,n}(x_i) \right)_{i=1}^s \stackrel{d}{\longrightarrow} N(0, (r_{ij})_{i,j=1}^s),
\]

(4.1)

where

\[
r_{ii} = \varrho_u(x_i) \int k^2(y)dy \quad \text{for } i = 1, \ldots, s \quad \text{and} \quad r_{ij} = 0 \quad \text{if } i \neq j.
\]

By strengthening the continuity assumption on \( f_u(\cdot) \) in some neighbourhood of the points \( x_1, \ldots, x_s \) the means \( \mathbb{E}\hat{\theta}_{u,n}(x_i) \) can be replaced by the true values \( \varrho_u(x_i), i = 1, \ldots, s \).

**Corollary 4.1** Let, in addition to the assumptions of Theorem 4.1, there exist a constant \( L_0 \geq 0 \) and some \( \varepsilon > 0 \) such that \( |f_u(x_i) - f_u(z)| \leq \ldots \)
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$L_0(x_i - z)$ for all $z \in b(x_i, \varepsilon)$, $i = 1, \ldots, s$ and let $nb_n^{1+2/d} \to 0$. Then

$$\sqrt{\frac{b_n^d}{|W_n|}} \left( \hat{g}_{u,n}(x_i) - g_u(x_i) \right)_{i=1}^s \xrightarrow{n \to \infty} \mathcal{N}(0, (r_{ij})_{i,j=1}^s) \quad (4.2)$$

and

$$\sqrt{\frac{b_n^d}{|W_n|}} \left( \sqrt{\hat{g}_{u,n}(x_i)} - \sqrt{g_u(x_i)} \right)_{i=1}^s \xrightarrow{n \to \infty} \mathcal{N}(0, \sigma^2 I_s), \quad (4.3)$$

where $\sigma^2 = (1/4) \int k^2(y) \, dy$ and $I_s$ denotes the $s$-dimensional unit matrix.

Proof of Theorem 4.1 Let $u \in S^{d-1}$ and the points $x_1, \ldots, x_s \neq o$ be fixed. To prove the assertion (4.1), we need (using the well-known Cramér-Wold method) to show that, for all $a_1, \ldots, a_s \in \mathbb{R}^1$ with $a_1^2 + \cdots + a_s^2 \neq 0$, the one-dimensional CLT

$$S_n := \sqrt{\frac{b_n^d}{|W_n|}} \sum_{i=1}^s a_i (\hat{g}_{u,n}(x_i) - E\hat{g}_{u,n}(x_i)) \xrightarrow{n \to \infty} \mathcal{N}(0, \int k^2(y) \, dy \sum_{i=1}^s a_i^2 g_u(x_i)) \quad (4.4)$$

holds. To shorten the notation we introduce the function

$$g_n(y) := \sum_{i=1}^s a_i k\left( \frac{y - x_i}{b_n} \right).$$

We split up the sampling window $W_n = [0, n]^d$ into unit cubes

$$E_z := [0, 1]^d + z, \quad z \in V_n := \{0, 1, \ldots, n - 1\}^d, \quad o := (0, \ldots, 0) \quad (4.5)$$

and define the triangular array of random variables

$$X_{nz} := \frac{1}{\sqrt{\frac{b_n^d}{|W_n|}}} \sum_{y_1, y_2 \in o \Psi_u} 1_{E_z} (y_1) g_n(y_2 - y_1), \quad z \in V_n, \quad (4.6)$$

which form a stationary random field for each $n \geq 1$ (due to the stationarity of $\Psi_u$). Obviously,

