CONTACT AND CHORD LENGTH DISTRIBUTION OF A STATIONARY VORONOI TESSELLATION

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Abstract

We give formulae for different types of contact distribution functions for stationary (not necessarily Poisson) Voronoi tessellations in \( \mathbb{R}^d \) in terms of the Palm void probabilities of the generating point process. Moreover, using the well-known relationship between the linear contact distribution and the chord length distribution we derive a closed form expression for the mean chord length in terms of the two-point Palm distribution and the pair correlation function of the generating point process. The results obtained are specified for Voronoi tessellations generated by Poisson cluster and Gibbsian processes, respectively.

Keywords: Voronoi tessellation; chord length; contact distribution; motion-invariant point process; two-point Palm distribution; Poisson cluster process; Gibbsian process

AMS 1991 Subject Classification: Primary 60D05, 60G55
Secondary 60G10, 60G60

1. Introduction and results

A tessellation \( T = \{C_i, i \geq 0\} \) in \( \mathbb{R}^d \) is an aggregate of countably many space-filling (up to a Lebesgue null set) and non-overlapping open sets (called cells). For reasons of consistency, one requires the property of local finiteness of \( T \), i.e.

\[
\#\{i \geq 0 : C_i \cap B \neq \emptyset\} < \infty \quad \text{for all bounded } B \subset \mathbb{R}^d.
\]  

(1.1)

This condition entails that the skeleton \( \partial T \) (= union of the cell boundaries \( \partial C_i, i \geq 0 \)) is a closed set in \( \mathbb{R}^d \) which determines \( T \) uniquely. Consequently, a (stationary) random tessellation \( T = \{C_i, i \geq 0\} \) defined over a hypothetical probability space \( [\Omega, \mathcal{F}, \mathbb{P}] \) satisfying (1.1) \( \mathbb{P}\)-a.s. (almost surely) can be described completely by the (stationary) random closed set \( \partial T = \bigcup_{i \geq 0} \partial C_i \); see [23], [22].

Throughout this paper we are concerned with Voronoi (or Dirichlet, or Thiessen) tessellations \( V(\Psi) \) generated by a stationary simple point process \( \Psi = \sum_{i \geq 0} \delta_{X_i} \) in \( \mathbb{R}^d \). In addition the isotropy of \( \Psi \) is required to derive the mean chord length formula in Theorem 2. Some basic facts of general point process theory and a list of notations are summarized in the Appendix. A cell \( C_i(\psi) \) of a Voronoi tessellation (briefly VT) \( V(\psi) \) with respect to the non-random countable support \( \{x_i, i \geq 0\} \) (called nuclei) of the locally finite counting measure \( \psi = \sum_{i \geq 0} \delta_{x_i} \) is defined to be the set of all points in \( \mathbb{R}^d \) which are closest to the nucleus \( x_i \), i.e.

\[
C_i(\psi) = \{x \in \mathbb{R}^d : \|x - x_i\| < \|x - x_j\|, j \neq i\}.
\]  

(1.2)

Received 26 March 1996; revision received 10 March 1997.

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By the construction of $V(\Psi) = \{C_i(\Psi), i \geq 0\}$ all cells are $d$-polytopes and (1.1) is $\mathbb{P}$-a.s. satisfied. For convenience, let $x_0 = x_0(\psi)$ denote that (unique) atom of $\psi$ nearest to the origin and let $C_0(\psi)$ be the Voronoi cell that contains the nucleus $x_0(\psi)$ and the origin $o$.

It should be mentioned that some authors use the term VT $V(\Psi)$ merely relative to a stationary Poisson point process $\Psi$ which will be called here Poisson–VT. For further details and the historical background the reader is referred to Miles [16], Miles and Maillardet [18], Möller [19], Möller [20], Okabe et al. [22] and Stoyan et al. [23] (in particular Chapter 10 and references therein). Because only a few explicit formulae of distributional characteristics of (Poisson–)VTs are known (see e.g. Muche and Stoyan [21], Heinrich and Muche [10], Last and Schassberger [14]), large-scale simulation studies are often the only way out to obtain—at least approximately—the desired density and distribution functions, see e.g. Hahn and Lorz [4], Heinrich and Schüle [11].

To be definite, we recall the definition of length of the ‘typical’ chord in a tessellation. Let $h(\theta)$ be the straight line through the origin $o$ passing the $d$-dimensional unit sphere $S^{d-1}$ in $s(\theta)$, where $s(\theta) = (s_1(\theta), \ldots, s_d(\theta))$ is given in spherical coordinates by

$$
\begin{align*}
  s_1(\theta) &= \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-2} \sin \vartheta_{d-1} \\
  s_2(\theta) &= \cos \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-2} \sin \vartheta_{d-1} \\
  &\vdots \\
  s_{d-1}(\theta) &= \cos \vartheta_{d-2} \sin \vartheta_{d-1} \\
  s_d(\theta) &= \cos \vartheta_{d-1}
\end{align*}
$$

for $\vartheta = (\vartheta_1, \ldots, \vartheta_{d-1}) \in \Theta_{d-1} = [-\pi, \pi] \times [0, \pi]^{d-2}$.

The countable point set $h(\theta) \cap \partial V(\Psi) = \{Y_j(\theta), j \in \{-1, 0, 1, \ldots\}\}$ defines a stationary simple (one-dimensional) point process $\Psi(\theta) = \sum_{j=-\infty}^{\infty} \delta_{Y_j(\theta)}$ the atoms of which are assumed lexicographically ordered ($<\cdot$) such that $Y_{j-1}(\theta) < Y_j(\theta)$ and $h(\theta) \cap \partial C_0(\Psi) = \{Y_{-1}(\theta), Y_0(\theta)\}$.

