CENTRAL LIMIT THEOREM FOR A CLASS OF RANDOM MEASURES ASSOCIATED WITH GERM-GRAIN MODELS

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Abstract

The germ-grain model is defined as the union of independent identically distributed compact random sets (grains) shifted by points (germs) of a point process. The paper introduces a family of stationary random measures in $\mathbb{R}^d$ generated by germ-grain models and defined by the sum of contributions of non-overlapping parts of the individual grains. The main result of the paper is the central limit theorem for these random measures, which holds for rather general independently marked germ-grain models, including those with non-Poisson distribution of germs and non-convex grains. It is shown that this construction of random measures includes those random measures obtained by positively extended intrinsic volumes. In the Poisson case it is possible to prove a central limit theorem under weaker assumptions by using approximations by $m$-dependent random fields. Applications to statistics of the Boolean model are also discussed. They include a standard way to derive limit theorems for estimators of the model parameters.

Keywords: $\beta$-mixing; Boolean model; germ-grain model; intrinsic volumes; $m$-dependent random field; random measure; random set; weak dependence

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1. Introduction

The ergodic theorem for spatial processes [23] establishes the existence of spatial averages when a sampling window expands unboundedly in all directions. In many applications such averages can be interpreted as intensities of stationary random measures concentrated on random closed sets in $\mathbb{R}^d$. For example, a point process corresponds to a random counting measure, a random closed set with non-empty interior gives rise to the random volume measure. Further examples are provided by geometric measures, e.g., surface and curvature measures of random sets. The relationships between intensities of such measures and parameters of the underlying random set or point process are the basis of the method of moments in statistics of stationary random sets [21, 27, 31].

A simple example states that, for any ergodic random closed set $Z$ in $\mathbb{R}^d$, the volume fraction inside window $W$ (the part of the window covered), converges almost surely to its expectation, called the volume fraction of $Z$, while $W$ is assumed to expand to the whole space in a ‘regular way’ [4, p. 332]. The corresponding central limit theorem was established

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in [1, 17, 18]. In fact, it is related to weak dependence properties of the indicator random field \( \zeta(x) = 1_{\mathcal{L}(x)} \) and limit theorems for additive functionals of random fields. Similar results are known for counting measures generated by point processes [14, 15].

For more general functionals (which include, e.g., the boundary length or surface area) the convergence of spatial averages was considered in [23] and [33]. However, until now a general approach to derive limit theorems for such functionals seems to be unknown. It should be noted that these limit theorems are very important for the construction of confidence intervals for estimators in statistics of stationary random sets. Variances for functionals related to surface and fibre processes has been found in [2, 3]. However, in the latter case the whole situation resembles ones with marked point processes, since the individual fibres are directly observable, see [27, Chapter 9].

Central limit theorems for geometric functionals other than volume fractions and the number of points were proved in [22] and [20] for the convexity number and the surface measure of the Boolean model \( \mathcal{Z} \). In this paper we consider more general models of random sets and geometric functionals and prove the corresponding central limit theorem. Also, we do not need the boundedness of the grain assumed in [20] for the case of surface measures.

It is known that ergodicity and first moment assumptions are not sufficient to prove a central limit theorem. One has to impose further conditions on both the random sets and the corresponding random measures. In the present paper we are interested in a class of random measures associated with independently marked germ-grain models. To begin with, we recall the definition of germ-grain models and introduce a class of random measures associated with them (Section 3). The germ-grain model is composed from a sequence of independent identically distributed random sets (grains) shifted by the points (germs) of a point process. Germ-grain models with compact grains are widely used in stochastic geometry and generalise the well-known Boolean model, which appears if the point process of germs is Poisson. Roughly speaking, these random measures associated with germ-grain models are defined through sums of contributions of the exposed (not covered by all other grains) parts of the individual grains. In the simplest case, this construction can be used to obtain the surface area of the germ-grain model.

Statistical properties of the random measures introduced here are studied in Sections 4 and 5. In particular, higher-order mixed moment measures are given in Section 4 and the absolute regularity (or \( \beta \)-mixing) is proved in Section 5. We show that the \( \beta \)-mixing property of the germ process implies \( \beta \)-mixing of the germ-grain model and the associated random measure, provided that the diameter of the typical grain has sufficiently high moments. Section 6 establishes the central limit theorem for the associated random measures. Section 7 deals with the Boolean model for which the central limit theorem holds under very mild (in fact, optimal) assumptions.

It is shown in Appendix that the random measures introduced here yield, in particular, the positive extensions of the intrinsic volumes defined by Matheron [19, Section 4.7]. Mean value formulae for extended intrinsic volumes are very important in statistics of the Boolean model. Therefore, the general results derived in this paper give rise to new mean-value formulae and provide a unified approach to limit theorems for spatial averages. A number of examples are considered in Section 8. Section 9 outlines several possible generalisations.
2. List of basic notation

\( [\mathbb{R}^d, \mathfrak{B}^d] \) \( d \)-dimensional Euclidean space with Borel \( \sigma \)-algebra;

\( \mathfrak{B}^d \) family of bounded Borel sets in \( \mathbb{R}^d \);

\( \|x\| \) Euclidean norm of \( x \in \mathbb{R}^d \);

\( \|K\| = \sup\{\|x\| : x \in K\}, K \in \mathfrak{B}^d \);  

\( |B| \) \( d \)-dimensional Lebesgue measure (or volume) of \( B \in \mathfrak{B}^d \);

\( B_r(x) = \{y \in \mathbb{R}^d : \|x - y\| \leq r\} \) for \( x \in \mathbb{R}^d \) and \( r > 0 \);

\( 1_B(x) = 1 \) if \( x \in B \) and = 0 otherwise (indicator function of \( B \))

\( F^c \) complement of set \( F \) in the underlying space;

\( \partial F \) boundary of \( F \subset \mathbb{R}^d \);

\( [\mathcal{F}, \sigma_f] \) family of closed sets in \( \mathbb{R}^d \) with the \( \sigma \)-algebra generated by the hit-or-miss topology, see [19, 27];

\( [\mathcal{K}, \sigma_k] \) family of compact sets in \( \mathbb{R}^d \) with \( \sigma \)-algebra \( \sigma_k = \{A \cap \mathcal{K} : A \in \sigma_f\} \);

\( \tilde{K} = \{-x : x \in K\} \) for \( K \in \mathcal{K} \);

\( \oplus \) Minkowski addition, i.e., \( F_1 \oplus F_2 = \{x + y : x \in F_1, y \in F_2\} \), \( F_1, F_2 \subset \mathbb{R}^d \);

\( \mathcal{C} \) family of convex compact sets in \( \mathbb{R}^d \);

\( [\Omega, \mathfrak{A}, \mathbb{P}] \) hypothetical common probability space on which all occurring random elements are defined;

\( \mathbb{E} \) expectation with respect to \( \mathbb{P} \);

\( \delta_x \) Dirac measure concentrated at point \( x \).

3. Germ-grain models and associated random measures

A stationary random closed set \( Z \) is a random element in space \( [\mathcal{F}, \sigma_f] \) such that \( Z \) has the same distribution as \( Z + x \) for all \( x \in \mathbb{R}^d \), see [19, 27]. The most important model of stationary random sets is the germ-grain model. This model is defined by means of a stationary marked point process

\[
\Psi_m = \sum_{i \geq 1} \delta_{[X_i; Z_i]}
\]

in \( \mathbb{R}^d \) with the mark space \( [\mathcal{K}, \sigma_k] \), where the stationarity is understood to be with respect to its first component, see [27] for further details on marked point processes. The corresponding germ-grain model \( Z \) is the set-theoretic union

\[
Z = \bigcup_{i \geq 1} (X_i + Z_i).
\]

The points \( X_i \) are called germs, while the sets \( Z_i \) are called grains, see [8, 27]. The model exists as soon as \( Z \) is closed and \( Z \neq \mathbb{R}^d \), see [11].
Throughout we assume that the corresponding unmarked point process

\[ \Psi(\cdot) = \Psi_m(\cdot) \times K = \sum_{i \geq 1} \delta_{X_i} \]

is simple and has a positive and finite intensity \( \lambda = E[\Psi([0,1]^d)] \). Remember that \( \Psi \) is a random element in the space \( N \) of locally finite counting measures \( \psi \) on \( \mathbb{R}^d \), where \( N \) is equipped with the \( \sigma \)-algebra \( \mathfrak{M} \) generated by sets of the form \( \{ \psi \in N : \psi(B) = k \} \) for \( B \in \mathcal{B}_0^d \) and \( k = 0, 1, \ldots \). The distribution of \( \Psi \) is denoted by \( P \), so that \( P(Y) = \mathbf{P}(\Psi \in Y) \) for each \( Y \in \mathfrak{M} \).

In this paper we consider only independently marked point processes, where \( \{Z_i, i \geq 1\} \) is a sequence of i.i.d. copies of a random compact set \( Z_0 \), independent of \( \Psi \). The random compact set \( Z_0 \) is called the typical grain. Note that \( Z_0 \) is a random element in the space \( \mathcal{K}, \sigma_k \) with distribution denoted by \( Q \). Then the random set \( (3.1) \) is closed and different from \( \mathbb{R}^d \) (so the germ-grain model exists) if

\[ E[Z_0 \oplus B_r(0)] < \infty \quad \text{for some} \ r > 0 \]  

(e.g., if \( E\|Z_0\|^d < \infty \)), see [11]. If the point process of germs \( \Psi \) is Poisson, \( Z \) is said to be a Boolean model [19, 27].

Germ-grain models give rise to a number of random measures on \( \mathcal{B}_0^d \), see [19]. In the simplest case it is possible to define a random measure \( \eta \) by \( \eta(W) = |Z \cap W| \), \( W \in \mathcal{B}_0^d \). Another random measure can be defined by taking the surface area of \( Z \) inside \( W \). Further random measures are the Minkowski random measures introduced in [19]. It is worthwhile to note that many interesting measures associated with \( Z \) can be decomposed into the sum with respect to all individual grains. In this sum each grain contributes to the resulting measure with some (possibly random) weight. For further information on general random measures the reader is referred to [16], [4, Chapter 6] and [27].

Below we present a unified approach to random measures associated with germ-grain models. Let \( \mathcal{K}, \mathcal{K} \) be a compact metric space with the corresponding Borel \( \sigma \)-algebra (in many examples \( \mathcal{K} \) is the unit sphere in \( \mathbb{R}^d \)). Furthermore, let \( [\Sigma, \mathcal{B}(\Sigma)] \) be the product space \( \mathcal{K} \times \mathbb{R}^d \) with the corresponding product \( \sigma \)-algebra. By \( M(\Sigma) \) we denote the family of locally finite measures on \( \Sigma \) equipped with the \( \sigma \)-algebra \( \mathfrak{M}(\Sigma) \) generated by sets of the form \( \{ \mu \in M(\Sigma) : a \leq \mu(S) \leq b \} \) for \( S \in \mathcal{B}(\Sigma) \) and \( 0 \leq a < b < \infty \).

Let

\[ H_k : \mathcal{K} \mapsto M(\Sigma) \]

be a \( (\sigma_k, \mathfrak{M}(\Sigma)) \)-measurable mapping. For each \( K \in \mathcal{K} \), \( H_K(\cdot) \) is a finite measure on \( \mathcal{B}(\Sigma) \). For the sake of convenience we write \( H_K(\Gamma, W) \) instead of \( H_K(\Gamma \times W) \) for \( \Gamma \in \mathfrak{U} \) and \( W \in \mathcal{B}_0^d \). The measure-valued function \( H \) is the basic object used to construct random measures associated with the germ-grain model.

We assume that the map \( H \) satisfies the following conditions

\[ H_K(\Gamma, W) = H_K(\Gamma, W \cap K), \]  

and

\[ H_{K+x}(\Gamma, W + x) = H_K(\Gamma, W) \]

for all \( K \in \mathcal{K}, \Gamma \in \mathfrak{U}, W \in \mathcal{B}_0^d \), and \( x \in \mathbb{R}^d \).
Central limit theorem for germ-grain models

In the definition of $h(f, W)$ the $i$th summand $H_{X_i + Z_i}(f, W \setminus \Xi_i)$ depends only on the part of the grain $X_i + Z_i$ which lies in $W$ and is not covered by other grains (this part is shown in dark). In many cases, the $i$th summand depends only on the outer (visible) boundary of the $i$th grain.

Moreover, we assume that

$$
E H_{Z_0}(\mathbb{U}, \mathbb{R}^d) < \infty,
$$

so that

$$
\overline{H}(\Gamma, W) = E H_{Z_0}(\Gamma, W)
$$

is a finite deterministic measure on $\mathcal{B}(\Sigma)$.

**Remark 3.1.** It is also possible to consider signed measures $H$. Then the finiteness condition must be replaced by the finiteness of the expected total variation of $H_{Z_0}$.

We are now in a position to define a locally finite random measure $\eta(\Gamma, W)$ on $\mathcal{B}(\Sigma)$ by

$$
\eta(\Gamma, W) = \sum_{i: i \geq 1} H_{X_i + Z_i}(\Gamma, W \setminus \Xi_i), \quad \Gamma \in \mathcal{U}, \ W \in \mathcal{B}_0^d.
$$

where

$$
\Xi_i = \bigcup_{j: j \neq i} (X_j + Z_j), \quad i \geq 1.
$$

Roughly speaking, $H_{X_i + Z_i}(\Gamma, W \setminus \Xi_i)$ is a contribution of the exposed part of the $i$th grain inside the window $W$ and satisfying some conditions specified by $\Gamma \subseteq \mathcal{U}$, see Figure 1.

The local finiteness of $\eta$ results from conditions which ensure the existence of the germ-grain model, for example, (3.2) is sufficient. Obviously, $\eta(\Gamma, W + x)$ equals $\eta(\Gamma, W)$ in distribution for all $\Gamma \in \mathcal{U}$, $W \in \mathcal{B}_0^d$, and $x \in \mathbb{R}^d$, i.e., $\eta(\Gamma, \cdot)$ is a stationary random measure on $\mathcal{B}_0^d$.

To avoid confusion, we put together our basic assumptions.

**Basic assumptions.**

(i) $\Psi_m$ is a simple stationary independently marked point process. The corresponding unmarked point process $\Psi$ has a finite positive intensity $\lambda$.

(ii) The measurable mapping $K \mapsto H_K(\cdot)$ satisfies (3.3) and (3.4).
(iii) The typical grain $Z_0$ satisfies (3.2) and (3.5).