$$S_n = \sum_{z \in V_n} (X_{nz} - E X_{nz}). \quad (4.7)$$
In the first step we prove (4.1) for BM's with strictly bounded typical grain $\Xi_0$, say $\Xi_0 \subseteq b(k(u, \Xi_0), \rho)$ $P$-a.s. for some $\rho > 0$. Since, by Condition (K), the function $g_n(\cdot)$ vanishes outside of $\bigcup_{i=1}^{s} b(x_i, b_n R)$, the diameter of its support is bounded by some constant $\rho^*$. Therefore, by definition of the BM (1.1), the random field $\{X_{nz}, z \in V_n\}$ turns out to be $m$-dependent with $m = [2(\rho + \rho^*)] + 1$, see e.g., Heinrich [7, 8]. A CLT for a triangular scheme of $m_n$-dependent random fields proved in [7] can be employed to show $S_n/\sqrt{\text{Var} S_n} \rightarrow_{n \to \infty} N(0, 1)$. To do this, using that additionally the random variables $X_{nz}$ are identically distributed, we have to verify the relation

$$\frac{|W_n|}{\text{Var} S_n} \text{Var} X_{no} \leq c_3 < \infty \quad \text{for some positive constant } c_3 \quad (4.8)$$

and the Lindeberg condition

$$L_n(\varepsilon) := \frac{|W_n|}{\text{Var} S_n} E(X_{no} - E X_{no})^2 1_{\{|X_{no} - E X_{no}| \geq \varepsilon \sqrt{\text{Var} S_n} \}} \rightarrow_0 0 \quad (4.9)$$

for any $\varepsilon > 0$.

Taking into account the definition of $S_n$ (see (4.4)) and applying statement (I) and relation (3.5) of Theorem 3.1, we find that

$$\text{Var} S_n \rightarrow_{n \to \infty} \int k^2(y)dy \sum_{i=1}^{s} a_i^2 g_u(x_i). \quad (4.10)$$

Without loss of generality, we assume that the r.h.s. of (4.10) is strictly positive; otherwise, we have $S_n \rightarrow_{n \to \infty} 0$ in probability $P$.

Once more examining the steps in the proof of Theorem 3.1, where the indicator function $1_{W_n}(\cdot)$ is replaced by $1_{E_n}(\cdot)$, it follows together with the continuity of $g_u(\cdot)$ at $x_1, \ldots, x_s$ that

$$\lim_{n \to \infty} |W_n|/\text{Var} X_{no} = \int k^2(y)dy \sum_{i=1}^{s} a_i^2 g_u(x_i).$$

This combined with (4.10) shows the validity of (4.8). It remains to verify (4.9). By the inequality $E|X - E X|^p \leq 2^p E|X|^p$ and the above
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For any \( p > 2 \) that

\[
L_n(\varepsilon) \leq \frac{|W_n| E|X_{n_0} - E X_{n_0}|^p}{\varepsilon^{p - 2} (\text{Var} \, S_n)^{p/2}} \leq \frac{2|W_n|}{\varepsilon^{p - 2} (b^d_n |W_n| \text{Var} \, S_n)^{p/2}} E \left| \sum_{y_1, y_2 \in \Psi_n} 1_{E_0} (y_1) b_n (y_2 - y_1) \right|^p.
\]

We shall next show that the \( p \)-th absolute moment on the r.h.s. of the last line is bounded in \( n \geq 1 \). Since, in view of Condition (K), \( |g_n(y)| \leq M \sum_{i=1}^S |a_i| b(x_i, Rb_n) \), it suffices to replace \( g_n(\cdot) \) by the indicator function \( 1_{b(x_i, Rb_n)}(\cdot) \):

\[
E \left| \sum_{y_1, y_2 \in \Psi_n} 1_{E_0} (y_1) b_n (y_2 - y_1) \right|^p \\
\leq E \left| \sum_{y_2 \in \Psi_n} 1_{E_0} (y_1) \Psi_n (b(y + x_i, Rb_n)) \right|^p \\
\leq E (\Psi_n (E_0 + b(x_i, Rb_n)))^p \leq E (\Psi_n (E_0 + b(o, Rb_n)))^{2p}.
\]

Since, by definition of \( (2.1) \), \( \Psi_n(\cdot) \leq \Pi(\cdot) \), the last term is bounded by the \( 2p \)-th moment of the Poisson distributed random variable \( \Pi(E_0 + b(o, 1)) \) (if \( Rb_n \leq 1 \)).