The distribution function (briefly DF) $P(\|Y_0(\theta)\| \leq r)$ coincides with the linear contact DF $H_l(\theta)(r) = P(\partial C_0(\Psi) \cap r l(\theta) \neq \emptyset)$, $r \geq 0$, where $l(\theta) = \{us(\theta) : 0 \leq u \leq 1\}$. On the other hand, the DF $L_\theta(x)$ of the length of the ‘typical’ chord is defined to be the probability of the event $\{\|Y_j(\theta) - Y_{j-1}(\theta)\| \leq x\}$ for a ‘randomly chosen’ index $j$. From the theory of point processes on the real line it is well known that $H_l(\theta)$ and $L_\theta$ are linked by the relationship

$$
H_l(\theta)(r) = \frac{1}{L(\theta)} \int_{0}^{r} (1 - L_\theta(x)) \, dx,
$$

where $L(\theta) = \int_{0}^{\infty} (1 - L_\theta(x)) \, dx$ is the mean chord length in direction $s(\theta)$.

By the simplicity of the point process $\Psi(\theta)$, Korolyuk’s theorem (see Daley and Vere-Jones [2], p. 45) reveals that its intensity $\lambda(\theta)$ coincides with the limit

$$
\lambda(\theta) = \frac{1}{L(\theta)} = \lim_{r \downarrow 0} \frac{H_l(\theta)(r)}{r}.
$$

The main goal of the present paper is to derive formulae for various types of contact DF's (including the linear contact DF) and the mean chord length of a stationary VT in terms of the one- and two-point Palm distribution of the generating point process.

The paper is organized as follows. Next in this section we formulate the main results and give some corollaries concerning special VTs. The results on contact DFs of stationary VTs
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are proved in Section 2. Section 3 gives the proof of the mean chord length formula stated in Theorem 2. As mentioned above Section 4 contains some definitions and results on point processes to facilitate the reading of the paper.

**Theorem 1.** Let \( \Psi = \sum_{i \geq 0} \delta_{X_i} \sim P \) be a stationary simple point process on \( \mathbb{R}^d \) with intensity \( \lambda > 0 \). Further, let \( B \subset \mathbb{R}^d \) be a compact, star-shaped set containing the origin \( o \).

Then the contact DF \( H_B(r) = \mathbb{P}(\partial C_0(\Psi) \cap rB \neq \emptyset) \) takes the form

\[
1 - H_B(r) = \lambda \int_{\mathbb{R}^d} P_0^i(\{\psi : \psi(\cup_{y \in \partial^* B} b(x - y, \|x - y\|) \cup b(x, \|x\|) = 0\}) \, dx
\]

where \( S(\psi, X) = \{x \in \mathbb{R}^d : \psi(\cup_{y \in X} b(x - y, \|x - y\|) \cup b(x, \|x\|) = 0\} \) for \( X \in \mathcal{B}_d \) and \( \partial^* B = \{tB(\psi) : \psi \in \Theta_{d-1}\} \) with \( tB(\psi) = \sup\{t \geq 0 : ts(\vartheta) \in B\} \).

In practice one prefers the standard types of contact DFs such as the spherical, the (cubic) quadratic and the linear contact DF which refer to the special 'structuring sets' \( B = b(o, 1) \), \( B = \{x : \langle x, x \rangle = 1\} \) and \( B = l(\vartheta) \), respectively. In particular the linear contact DF takes a comparatively simple form.

**Corollary 1.** Let \( \Psi = \sum_{i \geq 0} \delta_{X_i} \sim P \) be a stationary simple point process on \( \mathbb{R}^d \) with intensity \( \lambda > 0 \). Then, for any \( \vartheta \in \Theta_{d-1} \),

\[
1 - H_{l(\vartheta)}(r) = \lambda \int_{\mathbb{R}^d} P_0^i(\{\psi : \psi(b(x - rs(\vartheta), \|x - rs(\vartheta)\|) \cup b(x, \|x\|) = 0\}) \, dx.
\]

From (1.4) it is pretty clear that the DF \( H_{l(\vartheta)}(r) \) possesses a non-increasing density \( H_{l(\vartheta)}'(r) \) such that \( L_{l(\vartheta)}(r) = 1 - \tilde{L}(\vartheta) \lambda H_{l(\vartheta)}'(r) \). If the generating point process \( \Psi \sim P \) is additionally isotropic, then \( \partial V(\Psi) \) becomes a motion-invariant random closed set and so \( H_{l(\vartheta)}(r) \) and \( \lambda(\vartheta) \) do not depend on \( \vartheta \in \Theta_{d-1} \). Note that the behaviour of the linear contact DF of a Poisson VT was studied by Muche and Stoyan [21]. The following result provides the explicit form of the limit (1.5) for general motion-invariant VTs.

**Theorem 2.** Let \( \Psi = \sum_{i \geq 0} \delta_{X_i} \sim P \) be a simple, stationary and isotropic, second-order point process on \( \mathbb{R}^d \) with intensity \( \lambda > 0 \). Further, let the second-order product density \( \varrho^{(2)}(x) \) (or the pair correlation function \( g(r) = \varrho^{(2)}(x)/\lambda \) for \( x \in \mathbb{R}^d \) with \( \|x\| = r \) exist. Then the mean chord length \( L \) of the Voronoi tessellation \( V(\Psi) \) is given by

\[
\frac{1}{L} = \frac{\lambda^2}{2} \int_0^\infty \int_{\Theta_{d-1}} \int_{\Theta_{d-1}} P_{rs(\vartheta)}^{(2)}(\{\psi : \psi(b(o, r) = 0\}) g(r \|s(\vartheta) - s(\theta)\|) \\
\times \cos \theta_{d-1} - \cos \theta_{d-1} - 1 \prod_{i=2}^{d-1} \sin \theta_i \sin \theta_i r^{2(d-1)} d(\theta_1, \ldots, \theta_{d-1}) d(\theta_1, \ldots, \theta_{d-1}) dr.
\]