Consider the family of measures $H_K(\cdot)$ which admit integral representation

$$
H_K(\Gamma, W) = \int_{\Gamma} \mathbf{1}_W(\ell(u, K)) \Upsilon_K(du), \quad K \in \mathcal{K},
$$

where $\Upsilon_K : \mathcal{K} \mapsto M(\mathbb{U})$ is a $(\sigma, M(\mathbb{U}))$-measurable mapping. Here $M(\mathbb{U})$ is the set of finite measures on $[\mathbb{U}, \mathcal{U}]$, and $M(\mathbb{U})$ is the $\sigma$-algebra generated by sets of the form $\{\mu \in M(\mathbb{U}) : a \leq \mu(U) \leq b\}$ for $U \in \mathcal{U}$ and $0 \leq a < b < \infty$. Assume that $\Upsilon_K(\cdot)$ is invariant under translations of $K$, i.e., $\Upsilon_K(\cdot) = \Upsilon_{K+x}(\cdot)$, and the mapping $\ell : \mathbb{U} \times \mathcal{K} \mapsto \mathbb{R}^d$ is $(\mathcal{F} \otimes \sigma_K, \mathcal{B}^d)$-measurable and satisfies the conditions

$$
\ell(u, K, x) \in K \quad \text{and} \quad \ell(u, K, x + x) = \ell(u, K) + x
$$

for all $K \in \mathcal{K}, \ u \in \mathbb{U}$ and $x \in \mathbb{R}^d$. Then both (3.3) and (3.4) hold; $H_K(\Gamma, \mathbb{R}^d) = \Upsilon_K(\Gamma)$ for all $K \in \mathcal{K}$, and (3.7) can be written as

$$
H_K(\Gamma, \mathbb{R}^d) = \int_{\mathbb{R}^d} f(\ell(u, K)) \Upsilon_K(du). \quad (3.10)
$$

In this case the following useful lemma holds.

**Lemma 3.1.** If $H_K(\cdot)$ admits integral representation (3.9), then, for each measurable function $f : \mathbb{R}^d \mapsto [0, \infty),$

$$
\mathbb{E} \int_W f(y) H_{Z_0}(\Gamma, dy) = \int_W f(y) H(\Gamma, dy) = \int_{\mathcal{K}} \int_{\Gamma} \mathbf{1}_W(\ell(u, K)) f(\ell(u, K)) \Upsilon_K(du) Q(dK).
$$

4. Moment measures of $\eta$

Below we will find the moment measures of the random measure $\eta$ given by (3.7). Further $P_{x_1, \ldots, x_k}^t(\cdot)$ denotes the $k$th-order (reduced) Palm distribution of $\Psi$ with respect to $x_1, \ldots, x_k \in \mathbb{R}^d$ (see [16] and [27, pp. 121–124]), and

$$
G_{x_1, \ldots, x_k}^t[f] = \int_{N} \prod_{i : z_i \in \psi} f(z_i) P_{x_1, \ldots, x_k}^t(d\psi)
$$

is the probability generating functional with respect to the Palm distribution $P_{x_1, \ldots, x_k}^t$.

For each compact set $K$ write

$$
\tau_K(z) = \mathbb{P}[z \notin \tilde{Z}_0 \oplus K] = \mathbb{P}[Z_0 \cap (K - z) = \emptyset],
$$

and write shortly $\tau_y$ instead of $\tau_{\{y\}}$.

The following lemma gives the first moment measure of $\eta$.

**Lemma 4.1.** Under the basic assumptions,

$$
\mathbb{E} \eta(\Gamma, W) = \lambda(\Gamma) |W|.
$$

where

$$
\lambda(\Gamma) = \lambda \int_{\mathbb{R}^d} G_{0}^t[\tau_y] \tilde{H}(\Gamma, dy).
$$
In particular, if $\Psi$ is a stationary Poisson process with intensity $\lambda$, then
\[ \lambda(\Gamma) = \lambda \exp\{-\lambda E|Z_0|\} \overline{H}(\Gamma, \mathbb{R}^d) = \lambda (1 - p) \overline{H}(\Gamma, \mathbb{R}^d), \] (4.4)
where $p = P\{0 \in Z\}$ is the volume fraction of $Z$.

Proof. Note that
\[ H_{X_i+Z_i}(\Gamma, W) = H_{Z_i}(\Gamma, W - X_i) = \int_{\mathbb{R}^d} 1_W(X_i + y) H_{Z_i}(\Gamma, dy). \]
Using the refined Campbell theorem [4, p.116] and the independent marking we get
\[ E\eta(\Gamma, W) = \mathbb{E} \sum_{i: \xi_i \in \Psi} \int_{\mathbb{R}^d} 1_W(X_i + y) \prod_{j:j \neq i} (1 - 1_{X_j+Z_j}(X_i + y)) H_{Z_i}(\Gamma, dy) \]
\[ = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_W(x + y) \times \prod_{j:j \neq i} \mathbb{E}(1 - 1_{Z_0}(x_j - x - y)) P_x^i(dy) H_K(\Gamma, dy) Q(dK) dx \]
\[ = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_W(x + y) \times \prod_{j:j \neq i} \mathbb{E}(1 - 1_{Z_0}(x_j - y)) P_0^i(dy) H_K(\Gamma, dy) Q(dK) dx. \]

By definition of the probability generating functional $G_0^i$ with respect to the Palm distribution $P_0^i$,
\[ G_0^i[\tau_y] = \int_{N} \prod_{j:j \neq i} \mathbb{E}(1 - 1_{Z_0}(x_j - y)) P_0^i(dy) H_K(\Gamma, dy) Q(dK). \]
Now (4.2) and (4.3) are easy to derive, since
\[ E\eta(\Gamma, W) = \lambda |W| \int_{\mathbb{R}^d} G_0^i[\tau_y] H_K(\Gamma, dy) Q(dK) = \lambda |W| \int_{\mathbb{R}^d} G_0^i[\tau_y] \overline{H}(\Gamma, dy). \]
If $\Psi$ is a Poisson process, then (4.4) follows from Slivnyak’s theorem [4, p.459] and the explicit formula for the probability generating functional [4, p.225].

Remark 4.1. If $H$ admits integral representation (3.9), then, by Lemma 3.1,
\[ \lambda(\Gamma) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_0^i[\tau_{\ell(u,K)}] \gamma_K(du) Q(dK). \] (4.5)
We proceed with the following general result which gives higher-order mixed moment measures of $\eta$. Note that the $k$th-order factorial moment measure of $\Psi$ is defined by
\[ \alpha^{(k)}(B_1 \times \cdots \times B_k) = \mathbb{E} \sum_{x_1, \ldots, x_k \in \Psi} 1_{B_1}(x_1) \cdots 1_{B_k}(x_k), \]
where $\sum_{x_1, \ldots, x_k \in \Psi}$ designates the sum over all $k$-tuples of pairwise distinct atoms of $\Psi \in N$.\[\]
Lemma 4.2. In addition to the above basic assumptions let the kth-order factorial moment measure \( \alpha^{(k)} \) of \( \Psi \) exist, \( \mathbb{E} H_{Z_0}^{k} (\mathbb{U}, \mathbb{R}^d) < \infty \), and, for some \( r > 0 \),
\[
\int (\mathbb{R}^d)^{\nu} \prod_{j=1}^{\nu} \mathbb{E}[H_{Z_0}^{k_j} (\mathbb{U}, B_r(-x_j))] \alpha^{(\nu)}(d(x_1, \ldots, x_k)) < \infty
\]  
for all \( 1 \leq \nu \leq k \) and positive numbers \( k_1, \ldots, k_\nu \) satisfying \( k_1 + \cdots + k_\nu = k \). Then, for every \( \Gamma_1, \ldots, \Gamma_\nu \in \mathbb{U} \) and \( W_1, \ldots, W_\nu \in \mathfrak{B}_{\nu}^{d} \),
\[
\mathbb{E}[\eta(\Gamma_1, W_1) \ldots \eta(\Gamma_\nu, W_\nu)]
\]

\[
= \sum_{v=1}^{k} \sum_{I_1 \cup \cdots \cup I_v = \{1, \ldots, k\}} \int (\mathbb{R}^d)^{\nu} \int (\mathbb{R}^d)^{\nu} \prod_{j=1}^{\nu} \prod_{i_j \in I_j} W_{i_j} (x_j + y_{i_j})
\]

\[
\times \prod_{i_j \in I_j} \prod_{i_j \in I_j} (1 - \mathbf{1}_{K_i + y_{i_j}} (x_i - x_j)) G_{x_1, \ldots, x_k} \left[ \tau_{i_{j_1}, \ldots, i_{j_\nu}} : 1 \leq j \leq \nu, i_j \in I_j \right]
\]

\[
\times \prod_{j=1}^{\nu} \prod_{i_j \in I_j} H_{K_j} (\Gamma_{i_j}, dy_{i_j}) \prod_{j=1}^{\nu} Q(dK_j) \alpha^{(\nu)}(d(x_1, \ldots, x_\nu)).
\]

Here the sum \( \sum_{I_1 \cup \cdots \cup I_v = \{1, \ldots, k\}} \) is taken over all non-empty partitions of the set \( \{1, \ldots, k\} \) into \( v \in \{1, \ldots, k\} \) subsets \( I_1, \ldots, I_v \).

Remark 4.2. If \( \Psi \) is a stationary Poisson process, then \( \alpha^{(\nu)}(d(x_1, \ldots, x_\nu)) = \lambda^\nu dx_1 \ldots dx_\nu \), condition (4.6) follows from \( \mathbb{E} H_{Z_0}^{k} (\mathbb{U}, \mathbb{R}^d) < \infty \), and
\[
G_{x_1, \ldots, x_k}^{1} (\tau_{x_j + y_{i_j}} : 1 \leq j \leq \nu, i_j \in I_j) = \exp \left\{ -\lambda \mathbb{E} \left[ \bigcup_{j=1}^{\nu} \bigcup_{i_j \in I_j} (Z_0 + x_j + y_{i_j}) \right] \right\}.
\]

Proof: For any \( k \)-tuple of functions \( f_j : \mathbb{R}^d \mapsto \mathbb{R}^1 \), \( j = 1, \ldots, k \), with bounded supports the identity
\[
\sum_{x_1, \ldots, x_\nu \in \Psi} f_1(x_1) \ldots f_k(x_k) = \sum_{v=1}^{k} \sum_{I_1 \cup \cdots \cup I_v = \{1, \ldots, k\}} \sum_{y_1, \ldots, y_\nu \in \Psi} \prod_{j=1}^{\nu} \prod_{i_j \in I_j} f_{i_j}(y_{i_j})
\]

enables us to write
\[
\mathbb{E}[\eta(\Gamma_1, W_1) \ldots \eta(\Gamma_\nu, W_\nu)]
\]

\[
= \sum_{v=1}^{k} \sum_{I_1 \cup \cdots \cup I_v = \{1, \ldots, k\}} \mathbb{E} \left[ \int (\mathbb{R}^d)^{\nu} \prod_{q_1, \ldots, q_\nu \geq 1} \prod_{i_j \in I_j} W_{i_j} (x_{q_j} + y_{i_j})
\]

\[
\times \prod_{j=1}^{\nu} \prod_{p_j \in I_j} \prod_{i_j \in I_j} (1 - \mathbf{1}_{Z_{p_j} - y_{i_j}} (X_{q_j} - X_{p_j})) \prod_{j=1}^{\nu} \prod_{i_j \in I_j} H_{Z_{q_j}} (\Gamma, dy_{i_j}) \right].
\]

By simple manipulations it is seen that, for fixed \( q_1, \ldots, q_\nu \) and \( I_1, \ldots, I_\nu \), the product
\[
\prod_{j=1}^{\nu} \prod_{p_j \in I_j} \prod_{i_j \in I_j} (1 - \mathbf{1}_{Z_{p_j} - y_{i_j}} (X_{q_j} - X_{p_j})) \prod_{j=1}^{\nu} \prod_{p_j \in I_j} \prod_{i_j \in I_j} (1 - \mathbf{1}_{Z_{p_j} + y_{i_j}} (X_{p_j} - X_{q_j}))
\]

enables us to write
\[
\mathbb{E}[\eta(\Gamma_1, W_1) \ldots \eta(\Gamma_\nu, W_\nu)]
\]
equals
\[
\prod_{j=1}^{V} \prod_{i,j \in I_j} \prod_{j \neq i} (1 - \mathbf{1}_{Z_{q_i} + y_{ij}}(X_{q_i} - X_{q_j})) \prod_{p \geq 1} \prod_{i,j \in I_j} \prod_{j \neq i} (1 - \mathbf{1}_{Z_{p} + y_{ij}}(X_{p} - X_{q_j})).
\]

Exploiting the independence of marks associated with distinct atoms and the independence between the marks and the point process \(\Psi\) combined with a multiple use of Fubini’s theorem we find that

\[
E\left[ \sum_{q_1, \ldots, q_r \geq 1} \int_{(\mathbb{R}^d)^r} \prod_{j=1}^{V} \prod_{i,j \in I_j} \mathbf{1}_{W_{q_j}}(x_{q_j} + y_{ij}) \prod_{i,j \neq i} (1 - \mathbf{1}_{Z_{q_i} + y_{ij}}(X_{q_i} - X_{q_j})) \prod_{j=1}^{V} \prod_{i,j \in I_j} H_{Z_{q_j}}(\Gamma, dy_{ij}) \right]
\]
\[
\times \int_{N} \prod_{y \in \Psi} E\left[ \sum_{j=1}^{V} \prod_{i,j \in I_j} (1 - \mathbf{1}_{Z_{0} + y_{ij}}(y - x_j)) \prod_{j=1}^{V} \prod_{i,j \in I_j} H_{Z_{0}}(\Gamma, dy_{ij}) \right]
\]
\[
\times P_{x_1, \ldots, x_v}(d\psi) E\left[ \prod_{j=1}^{V} \prod_{i,j \in I_j} H_{Z_{0}}(\Gamma, dy_{ij}) \right]
\]
\[
\times \alpha^{(v)}(d(x_1, \ldots, x_v)).
\]

Here we have also used the relationship

\[
\int_{N} \sum_{x_1, \ldots, x_v \in \Psi} f(x_1, \ldots, x_v, \psi - \delta_{x_1} - \cdots - \delta_{x_v}) P(d\psi)
\]
\[
= \int_{(\mathbb{R}^d)^v} \int_{N} f(x_1, \ldots, x_v, \psi) P_{x_1, \ldots, x_v}(d\psi) \alpha^{(v)}(d(x_1, \ldots, x_v)),
\]

which holds for any \((\mathcal{B}^d)^v \otimes \mathcal{N}\)-measurable function \(f : (\mathbb{R}^d)^v \times N \rightarrow \mathbb{R}^1\).

Finally, since

\[
E\left[ \prod_{j=1}^{V} \prod_{i,j \in I_j} (1 - \mathbf{1}_{Z_{0} + y_{ij}}(y - x_j)) \right] = P \left\{ y \notin \bigcup_{j=1}^{V} \bigcup_{i,j \in I_j} (\tilde{Z}_0 + y_{ij} + x_j) \right\},
\]

the definition of the probability generating functional \(G_{x_1, \ldots, x_v}^{(v)}(\tau_{[x_j + y_{ij}; 1 \leq j \leq v, i \in I_j]}\right)\) yields (4.7). The liberal use of Fubini’s theorem is justified because the right-hand side of (4.7) is bounded by

\[
\sum_{v=1}^{k} \sum_{k_1 + \cdots + k_v = k} \frac{k!}{k_1! \cdots k_v!} \int_{(\mathbb{R}^d)^v} \prod_{j=1}^{V} E[H_{Z_{0}}^{k_j}(\bigcup_{j=1}^{k} B_{\epsilon}(-x_j))] \alpha^{(v)}(d(x_1, \ldots, x_v)),
\]
where $r > 0$ is chosen such that $W_1 \cup \cdots \cup W_k \subseteq B_r(0)$. Now (4.6) implies that this sum is finite.