Finally, noting that

\[
\frac{|W_n|}{(b^d_n |W_n|)^{p/2}} = (nb^{1+2/(p-2)})^{-d(p-2)/2} \xrightarrow{n \to \infty} 0 \quad \text{if} \quad p > 2(1 + \gamma)/\gamma,
\]

the conditions (4.8) and (4.9) are completely verified. Thus, (4.4) follows from Theorem 2 in [7].

In the second step we drop the strict boundedness of \( \Xi_0 \) and approximate the BM \( \Xi(\lambda, Q) \) satisfying the condition \( E|\Xi_0|^2 < \infty \) by a "truncated" BM \( \Xi^{(\rho)}(\lambda, Q) = \bigcup_{\geq 1} (X_i + \Xi^{(\rho)}) \) which is generated by a original Poisson germ process \( \Pi(\lambda) \) and the "truncated" typical grain \( \Xi^{(\rho)}(\rho) = \Xi_0 \cap b(l(u, \Xi_0), \rho) \) for (sufficiently large) \( \rho > 0 \). Using the above-proved CLT (4.4) for \( \Xi^{(\rho)}(\lambda, Q) \), Chebychev’s inequality and standard arguments from weak convergence theory (see Billingsley [1]), we establish the CLT (4.4) for \( \Xi(\lambda, Q) \) by showing that, for given \( \varepsilon > 0 \),
exists some \( n_0(\varepsilon) \) such that

\[
\lim_{\rho \to -\infty} \sup_{n \geq n_0(\varepsilon)} (b_n^d |W_n| \text{Var}(\hat{\varphi}_{u,n}(x) - \hat{\varphi}_{u,n}(x))) \\
\leq \varepsilon \quad \text{for} \quad x \in \{x_1, \ldots, x_r\},
\]

(4.11)

where \( \varphi_{u,n}(x) \) denotes the kernel estimator (2.9) for the truncated model \( \Xi^{(\rho)}(\lambda, Q) \). Further, let \( \Psi_{u}^{(\rho)} \) denote the corresponding point process (2.1) of exposed tangent points associated with the truncated BM \( \Xi^{(\rho)}(\lambda, Q) \). According to the definition of \( \Xi^{(\rho)} \) we have \( l(u, \Xi_0) = l(u, \Xi^{(\rho)}) \) so that \( s(\Psi_u) \subseteq s(\Psi_u^{(\rho)}) \) (i.e., in the truncated BM less tangent points \( l(u, \Xi_0) + X_j \) are covered than in the original BM). We first rewrite the difference \( \varphi_{u,n}(x) - \hat{\varphi}_{u,n}(x) \) as two sums

\[
\sum_{x_1, x_2}^{(1)} = \sum_{x_1 \in s(\Psi_u^{(\rho)}(u)), x_2 \in s(\Psi_u^{(\rho)})} \quad \text{and} \quad \sum_{x_1, x_2}^{(2)} = \sum_{x_1 \in s(\Psi_u^{(\rho)}(u)), x_2 \in s(\Psi_u^{(\rho)}(u)), u \in \Psi_u^{(\rho)}}
\]

and after applying an elementary inequality we get

\[
b_n^d |W_n| \text{Var}(\hat{\varphi}_{u,n}(x) - \hat{\varphi}_{u,n}(x)) \\
\leq \frac{2}{b_n^d |W_n|} \sum_{i=1}^{2} \text{Var}\left( \sum_{x_1, x_2}^{(i)} 1_{W_n}(x_i) k \left( \frac{x_2 - x_1 - x}{b_n} \right) \right).
\]

Both variances on the r.h.s. can be treated in like manner as the covariance (3.6) above. The crucial point is the observation that the third- and fourth-order sums, which correspond to the summands \( A_3(x, y), \ldots, A_7(x, y) \) in (3.6), can be estimated without regarding the magnitude of the truncation parameter \( \rho \).