**Corollary 2.** (Gilbert [3], Miles [17]) In the special case of a stationary Poisson process with intensity \( \lambda \) we have

\[
\tilde{L} = \lambda^{-1/d} \frac{d \Gamma(d - \frac{1}{2})(\Gamma((d + 1)/2))^2}{2 \Gamma(d)\Gamma(2 - (1/d))(\Gamma((d/2) + 1))^{(2d-1)/d}} \quad \text{for } d \geq 1.
\]
The expression for the mean chord length in Theorem 2 can be further simplified by using the isotropy of the point process which in turn implies the invariance of the term

\[ P_{rs(\theta),rs(\theta)}(\{\psi : \psi(b(o,r)) = 0\})g(r\|s(\theta) - s(\theta)\|) \]  

under rotation of the pair \( s(\theta), s(\theta) \) about the origin. Especially in the planar case the one-parametric representation \( s(\theta) = (\sin \theta, \cos \theta), -\pi \leq \theta \leq \pi \), allows us to express \( \bar{L} \) as a double integral. Moreover, since the segment process of edges of \( \partial V(\Psi) \) forms a motion-invariant planar fibre process, we can apply the well-known stereological relationship

\[ 2L_A = \pi \lambda(\theta) \text{ for any } \theta \in [-\pi, \pi] \text{ with } L_A = E \mathcal{H}^1(\partial V(\Psi) \cap [0, 1]^d), \]

where \( \mathcal{H}^1(\cdot) \) denotes the one-dimensional Hausdorff measure in \( \mathbb{R}^2 \), see [23]. In this way we can express the ‘edge length density’ \( L_A \) in terms of \( \bar{L} \).

**Corollary 3.** For any simple motion-invariant planar point process \( \Psi = \sum_{i:j \geq 0} \delta X_i \sim P \) with intensity \( \lambda > 0 \) we have

\[ \frac{1}{L} = \frac{2L_A}{\pi} = 8\lambda^2 \int_0^\infty \int_0^\pi P_{rs(\theta),rs(\theta)}(\{\psi : \psi(b(o,r)) = 0\})g\left(2r\sin\frac{\theta}{2}\right)r^2\sin\frac{\theta}{2} d\theta dr. \]

In the remaining part of this section we sketch the application of the above results to some special non-Poisson VTs. For the sake of brevity, we only give the Palm void probabilities and the shape of (1.6) for three types of point processes.

**1.1. Gibbsian point process with local energy \( E(x, \psi) \)**

This class of (stationary) point processes is characterized by the identity

\[ \int_N f(\psi)e^{-E(o,\psi)}P(d\psi) = \lambda \int_N f(\psi)P_0^r(d\psi) \]

for any non-negative measurable function \( f : N \mapsto \mathbb{R}^1 \), see [23]. Substituting \( f(\psi) = |S(\psi, r\delta^*B)| \) into the second formula of Theorem 1 provides an expression for the contact DF \( H_B(r) \) as mean value with respect to the stationary distribution \( P \):

\[ 1 - H_B(r) = \int_N |S(\psi, r\delta^*B)| e^{-E(o,\psi)}P(d\psi). \]

Both of the following two-point processes are Poisson cluster processes generated by a stationary Poisson cluster centre process having intensity \( \lambda_c \) and an a.s. finite typical cluster, see [2]. We give a closed-form expression for the desired one- and two-point Palm probabilities in terms of the characteristics defining the distribution of the typical cluster. These formulae can be deduced from the representation of the \( n \)-point Palm distribution of a general Poisson cluster process proved in Heinrich [7].

**1.2. Gauss–Poisson process**

The typical cluster of this Poisson cluster process contains either only the cluster centre (with probability \( 1 - p \)) or an additional second point (with probability \( p \)) which possesses a density \( q(x) \), for details see [2], p. 247.

For any bounded Borel set \( A \subset \mathbb{R}^d \), we have

\[ \lambda P_0^r(\{\psi : \psi(A) = 0\}) = \lambda_c \left(1 + p - p \int_A (q(x) + q(-x)) \, dx\right) P(\{\psi : \psi(A) = 0\}) \]
with \( \lambda = (1 + p)\lambda_c \) and the void probability

\[
P(\{\psi : \psi(A) = 0\}) = \exp \left\{ -\lambda_c \left( (1 + p)|A| - p \int_{\mathbb{R}^d} |A \cap (A + x)|q(x) \, dx \right) \right\}.
\]

Note that the Gauss–Poisson process is isotropic if \( q(x) \) only depends on \( \|x\| \) implying in particular that \( q(-x) = q(x) \). Under the latter assumption the two-point Palm void probability takes the form

\[
P_{x,y}^1(\{\psi : \psi(b(o, r)) = 0\}) = P(\{\psi : \psi(b(o, r)) = 0\})
\times \left\{ 2p\lambda_c q(y - x) + \lambda_c^2 \left( 1 + p - 2p \int_{b(x,r)} q(z) \, dz \right) \left( 1 + p - 2p \int_{b(y,r)} q(z) \, dz \right) \right\}.
\]

### 1.3. Neyman–Scott process

In this case the random number and the positions of the cluster members are stochastically independent. The cluster size distribution is given by the probability generating function \( g(\cdot) \) and the cluster members are i.i.d. random vectors with the common probability density \( f(\cdot) \), see [2], p. 245.