If the grain $Z_0$ is Hausdorff rectifiable ($H^m$-rectifiable) [35] with $m < d$, then $Z$ represents a particular class of the random processes of Hausdorff rectifiable closed sets in $\mathbb{R}^d$. This concept includes the well-known fibre and surface processes studied in [27]. Lemma 4.2 can be used to express the corresponding moment measures found in [35].

The second-order moment measures of $\eta$ depend on the second-order moment measure

$$
\overline{H}(\Gamma_1, W_1; \Gamma_2, W_2) = E[H_{Z_0}(\Gamma_1, W_1)H_{Z_0}(\Gamma_2, W_2)], \quad W_1, W_2 \in \mathcal{B}_0^d,
$$

of $H_{Z_0}(\cdot, \cdot)$, and the measure

$$
\Phi(x; \Gamma_1, W_1; \Gamma_2, W_2) = E[H_{Z_1}(\Gamma_1, (Z_2^x + x) \cap W_1)H_{Z_2}(\Gamma_2, (Z_1^x - x) \cap W_2)]
$$

defined for two independent grains $Z_1$ and $Z_2$ having the same distribution as $Z_0$. Furthermore, $\alpha^{(2)}_{\text{red}}$ (resp. $\gamma^{(2)}_{\text{red}}$) is the reduced second-order factorial moment (resp. reduced covariance) measure of $\psi$, i.e., $\alpha^{(2)}_{\text{red}}(B) = \int \psi(B)P_0^1(d\psi)$ and $\gamma^{(2)}_{\text{red}}(B) = \alpha^{(2)}_{\text{red}}(B) - \lambda |B|$ for $B \in \mathcal{B}_0^d$.

**Corollary 4.1.** If $\Psi$ is a second-order point process, $E\overline{H}_{Z_0}(\cdot, \cdot, \cdot) < \infty$ and

$$
\int_{(\mathbb{R}^d)^2} \overline{H}(U, W - x_1)\overline{H}(U, W - x_2)\alpha^{(2)}(d(x_1, x_2)) < \infty,
$$

then

$$
E[\eta(\Gamma_1, W)\eta(\Gamma_2, W)] = \lambda \int_{(\mathbb{R}^d)^2} \gamma_W(y_2 - y_1)G^1_{0}[\tau_{[y_1, y_2]}]\overline{H}(\Gamma_1, dy_1; \Gamma_2, dy_2) + \lambda \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^2} \gamma_W(x + y_2 - y_1)G^1_{0}[\tau_{[y_1, x + y_2]}]\Phi(x; \Gamma_1, dy_1; \Gamma_2, dy_2)\alpha^{(2)}_{\text{red}}(dx),
$$

where $\gamma_W(x) = |W \cap (W + x)|$ is the set-covariance function of $W$.

### 5. Absolute regularity of the random measure $\eta(\Gamma, \cdot)$

The absolute regularity coefficient (or $\beta$-mixing, or weak Bernoulli coefficient) $\beta(\mathcal{X}, \mathcal{Y})$ between any two sub-$\sigma$-algebras $\mathcal{X}, \mathcal{Y} \subset \mathcal{A}$ is defined by

$$
\beta(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \sup_k \sum_i \left| \mathbf{P}(A_k \cap B_l) - \mathbf{P}(A_k)\mathbf{P}(B_l) \right|,
$$

see [29]. The supremum is taken over all pairs of finite partitions $\{A_k\}$ and $\{B_l\}$ of $\Omega$ such that $A_k \in \mathcal{X}$ and $B_l \in \mathcal{Y}$. Standard measure-theoretic arguments ensure that the supremum in (5.1) does not change its value if the $A_k$'s and $B_l$'s belong to semi-algebras $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ generating $\mathcal{X}$ and $\mathcal{Y}$ respectively.

To be general enough, we consider the random measure

$$
\eta(f, \cdot) = \sum_{i \geq 1} \int f(u)H_{Z_i}(du, (\cdot) \setminus \Xi_i)
$$

(5.2)
for any measurable, bounded function \( f : \mathbb{U} \mapsto [0, \infty) \). For any \( B \in \mathcal{B}^d \) define sub-\( \sigma \)-algebras \( \mathcal{A}_{\eta(f)}(B) \) (respectively \( \mathcal{A}_{\psi}(B) \)) generated by events \( \{ \eta(f, B') \in [a, b) \} \) (respectively \( \{ \psi(B') = k \} \)) for \( B' \in \mathcal{B}_0^d, B' \subseteq B, \Gamma \in \mathcal{U}, 0 \leq a < b < \infty \), and \( k = 0, 1, 2, \ldots \). Furthermore, define the restricted germ-grain model

\[
Z(B) = \bigcup_{i: X_i \in B} (X_i + Z_i).
\]

The following lemma will serve as a cornerstone for the later proof of the asymptotic normality of \( \eta(f, W) \).

**Lemma 5.1.** For any two pairs of bounded Borel sets \( F, G \) and \( F', G' \) such that \( F \subseteq F' \), \( G \subseteq G' \) and \( F \cap G = \emptyset \) we have

\[
\beta(\mathcal{A}_{\eta(f)}(F), \mathcal{A}_{\eta(f)}(G)) \leq \beta(\mathcal{A}_{\psi}(F'), \mathcal{A}_{\psi}(G')) + 2P[Z(\tilde{F}') \cap F \neq \emptyset] + 2P[Z(\tilde{G}') \cap G \neq \emptyset].
\]  

**Remark 5.1.** Estimate (5.3) is in the spirit of Mase [17] who derived upper bounds of the \( \alpha \)-mixing coefficient of germ-grain models. Explicit estimates of the \( \beta \)-mixing coefficient of Voronoi tessellations and Poisson cluster processes are given in [13].

**Proof.** Let \( F_1, \ldots, F_p \) (resp. \( G_1, \ldots, G_q \)) form an arbitrary partition of \( F \) (resp. \( G \)). Define the following dissections of the probability space \( \Omega \) into disjoint events:

\[
A_k = \{ \eta(f, F_i) \in J_k^{(1)}, \ldots, \eta(f, F_p) \in J_k^{(p)} \},
\]

\[
B_l = \{ \eta(f, G_i) \in J_l^{(1)}, \ldots, \eta(f, G_q) \in J_l^{(q)} \}
\]

for \( k = (k_1, \ldots, k_p) \) and \( l = (l_1, \ldots, l_q) \) with \( k_i = 1, \ldots, m_i, \ i = 1, \ldots, p \) and \( l_j = 1, \ldots, n_j, \ j = 1, \ldots, q \). The intervals \( J_k^{(i)}, k_i = 1, \ldots, m_i, \) (resp. \( J_l^{(j)}, l_j = 1, \ldots, n_j \)) are pairwise disjoint and

\[
J_k^{(i)} \cup \cdots \cup J_n^{(i)} = [0, \infty) \quad \text{(resp. } J_k^{(i)} \cup \cdots \cup J_n^{(i)} = [0, \infty)\text{)}
\]

for any \( i = 1, \ldots, p \) (resp. \( j = 1, \ldots, q \)). According to the definition of the absolute regularity coefficient \( \beta(\mathcal{A}_{\eta(f)}(F), \mathcal{A}_{\eta(f)}(G)) \) we have

\[
\beta(\mathcal{A}_{\eta(f)}(F), \mathcal{A}_{\eta(f)}(G)) = \frac{1}{2} \sup \sum_k \sum_l |P(A_k \cap B_l) - P(A_k)P(B_l)|,
\]

where the supremum ranges over all partitions \( F_1, \ldots, F_p, G_1, \ldots, G_q, J_1^{(i)}, \ldots, J_n^{(i)} \) for \( i = 1, \ldots, p \) and \( J_1^{(j)}, \ldots, J_n^{(j)} \) for \( j = 1, \ldots, q \). We compare the events \( A_k \) with the corresponding events \( \tilde{A}_k = \{ \eta_{F}(f, F_i) \in J_k^{(1)}, \ldots, \eta_{F}(f, F_p) \in J_k^{(p)} \} \) arising from the ‘truncated’ random measure \( \eta_{\tilde{F}}(f, \cdot) \) given by

\[
\eta_{\tilde{F}}(f, \cdot) = \sum_{i: \tilde{X}_i \geq 1} \mathbf{1}_{\tilde{F}}(X_i) \int_{\mathcal{U}} f(u)H_{\tilde{Z}_i}(du, \cdot) \cup \Xi_i).
\]

Note that \( \eta(f, F') = \eta_{\tilde{F}}(f, F') \) for all \( F' \subseteq F \), as soon as \( Z(\tilde{F}') \cap F = \emptyset \). Then (3.3) yields

\[
\{Z(\tilde{F}') \cap F = \emptyset \} \cap \tilde{A}_k \subseteq A_k \quad \text{and} \quad \{Z(\tilde{F}') \cap F = \emptyset \} \cap A_k \subseteq \tilde{A}_k.
\]
Therefore,
\[ A_k \triangle \tilde{A}_k \subseteq \{Z(\tilde{F}^c) \cap F \neq \emptyset\} \cap (A_k \cup \tilde{A}_k), \]
where \( A \triangle B = (A \cap B^c) \cup (A^c \cap B) \) designates the symmetric difference. Thus,
\[ \sum_k \mathbb{P}(A_k \triangle \tilde{A}_k) \leq 2\mathbb{P}[Z(\tilde{F}^c) \cap F \neq \emptyset]. \]

Analogously,
\[ \sum_l \mathbb{P}(B_l \triangle \tilde{B}_l) \leq 2\mathbb{P}[Z(\tilde{G}^c) \cap G \neq \emptyset] \]
for the events \( \tilde{B}_l = \{\eta_G(f, G_1) \in J^{(1)}_l, \ldots, \eta_G(f, G_q) \in J^{(q)}_l\} \). After elementary manipulations we get
\[ \sum_k \sum_l |\mathbb{P}(A_k \cap B_l) - \mathbb{P}(A_k) \mathbb{P}(B_l) - (\mathbb{P}(A_k \cap \tilde{B}_l) - \mathbb{P}(\tilde{A}_k) \mathbb{P}(\tilde{B}_l))| \]
\[ \leq 2 \sum_k \mathbb{P}(A_k \triangle \tilde{A}_k) + 2 \sum_l \mathbb{P}(B_l \triangle \tilde{B}_l) \]
\[ \leq 4\mathbb{P}[Z(\tilde{F}^c) \cap F \neq \emptyset] + 4\mathbb{P}[Z(\tilde{G}^c) \cap G \neq \emptyset]. \quad (5.4) \]

Note that \( \tilde{A}_k \) and \( \tilde{B}_l \) are conditionally independent given \( \Psi \). Using arguments similar to [34] and the formula (4.6) of [13] we obtain that
\[ \sum_k \sum_l |\mathbb{P}(\tilde{A}_k \cap \tilde{B}_l) - \mathbb{P}(\tilde{A}_k) \mathbb{P}(\tilde{B}_l)| \leq 2\beta(\mathfrak{A}_\psi(\tilde{F}), \mathfrak{A}_\psi(\tilde{G})). \]
The latter estimate combined with (5.4) completes the proof of Lemma 5.1.

Next we specify (5.3) for
\[ F = [-a, a]^d, \quad G = \mathbb{R}^d \setminus [-b, b]^d, \]
\[ \tilde{F} = -(a + \Delta), a + \Delta]^d, \quad \tilde{G} = \mathbb{R}^d \setminus -(b - \Delta), b - \Delta]^d \]
with \( 0 < a < b < \infty \) and \( \Delta = (b - a)/4 \). Note that by a simple approximation argument inequality (5.3) remains valid for unbounded \( G \) and \( \tilde{G} \).

**Lemma 5.2.** Let \( \Psi \) be a stationary point process with intensity \( \lambda > 0 \). Assume \( \mathbb{E}\|Z_0\|^d < \infty \). Then, for the \( F, \tilde{F} \) and \( G, \tilde{G} \) defined above,
\[ \mathbb{P}[Z(\tilde{F}^c) \cap F \neq \emptyset] \leq \lambda d2^d \left(1 + \frac{a}{\Delta}\right)^{d-1} \int_{\Delta}^\infty x^d dD(x) \]
and
\[ \mathbb{P}[Z(\tilde{G}^c) \cap G \neq \emptyset] \leq \lambda d2^d \left(1 + \frac{3a}{\Delta}\right)^{d-1} \int_{\Delta}^\infty x^d dD(x), \]
where \( D(x) = \mathbb{P}[\|Z_0\| \leq x], \ x \geq 0 \).
Proof. By the definition of the germ-grain model (3.1) we obtain that

$$\mathbb{P}(Z(\tilde{F}^c) \cap F \neq \emptyset) = \frac{1}{N} \left[ \prod_{i : x_i \in \tilde{F}} (1 - (1 - 1_{\tilde{F}}(x_i))) \mathbb{P}((Z_0 + x_i) \cap F \neq \emptyset) \right] \mathbb{P}(d\psi).$$

Using the elementary inequality $1 - \prod (1 - x_i) \leq \sum x_i$, $0 \leq x_i \leq 1$, and applying Campbell’s and Fubini’s theorem leads to

$$\mathbb{P}(Z(\tilde{F}^c) \cap F \neq \emptyset) \leq \frac{1}{N} \int_{\tilde{F}} \mathbb{P}(x \in Z_0 + \tilde{F}) \, dx = \lambda \mathbb{E}[\tilde{F}^c \cap (F + Z_0)].$$

Since $Z_0 + F \subseteq \{-(a + ||Z_0||), a + ||Z_0||\}^d$, we can continue with

$$\mathbb{P}(Z(\tilde{F}^c) \cap F \neq \emptyset) \leq \lambda \mathbb{E}\left[\{-(a + \Delta), a + |\Delta|\}^d \setminus \{-(a + ||Z_0||), a + ||Z_0||\}\right]$$

$$= \lambda 2^d \int_{\Delta} ((a + x)^d - (a + \Delta)^d) \, dD(x)$$

$$\leq \lambda d 2^d \left(\frac{a + \Delta}{\Delta}\right)^{d-1} \int_{\Delta} x^d \, dD(x).$$

Similarly,

$$\mathbb{P}(Z(\tilde{G}^c) \cap G \neq \emptyset) \leq \lambda \mathbb{E}\left[\{-(b - \Delta), b - |\Delta|\}^d \setminus \{-(b + ||Z_0||), b + ||Z_0||\}\right]$$

$$\leq \lambda d 2^d \left(\frac{b - \Delta}{\Delta}\right)^{d-1} \int_{\Delta} x^d \, dD(x).$$

Thus, by $b - \Delta = a + 3\Delta$, the proof of Lemma 5.2 is completed.