In order to calculate these terms we may use the Campbell-type identity

\[
E \sum_{x_i \in s(\Psi_u^{(\rho)}(u)), i=1, \ldots, l} f(x_1, \ldots, x_l) \\
= \int f(x_1, \ldots, x_l) p_u^{[\rho_1, \ldots, \rho_l]}(x_1, \ldots, x_l) d(x_1, \ldots, x_l)
\]

for any absolutely integrable function \( f: (\mathbb{R}^d)^l \mapsto \mathbb{R}^1 \) and any choice \( \rho_i \in \{\rho, \infty\}, \quad i = 1, \ldots, l \). Here the (Radon-Nikodym) density
\( p^{[p_1, \ldots, p_l]}_u(x_1, \ldots, x_l) \) is formally defined by

\[
p^{[p_1, \ldots, p_l]}_u(x_1, \ldots, x_l) := \lim_{\varepsilon_1, \ldots, \varepsilon_l \to 0} \frac{\mathbb{E} \prod_{i=1}^l \Psi^{(p_i)}_u(b(x_i, \varepsilon_i))}{\prod_{i=1}^l |b(o, \varepsilon_i)|}
\]

for \( p_1, \ldots, p_l \in \{\rho, \infty\} \),

and it can be shown that \( p^{[p_1, \ldots, p_l]}_u(x_1, \ldots, x_l) \) takes the form

\[
\lambda' \exp\left\{ -\lambda \mathbb{E} \bigcup_{i=1}^l (\Xi_0^{(p_i)} - x_i) \big| \right\}
\]

\[
\int K \cdots \int K \prod_{i, j \neq i}^l (1 - 1_{K_{i,j}^{(p_i)} - l(u, K_i)}(x_j - x_i)) Q(dK_1) \cdots Q(dK_l).
\]

Using this formula after lengthy calculations (quite similar to those performed in Section 3) we arrive at

\[
\begin{align*}
\left| \text{Var} \left( \sum_{x_1, x_2} \left( \sum_{i=1}^l \mathbb{E}^{(i)} \left( \frac{x_2 - x_1 - x}{b_n} \right) \right) \right) \\
= - \mathbb{E} \sum_{x_1, x_2} \left( \sum_{i=1}^l \mathbb{E}^{(i)} \left( \frac{x_2 - x_1 - x}{b_n} \right) \right) \\
- \mathbb{E} \sum_{x_1, x_2} \mathbb{E}^{(i)} \left( \frac{x_2 - x_1 - x}{b_n} \right) \mathbb{E}^{(i)} \left( \frac{x_1 - x_2 - x}{b_n} \right) \right| \leq c_4 b_n^2 |W_n|
\end{align*}
\]

for \( i = 1, 2 \), where the constant \( c_4 \) depends on \( \lambda, \mathbb{E}|\Xi_0| \) and \( \mathbb{E}|\Xi_0|^2 \) but not on \( \rho \).

Finally, it remains to treat the four terms of type \( \mathbb{E} \Sigma^{(1)} \ldots \) and \( \mathbb{E} \Sigma^{(2)} \ldots \) all of which are expressible by means of \( p^{[p_1, p_2]}_u(x_1, x_2) \) with \( p_1, p_2 \in \{\rho, \infty\} \). From the above formula for \( l = 2 \) we see that \( p^{[p_1, p_2]}_u(x_1, x_2) = p^{[p_1, p_2]}_u(o, x_2 - x_1) \) and, for \( p_1 = \rho, p_2 = \infty \), we have

\[
\begin{align*}
p^{[\rho, \rho]}_u(o, y) - p^{[\rho, \infty]}_u(o, y) &= \lambda^2 \left( \exp\left\{ -\lambda \mathbb{E} \Xi_0^{(\rho)} + (\Xi_0^{(\rho)} + y) \right\} \right) \\
&= \mathbb{P}(y \notin \Xi_0^{(\rho)} - l(u, \Xi_0)) \\
&\quad - \exp\left\{ -\lambda \mathbb{E} \Xi_0 \cup (\Xi_0^{(\rho)} + y) \right\} \mathbb{P}(y \notin \Xi_0 - l(u, \Xi_0)) \\
&\quad - \mathbb{P}(y \notin \Xi_0^{(\rho)} - l(u, \Xi_0)).
\end{align*}
\]
This yields