It is well known that

\[
\lambda^*P_{\psi}^1(\{\psi : \psi(A) = 0\}) = \lambda_c \int_{\mathbb{R}^d} \left( 1 - \int_{\mathbb{R}^d} (1 - \int_{A} f(u + v) \, du) \right) f(v) \, dv P(\{\psi : \psi(A) = 0\})
\]

with \( \lambda = g'(1)\lambda_c \) and the void probability

\[
P(\{\psi : \psi(A) = 0\}) = \exp \left\{ -\lambda_c \int_{\mathbb{R}^d} \left( 1 - \int_{\mathbb{R}^d} (1 - \int_{A} f(u + v) \, du) \right) \right\} \, dv, \quad A \in \mathcal{B}^d.
\]

If the density \( f(x) \) depends only on the distance \( \|x\| \), then the Neyman–Scott process is motion-invariant and with the abbreviation \( F(v, r) = \int_{b(v,r)} f(u) \, du \) we have

\[
P_{x,y}^1(\{\psi : \psi(b(o, r)) = 0\}) = \lambda_c \int_{\mathbb{R}^d} \left( 1 - F(v, r) \right) f(x + v) f(y + v) \, dv
\]

\[
+ \lambda_c^2 \int_{\mathbb{R}^d} \left( 1 - F(v, r) \right) f(x + v) \, dv \int_{\mathbb{R}^d} \left( 1 - F(v, r) \right) f(y + v) \, dv
\]

\[
\times P(\{\psi : \psi(b(o, r)) = 0\}).
\]

There are further ‘Poisson-based’ point processes such as Cox processes and certain dependently thinned Poisson processes for which the above Palm probabilities are available at least in principle.

**Example.** (Mean chord length of a planar Gauss–Poisson–Voronoi tessellation.)
Let \( \Psi = \sum_{i \geq 0} \delta_{X_i} \) be a Gauss–Poisson process in \( \mathbb{R}^2 \) as defined above with the density \( q(x) = \chi_{b(o,R)}(x)/\pi R^2 \), i.e. the second point around the cluster centre is uniformly distributed on the disc \( b(o, R), R > 0 \). From the above formula of the two-point Palm void probability for this particular Gauss–Poisson process we are able to derive the subsequent formula for the
Figure 1: Plot of the functions $\tilde{L}(\lambda_c, p, R)$ with $\lambda_c = 1/(1 + p)$.

The reciprocal value of the mean chord length $\tilde{L}(\lambda_c, p, R)$ (which coincides with $2L_A/\pi$) of the corresponding Voronoi tessellation:

$$
\frac{1}{\tilde{L}(\lambda_c, p, R)} = 16 \int_0^\infty \exp \left\{ -\lambda_c (1 + p) \pi r^2 + \frac{2\lambda_c p}{R^2} \int_0^{R \wedge (2r)} |b(o, r) \cap b(\rho s, r)| \rho \, d\rho \right\} \times \left[ \frac{2\lambda_c pr^2}{\pi R^2} \left( 1 - \sqrt{1 - \left( \frac{R \wedge (2r)}{2r} \right)^2} \right) \right.
$$

$$
+ \lambda_c^2 r^2 \left( 1 + p - \frac{2p}{\pi R^2} |b(o, R) \cap b(rs, r)| \right)^2 \right] dr,
$$

where $s = (0, 1)$.

It is easily checked that

$$
\lim_{R \to \infty} \tilde{L}(\lambda_c, p, R) = \frac{\pi}{4\sqrt{\lambda_c (1 + p)}} \text{ and } \lim_{R \to 0} \tilde{L}(\lambda_c, p, R) = \frac{\pi}{4\sqrt{\lambda_c (1 + p/2)}}.
$$

On the other hand, we have $\tilde{L}(\lambda_c, p, 0) = \pi/4\sqrt{\lambda_c}$ for any $0 \leq p \leq 1$ showing that $\tilde{L}(\lambda_c, p, 0) \neq \tilde{L}(\lambda_c, p, 0 + 0)$ for $0 < p \leq 1$. The dependence of $\tilde{L}(\lambda_c, p, R)$ on the values $(p, R)$ in case of constant cell intensity $\lambda = \lambda_c (1 + p) = 1$ is seen in Table 1 and Figure 1.
2. Contact distribution of a stationary Voronoi tessellation

To begin with we recall the definition of contact DF $H_B$ of a general stationary random closed set $Z$, see e.g. [23]. For some set $B \in \mathfrak{B}_o^d$ containing the origin $o$, the contact DF $H_B$ is defined by

$$ H_B(r) = \mathbb{P}(Z \cap rB \neq \emptyset | o \notin Z) = 1 - \frac{\mathbb{P}(Z \cap rB = \emptyset)}{\mathbb{P}(o \notin Z)} \quad \text{for } r \geq 0. $$

In order to ensure that $H_B(r)$ becomes a non-decreasing and right-continuous function, the so-called ‘structuring set’ $B$ is assumed star-shaped relative to the origin $o$ (i.e. $rB \subseteq B$ for $r \in [0,1]$) and compact. Conditions ensuring $H_B(\infty) = 1$ and a Glivenko–Cantelli theorem for the empirical counterpart of $H_B$ for general random closed sets are given in [8]. Further regularity properties, e.g. absolute continuity, and Kaplan–Meier estimators of $H_B$ are studied in [6].

In the particular case $Z = \bigcup_{i \geq 0} \partial C_i(\Psi)$ we may write

$$ 1 - H_B(r) = \mathbb{P}(rB \subseteq C_0(\Psi)) = \mathbb{P}(\partial C_0(\Psi) \cap rB = \emptyset). $$

Obviously, in this case $H_B$ is a proper DF.
Proposition 1. Let \( \Psi \sim P \) be a stationary simple point process on \( \mathbb{R}^d \). Then
\[
Y_0 = \{ \psi \in N : \exists x, y \in \mathbb{R}^d \text{ with } x \neq y, \|x\| = \|y\| \text{ and } \psi((x)) \geq 1, \psi((y)) \geq 1 \}
\]
belongs to \( \mathcal{M} \) and has \( P \)-measure zero, and the relation
\[
P(\Psi(b(o, R) \geq k) - \int_{b(o, R)}^\infty P_Y(\{\psi \in \mathbb{R}^d : \psi(b(x, \|x\|)) = k - 1\}) \, dx \quad (2.1)
\]
holds for every \( k = 1, 2, \ldots \) and \( R > 0 \).