6. Limit theorems for associated random measures

The ergodic theorem of Nguyen and Zessin [23] can be used to establish the spatial strong law of large numbers for the measure $\eta(\Gamma, W)$ if $W \uparrow \mathbb{R}^d$, i.e., $W$ expands infinitely in all directions in a regular way, see [4, p. 332].

Theorem 6.1. In addition to the basic assumptions assume that $\Psi$ is ergodic (under $d$-dimensional shifts, see [4, p. 341]). Then, for any $\Gamma \in \mathbb{K}$,

$$|W|^{-1} \eta(\Gamma, W) \to \lambda(\Gamma) \quad a.s. \quad as \quad W \uparrow \mathbb{R}^d$$

with $\lambda(\Gamma)$ defined in Lemma 4.1.

Proof. First, note that for any $K \in \mathbb{K}$ we have

$$\int_{\mathbb{R}^d} H_K(\Gamma, W + x) \, dx = |W| H_K(\Gamma, \mathbb{R}^d).$$

In [11] it was proved that the ergodicity of $\Psi$ entails the ergodicity of $Z$ provided (3.2) is valid. This, in turn, implies that the spatial stochastic process $\eta(\Gamma, F)$, $F \in \mathbb{B}_d^d$, $(\Gamma$ is fixed) is ergodic under $d$-dimensional shifts. In order to apply Corollary 4.20 in [23], it is necessary to
bound the family $\eta(\Gamma, F)$, $F \subseteq [0, 1]^d$, $F \in \mathfrak{B}_0^d$, by some integrable random variable $Y$ being independent of $F$. By (3.3) and (3.4),

$$\eta(\Gamma, F) \leq \sum_{i,j \geq 1} H_{Z_1}(\Gamma, [0, 1]^d - X_i) = Y.$$ Together with Campbell’s theorem,

$$EY \leq \lambda \int_{[0, 1]^d} E H_{Z_0+X}([0, 1]^d) \, dx = \lambda \overline{H}(\Gamma, [0, 1]^d) < \infty$$

proving (6.1).

This theorem, in particular, yields the almost sure convergence of spatial intensities for extensions of the intrinsic volumes, see also [11] and [33].

Below we formulate a central limit theorem for the finite-dimensional distributions of the set-indexed sequence

$$\tilde{\eta}(\Gamma, W_n) = \frac{|W_n|^{1/2}}{\lambda(\Gamma)} \left( \eta(\Gamma, W_n) - \lambda(\Gamma) \right), \Gamma \in \mathcal{U},$$

where $W_n$ denotes the cube $[-n, n]^d$. For this, we need a suitable formulation of the $\beta$-mixing condition imposed on the underlying point process $\Psi$. Assume that

$$\beta(\mathfrak{A}_\psi([-a, a]^d), \mathfrak{A}_\psi([-a - \Delta]^d, [a + \Delta]^d)) \leq \left( \frac{a}{\min(a, \Delta)} \right)^{d-1} \beta_\psi(\Delta)$$

for any $a, \Delta \geq 1$, where $\beta_\psi(\cdot)$ is a non-increasing function on $[1, \infty)$. Condition (6.3) expresses the degree of weak dependence (in terms of the $\beta$-mixing coefficient) between the behaviour of the point process $\Psi$ inside $[-a, a]^d$ and outside $[-(a + \Delta), a + \Delta]^d$. Following the general concept of spatial mixing it is quite natural and even necessary that the mixing concept depends on both the distance and the size (volume or surface) of the two separated support sets, where at least one of them must be bounded, see also [5].

**Theorem 6.2.** In addition to the above basic assumptions suppose that there exists $\delta > 0$ such that

$$E \left[ \sum_{i,j \geq 1} H_{X_i+Z_j}([0, 1]^d) \right]^{2+\delta} < \infty,$$

$$E \| Z_0 \|^{2d(1+\delta)/(\delta+\varepsilon)} < \infty \text{ for some } \varepsilon > 0,$$

$$\sum_{n=1}^{\infty} n^{-d-1} (\beta_\psi(n))^{\delta/(2+\delta)} < \infty.$$

(If $\sum_{i,j \geq 1} H_{X_i+Z_j}([0, 1]^d) \leq c$ with probability 1 for some constant $c < \infty$, then put $\delta = \infty$ and $\varepsilon = 0$ in (6.5) and (6.6).)

Then, for any $k$-tuple $\Gamma_1, \ldots, \Gamma_k \in \mathcal{U}$, the random vector $(\tilde{\eta}(\Gamma_1, W_n), \ldots, \tilde{\eta}(\Gamma_k, W_n))$ converges in distribution as $n \to \infty$ to a $k$-dimensional centred Gaussian random vector $(\xi_1, \ldots, \xi_k)$ with covariances $E \xi_i \xi_j = \sigma^2(\Gamma_i, \Gamma_j)$, $1 \leq i \leq j \leq k$, given by
\[\sigma^2(\Gamma_i, \Gamma_j) = \lambda \int_{\mathbb{R}^d} G_0'[\tau_{(y_1,y_2)}] \overline{H}(\Gamma_i, dy_1; \Gamma_j, dy_2) \]
\[+ \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{0,x}'[\tau_{(y_1,x+y_2)}] \Phi(x; \Gamma_i, dy_1; \Gamma_j, dy_2) \gamma_{\text{red}}^{(2)}(dx) \]
\[+ \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{0,x+y_1-y_2}'[\tau_{(y_1,x+y_1)}] \]
\[- G_0'[\tau_{y_1}] G_0'[\tau_{y_2}] \Phi(x, \Gamma_i, dy_1) \Phi(-x, \Gamma_j, dy_2) dx \]
\[- \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_0'[\tau_{y_1}] G_0'[\tau_{y_2}] \Phi(x, \Gamma_i, dy_1) \Phi(-x, \Gamma_j, dy_2) dx, \quad (6.7)\]

where the function \(\tau\) and the measures \(\overline{H}(\Gamma_i, dy_1; \Gamma_j, dy_2)\) and \(\Phi(x; \Gamma_i, dy_1; \Gamma_j, dy_2)\) are defined in (4.1), (4.8) and (4.9) respectively, and
\[
\Phi(x, \Gamma, W) = \mathbb{E}[HZ_0(\Gamma, (Z_0^c + x) \cap W)], \quad (6.8) \\
\hat{\Phi}(x, \Gamma, W) = \overline{H}(\Gamma, W) - \Phi(x, \Gamma, W). \quad (6.9)
\]

In (6.7) all integrals converge absolutely as a consequence of the mixing and moment conditions (6.4)-(6.6).

**Remark 6.1.** Conditions (6.4)-(6.6) ensure the convergence of covariances, i.e.,
\[
\mathbb{E}[\hat{\eta}(\Gamma_i, W_n)\hat{\eta}(\Gamma_j, W_n)] \to \sigma^2(\Gamma_i, \Gamma_j) \quad \text{as} \quad n \to \infty.
\]

**Remark 6.2.** Let \(l\) denote the smallest integer greater than or equal to \(2 + \delta\). If the total variation of the \(k\)-th order reduced factorial moment measure \([4, \text{p. 357}] \gamma_{\text{red}}^{(k)}\) if finite for \(k = 2, \ldots, l\), and \(\mathbb{E}H_{Z_0}^\rho(\mathbb{U}, \mathbb{R}^d) < \infty\), then (6.4) holds. This is immediately seen from Lemma 4.2.

**Remark 6.3.** The \(\beta\)-mixing condition (6.6) can be verified for quite a few classes of point processes under mild additional assumptions. For example, in the special case of a Poisson cluster process \(\Psi\) (which is a Boolean model with a random discrete almost surely finite typical grain or cluster \(Z_c\)) we have by Lemmas 5.1 and 5.2 that
\[
\beta_{\Psi}(t) \leq 4\lambda d 6^{d-1} \mathbb{E}[\rho_c^d I_{\{\rho_c \geq t/2\}}] 
\]
with \(\rho_c = \sup\{\|x\| : x \in Z_c\}\).

Similar estimates of the \(\beta\)-mixing rate are known for
- dependently thinned (Poisson) point processes (e.g. soft- and hard-core processes) as defined by Matérn, and their generalisations, see [27];
- Gibbs point processes satisfying Dobrushin’s uniqueness conditions;
- point processes generated by a (Poisson-) Voronoi tessellation of \(\mathbb{R}^d\) (e.g., vertices, midpoint of edges), see [13].

**Proof of Theorem 6.2.** According to the Cramér-Wold device we need to prove that, for any \((a_1, \ldots, a_k) \in \mathbb{R}^k \setminus \{0\}\), the sum
\[
S_n = \sum_{j=1}^k a_j \hat{\eta}(\Gamma_j, W_n)
\]
converges weakly as \( n \to \infty \) to a Gaussian random variable \( \xi \) with mean zero and variance
\[
\sigma^2 = \sum_{i,j=1}^k a_i a_j \sigma^2(\Gamma_i, \Gamma_j).
\]
In order to apply a central limit theorem for stationary \( \beta \)-mixing random fields in [13] we rewrite the above sum as
\[
|W_n|^{1/2} S_n = \sum_{z \in I_n} X_z \quad \text{with} \quad X_z = \sum_{j=1}^k a_j(\eta(\Gamma_j, E_z) - \lambda(\Gamma_j)),
\]
where \( E_z = [0, 1]^d + z \) for \( z \in I_n = \{-n, \ldots, 0, \ldots, n-1\}^d \).

From the simple estimate
\[
\mu(U, [0, 1]^d) = \sum_{i : i \geq 1} z \in \eta([0, 1]^d)
\]
together with (6.4) we deduce that \( \mathbb{E}|X_0|^{2 + \delta} < \infty \). In view of Lemmas 5.1 and 5.2 combined with (6.3) we obtain
\[
\beta(\mathbb{A}_\eta(f), (\mathbb{A}_\eta(f), \mathbb{B}_\eta(f))([0, 1]^d \setminus [-(a + \Delta), a + \Delta]^d))
\]
\[
\leq \left( \frac{4a + \Delta}{2\Delta} \right)^d + \lambda d^2 + 1 \int_{\Delta/4}^{\infty} x^d dD(x)
\]
for all \( a, \Delta \geq 1 \) and any bounded measurable function \( f : \mathbb{U} \to [0, \infty) \). The right-hand side
of the latter inequality can be bounded by a term of the form
\[
\left( \frac{a}{\min(a, \Delta)} \right)^d \beta_\eta(\Delta)
\]
with
\[
\beta_\eta(\Delta) = c_1 \psi(\Delta/2) + c_2 \int_{\Delta/4}^{\infty} x^d dD(x)
\]
and constants \( c_1 \) and \( c_2 \) depending only on \( d \). In order to verify the \( \beta \)-mixing condition needed in the central limit theorem in [13] we have to ensure that
\[
\sum_{n=1}^{\infty} n^{d-1} (\beta_\eta(n))^{\delta/(2 + \delta)} < \infty \quad \text{and} \quad n^d \beta_\eta(n) \to 0 \quad \text{as} \quad n \to \infty.
\]
In turn, this follows from (6.5), (6.6) and the fact that \( \beta_\eta(\cdot) \) is a non-increasing function.

If \( |X_z| \leq c, z \in I_n \), for some constant \( c \), we need the convergence of the series
\[
\sum_{n=1}^{\infty} n^{d-1} \beta_\eta(n)
\]
and
\[
\sum_{n=1}^{\infty} n^{d-1} \psi(n/2) + \sum_{n=1}^{\infty} n^{d-1} \int_{n/4}^{\infty} x^d dD(x).
\]
This results from (6.6) for \( \delta = \infty \) and \( \mathbb{E}\|Z_0\|^{2d} < \infty \), so that the proof of Theorem 6.2 is completed.
7. A central limit theorem for random measures generated by Boolean models

From now on, let $\Psi$ be a stationary Poisson process with intensity $\lambda > 0$, i.e. the corresponding germ-grain model $Z$ is a Boolean model [19, 27]. In this case, by Slivnyak’s theorem, $\gamma_{\text{red}}^{(2)}(\cdot)$ vanishes identically and

$$G_{x_1, x_2}(r_{[0, v]}) = \exp[-\lambda E|Z_0 \cup (Z_0 + v)|] = q(v)(1 - p)^2,$$

(7.1)

where $p = P[0 \in Z] = 1 - \exp[-\lambda E|Z_0|]$ is the volume fraction of $Z$,

$$q(v) = \frac{C(v) - p^2}{(1 - p)^2} + 1 = \exp[\lambda E|Z_0 \cap (Z_0 + v)|], \ v \in \mathbb{R}^d,$$

(7.2)

and $C(v) = P[0 \in Z, v \in Z]$ is the covariance function of $Z$. Using these formulae together with (6.8) we can simplify the covariances $\sigma^2(\Gamma_i, \Gamma_j)$ in Theorem 6.1 as follows:

$$\sigma^2(\Gamma_i, \Gamma_j) = \lambda(1 - p)^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q(y_1 - y_2) \Phi(x, \Gamma_i, dy_1; \Gamma_j, dy_2)
\quad + \lambda^2(1 - p)^2 \int_{\mathbb{R}^d} (q(x)\Phi(x, \Gamma_i)\Phi(-x, \Gamma_j) - \overline{H}(\Gamma_i, \mathbb{R}^d)\overline{H}(\Gamma_j, \mathbb{R}^d)) \ dx,$$

(7.3)

where

$$\Phi(x, \Gamma) = \Phi(x, \Gamma, \mathbb{R}^d) = E[H_{Z_0}(\Gamma, Z_0^c + x)].$$

Taking into account the inequalities

$$\int_{\mathbb{R}^d} (q(x) - 1) \ dx \leq \lambda E|Z_0|^2 \exp[\lambda E|Z_0|]$$

and

$$\int_{\mathbb{R}^d} |\Phi(x, \Gamma_i)\Phi(-x, \Gamma_j) - \overline{H}(\Gamma_i, \mathbb{R}^d)\overline{H}(\Gamma_j, \mathbb{R}^d)| \ dx
\quad \leq E|Z_0|H_{Z_0}(\Gamma_i, \mathbb{R}^d) + E|Z_0|H_{Z_0}(\Gamma_j, \mathbb{R}^d),$$

we conclude that $\sigma^2(\Gamma_i, \Gamma_j), 1 \leq i, j \leq k$, are finite whenever

$$E|Z_0|^2 < \infty \quad \text{and} \quad E H_{Z_0}^2(\Gamma_i, \mathbb{R}^d) < \infty \quad \text{for} \quad i = 1, \ldots, k.$$

(7.4)

The following Theorem 7.1 restates the main result of the preceding section in case of a stationary Poisson process of germs under considerably relaxed conditions. In fact, these conditions are optimal because they are necessary to ensure the existence of the covariance matrix. This improvement results from a suitable (although somewhat laborious) approximation technique by m-dependent fields which is quite different from that used in [9, 10] and [12].