\[ \mathbb{E} \sum_{x_1, x_2}^{(1)} 1_{W_n}(x_1) k^2 \left( \frac{x_2 - x_1 - x}{b_n} \right) \]

\[ = |W_n| \int k^2 \left( \frac{y - x}{b_n} \right) (p_u^{[\rho, \sigma]}(o, y) - p_u^{[\rho, \infty]}(o, y)) dy \]

\[ \leq \lambda^2 b_n^d |W_n| \int k^2(z) d\gamma(\lambda \gamma(I_0) \gamma(I_0^{(\rho)})) \]

\[ + \mathbb{P}(\mathbb{X} + l(u, \Xi_0) \in (\Xi_0 \setminus \Xi_0^{(\rho)}) \oplus b(o, b_n R)). \]

The same bound can be shown to hold for the second expectation term in (4.12) for \( i = 1 \) and also for both of the corresponding terms in case \( i = 2 \). Since \( b_n R \leq 1 \) for \( n \geq n_0 \) and \( K \setminus K^{(\sigma)} \downarrow \emptyset \) as \( \sigma \uparrow \infty \) for any convex \( K \in \mathcal{K} \), we may apply Lebesgue’s dominated convergence theorem and the continuity of the measure \( \mathbb{P} \) to verify that the terms \( \mathbb{E} \sum^{(1)} \cdots \) and \( \mathbb{E} \sum^{(2)} \cdots \) become arbitrarily small when \( \rho \) is large enough. Combining this with (4.12) and \( b_n \rightarrow n \rightarrow \infty \) 0 we obtain (4.11). This completes the proof of Theorem 4.1.

**Proof of Corollary 4.1** By the Lipschitz-continuity of \( f_u(\cdot) \) in some neighbourhood of \( x_1, \ldots, x_s \), we get the inequality (3.3) at these points. Therefore,

\[ \sqrt{b_n^d |W_n|} \|\hat{\varrho}_{u,n}(x_i) - \varrho_u(x_i)\| \leq c_1 \int \|z\| k(z) dz \sqrt{b_n^{d+2} n^d} \]

for \( i = 1, \ldots, s \). This and together with \( n b_n^{1+2/d} \rightarrow n \rightarrow \infty 0 \) proves (4.2).

To prove (4.3) we use the simple algebraic identity

\[ \sqrt{\hat{\varrho}_{u,n}(x_i) - \varrho_u(x_i)} = \frac{\hat{\varrho}_{u,n}(x_i) - \varrho_u(x_i)}{\sqrt{\hat{\varrho}_{u,n}(x_i)} + \sqrt{\varrho_u(x_i)}} \quad \text{for } i = 1, \ldots, s \]

and the convergence \( \sqrt{\hat{\varrho}_{u,n}(x_i)} \rightarrow n \rightarrow \infty \sqrt{\varrho_u(x_i)} \) in probability \( \mathbb{P} \) which follows from part (I) of Theorem 3.1.

Thus, applying a well-known property of weak convergence of random variables (see Billingsley [1]), we obtain with the notation of
the proof of Theorem 4.1 that the weak limits of the sequences

\[ \sqrt{b_n^d |W_n|} \sum_{i=1}^{s} a_i \left( \sqrt{\hat{\mu}_{u,n}(x_i)} - \sqrt{\mu_u(x_i)} \right) \] and

\[ \sqrt{b_n^d |W_n|} \sum_{i=1}^{s} \frac{a_i}{2 \sqrt{\mu_u(x_i)}} \left( \hat{\mu}_{u,n}(x_i) - \mu_u(x_i) \right) \]

coincide (if at least one of them exists). In fact, the second sequence converges weakly to a Gaussian random variable \( N(0, (1/4) \int k^2(\gamma)dy \sum_{i=1}^{s} a_i^2) \) as it was stated in Theorem 4.1. Thus, once more by Cramér-Wold, we obtain (4.3).