Proof. The first assertion is proved in [2], p. 479, whilst the second one is obtained by application of the 'refined' Campbell theorem, see Heinrich and Stoyan [12].

Proof of Theorem 1. For any \( \psi \in Y_0^c = N \setminus Y_0 \), let \( x_0 = x_0(\psi) \) denote the atom of \( \psi \) that is closest to the origin \( o \) so that the associated Voronoi cell \( C_0(\psi) \) defined by (1.2) contains \( o \). Further, let \( X \) be a compact set of \( \mathbb{R}^d \) containing \( o \).

Then, by the definition of the Voronoi tessellation \( V(\psi) \), it is easily checked for \( \psi \in Y_0^c \) that
\[
X \subset C_0(\psi) \Leftrightarrow \|x - x_0(\psi)\| < \|x - x_j(\psi)\|, \forall j \geq 1, \forall x \in X
\]
\[
\Leftrightarrow x_j(\psi) \notin b(x, \|x - x_0(\psi)\|), \forall j \geq 1, \forall x \in X
\]
\[
\Leftrightarrow (\psi - \delta_{x_0}(\psi))(b(x, \|x - x_0(\psi)\|)) = 0, \forall x \in X
\]
\[
\Leftrightarrow (\psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - x_0(\psi)\|)) = 0.
\]

This enables us to rewrite the indicator function of the set \( \{ \varphi \in Y_0^c : X \subset C_0(\varphi) \} \) as follows:
\[
I_{\{\varphi \in Y_0^c : (\psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - x_0(\varphi)\|)) = 0\}}(\psi)
\]
\[
= \sum_{y : \psi(y) > 0} I_{y : \psi(y) > 0} \left( \begin{array}{l}
I_{\varphi \in Y_0^c : (\psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - y\|)) = 0}
\end{array} \right) \psi(y) \quad (A)
\]
\[
= \sum_{y : \psi(y) > 0} I_{\varphi \in Y_0^c : (T_y \psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - y\|)) = 0} \psi(y). \quad (B)
\]

Here, (A) holds since \( o \in X \) and (B) is a consequence of the identity \((\psi - \delta_{x_0})(\cdot) = (T_y \psi - \delta_{x_0})(\cdot - y)\).

\[
P(X \subset C_0(\psi)) = \int_{Y_0^c} \sum_{y : \psi(y) > 0} I_{\varphi \in N : (\psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - y\|)) = 0} (T_y \psi - \delta_{x_0}) P(d\psi)
\]
\[
= \lambda \int_{[0, R]} I_{\|\psi\| > 0} \int_{N} I_{\varphi \in N : (\psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - y\|)) = 0} \psi \, P_0^1(d\psi) \, dy \quad (2.2)
\]
\[
= \lambda \int_{N} \int_{B} I_{\|\psi\| > 0} \, P_0^1(d\psi).
\]

The last line follows from \( I_{S(\psi, X)(y)} = I_{\varphi \in N : (\psi - \delta_{x_0}(\psi))(\cup_{x \in X} b(x, \|x - y\|)) = 0} \psi \) and by applying Fubini's theorem.

To accomplish the proof of Theorem 2 we have to show that, for any compact, star-shaped (relative to the origin \( o \)) set \( B \),
\[
\bigcup_{y \in B} b(x - y, \|x - y\|) = \bigcup_{y \in \partial^* B} b(x - y, \|x - y\|) \cup b(x, \|x\|) \quad (2.4)
\]
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with \( \partial^* B \) as defined in Theorem 1. Let \( t_B(\theta) = \sup\{t \geq 0 : ts(\theta) \in B\} \) denote the largest distance from the origin to a boundary point of \( B \) in direction \( s(\theta) \). In view of the star-shapedness, \( B = \{ts(\theta) : 0 \leq t \leq t_B(\theta), \theta \in \Theta_{d-1}\} \). Therefore, it suffices to verify that

\[
\bigcup_{0 \leq t \leq t_B(\theta)} C(x, \theta, t) = C(x, \theta, t_B(\theta)),
\]

where the ‘cap’ \( C(x, \theta, t) \) is defined by \( C(x, \theta, t) = b(x - ts(\theta), \|x - ts(\theta)\|) \setminus b(x, \|x\|) \). But the relation (2.5) is immediately seen from the following result.

**Proposition 2.** For any \( x \in \mathbb{R}^d \) and \( \theta \in \Theta_{d-1} \) we have

\[
C(x, \theta, t_1) \subseteq C(x, \theta, t_2) \quad \text{if} \quad 0 \leq t_1 \leq t_2.
\]

**Proof.** Let \((x, y)\) denote the scalar product of \( x, y \in \mathbb{R}^d \). Then

\[
y \in C(x, \theta, t_1) \iff \|y - x\| \geq \|x\| \text{ and } \|y - x + t_1s(\theta)\| \leq \|x - t_1s(\theta)\|
\]

\[
\iff \|y\|^2 > 2(x, y) \quad \text{and} \quad \|y - r\|^2 + 2r(y, s(\theta)) \leq 2(x, y)
\]

implying that \((y, s(\theta)) < 0\). Therefore, the previous lines are valid for every \( t_2 > t_1 \).

Finally, combining (2.2), (2.3) and (2.4) completes the proof of Theorem 1.