**Theorem 7.1.** If $\Psi$ is a stationary Poisson process with intensity $\lambda$ and (7.4) is satisfied, then $(\hat{\eta}(\Gamma_1, W_n), \ldots, \hat{\eta}(\Gamma_k, W_n))$ converges in distribution as $n \to \infty$ to a Gaussian centred random vector $(\xi_1, \ldots, \xi_k)$ with the covariances $E\xi_i \xi_j = \sigma^2(\Gamma_i, \Gamma_j), 1 \leq i, j \leq k$, given by (7.3).
Proof. We only need to consider the univariate case \( k = 1 \) for some fixed \( \Gamma = \Gamma_1 \in \mathcal{U} \). With the notation introduced in the proof of Theorem 6.2 we put

\[
S_n = |W_n|^{-1/2} \sum_{z \in I_n} [X_z - \lambda(1-p)H(\Gamma, \mathbb{R}^d)] \quad \text{with} \quad X_z = \eta(\Gamma, E_z).
\]

Setting \( F_z = E_z \oplus W_m \) for some fixed integer \( m \geq 1 \) and \( z \in \mathbb{Z}^d = \{0, \pm 1, \pm 2, \ldots\}^d \) we decompose \( S_n = S_n^{(m)} + S_n^{(m)} + S_n^{(m)} \) by splitting \( X_z \) into three random variables:

\[
X_z^{(m)} = \sum_{i \geq 1} 1_{F_z}(X_i) HZ_i + X_i(\Gamma, E_z \setminus \Xi_i(F_z)) \quad \text{with} \quad \Xi_i(B) = \bigcup_{j \neq i, j \in B} (Z_j + X_j),
\]

\[
X_{z,1}^{(m)} = \sum_{i \geq 1} 1_{F_z}(X_i) HZ_i + X_i(\Gamma, E_z \setminus \Xi_i(F_z)),
\]

\[
X_{z,2}^{(m)} = \sum_{i \geq 1} (HZ_i + X_i(\Gamma, E_z \setminus \Xi_i)) - HZ_i + X_i(\Gamma, E_z \setminus \Xi_i(F_z))).
\]

We first note that, by our assumptions, the independently marked Poisson counting measures

\[
\Psi_z = \sum_{i \geq 1} 1_{F_z}(X_i) \delta_{[X_i, Z_i]}, \quad z \in \mathbb{Z}^d,
\]

can be considered as a family of independent identically distributed random elements taking values in some measurable space \([N_{\text{mark}}, \mathcal{N}_{\text{mark}}]\) of marked counting measures defined on \([0, 1]^d \times \mathcal{K}\), see [6] for details. Therefore, having in mind the properties of \( H_K(\cdot), K \in \mathcal{K} \), it is easily seen that the random variables \( X_z^{(m)}, z \in I_n \), constitute a stationary \( 2m \)-dependent random field which allows a block representation

\[
X_z^{(m)} = g(\Psi_y; y \in \{-m, \ldots, m\}^d + z), \quad z \in \mathbb{Z}^d,
\]

where \( g : N_{\text{mark}}^{(2m+1)} \rightarrow \mathbb{R}^1 \) is a \( N_{\text{mark}}^{(2m+1)} \)-measurable function. Applying the central limit theorem for this type of weakly dependent fields, see, e.g. [9], yields the weak convergence of \( S_n^{(m)} \) (as \( n \rightarrow \infty \)) to a centred Gaussian random variable with variance

\[
\sigma_m^2 = \sum_{z \in \{-m, \ldots, m\}^d} \text{Cov}(X_0^{(m)}, X_z^{(m)}),
\]

if \( \mathbf{E}(X_0^{(m)})^2 < \infty \). The latter holds for any \( m \geq 1 \), if \( \mathbf{E}|Z_0|^2 < \infty \) and \( \mathbf{E} H_{Z_0}^2(\Gamma, \mathbb{R}^d) < \infty \). In order to prove the asymptotic normality of \( S_n \) it remains to verify that

\[
\sup_{n \geq 1} \mathbf{E}(X_{n,i}^{(m)})^2 \leq s_i^{(m)} = \sum_{z \in \mathbb{Z}^d} |\text{Cov}(X_{0,i}^{(m)}, X_z^{(m)})| \longrightarrow 0 \quad \text{as} \quad m \rightarrow \infty \quad \text{for} \quad i = 1, 2.
\]

After straightforward calculations similar to those leading to (7.3) followed by some obvious
7. A central limit theorem for random measures generated by Boolean models

From now on, let $\Psi$ be a stationary Poisson process with intensity $\lambda > 0$, i.e. the corresponding germ-grain model $Z$ is a Boolean model [19, 27]. In this case, by Slivnyak’s theorem, $\gamma^{(2)}_{\text{red}}(\cdot)$ vanishes identically and

$$G_{x_1,x_2}^{(2)}[\tau(0,v)] = \exp\{-\lambda E|Z_0 \cup (Z_0 + v)|\} = q(v)(1 - p)^2,$$

where $p = P[0 \in Z] = 1 - \exp\{-\lambda E|Z_0|\}$ is the volume fraction of $Z$,

$$q(v) = \frac{C(v) - p^2}{(1 - p)^2} + 1 = \exp[\lambda E|Z_0 \cap (Z_0 + v)|], \quad v \in \mathbb{R}^d,$$

and $C(v) = P[0 \in Z, v \in Z]$ is the covariance function of $Z$. Using these formulae together with (6.8) we can simplify the covariances $\sigma^2(\Gamma_i, \Gamma_j)$ in Theorem 6.2 as follows:

$$\sigma^2(\Gamma_i, \Gamma_j) = \lambda(1 - p)^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q(y_1 - y_2)H(\Gamma_i, dy_1; \Gamma_j, dy_2)$$

$$+ \lambda^2(1 - p)^2 \int_{\mathbb{R}^d} q(x)\Phi(\Gamma_i, \Gamma_j)\Phi(-x, \Gamma_j) - H(\Gamma_i, \mathbb{R}^d)H(\Gamma_j, \mathbb{R}^d) \, dx,$$

(7.3)

where

$$\Phi(\Gamma, \Gamma') = \Phi(\Gamma, \Gamma', \mathbb{R}^d) = E[H_{Z_0}(\Gamma, Z_0 + x)].$$

Taking into account the inequalities

$$\int_{\mathbb{R}^d} (q(x) - 1) \, dx \leq \lambda E|Z_0|^2 \exp[\lambda E|Z_0|]$$

and

$$\int_{\mathbb{R}^d} |\Phi(\Gamma_i, \Gamma_j)\Phi(-x, \Gamma_j) - H(\Gamma_i, \mathbb{R}^d)H(\Gamma_j, \mathbb{R}^d)| \, dx$$

$$\leq E|Z_0|H_{Z_0}(\Gamma_i, \mathbb{R}^d) + E|Z_0|H_{Z_0}(\Gamma_j, \mathbb{R}^d),$$

we conclude that $\sigma^2(\Gamma_i, \Gamma_j), 1 \leq i, j \leq k$, are finite whenever

$$E|Z_0|^2 < \infty \quad \text{and} \quad E H_{Z_0}^2(\Gamma_i, \mathbb{R}^d) < \infty \quad \text{for} \quad i = 1, \ldots, k.$$  

(7.4)

The following Theorem 7.1 restates the main result of the preceding section in case of a stationary Poisson process of germs under considerably relaxed conditions. In fact, these conditions are optimal because they are necessary to ensure the existence of the covariance matrix. This improvement results from a suitable (although somewhat laborious) approximation technique by $m$-dependent fields which is quite different from that used in [9, 10] and [12].

**Theorem 7.1.** If $\Psi$ is a stationary Poisson process with intensity $\lambda$ and (7.4) is satisfied, then $(\hat{\eta}(\Gamma_1, W_n), \ldots, \hat{\eta}(\Gamma_k, W_n))$ converges in distribution as $n \to \infty$ to a Gaussian centred random vector $(\xi_1, \ldots, \xi_k)$ with the covariances $E\xi_i\xi_j = \sigma^2(\Gamma_i, \Gamma_j), 1 \leq i, j \leq k$, given by (7.3).
estimates we obtain
\[
\text{Cov}(X_{m,1}^{(m)}, X_{m,1}^{(m)}) \leq \lambda \int_{F_0 \cap F_1} E[H_{Z_0}(\Gamma, E_0 - x)H_{Z_0}(\Gamma, E_2 - x)] \, dx + \lambda^2 \int_{F_0} \int_{F_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{E_0}(x_1 + y_1)1_{E_2}(x_2 + y_2) \\
\times (a(\mathbb{R}^d, K_2) + b(\mathbb{R}^d, K_1) + 1 - \exp(-w(F_0, F_2))) \\
\times H_{K_1}(\Gamma, dy_1)H_{K_2}(\Gamma, dy_2)Q(dK_1)Q(dK_2) \, dx \, dx_1,
\]
where
\[
a(F_0, K_2) = 1_{F_0 \cap (K_2 + x_1 + y_1)}(x_1), \quad b(F_2, K_1) = 1_{F_2 \cap (K_1 + x_2 + y_2)}(x_1),
\]
and
\[
w(F_0, F_2) = \lambda E[|F_0 \cap (\tilde{Z}_0 + x_1 + y_1) \cap F_2 \cap (\tilde{Z}_0 + x_2 + y_2)].
\]
Therefore,
\[
\text{Cov}(X_{m,1}^{(m)}, X_{m,1}^{(m)}) \leq \lambda \int_{F_0 \cap F_1} E[H_{Z_0}(\Gamma, E_0 - x)H_{Z_0}(\Gamma, E_2 - x)] \, dx + \lambda^2 \int_{F_0} \int_{F_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{E_0}(x_1 + y_1)1_{E_2}(x_2 + y_2) \\
\times (\exp(-\lambda E[F_0 - (x_1 + y_1)]) \cap \tilde{Z}_0] - \exp(-\lambda E[\tilde{Z}_0])] \times H_{Z_0}(\Gamma, dy_1)H_{Z_0}(\Gamma, dy_2) \, dx \\
+ \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{E_0}(x_1 + y_1)1_{E_2}(x_2 + y_2) \\
\times |I(\mathbb{R}^d, \mathbb{R}^d) - I(\mathbb{R}^d, F_0) - I(\mathbb{R}^d, F_2) + I(F_0, F_2)| \\
\times H_{K_1}(\Gamma, dy_1)H_{K_2}(\Gamma, dy_2)Q(dK_1)Q(dK_2) \, dx_1 \, dx_1,
\]
In view of our moment assumptions (7.4), it follows from the Lebesgue dominated convergence theorem that the right-hand side tends to zero as \(m \to \infty\).

Next we estimate the covariances occurring in \(s_2^{(m)}\). For notational simplicity write
\[
u(F_0) = \lambda E[(F_0 - (x_1 + y_1)) \cap \tilde{Z}_0], \quad v(F_2) = \lambda E[(F_2 - (x_2 + y_2)) \cap \tilde{Z}_0],
\]
and
\[
I(F_0, F_2) = \exp[-(\nu(F_0)+v(F_2))](1-\alpha(F_0, K_2))(1-b(F_2, K_1)) \exp[w(F_0, F_2) - 1].
\]
Then
\[
\text{Cov}(X_{m,2}^{(m)}, X_{m,2}^{(m)}) \leq \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{E_0}(x_1 + y_1)1_{E_2}(x_2 + y_2) \\
\times (\exp(-\lambda E[F_0 - (x_1 + y_1)]) \cap \tilde{Z}_0] - \exp(-\lambda E[\tilde{Z}_0])] \times H_{Z_0}(\Gamma, dy_1)H_{Z_0}(\Gamma, dy_2) \, dx \\
+ \lambda^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{E_0}(x_1 + y_1)1_{E_2}(x_2 + y_2) \\
\times |I(\mathbb{R}^d, \mathbb{R}^d) - I(\mathbb{R}^d, F_0) - I(\mathbb{R}^d, F_2) + I(F_0, F_2)| \\
\times H_{K_1}(\Gamma, dy_1)H_{K_2}(\Gamma, dy_2)Q(dK_1)Q(dK_2) \, dx \, dx_1.
\]
Therefore,

\[
\begin{align*}
\varsigma^{(m)}_2 & \leq \lambda^2 \mathbb{E}[(F_0^c \oplus E_0) \cap \tilde{Z}_0] \mathbb{E}H_{Z_0}^2(\Gamma, \mathbb{R}^d) \\
& + (\lambda \mathbb{E}[(F_0^c \oplus E_0) \cap \tilde{Z}_0])^2 (\lambda \mathbb{H}(\Gamma, \mathbb{R}^d \mathbb{E}|Z_0|^2 + 2 \mathbb{H}(\Gamma, \mathbb{R}^d \mathbb{E}|Z_0|H_{Z_0}(\Gamma, \mathbb{R}^d) \\
& + \mathbb{E}[(F_0^c \oplus E_0) \cap \tilde{Z}_0]|H_{Z_0}(\Gamma, \mathbb{R}^d)(1 + 2\lambda \mathbb{E}|Z_0|)\mathbb{H}(\Gamma, \mathbb{R}^d) \\
& + \mathbb{E}[(F_0^c \oplus E_0) \cap \tilde{Z}_0]|Z_0|(2\lambda + 3\lambda^2 \mathbb{E}|Z_0|(\mathbb{H}(\Gamma, \mathbb{R}^d))^2),
\end{align*}
\]

whence, arguing as above, we get \( \varsigma^{(m)}_2 \to 0 \) as \( m \to \infty \).