5. PROOF OF THEOREM 2.1

We first formulate a slight generalization of the CLT (2.15). For this, we define in analogy to (2.13) (after inverting (2.5)) the empirical counterpart of \( f_u(x) \) by

\[ \hat{f}_{u,n}(x) = \frac{(1 - \hat{p}_n)^2 \hat{\mu}_{u,n}(x)}{\lambda_{u,n}(1 - 2\hat{p}_n + \hat{C}_n(x))}, \quad x \neq o. \] (5.1)

The main step towards Theorem 2.1 consists proving asymptotic normality of this estimator.

**Lemma 5.1** Let the assumptions of Corollary 4.1 be satisfied. Then

\[ \sqrt{b_n^d |W_n|} \left( \sqrt{\hat{f}_{u,n}(x_i)} - \sqrt{f_u(x_i)} \right)_{i=1}^{s} \xrightarrow{n \to \infty} N(0, (\tau_{ij})_{i,j=1}^{s}), \] (5.2)

where

\[ \tau_{ii} = \int k^2(\gamma)dy/4\lambda^2(1 - 2p + C(x_i)) \]

for \( i = 1, \ldots, s \) and \( \tau_{ij} = 0 \) if \( i \neq j \).

**Proof of Lemma 5.1** Let \( x \in \{x_1, \ldots, x_s\} \) be fixed. Inserting the abbreviations

\[ p(x) := 2p - C(x) \quad \text{and} \quad \hat{C}_n(x) := 2\hat{p}_n - \hat{C}_n(x) \]
into the formulae (2.5) and (5.1), respectively, we may split the difference \( \hat{f}_{un}(x) - f_u(x) \) as follows:

\[
\hat{f}_{un}(x) - f_u(x) = T_n^{(1)}(x) - T_n^{(2)}(x) + T_n^{(3)}(x) - T_n^{(4)}(x)
\]

\[
= \frac{(1-p)^2}{\lambda_u^2(1-p(x))} (\hat{\theta}_{un}(x) - \theta_u(x))
\]

\[
- \frac{\hat{\theta}_{un}(x)(2 - \hat{p}_n - p)}{\lambda_u^2(1-p(x))} (\hat{p}_n - p)
\]

\[
+ \frac{(1 - \hat{p}_n)^2 \hat{\theta}_{un}(x)}{\lambda_u^2(1 - \hat{p}_n(x))(1-p(x))} (\hat{p}_n(x) - p(x))
\]

\[
- \frac{\hat{\theta}_{un}(x)(1 - \hat{p}_n)^2(\lambda_u + \hat{\lambda}_{un})}{\lambda_u^2(1 - p(x))} (\hat{\lambda}_{un} - \lambda_u).
\]

From (2.10) and (2.11) it is rapidly seen that \( \hat{p}_n(x) = \hat{p}_n(x) - \frac{|W_n \cap (\Xi(\lambda, Q) - x)|}{|W_n|} + \hat{p}_n \), where the ratio

\[
\hat{p}_n(x) = \frac{|W_n \cap (\Xi(\lambda, Q) \cup (\Xi(\lambda, Q) - x))|}{|W_n|}
\]

can be regarded as an unbiased estimator of the volume fraction \( p(x) = P(o \in \Xi(\lambda, Q) \cup (\Xi(\lambda, Q) - x)) \) of a stationary BM generated by \( \Pi_\lambda \) and the typical grain \( \Xi_0 \cup (\Xi_0 - x) \). Standard manipulations similar to those used in the proof of Theorem 3.2 yield

\[
|W_n| \text{Var} \hat{p}_n = \int \left| \frac{|W_n \cap (W_n - y)|}{|W_n|} \right| \left\{ \exp\{-2\lambda E|\Xi_0 \cup (\Xi_0 - y) \cup (\Xi_0 - x - y)|\} - \exp\{-2\lambda E|\Xi_0 \cup (\Xi_0 - x)|\} \right\} dy
\]

\[
\leq \lambda(1-p)E|\Xi_0 \cup (\Xi_0 - x)|^2.
\]

Therefore,

\[
|W_n| \text{Var} \hat{p}_n(x) \leq 3|W_n|(\text{Var} \hat{p}_n(x) + 2\text{Var} \hat{p}_n(o)) \leq 18\lambda(1-p)E|\Xi_0|^2.
\]