3. Proof of Theorem 2

Let \( s \) denote the unit vector \( s((0, \ldots, 0)) = (0, \ldots, 0, 1) \).

**Proposition 3.** For any \( x = (x_1, \ldots, x_d) \neq 0 \) and \( r > 0 \) we have

\[
\partial b(o, \|x\|) \cap \partial b(rs, \|x - rs\|) = \partial b(o, \|x\|) \cap \{y \in \mathbb{R}^d : (y, s) = x_d\}.
\]

This point set is a \((d - 2)\)-dimensional hypersphere on \( \partial b(o, \|x\|) \) not depending on \( r > 0 \).

**Proof.** By definition of the Euclidean norm and the scalar product,

\[
\partial b(o, \|x\|) \cap \partial b(rs, \|x - rs\|) = \{y \in \mathbb{R}^d : \|y\|^2 = \|x\|^2 \text{ and } \|y - rs\|^2 = \|x - rs\|^2\}
\]

\[
= \{y \in \mathbb{R}^d : \|y\| = \|x\| \text{ and } (y, s) = (x, s) = x_d\}.
\]

Corollary 1 applied to \( l = l((0, \ldots, 0, 1)) \) yields

\[
1 - H_l(r) = \lambda \int_{\mathbb{R}^d} P_0^l((\psi : \psi(b(x, \|x\|)) = 0, \psi(b(x - rs, \|x - rs\|) \setminus b(x, \|x\|)) = 0)) \, dx,
\]

whence together with

\[
\lambda \int_{\mathbb{R}^d} P_0^l((\psi : \psi(b(x, \|x\|)) = 0)) \, dx = 1,
\]

which results from (2.1) for \( k = 1 \) and \( R \to \infty \), it follows that

\[
H_l(r) = \lambda \int_{\mathbb{R}^d} P_0^l((\psi : \psi(b(x, \|x\|)) = 0, \psi(b(x - rs, \|x - rs\|) \setminus b(x, \|x\|)) \geq 1)) \, dx.
\]

(3.2)
After introducing spherical coordinates $x = \rho s(\theta)$ with $\rho = \|x\| \geq 0$ and $\theta = (\theta_1, \ldots, \theta_{d-1}) \in \Theta_{d-1}$ and using the rotation invariance of the Palm distribution $P_\rho$, which implies that

$$P_\rho^1(\{\psi : \psi(B + x) = 0\}) = P_\rho^1(\{\psi : \psi(-B - x) = 0\}) = P_\rho^1(\{\psi : \psi(-B) = 0\})$$

for any $x \in \mathbb{R}^d$ and $B \in \mathfrak{B}_d$, we can rewrite the right-hand side of (3.2) as follows:

$$\lambda \int_0^\infty \int_{\Theta_{d-1}} P_{\rho s(\theta)}^1(\{\psi : \psi(b(o, \rho) = 0, \psi(S(\rho, r, \theta) \geq 1)\rho^{d-1} \prod_{i=2}^{d-1} (\sin \theta_i)^{-1} d\theta d\rho,$$

(3.3)

where

$$S(\rho, r, \theta) = b(rs, R(\rho, r, \theta)) \setminus b(o, \rho)$$

with

$$R(\rho, r, \theta) = \|\rho s(\theta) - rs\| = \sqrt{\rho^2 + r^2 - 2rp \cos \theta_{d-1}}.$$

The latter equality is easily seen from (1.3). The projection of the 'cap' $S(\rho, r, \theta)$ onto the $(x_1, x_d)$-plane is illustrated in Figure 2.

Further notice that, in analogy to Proposition 1, the existence of the product density $\varrho^{(2)}(x)$ implies that $P_o^1(Y_0) = 0$, or in other words that $\psi(\partial b(o, \|y\|) \leq 1), \forall y \in \mathbb{R}^d$ for $P_o^1$-almost every $\psi \in N$. For each such realization $\psi \in N$ the following decomposition holds:

$$\mathbf{1}_{\{\psi : \psi(S(\rho, r, \theta) \geq 1)\}}(\psi) = \sum_{y \in S(\rho, r, \theta) : \psi(|y|) > 0} \mathbf{1}_{\{\psi : \psi(b(o, \|y\|) = 0)\}}(\psi).$$

Since the two-point (reduced) Palm distribution $P_{x,y}^1$ coincides with the iterated Palm distribution $(P_{x,y}^1)_y$, see Hanisch [5] or Kallenberg [13], we find by applying the corresponding
Contact and chord length distribution

For better understanding of this formula we recall the fact that \( q(\cdot - x) \) is the Lebesgue density (intensity function) of the intensity measure with respect to \( P_x \).

Since \( \Psi \sim P \) is assumed isotropic, \( q(z) \) equals \( g(||z||) \), where \( g(\cdot) \) denotes the corresponding pair correlation function. Next we represent the set \( S(p, r, e) \) by spherical coordinates. Proposition 3 and standard relations of planar trigonometry yield

\[
S(p, r, e) = \{ z : \rho < r \leq r + R(p, r, \theta), (\theta_1, \ldots, \theta_{d-2}) \in \Theta_{d-2}, 0 \leq \theta_{d-1} \leq \arccos((r^2 - \rho^2 + 2\rho r \cos \theta_{d-1})/2r) \}.
\]

Hence, after passing to spherical coordinates \( y = \tau s(\theta) \) in (3.4),

\[
P_{x, ot} f_{t, lt : \tau \in z(\theta)} = \int_{S(p, r, e)} g(||s(\theta) - \rho s(\theta)||) \tau^{d-1} \prod_{i=2}^{d-1} (\sin \theta_i)^{-1} d\theta_1 \ldots d\theta_{d-2} d\tau.
\]