**Corollary 7.1.** If \( \Psi \) is a stationary Poisson process with intensity \( \lambda \), and \( H \) admits integral representation (3.9) such that \( \mathbb{E}|Z_0|^2 < \infty \) and \( \mathbb{E}Y_{Z_0}^2(\Gamma_i) < \infty \) for \( i = 1, \ldots, k \), then \( (\tilde{\eta}(\Gamma_1, W_n), \ldots, \tilde{\eta}(\Gamma_k, W_n)) \) converges in distribution as \( n \to \infty \) to a Gaussian centred random vector \( (\xi_1, \ldots, \xi_k) \) with

\[
\mathbb{E}\xi_i \xi_j = \sigma^2(\Gamma_i, \Gamma_j) = \lambda(1-p)^2 \mathbb{E} \left[ \int_{\Gamma_j} \int_{\Gamma_j} q(\zeta_{u_1, u_2}) \gamma_{Z_0}(du_1) \gamma_{Z_0}(du_2) \right] \\
+ \lambda^2(1-p)^2 \mathbb{E} \left[ \int_{\mathbb{R}^d} q(x) \Phi(x, \Gamma_i) \Phi(-x, \Gamma_j) - \mathbb{E} \gamma_{Z_0}(\Gamma_i) \gamma_{Z_0}(\Gamma_j) \right] dx,
\]

where \( q_{u_1, u_2} = \ell(u_1, Z_0) - \ell(u_2, Z_0) \) and

\[
\Phi(x, \Gamma) = \mathbb{E} \left[ \int_{\Gamma} (1 - \mathbb{1}_{Z_0 + \ell(u, Z_0)}(x)) \gamma_{Z_0}(du) \right].
\]

In particular, if \( \mathbb{U} = \{u_1, \ldots, u_k\} \) consists of \( k \) distinct points, \( \Gamma_i = \{u_i\}, 1 \leq i \leq k \), and \( \gamma_{Z_0} = \delta_{u_1} + \cdots + \delta_{u_k} \) is a deterministic counting measure, then

\[
\sigma^2(\Gamma_i, \Gamma_j) = \lambda(1-p)^2 \mathbb{E}[q(\zeta_{u_1, u_2})] + \lambda^2(1-p)^2 \int_{\mathbb{R}^d} q(x) \varphi_{u_1}(x) \varphi_{u_j}(-x) - 1) \right) dx
\]

for \( 1 \leq i, j \leq k \), where \( \varphi_u(x) = \mathbb{P}(x \notin Z_0 + \ell(u, Z_0)) \).

**8. Examples and statistical applications**

In this section we consider only Boolean models. Then the Poisson germ process is determined by only one parameter (the intensity \( \lambda \)), while for the typical grain the mean values of geometric functionals are usually estimated. The law of large numbers for random measures associated with the Boolean model is widely used to estimate the model’s parameters \([21, 26, 31]\). The corresponding moment methods equations relate the intensities of random measures and the parameters of interest. In many cases the random measures involved admit the representation (3.7).

Consider a measure \( H_K(\cdot) \) on \([\Sigma, \mathcal{B}(\Sigma)]\) which satisfies our basic assumptions. This measure is used to define the random measure \( \eta \) associated with the underlying Boolean model. Then Lemma 4.1 and Theorem 6.1 yield

\[
\frac{\eta(\Gamma, W)}{|W|(1 - \rho_W)} \to \lambda \mathbb{H}(\Gamma, \mathbb{R}^d) \quad \text{as} \quad W \uparrow \mathbb{R}^d,
\]

(8.1)
where \( \hat{p}_W = |W \cap Z|/|W| \) estimates the volume fraction of \( Z \). To estimate \( \lambda \), we set \( \Gamma = \emptyset \) and take a measure \( H_K(\cdot) \) with a known expected total mass \( \overline{H}(\Gamma, \mathbb{R}^d) = \mathbb{E}H_Z(\Gamma, \mathbb{R}^d) \) (for example, any probability measure on \( \Sigma \) will do). Then the estimate of \( \lambda \) obtained can be plugged into (8.1) to estimate \( \overline{H}(\Gamma, \mathbb{R}^d) \) for another \( H_K(\cdot) \).

Below we will prove a central limit theorem for the random measure

\[
\tilde{\eta}(\Gamma, W_n) = \left| W_n \right|^{1/2} \left( \frac{\eta(\Gamma, W_n)}{|W_n|} - \lambda \overline{H}(\Gamma, \mathbb{R}^d) \right),
\]

(8.2)

where \( W_n = [-n, n]^d \) and \( n \rightarrow \infty \). The following limit theorem allows us to investigate asymptotic properties of the estimators obtained by the method of moments.

**Theorem 8.1.** Under the conditions of Theorem 7.1, the random vector \( \tilde{\eta}(\Gamma_1, W_n), \ldots, \tilde{\eta}(\Gamma_k, W_n) \) converges in distribution to a centred Gaussian random vector with the covariances

\[
\sigma^2(\Gamma_i, \Gamma_j) = \lambda \int_{\mathbb{R}^d} q(y_1 - y_2) \overline{H}(\Gamma_i, d y_1, d y_2; \Gamma_j, d y_2) + \lambda^2 \int_{\mathbb{R}^d} \Phi(x, \Gamma_i) \Phi(-x, \Gamma_j) q(x) d x,
\]

where \( i, j = 1, \ldots, k \), and \( \Phi(x, \Gamma) = \Phi(x, \Gamma, \mathbb{R}^d) = \mathbb{E}[H_Z(\Gamma, Z_0 + x)] \), see (6.9).

**Proof.** Similarly to [22, Theorem 5.5], it suffices to calculate the mixed moment of \( \eta(\Gamma, W_n) \) and \( \hat{p}_n = \hat{p}_W \). First, note that

\[
1 - \hat{p}_n = \left| W_n \right|^{-1} \int_{W_n} [1 - 1_Z(x)] d x = \left| W_n \right|^{-1} \int_{W_n} \prod_{j: j \geq 1} (1 - 1_{X_j + Z_j}(x)) d x.
\]

Therefore,

\[
\mathbb{E}[\eta(\Gamma, W_n)(1 - \hat{p}_n)] = \left| W_n \right|^{-1} \int_{W_n} \mathbb{E} \left[ \sum_{i, i \geq 1} \int_{\mathbb{R}^d} 1_{W_n}(X_i + y)(1 - 1_{X_i + Z_i}(x)) \times \prod_{j: j \neq i} \left\{(1 - 1_{X_j + Z_j}(x))(1 - 1_{X_j + Z_j}(X_i + y))\right\} H_{Z_i}(\Gamma, d y) \right] d x.
\]

The identity

\[
(1 - 1_{X_j + Z_j}(x))(1 - 1_{X_j + Z_j}(X_i + y)) = 1_{\tilde{Z}_j}(X_j - X_i + X_i - x)1_{\tilde{Z}_j}(X_j - X_i - y)
\]

together with the refined Campbell theorem yield

\[
\mathbb{E}[\eta(\Gamma, W_n)(1 - \hat{p}_n)] = \lambda \int_{\mathbb{R}^d} \gamma_{W_n}(v) \left| W_n \right|^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} 1_{W_n}(v + x)1_{K^c}(v - y) G_0^1[\tau_{(y-v,y)}] H_K(\Gamma, d u) Q(d K) d v d x.
\]

\[
= \lambda \int_{\mathbb{R}^d} \int_{\mathbb{K}} \int_{\mathbb{K}} 1_{W_n}(v + x)1_{K^c}(v - y) G_0^1[\tau_{(y-v,y)}] H_K(\Gamma, d u) Q(d K) d v d x.
\]

\[
= \lambda \int_{\mathbb{R}^d} \gamma_{W_n}(v) \left| W_n \right|^{-1} \int_{\mathbb{K}} \int_{\mathbb{K}} 1_{K^c}(v - y) G_0^1[\tau_{(y-v,y)}] H_K(\Gamma, d y) Q(d K) d v.
\]
The calculations above do not refer to the Poisson assumption. If $\Psi$ is Poisson, then (7.1) yields

$$
E[\eta(\Gamma, W_n)(1 - \tilde{p}_n)] = \lambda(1 - p)^2 \int_{\mathbb{R}^d} \gamma W_n(v)|W_n|^{-1} \int_{\mathbb{R}^d} (1 - 1_{Z_0 + y}(v)) \overline{H}(\Gamma, dy)q(v) dv.
$$

The proof can be easily accomplished by elementary calculations.

**Remark 8.1.** It should be noted that our conditions do not yield the convergence of the moments in (8.1). In fact, $E\tilde{\eta}(\Gamma, W_n)$ can be infinite.

**Corollary 8.1.** Assume that $H$ admits integral representation (3.9). Then, under the conditions of Theorem 7.1, the statement of Theorem 8.1 holds with the covariances

$$
\sigma^2(\Gamma_i, \Gamma_j) = \lambda \mathbb{E} \left[ \int_{\Gamma_i} \int_{\Gamma_j} q(\zeta_{u_1, u_2}) \gamma Z_0(du_1) \gamma Z_0(du_2) \right] + \lambda^2 \int_{\mathbb{R}^d} \tilde{\Phi}(x, \Gamma_i) \tilde{\Phi}(-x, \Gamma_j) q(x) dx,
$$

where $\zeta_{u_1, u_2} = \ell(u_1, Z_0) - \ell(u_2, Z_0)$, and

$$
\tilde{\Phi}(x, \Gamma) = \mathbb{E} \left[ \int_{\Gamma} 1_{Z_0 + \ell(u, Z_0)}(x) \gamma Z_0(du) \right].
$$

In particular, if $\Gamma = \{u_1, \ldots, u_n\}$, and $\gamma Z_0(\cdot)$ is the deterministic counting measure, then

$$
\sigma^2(u_i, u_j) = \lambda \mathbb{E}q(\zeta_{u_1, u_2}) + \lambda^2 \int_{\mathbb{R}^d} \tilde{\varphi}_{u_i}(x) \tilde{\varphi}_{u_j}(-x) q(x) dx, \quad (8.3)
$$

where $\tilde{\varphi}_u(x) = 1 - \varphi_u(x) = P(x \in \tilde{Z}_0 + \ell(u, Z_0))$.

Now we consider several examples of measures $H_K(\cdot)$ and discuss possible statistical applications of the above asymptotic theory.

**Example 8.1.** Let $H_K(\Gamma, W) = |K \cap W|$ be independent of $\Gamma$, so that $\Gamma$ consists of a single point. Then $\eta(\Gamma, W)$ is equal to the Lebesgue measure of the set of points covered by $Z_i + X_i$ for exactly one $i$. Clearly, $\eta$ cannot be computed if only the union-set $Z$ is observable. If $E|\gamma Z_0(\Gamma, \mathbb{R}^d)| = E|\gamma Z_0| < \infty$, then Theorem 6.1 yields

$$
|W|^{-1} \eta(\Gamma, W) \to \lambda(1 - p)E|\gamma Z_0| \quad \text{a.s. as} \quad W \uparrow \mathbb{R}^d.
$$

Furthermore, if $E|\gamma Z_0|^2 < \infty$, then $\tilde{\eta}(\Gamma, W_n)$ given by (6.2) satisfies the central limit theorem (Theorem 7.1) with the limiting variance given by

$$
\sigma^2 = \lambda(1 - p)^2 \mathbb{E} \left[ \int_{\gamma Z_0} \int_{\gamma Z_0} q(y_1 - y_2) dy_1 dy_2 \right] + \lambda^2 (1 - p)^2 \int_{\mathbb{R}^d} \left( q(x)E|\gamma Z_0 \cap (Z_0^c + x)| - E|\gamma Z_0 \cap (Z_0^c - x)| \right)^2 dx.
$$

Note that $\eta$ is defined by formula (3.7), which refers to the individual grains from the underlying germ-grain model. However, only observations of the union-set $Z$ are available for the statistical analysis. Most of the interesting examples appear in the case when $\Gamma$ is the unit sphere $S^{d-1}$ in $\mathbb{R}^d$, and $H$ admits the integral representation (3.9). The typical grain $Z_0$ is supposed to be almost surely convex.
**Example 8.2.** Let $H_{K} (\cdot)$ admit integral representation (3.9) with $\Upsilon_{Z_{0}}$ concentrated at a single point $u \in \mathbb{U}$ so that $\Upsilon_{Z_{0}} (\{u\}) = 1$. Then

$$H_{K} (\Gamma, W) = \mathbf{1}_{W} (\ell (u, K)) \mathbf{1}_{\Gamma} (u)$$

is a probability measure on $\Sigma$. Set $\Gamma = \mathbb{U}$. Then, the $i$th term in (3.7) is 1 if and only if $\ell (u, Z_{i} + X_{i}) \in W \setminus \Xi_{i}$. The latter means that the specific point, $\ell (u, Z_{i} + X_{i})$, of the $i$th grain is exposed, i.e., this specific point is not covered by the grains $(Z_{j} + X_{j})$ with $j \neq i$. Then, for all $\Gamma \ni u$, $\eta (\Gamma, W)$ is equal to the number of exposed specific points inside $W$, i.e.,

$$\eta (\Gamma, W) = \# \{ i : \ell (u, Z_{i}) + X_{i} \in W \setminus \Xi_{i} \}.$$

For example, if $\mathbb{U} = \mathbb{S}^{d-1}$ and $\ell (u, K)$ is the tangent point of $K$ in the direction $u$, then $\eta (\Gamma, W)$ is the number $N^{+} (u, W)$ of exposed tangent points in the direction $u$, see [22, 20]. Note that $\ell (u, K)$ is the lexicographical minimum of the support set $\mathcal{L} (u, K)$ defined as

$$\mathcal{L} (u, K) = \{ x \in \partial K : \langle u, x \rangle = -h (K, u) \},$$

where $\langle u, x \rangle$ is the scalar product and $h (K, u) = \sup \{ \langle u, x \rangle : x \in K \}$ is the support function of $K$.

Then Lemma 4.2 gives moment measures for the point process of tangent points. Since $\eta (\Gamma, W) = N^{+} (u, W)$ is observable, and

$$\hat{\lambda}_{W} = \frac{\eta (\Gamma, W)}{|W| (1 - \hat{p}_{W})} \to \lambda \text{ a.s. as } W \uparrow \mathbb{R}^{d},$$

it is possible to estimate the intensity of the Boolean model using the spatial intensity of $\eta$ (or the intensity of the point process of exposed tangent points). Corollary 8.1 yields a central limit theorem for the corresponding intensity estimator. Since $\hat{\varphi}_{u} (x) \hat{\varphi}_{u} (-x) = 0$ for almost all $x$, (8.3) yields

$$\sigma_{uu}^{2} = \frac{\lambda}{1 - p}. \tag{8.6}$$

This variance has been computed directly in [22]. If $\Upsilon_{Z_{0}} = \Upsilon_{0}$ is a deterministic probability measure on $\mathbb{U} = \mathbb{S}^{d-1}$, then $\eta (\mathbb{U}, W)$ is the weighted number of exposed tangent points considered in [20], so that Corollary 8.1 yields Theorem 3.1 of [20], which gives the asymptotic variance of the corresponding estimator.