In the same manner, starting from the definition of \( \lambda_u \theta_u^{(2)}(x) \) and taking into account the unbiasedness of (2.12) combined with (2.2),
we get that
\[ |W_n| \text{Var} \hat{\lambda}_{u,n} = \frac{1}{|W_n|} \mathbb{E} \sum_{x,y \in \Psi_u}^* 1_{W_n}(x)1_{W_n}(y) + \lambda_u \cdot |W_n| \lambda_u^2 \]
\[ = \lambda_u + \lambda_u \int \frac{|W_n \cap (W_n - y)|}{|W_n|} (\rho_u^{(2)}(y) - \lambda_u) dy \]
\[ \leq \lambda (1 - p) + \lambda^2 (1 - p) \int |f_u(y) - \exp\{ -\lambda E[\Xi_0 \cap (\Xi_0 - y)]\}| dy \]
\[ \leq \lambda (1 - p) (1 + 2\lambda E[\Xi_0] + \lambda^2 E[\Xi_0^2]). \]

By the way, these inequalities reprove the weak (or more precisely the \(L^2-\)) consistency of the estimators \(\hat{p}_n, \hat{\lambda}_n(*)\) and \(\hat{\lambda}_{u,n}\) (in fact they are strongly consistent) and they imply that
\[ \frac{b_d^d |W_n|(T_n^{(2)}(x) - T_n^{(3)}(x) + T_n^{(4)}(x))}{n} \to 0 \quad \text{in probability } P. \]

Hence
\[ \sqrt{b_d^d |W_n|} \left( \sqrt{f_{u,n}(x_i)} - \sqrt{f_{u}(x_i)} \right) \]
\[ \text{and} \quad \sqrt{b_d^d |W_n|} \left( \frac{T_n^{(1)}(x_i)}{2\sqrt{f_u(x_i)}} \right) \]

have the same weak limits. Thus, the assertion of Lemma 5.1 is proved by (4.2).

Next we specify (5.2) to a BM \(\Xi(\lambda, F)\) with spherical typical grain \(\Xi_0 = b(a, R_0)\), replace the points \(x_i\) by \(t_i u\) and use then relation (2.7). This yield the desired CLT (2.15) provided that the function \(f_u(\cdot)\) is locally Lipschitz in \(t_i u, \ldots, t_i u\).

To accomplish the proof of Theorem 2.1 we next show the local Lipschitz-continuity of the function \(f_u(\cdot)\) on the open segment \(\{tu: a < t < b\}\) whenever the DF of the diameter \(D_0 = 2R_0\) satisfies a Lipschitz condition on \((a, b)\).

**Lemma 5.2** Let \(\Xi(\lambda, F)\) be a stationary BM in \(\mathbb{R}^d\) with spherical typical grain \(\Xi_0 = b(a, R_0)\) having a diameter DF \(F\) satisfying \(\int_0^\infty r^d dF\)
\[ (r) < \infty \] and the Lipschitz condition (2.14).

Then, for any \(y \in b(a, \varepsilon)\) with \(0 < \varepsilon < (b - a) / (1 + \sqrt{2})\),
\[ |f_u(tu + y) - f_u(tu)| \leq (1 + \sqrt{2})L\|y\| \quad \text{for} \]
\[ t \in (a + (1 + \sqrt{2}) \varepsilon, b - (1 + \sqrt{2}) \varepsilon). \]
Proof of Lemma 5.2 From Proposition 2.1 and (2.14) we obtain

\[ |f_u(tu + y) - f_u(tu)| \leq L \left| \frac{\|tu + y\|^2}{\|tu + y, u\|} - t \right| = L \left| \frac{\|y\|^2 + t(y, u)}{t + (y, u)} \right|.
\]

Here we have used that

\[ \|tu + y\| \geq \langle tu + y, u \rangle \geq t - \|y\| \geq a + \sqrt{2} \varepsilon > 0. \]