Inserting this expression in (3.3) and changing the order of the integrals we obtain a formula for \( H_1(r) \) in terms of the two-point Palm distribution and the pair correlation function of \( \Psi \sim P \):

\[
H_1(r^*) = 2 \int_{\Theta_{d-2}} \int_{\Theta_{d-2}} F(\theta^*, \theta^*, r) \prod_{i=2}^{d-1} (\sin \theta_i)^{-1} d\theta^* d\theta^*,
\]

where \( \theta^* = (\theta_1, \ldots, \theta_{d-2}) \in \Theta_{d-2}, \theta^* = (\theta_1, \ldots, \theta_{d-2}) \in \Theta_{d-2} \) and

\[
F(\theta^*, \theta^*, r) = \int_0^{\pi} \int_0^{\infty} \int_0^{r + R(p, r, \theta)} \arccos(\tau^2 - \rho^2 + 2\rho r \cos \theta_{d-1}/2r) \]

\[
\times g(||s(\theta) - \rho s(\theta)||/(\rho r)^{d-1} (\sin \theta_{d-1})^{-1} d\theta_{d-1} d\tau d\rho.
\]

Here and in what follows we assume that \( d \geq 3 \). In the planar case the first integral stretches over \([-\pi, \pi]\) and the following steps remain valid with obvious changes.
Now, we are ready to determine the limit of the ratio $H_1(r)/r$ as $r \downarrow 0$. For this, let $\rho > r > 0$ be fixed and $r$ sufficiently small. By elementary manipulations we find that

$$R(\rho, r, \theta) + r = \rho + r(1 - \cos \theta_{d-1}) + \frac{2r^2 \rho \sin^2 \theta_{d-1}}{(R(\rho, r, \theta) + \rho - r)(R(\rho, r, \theta) + \rho + r)} \quad (3.6)$$

and, for $p < r < p + r(1 - \cos \theta_{d-1})$,

$$\frac{\tau^2 - p^2 + 2pr \cos \theta_{d-1}}{2r} = \frac{\tau - p}{r} + \cos \theta_{d-1} - c_1(r) \quad \text{with } 0 \leq c_1(r) \leq \frac{r^2}{2(\rho - r)} \quad (3.7)$$

By means of the estimates (3.6) and (3.7) it can be shown that (if it exists) the limit $\lim_{r \to 0} F(\theta^*, \theta^*, r)/r$ remains the same if we omit the remainder terms $c_1(r)$ and $c_2(r)$ in the above integral expression of $F(\theta^*, \theta^*, r)$.

Furthermore,

$$P_{\rho s(\theta), \tau s(\theta)}^I (\{ \psi : \psi(b(o, \rho) \cup T(\rho, \tau, \theta_{d-1})) = 0 \}) \leq P_{\rho s(\theta), \tau s(\theta)}^I (\{ \psi : \psi(b(o, \rho) \cup \Omega(\rho, r, \theta) \cap b(o, \tau)) = 0 \}) \leq P_{\rho s(\theta), \tau s(\theta)}^I (\{ \psi : \psi(b(o, \rho, \rho)) = 0 \}), \quad (3.8)$$

where $T(\rho, \tau, \theta_{d-1}) = \{ \sigma s(\theta) : \rho < \sigma \leq \tau, (\theta_1, \ldots, \theta_{d-2}) \in \Theta_{d-2}, 0 \leq \theta_{d-1} \leq \theta_{d-1} \}$.

After some rearrangements we get

$$\frac{1}{r} \int_0^\pi \int_0^{r + r(1 - \cos \theta_{d-1})} \int_0^\infty \int_0^\infty \arccos((\tau - \rho)/r + \cos \theta_{d-1})$$

$$P_{\rho s(\theta), \tau s(\theta)}^I (\{ \psi : \psi(b(o, \rho) \cup b(o, \tau) \cap T(\rho, \tau, \theta_{d-1})) = 0 \}) \times g(\| \rho s(\theta) - \tau s(\theta) \|)(\rho \tau)^{d-1} (\sin \theta_{d-1} \sin \theta_{d-1})^{d-2} d\theta_{d-1} d\rho d \theta_{d-1}$$

$$= \int_0^\pi \int_0^{r(1 - \cos \theta_{d-1})} \int_0^\infty \int_0^\infty \arccos(\tau + \cos \theta_{d-1}) f_0(\theta, \vartheta, \rho + rt, \rho)$$

$$\times (\sin \theta_{d-1} \sin \theta_{d-1})^{d-2} d\theta_{d-1} d\rho d \theta_{d-1}$$

$$= \int_0^\pi \int_0^{\theta_{d-1}} \int_0^\infty \int_0^{\theta_{d-1}} f_0(\theta, \vartheta, \rho + r(\cos \gamma - \cos \theta_{d-1}), \rho) \sin \gamma dy d\rho$$

$$\times (\sin \theta_{d-1} \sin \theta_{d-1})^{d-2} d\theta_{d-1} d\theta_{d-1},$$

where

$$f_0(\theta, \vartheta, \rho) = P_{\rho s(\theta), \tau s(\theta)}^I (\{ \psi : \psi(b(o, \rho) \cup T(\rho, \tau, \theta_{d-1})) = 0 \}) g(\| \rho s(\theta) - \tau s(\theta) \|)(\rho \tau)^{d-1}.$$