**Example 8.3.** Suppose that the support set $\mathcal{L} (u, Z_{0}) = \{ \ell (u, Z_{0}) \}$ is a singleton for all $u \in \mathbb{U} = \mathbb{S}^{d-1}$ and almost all realisations of $Z_{0}$. Let $k_{Z_{0}} (u)$ be a positive function which depends on $\partial Z_{0} \cap B_{\varepsilon} (\ell (u, Z_{0}))$ for arbitrarily small $\varepsilon > 0$. In particular, $k_{Z_{0}} (u)$ can be the absolute curvature of $\partial Z_{0}$ at the corresponding tangent point or a function of this curvature. The measure $H$ is given by the integral representation (3.9) with $\Upsilon_{Z_{0}} (\cdot)$ concentrated at $\{ u \}$ with mass $k_{Z_{0}} (u)$. Then

$$E \eta (\{ u \}, W) = \lambda |W| (1 - p) E k_{Z_{0}} (u),$$

whence the expected value of $k_{Z_{0}} (u)$ can be estimated if an estimator of $\lambda$ is available. For instance, if the grain is a random ball of radius $\xi$, then all moments of $\xi$ and also all expectations $E f (\xi)$ (if they exist) can be estimated.
If $\mathbb{E}|Z_0|^2 < \infty$ and $\mathbb{E}k_{Z_0}(u) < \infty$ for all $u$, then Corollary 8.1 is applicable with the limiting variance given by

$$\sigma^2_{\text{lin}}(k) = \lambda \mathbb{E}(k_{Z_0}(u)^2)/(1 - p).$$

**Example 8.4.** Let $d = 2$, let $\mathbb{U}$ be the unit circle, and let $H_K(\Gamma, \cdot)$ with $\Gamma \ni u$ be concentrated at the tangent point $\ell(u, K)$ with mass 1 if the support set $\mathcal{L}(u, K)$ is a singleton. Otherwise, $\mathcal{L}(u, K)$ is a segment and $H_K(\Gamma, \cdot)$ assigns the weights $1/2$ to its end-points, denoted by $\ell(u, K)$ and $\ell'(u, K)$. Note that $H_K(\cdot)$ does not admit integral representation (3.9).

The corresponding measure $\eta$ is still observable and can be used to estimate the intensity of the Boolean model, since, by Theorem 6.1, $\eta(\Gamma, W)/|W| \rightarrow \lambda(1 - p)$ almost surely as $W \uparrow \mathbb{R}^d$. Theorem 8.1 implies that the asymptotic variance of the estimator $\hat{\lambda}_W$ from (8.5) is equal to

$$\sigma^2_{\text{lin}} = \lambda \mathbb{E}[q(0) + q(\ell'(u, Z_0)) - q(\ell(u, Z_0))]/2,$$

where $q$ is defined in (7.2). This variance is less than the variance given by (8.6) of the usual tangent points estimator, since $q(v) \leq q(0) = 1/(1 - p)$ for $v \in \mathbb{R}^d$.

**Example 8.5.** Assume that $\mathbb{U} = \mathbb{S}^{d-1}$, and $H_{Z_0}(\Gamma, W) = \Theta_{d-1}(Z_0, \Gamma \times W)$ is the $(d - 1)$-dimensional generalised surface area measure of the typical grain, see Appendix. If $W$ is an open set, then, by Theorem A.1, $\eta(\Gamma, W)$ is equal to the surface area measure of the boundary of the germ-grain model $Z$ measured inside $W$. In particular, $\eta(\mathbb{S}^{d-1}, W)$ is equal to the surface area of $(\partial Z) \cap W$. Then the results above give the ergodic theorem and the central limit theorem for surface measures. In particular, the ergodic theorem yields an estimator of the mean surface area measure $\mu = \mathbb{E}\Theta_{d-1}(Z_0, \Gamma \times \mathbb{R}^d)$ of $Z_0$.

If the typical grain has no flat pieces on its boundary, then $H_K(\cdot)$ admits integral representation (3.9) with $\ell(u, K)$ being the tangent point and $\gamma_{Z_0}(\Gamma) = \Theta_{d-1}(Z_0, \Gamma \times \mathbb{R}^d)$ being the area measure of order $(d - 1)$, see [25, p. 203]. Then Theorem 7.1 yields the limit theorem for surface measures proved in [20] for the bounded grains. Note that it is possible to extend this example for non-convex grains with rectifiable boundaries. If the window $W$ is closed, then the situation is more complicated, since the parts of the boundary of $W$ covered by $Z$ contribute to $\eta$, see [20]. It was shown in [20] that the central limit theorem does not hold in this case.

The following example was inspired by Hall [7, Section 5.6], who considered the planar Boolean model with a circular typical grain of radius $\xi$ and statistic

$$\kappa(W) = \sum_{i;i \geq 1} \phi_i a(R_i),$$

where $\phi_i$ is the angular content of the exposed boundary of the $i$th grain (disk) within the window $W$, $R_i$ is the radius of the $i$th disk, and $a : [0, \infty) \rightarrow [0, \infty)$ is a function. In other words, $\kappa$ is the weighted sum of the angular contents of all protruded pieces of the boundary. It is shown in [7, p. 323] that

$$\mathbb{E}\kappa(W) = 2\pi \lambda|W|(1 - p)\mathbb{E}a(\xi),$$

if $\mathbb{E}\xi^2 < \infty$ and $\mathbb{E}|a(\xi)| < \infty$, where $\xi$ is the radius of the typical grain (disk). From this, it is possible to estimate $\mathbb{E}a(\xi)$. 
Example 8.6. Let $H_K(\Gamma, W) = \Theta_j(K, \Gamma \times W)f(K)$, where $f(K)$ is a translation-invariant positive functional on the space of convex compact sets and $\Theta_j$ is the $j$th generalised curvature measure, see [25] and (A.1). The corresponding measure $\eta(\cdot)$ is observable as soon as the value $f(K)$ is retrievable from any relatively open piece of the boundary $\partial K$. In particular, this is true if the typical grain is a ball. It follows from Lemma 4.1 that

$$E\eta(\Gamma, W) = \lambda |W| (1 - p) E[\Theta_j(Z_0, \Gamma \times \mathbb{R}^d)f(Z_0)],$$

which yields (8.8) as a special case for $d = 2$, $\Gamma = \mathbb{U} = S^{d-1}$, $j = 1$, and $f(B_\xi(x)) = \xi^{-1}a(\xi)$.

If $E|Z_0|^2 < \infty$ and $E[V_j(Z_0)f(Z_0)]^2 < \infty$ (where $V_j$ is the intrinsic volume, see [25, p. 210]), then Theorem 7.1 implies that the finite-dimensional distributions of $\tilde{\eta}(\Gamma, W_n)$ are asymptotically Gaussian with the covariance (7.3). These moment assumptions coincide with those imposed in [7, Theorem 5.3] for planar Boolean models with circular grains. In the latter case $H_{Z_0}(\cdot)$ admits integral representation (3.9) with $\Upsilon_{Z_0}(du) = a(\xi)du$ and $f(u, Z_0)$ being the tangent point of $Z_0$ in direction $u$. Then Corollary 7.1 yields the asymptotic variance of $\kappa(W)$

$$\sigma^2 = \lambda(1 - p)^2 \mathbf{E} \left[ a(\xi)^2 \xi^{-2} \int_{\partial Z_0} \int_{\partial Z_0} q(y_1 - y_2) dy_1 dy_2 \right]$$

$$+ \lambda^2(1 - p)^2 \int_{\mathbb{R}^d} (q(x)b(x)b(-x) - 4\pi^2(Ea(\xi))^2) dx,$$

where the integrals over curves are understood with respect to the 1-dimensional Hausdorff measure (curve length), and

$$b(x) = \mathbf{E} \left[ a(\xi)^{1/\xi} \int_{\partial Z_0 \cap (Z_0^* + x)} dy \right]$$

is the expected angular content of $\partial Z_0$ within $Z_0^* + x$ multiplied by $a(\xi)$.

Example 8.7. Let $Z_0$ be a random convex polytope in $\mathbb{R}^d$. Assume that $\mathbb{U} = S^{d-1}$ and $H_{Z_0}(\cdot)$ is the number of vertices of $Z_0$ lying inside $W$. Then $\eta(\mathbb{U}, W)$ gives the number of exposed vertices inside $W$. Theorem 6.1 yields

$$\eta(\mathbb{U}, W)/|W| \rightarrow \lambda(1 - p)Ev \quad \text{as} \quad W \uparrow \mathbb{R}^d,$$

where $v$ is the number of vertices of the typical grain $Z_0$. The central limit theorem is valid if both $v$ and $|Z_0|$ have finite second moments. In order to compute the limiting variance consider the vertices of $Z_0$ to be a point process $\Psi_{Z_0}$ with a finite total number of points. Its second-order moment measure is denoted by $\alpha^{(2)}_{Z_0}$. Then (7.3) yields

$$\sigma^2 = \lambda(1 - p)^2 \int_{(\mathbb{R}^d)^2} q(x - y)\alpha^{(2)}_{Z_0}(dx, dy)$$

$$+ \lambda^2(1 - p)^2 \int_{\mathbb{R}^d} (q(x)f(x)f(-x) - (Ev)^2) dx,$$

(8.9)

where

$$f(x) = \mathbf{E} \sum_{Y_i \in \Psi_{Z_0}} 1_{Z_0^*(Y_i - x)}.$$
Example 8.8. Let $H_{Z_0}^{(r)}(\Gamma, W) = \tilde{\mu}_r(Z_0, \Gamma \times W)$, see (A.1). Then the corresponding measure $\eta^{(r)}(\Gamma, W)$ coincides with $\tilde{\mu}_r(Z, \Gamma \times W)$. As in (A.1),

$$\eta^{(r)}(\Gamma, W) = \frac{1}{d} \sum_{j=0}^{d-1} r^{d-j}\binom{d}{j} \eta_j(\Gamma, W), \quad (8.10)$$

where $\eta_j(\Gamma, W), j = 0, \ldots, d - 1$, are measures for $H_K^j(\Gamma, W) = \Theta_j(K, \Gamma \times W), K \in \mathcal{C}$, defined by (3.7). We can use (8.2) to define the corresponding measures $\tilde{\eta}_j(\Gamma, W_n)$. Assume that $\mathbb{E}|Z_0 \oplus B_r(0)|^2 < \infty$ for some $r > 0$.

Note that, similarly to the standard Crâmer-Wold device, a random vector

$$(\tilde{\eta}_0(\Gamma, W_n), \ldots, \tilde{\eta}_{d-1}(\Gamma, W_n))$$

converges in distribution to $(\xi_0, \ldots, \xi_{d-1})$ as $n \to \infty$, if, for each $r > 0$,

$$\sum_{j=0}^{d-1} r^j \tilde{\eta}_j(\Gamma, W_n)$$

converges to $\sum_{j=0}^{d-1} r^j \xi_j$ weakly as $n \to \infty$. After applying this to the polynomial expansions (8.10) and (A.1), one can prove that the random vector $(\tilde{\eta}_0(\Gamma, W_n), \ldots, \tilde{\eta}_{d-1}(\Gamma, W_n))$ converges in distribution to a centred Gaussian random vector with the covariances given by

$$\sigma_{ij} = \lambda \int_{\mathbb{R}^d} q(y_1 - y_2) \overline{H}_{ij}(\Gamma, dy_1; \Gamma, dy_2) + \lambda^2 \int_{\mathbb{R}^d} \Phi_i(x, \Gamma)\Phi_j(-x, \Gamma)q(x)dx,$$

where $i, j = 0, \ldots, d - 1,$

$$\overline{H}_{ij}(\Gamma, W_1; \Gamma, W_2) = \mathbb{E}[\Theta_i(Z_0, \Gamma \times W_1)\Theta_j(Z_0, \Gamma \times W_1)],$$

and

$$\Phi_i(x, \Gamma) = \mathbb{E}[\Theta_i(Z_0, \Gamma \times (Z_0 + x))].$$

This example allows us to find the joint limit distribution for the estimators of the intensities of Minkowski measures of different orders.

9. Concluding remarks

To conclude with, we outline several possible generalisations. We give only the results for the first moments in the Poisson case, although a laborious application of the methods developed above (with evident changes) allows us to derive the corresponding limit theorems.

Let us consider a measure $H_{K_1, K_2}(\cdot)$ which depends on two compact sets $K_1$ and $K_2$ in such a way that, for all $x \in \mathbb{R}^d$,

$$H_{K_1+x, K_2+x}(\Gamma, W + x) = H_{K_1, K_2}(\Gamma, W) = H_{K_1, K_2}(\Gamma, W \cap K_1 \cap K_2). \quad (9.1)$$

Define

$$\eta(\Gamma, W) = \sum_{i,j \geq 1, i \neq j} H_{Z_i+x_i, Z_j+x_j}(\Gamma, W \setminus \Xi_{ij}),$$

where $\Xi_{ij}$ are...
where
\[ \mathbb{E}_{ij} = \bigcup_{k: k \neq i, j} (X_k + Z_k). \]

Then the first moment of \( \eta \) is given by
\[ \mathbf{E}_\eta(\Gamma, W) = \lambda |W| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_{0,z_1}^{1} [\tau_{y}] \overline{H}_z(\Gamma, dy) \alpha^{(2)}_{\text{red}}(dz). \]

where
\[ \overline{H}_z(\Gamma, W) = \mathbf{E} H_{Z_1, Z_2 + z}(\Gamma, W) \]
for two independent grains \( Z_1 \) and \( Z_2 \) (we assume that \( \overline{H}_z(\Gamma, \mathbb{R}^d) \) is finite for all \( z \in \mathbb{R}^d \)). If the point process of germs is Poisson, then
\[ \mathbf{E}_\eta(\Gamma, W) = \lambda^2 (1 - p) |W| \mathbf{E} \left[ \int_{\mathbb{R}^d} H_{Z_1, Z_2 + z}(\Gamma, \mathbb{R}^d) \, dz \right]. \]

**Example 9.1.** Let \( Z \) be the Boolean model with almost surely convex grains. For \( \Gamma = \bigcup \) consider the measure
\[ H_{K_1, K_2}(\Gamma, W) = 1_{\partial K_1 \cap \partial K_2 \cap W \neq \emptyset}, \quad K_1, K_2 \in \mathcal{C}, \]
which satisfies (9.1). Then,
\[ \overline{H}_z(\Gamma, \mathbb{R}^d) = \int_{\mathbb{R}^d} 1_{\partial Z_1 \cap (\partial Z_2 + z) \neq \emptyset} \, dz = |\partial Z_1 \oplus \partial \tilde{Z}_2|, \]
whence
\[ \mathbf{E}_\eta(\Gamma, W) = \lambda (1 - p) |W| \mathbf{E} |\partial Z_1 \oplus \partial \tilde{Z}_2|. \]

For instance, if \( Z_0 = M \) is a deterministic central symmetric convex set, then \( \partial M \oplus \partial \tilde{M} = 2M \) and (9.3) yields
\[ \mathbf{E}_\eta(\Gamma, W) = \lambda (1 - p) 2^d |W| |M|. \]

To give a geometric interpretation of the measure \( \eta \) in the planar case, remember that \( X_i + \partial Z_i \) and \( X_j + \partial Z_j \) are either disjoint or intersect at two points (they may touch with probability zero). Then \( 2\eta(\mathbb{R}^{d-1}, W) \) gives the number of such points in \( W \) (resulting from \( (X_i + \partial Z_i) \cap (X_j + \partial Z_j) \)) which are exposed (not covered by \( X_k + Z_k \) with \( k \neq i, j \)).