Thus, if \( t \geq (1 + \sqrt{2})\|y\|, \)

\[ \left| \frac{\|y\|^2 + t(y, u)}{t + (y, u)} \right| \leq \left| \frac{\|y\|(t + \|y\|)}{t - \|y\|} \right| \leq (1 + \sqrt{2})\|y\|.
\]

We complete the proof of Lemma 5.2 by noting that the inclusion

\[ a < \frac{\|tu + y\|^2}{t + (y, u)} < b \quad \text{holds, whenever} \]

\[ a + (1 + \sqrt{2})\|y\| < t < b - (1 + \sqrt{2})\|y\|. \]

6. SOME EXTENSIONS AND CONCLUDING REMARKS

To reduce the inherent instability of the empirical DF \( \hat{F}_{u,n}(t) \) one should consider instead a convex linear combination \( \hat{F}_{n}(t) := w_1 \hat{F}_{u_1,n}(t) + \cdots + w_m \hat{F}_{u_m,n}(t) \) for pairwise distinct directions \( u_1, \ldots, u_m \in S^{d-1} \). By means of Lemma 4.2 in Heinrich and Molchanov [11] the explicit form of the asymptotic variance of \( \hat{g}_n(x) = \sum_{i=1}^{m} w_i g_{u_i,n}(x) \) can be calculated, see the forthcoming paper [12]. In the particular case \( l(u_i, Z_0) \neq l(u_j, Z_0) \) P-a.s. for \( i \neq j \) we get \( \lim_{n \to \infty} b_n^d |W_n| \text{Var} \hat{g}_n(x) = \int k^2(y)dy \sum_{i=1}^{m} w_i^2 g_{u_i}(x) \). Combining this with (2.13) we minimize the variance of the Gaussian limit distribution by taking equal weights \( w_1 = \cdots = w_m = 1/m \) for any selection of \( u_1, \ldots, u_m \).

Consistency and asymptotic normality of \( \hat{F}_{n}(t) \) is obtained by repeating (nearly verbatim) the arguments used in the proofs of the Theorems 3.1 and 2.1.

To extend the above results to BM’s with a non-spherical typical grain we need to express certain marginal DF’s of the distribution \( Q \) in
terms of the function $f_u(x)$ on a subset of points $(x, u) \in \mathbb{R}^d \times S^{d-1}$. Under appropriate continuity assumptions Lemma 5.1 provides asymptotic normality of the corresponding empirical DF’s defined by substituting $\lambda_u, p$ and $C(x)$ by (2.12), (2.10) and (2.11), respectively. In this way, we are in a position to treat a stationary BM with the typical grain $\Xi_0 = R_0K$, where $K$ is a fixed convex body in $\mathbb{R}^d$ containing the origin $o$ and $R_0$ is a random scaling factor with DF $F_0$. Let $u_0 \in S^{d-1}$ be chosen such that $-(u_0, l(u_0, K)) = \|l(u_0, K)\| = \sup \{t > 0: tu_0 \in K\}$. Define $\delta(u_0, K) := \|l(u_0, K)\| + \sup \{t > 0: tu_0 \in K\}$. Then, for any $t > 0$,

$$f_{u_0}(tu_0) = \mathbb{P}(tu_0 \notin \Xi_0 - l(u_0, \Xi_0) = \mathbb{P}(tu_0 \notin R_0(K - l(u_0, K))) = F_0(t/\delta(u_0, K)).$$

Therefore, together with (5.1) we may define $\hat{F}_0(t) = \hat{f}_{u_0}(t \delta(u_0, K)u_0)$, $t > 0$, providing an empirical DF of $R_0$ for which Lemma 5.1 is applicable.

Analogously, one can express the empirical counterparts of the DF’s of the random half-axes $A_1, \ldots, A_d$ of a $d$-dimensional ellipsoid $\Xi_0$ with fixed location around $o$ by means of $\hat{f}_u(x)$ provided the non-random directions of the half-axes are known.

Much more information about the distribution of the typical grain $\Xi_0$ can be gained by considering “multiply directed” exposed tangent point processes. This has been outlined already by Molchanov [7]. The asymptotic behaviour of the corresponding higher-order kernel product density estimators is studied in [12].

References