**Example 9.2.** Let \( Z \) be the planar Boolean model with almost surely strictly convex typical grain \( Z_0 \) (so that \( \partial Z_0 \) does not contain a segment for almost all realisations of \( Z_0 \)). Then \( \mathcal{L}(u, Z_0) = \{\ell(u, Z_0)\} \) almost surely for all \( u \in \mathbb{U} = \mathbb{S}^{d-1} \). For strictly convex \( K_1 \) and \( K_2 \) set
\[ H_{K_1, K_2}(\{u\}, W) = 1_{K_1 \cap K_2 \cap W \neq \emptyset} 1_{\ell(-u, K_1 \cap K_2) \in W} 1_{\ell(-u, K_1) \notin K_2} 1_{\ell(-u, K_2) \notin K_1}, \quad u \in \mathbb{U}. \]

Then \( \eta(\{u\}, W) \) equals the number of negative tangent points in direction \( u \), see [27, p. 241]. The difference between the number of positive tangent points (Example 8.2) and \( \eta(\{u\}, W) \) is equal to the Euler-Poincaré characteristic of \( Z \cap \text{Int} W \), see [26, 27].
Now (9.2) yields

\[ E[I(u), W] = \lambda^2 (1 - p) |W| \mathbf{E} \int_{\mathbb{R}^d} 1_{Z_1 \cap (Z_2 + z) \neq \emptyset} \mathbf{1}_{t(-u, Z_1) \notin Z_2 + z} \mathbf{1}_{t(-u, Z_2) + z \notin Z_1} \, dz \]

\[ = \lambda^2 (1 - p) |W| \mathbf{E} [\int (Z_1 + \tilde{Z}_2) \setminus (\ell(-u, Z_1) \oplus \tilde{Z}_2) \cup (Z_1 \oplus \ell(u, \tilde{Z}_2))] / \]

\[ = \lambda^2 (1 - p) |W| (\mathbf{E} |Z_1 + \tilde{Z}_2| - \mathbf{E} |Z_1| - \mathbf{E} |Z_2|), \]

since \( \ell(-u, Z_1) \in Z_1 \) and \( \ell(u, \tilde{Z}_2) \in \tilde{Z}_2 \). It is known ([25, p. 275] and [32]) that

\[ |Z_1 + \tilde{Z}_2| = |Z_1| + |Z_2| + 2A(Z_1, \tilde{Z}_2), \]

where

\[ A(Z_1, \tilde{Z}_2) = \frac{1}{2} \int_{\mathbb{S}^1} h(Z_1, u) \Theta_1(\tilde{Z}_2, du \times \mathbb{R}^d) \]

is a so-called mixed area of \( Z_1 \) and \( \tilde{Z}_2 \). Since \( Z_1 \) and \( Z_2 \) are independent and have the same distribution as \( Z_0 \),

\[ \mathbf{E}A(Z_1, \tilde{Z}_2) = A(\mathbf{E}Z_0, \mathbf{E}\tilde{Z}_0), \]

where \( \mathbf{E}Z_0 \) is the Aumann expectation of \( Z_0 \), which satisfies \( \mathbf{E}h(Z_0, u) = h(\mathbf{E}Z_0, u) \) and \( \mathbf{E}\Theta_1(\tilde{Z}_0, du \times \mathbb{R}^d) = \Theta_1(\mathbf{E}\tilde{Z}_0, du \times \mathbb{R}^d) \), see [28, 31, 32]. Therefore,

\[ E[I(u), W] = \lambda^2 (1 - p) |W| A(\mathbf{E}Z_0, \mathbf{E}\tilde{Z}_0). \]

From this we immediately obtain the spatial density of the Euler-Poincaré characteristic:

\[ \frac{\chi(Z \cap W)}{|W|} \rightarrow \lambda (1 - p) (1 - \lambda A(\mathbf{E}Z_0, \mathbf{E}\tilde{Z}_0)) \text{ a.s. as } W \uparrow \mathbb{R}^d, \]

which was first derived by Weil [31]. For general convex \( Z_0 \) this result can be obtained by approximations with strictly convex sets. The technique above can be applied to prove the central limit theorem for the Euler-Poincaré characteristics. It will be considered elsewhere.

Clearly, it is possible to consider also vector-valued measures \( H_K(\cdot) \). In fact, many results can be generalised also for the germ-grain model \( Z \) generated by a not necessarily independently marked point process. Unfortunately, in this case formulas for the variances are getting incomprehensible. Note that other generalisations are possible for independently marked point processes with other spaces of marks, for example, when the points are marked by functions, measures or capacities.

**Appendix A. Random measures and geometric functionals**

Curvature measures are very important functionals defined on the family of convex sets, see [25]. The positive extension of curvature measures onto the convex ring \( \mathcal{R} \) (the family of finite unions of convex compact sets) is defined as follows (see [24, 25, 30] for further details). For given \( F \in \mathcal{R} \) and \( x \in \mathbb{R}^d \), a point \( q \in F \) is called a projection of \( x \) onto \( F \) if there exists a neighbourhood \( G \) of \( q \) such that \( q \) is the nearest point to \( x \) among all points from \( F \cap G \). Let
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\( \Pi(F, x) \) be the set of all projections of \( x \). For \( \Gamma \subseteq \mathbb{S}^{d-1} \) and \( r > 0 \) let \( \bar{c}_r(F, \Gamma \times W, x) \) be the number of points \( q \in \Pi(F, x) \cap W \) such that \( 0 < \|x - q\| \leq r \) and the direction of \( x - q \) belongs to \( \Gamma \). Then

\[
\bar{c}_r(F, \Gamma \times W) = \int_{\mathbb{R}^d} \bar{c}_r(F, \Gamma \times W, x) \, dx = \frac{1}{d} \sum_{j=0}^{d-1} r^{d-j} \binom{d}{j} \bar{c}_j(F, \Gamma \times W), \tag{A.1}
\]

see [24]. If \( K \in \mathcal{C} \), then we write \( \Theta_j(K, \Gamma \times W) \) instead of \( \bar{c}_j(F, \Gamma \times W) \), which is called the \( j \)th generalised curvature measure of \( K \). If \( F \in \mathcal{R} \), the coefficients \( \Theta_j(F, \Gamma \times W) \) in (A.1) are called the positive extensions of the curvature measures.

Formula (3.7) suggests another way to define functionals on the convex ring. Let \( H_K(\cdot) \) be a measure on \([\Sigma, \mathcal{B}(\Sigma)]\), where \( K \in \mathcal{C} \). Then

\[
\eta_F(\Gamma, W) = \sum_{i=1}^{n} H_{K_i}(\Gamma, W \setminus \cup_{1 \leq j \leq n, j \neq i} K_j). \tag{A.2}
\]

extends \( H \) for \( F = \bigcup_{i=1}^{n} K_i \). In general, this extension depends on the decomposition of \( F \) into the union of convex sets. Indeed, for any convex \( F \) we obtain \( \eta_F(\Gamma, W) = 0 \) by using the trivial representation \( F = F \cup F \). However, as it will be shown, such a situation is not possible if \( F \) is a realisation of a germ-grain model satisfying rather weak assumptions.

If \( F \in \mathcal{R} \), then \( x \in \partial F \) is said to be an exposed positive tangent point if \( x = \Pi(F, v) \) for some \( v \notin F \). The set of all exposed positive tangent points of \( F \in \mathcal{R} \) is denoted by \( \partial^+ F \) and is said to be the positive boundary of \( F \). The set-difference \( \partial F \setminus \partial^+ F \) comprises sets of dimensions not greater than \((d - 2)\). Note that \( \partial^+ F \) contains the set

\[
\mathcal{D}(K_1, \ldots, K_n) = \bigcup_{i=1}^{n} (\partial K_i \setminus \cup_{1 \leq j \leq n, j \neq i} K_j)
\]

for each decomposition \( F = \bigcup_{i=1}^{n} K_i \) of \( F \) into the union of convex compact sets.

Let us use (A.2) to extend onto \( \mathcal{R} \) the measure \( H_K(\Gamma, W) = \bar{c}_r(K, \Gamma \times W) \), where \( \mathcal{R} = \mathbb{S}^{d-1} \). For the moment, we assume that \( F = \bigcup_{i=1}^{n} K_i \) with

\[
\mathcal{D}(K_1, \ldots, K_n) = \partial^+ F. \tag{A.3}
\]

This means that the positive boundary of \( F \) is equal to the union of all ‘visible’ (or exposed) boundaries of individual grains. A similar condition appears in [36] when considering unions of sets of positive reach. By (A.1) and (A.3),

\[
\bar{c}_r(F, \Gamma \times W) = \sum_{i=1}^{n} H_{K_i}(\Gamma, W \setminus \cup_{1 \leq j \leq n, j \neq i} K_j).
\]

Expanding both sides in the polynomials (A.1) shows that the basic formula (A.2) applied to the curvature measure \( H_K(\Gamma, W) = \Theta_j(K, \Gamma \times W) \), \( K \in \mathcal{C} \), gives its positive extension onto the convex ring, i.e., \( \eta_F(\Gamma, W) = \Theta_j(F, \Gamma \times W) \).

In the following we give conditions, when the identity (A.3) holds for the germ-grain model (3.1). First, define the set-theoretic limit

\[
\mathcal{D}(X_i + Z_i; i \geq 1) = \lim_{n \to \infty} \mathcal{D}(X_{i_1} + Z_{i_1}, \ldots, X_{i_k(n)} + Z_{i_k(n)}) = \bigcup_{i \geq 1} (\partial (X_i + Z_i) \setminus \Xi_i),
\]

where \( \{i_1, \ldots, i_k(n)\} = \{i \geq 1 : X_i \in B_n(0)\} \).
Theorem A.1. Let \( \Psi \) be a stationary second-order point process with second-order reduced factorial moment measure \( \phi^{(2)}_{\text{red}}(\cdot) \), which is absolutely continuous with respect to the Lebesgue measure in \( \mathbb{R}^d \). Assume that the typical grain \( Z_0 \) is almost surely compact and convex. Then, for \( Z \) defined in (3.1), we have

\[
P(\mathcal{D}(X_i + Z_i; i \geq 1) = \partial^+ Z) = 1.
\]

Clearly, the conditions of Theorem A.1 hold for each Boolean model with a convex typical grain.

**Proof.** Let \( K_1 \) and \( K_2 \) be two convex compact sets. Then \( \partial \mathcal{D}(K_1, K_2) \neq \partial^+(K_1 \cup K_2) \) implies that either \( \mathcal{L}(u, K_1) \cap \mathcal{L}(u, K_2) \neq \emptyset \) or \( \mathcal{L}(u, K_1) \cap \mathcal{L}(-u, K_2) \neq \emptyset \) for some \( u \in \mathbb{S}^{d-1} \). Thus,

\[
L(K_1, K_2) = \{(x, y) : \mathcal{D}(K_1 + x, K_2 + y) \neq \partial^+ ((K_1 + x) \cup (K_2 + y))\}
\subseteq \bigcup_{u \in \mathbb{S}^{d-1}} \{(x, y) : y - x \in (\mathcal{L}(u, K_1) \oplus \mathcal{L}(u, K_2)) \cup (\mathcal{L}(u, K_1) \oplus \mathcal{L}(-u, K_2))\}.
\]

It follows from Theorem 1.7.5 of [25] that

\[
\mathcal{L}(u, K_1) \oplus \mathcal{L}(-u, K_2) = \mathcal{L}(u, K_1) \oplus \mathcal{L}(u, \tilde{K}_2) = \mathcal{L}(u, K_1 \oplus \tilde{K}_2).
\]

Thus,

\[
L(K_1, K_2) \subseteq \{(x, y) : y - x \in \partial(K_1 \oplus \tilde{K}_2) \cup \Lambda(K_1, K_2)\},
\]

where

\[
\Lambda(K_1, K_2) = \bigcup_{u \in \mathbb{S}^{d-1}} [\mathcal{L}(u, K_1) \oplus \mathcal{L}(u, K_2)],
\]

see [25, p. 86]. The technique described in [25, Section 2.3] can be used to prove that the Lebesgue measure of \( \Lambda(K_1, K_2) \) is equal to zero. First, the equality

\[
\mathcal{L}(u, K_1 \oplus B_1(0)) = \mathcal{L}(u, K_1) + u
\]
allows us to consider sets \( K_1 \) and \( K_2 \) which contain a ball of radius 1. Therefore, Lemma 2.3.9 of [25] yields

\[
\Lambda(K_1, K_2) \subseteq \bigcup_{i=1}^{m} ((C_i + \tilde{C}_i) + \alpha_i),
\]

where \( \alpha_i \in \mathbb{R}^d \) and \( C_1, \ldots, C_m \) are caps of \( K_1 \oplus K_2 \) covering the boundary of \( K_1 \oplus K_2 \). (A cap is defined to be a non-empty intersection of the convex set with a closed half-space.) Furthermore, Theorem 2.3.2 [25] gives a possibility of choosing these caps in such a way that

\[
\sum_{i=1}^{m} |C_i| < \varepsilon
\]
Central limit theorem for germ-grain models

for any given $\varepsilon > 0$. Note that $|C_i \oplus \hat{C}_i| \leq (d + 1)^d |C_i|$. The latter follows from the fact that $\hat{C} \subset dC$ for any convex compact set $C$ with non-empty interior and having its centroid at the origin [25, p. 81]. Thus, the Lebesgue measure of the set in the right-hand side of (A.5) can be made arbitrarily small. By (A.4), $L(K_1, K_2) \subseteq \{(x, y) : y - x \in \hat{L}(K_1, K_2)\}$ with $|\hat{L}(K_1, K_2)| = 0$.

Now consider the germ-grain model $Z$ defined by (3.1). Notice that

$$\mathcal{D}(X_i + Z_i; i \geq 1) \neq \partial^+ Z_i \subseteq \bigcup_{i \neq j} \{\mathcal{D}(Z_i + X_i, Z_j + X_j) \neq \partial^+ ((Z_i + X_i) \cup (Z_j + X_j))\}

= \bigcup_{i \neq j} \{(X_i - X_j) \in \hat{L}(Z_i, Z_j)\}.$$

The probability of the event in the right-hand side equals the limit (as $n \to \infty$) of the probabilities

$$P\left\{\bigcup_{i \neq j; X_i, X_j \in B_n(0)} \{X_i - X_j \in \hat{L}(Z_i, Z_j)\}\right\}
\leq \int_{B_n(0) \times B_n(0)} \int_{\mathcal{K} \times \mathcal{K}} 1_{\hat{L}(K_1, K_2)}(x - y) Q(dK_1) Q(dK_2) \alpha^{(2)} \alpha^{(2)}(d(x, y))$$

$$= \lambda \int_{\mathcal{E} \times \mathcal{E}} \int_{B_n(0)} \alpha^{(2)}_{\text{red}}(\hat{L}(K_1, K_2) \cap B_n(y)) dy Q(dK_1) Q(dK_2).$$

By the assumptions of Theorem A.1, $\alpha^{(2)}_{\text{red}}(\hat{L}(K_1, K_2)) = 0$, so that the latter integral is equal to zero for every $n \geq 1$. This completes the proof of Theorem A.1.

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References