Since, for fixed $\theta, \vartheta \in \Theta_{d-1}$, the function $(\tau, \rho) \mapsto f_0(\theta, \vartheta, \rho)$ is measurable and Lebesgue integrable on $D_\delta = \{ (x, y) : x > 0, y > 0, |x - y| < \delta \}$ for some $\delta > 0$, we can approximate $f_0(\theta, \vartheta, \ldots)$ by continuous functions on $D_\delta$ with compact support in $L^1(D_\delta)$-norm.
The same approximation argument applies when \( f_0(\theta, \varphi, \tau, \rho) \) is replaced by the function
\[
f_1(\theta, \varphi, \tau, \rho) = P_{\rho s(\theta), r s(\theta)}^\perp(\{\psi : \psi(b(o, \rho)) = 0\}) g(\|\rho s(\theta) - r s(\theta)\|)(\rho \tau)^{d-1}.
\]
Hence, by (3.6) and having in mind that \( f_0(\theta, \varphi, \tau, \rho) = f_1(\theta, \varphi, \tau, \rho) \) and using the symmetry \( f_i(\theta, \varphi, \tau, \rho) = f_i(\varphi, \theta, \tau, \rho) \) for \( i = 1, 2 \), we obtain
\[
\lim_{r \downarrow 0} \frac{F(\theta^*, \varphi^*, r)}{r} = \int_0^\pi \int_0^{\theta_{d-1}} \int_0^{\theta_{d-1}} \int_0^{\theta_{d-1}} f_0(\theta, \varphi, \tau, \rho) \sin \gamma \, d\gamma \, d\rho (\sin \theta_{d-1} \sin \varphi_{d-1})^{d-2} \, d\theta_{d-1} \, d\varphi_{d-1} = \frac{1}{2} \int_0^\pi \int_0^{\pi} \int_0^{\infty} f_0(\theta, \varphi, \tau, \rho) \cos \theta_{d-1} \, d\rho \, (\sin \theta_{d-1} \sin \varphi_{d-1})^{d-2} \, d\theta_{d-1} \, d\varphi_{d-1}.
\]
Finally, divide both sides of (3.5) by \( r \) and let \( r \downarrow 0 \). Interchanging integration and passage to the limit and inserting the latter expression of the limit \( \lim_{r \downarrow 0} F(\theta^*, \varphi^*, r)/r \) into the integral on the right-hand-side of (3.5), we obtain the asserted relation of Theorem 2. This completes the proof of Theorem 2.

**Proof of Corollary 2.** From the independence properties of a Poisson process
\[
P_{\rho s(\theta), r s(\theta)}^\perp(\{\psi : \psi(b(o, \rho)) = 0\}) = P(\Psi(b(o, \rho)) = 0) = \exp(-\lambda \omega_d r^d)
\]
and \( g(r||s(\theta) - s(\theta)||) = 1 \) for any \( r > 0 \) and \( s(\theta), s(\theta) \in S^{d-1} \), see Section 4.
Hence, from Theorem 2,
\[
\frac{1}{L} = \frac{\lambda^2}{2} \int_0^\pi \int_0^\pi |\cos \theta_{d-1} - \cos \varphi_{d-1}|(\sin \theta_{d-1} \sin \varphi_{d-1})^{d-2} \, d\theta_{d-1} \, d\varphi_{d-1}
\]
\[
\times (\Gamma((d - 1)/d))^{2} \int_0^\infty e^{-\lambda \omega_d r^d} \, r^{2(d-1)} \, dr
\]
\[
= 2(d - 1)(\lambda \omega_d)^{2} \int_0^\pi \sin^{2d-3} \varphi \, d\varphi \frac{\Gamma(2 - (1/d))}{d(\lambda \omega_d)^{(2d-1)/d}}
\]
\[
= \lambda^{1/d} 2\Gamma(d)(\Gamma(2 - (1/d))(\Gamma((d/2) + 1))^{(2d-1)/d}
\]
\[
\frac{d \Gamma((d - 1)/2)(\Gamma((d + 1)/2)^2).
\]
The latter formula was already derived in [3] and [17] using quite different techniques which in turn differ from our approach.

**Proof of Corollary 3.** Since \( s(\theta) = (\sin \theta, \cos \theta) \) for \( \theta \in [-\pi, \pi] \), we immediately get
\[
\|s(\theta) - s(\theta)\| = 2 \sin \frac{|\theta - \varphi|}{2}
\]
and
\[
P_{\rho s(\theta), r s(\theta)}^\perp(\{\psi : \psi(b(o, \rho)) = 0\}) = P_{\rho s(\theta), r s(\theta)}^\perp(\{\psi : \psi(b(o, \rho)) = 0\}).
\]
Therefore, we deduce from Theorem 2 that
Elementary considerations show that the inner integral over \([-z - \vartheta, n - r]\) can be replaced by an integral over \([-2, z]\). Finally, interchanging the inner integrals and using the formula
\[
\int_{-\pi}^{\pi} |\cos(\theta + \vartheta) - \cos \vartheta| r^2 \, d\theta \, d\vartheta = 8 \sin \frac{\theta}{2}
\]
combined with the obvious symmetry \(P_{rs(0),rs(\vartheta)}(\ldots) = P_{rs(0),rs(-\vartheta)}(\ldots)\) yields the desired formula stated in Corollary 3.

**Appendix**

We first list the notation used throughout the paper:

- \(\|x\|\) = Euclidean norm in \(\mathbb{R}^d\)
- \(b(x, r) = \{y \in \mathbb{R}^d : \|y - x\| < r\}\) : (closed) \(d\)-ball with centre \(x\) and radius \(r\)
- \(\mathcal{B}\{\mathbb{B}_d\} = \sigma\)-field (ring) of (bounded) Borel sets in \(\mathbb{R}^d\)
- \(|B|\) = \(d\)-dimensional Lebesgue measure of \(B \in \mathcal{B}_d\)
- \(\partial B\) = boundary of \(B \in \mathcal{B}_d\)
- \(\omega_d = |b(o, 1)| = \pi^{d/2}/\Gamma\left(\frac{d}{2} + 1\right)\)
- \(N\) = set of locally finite counting measures \(\psi\) on \(\mathbb{R}^d\)
- \(\mathfrak{M}\) = \(\sigma\)-field generated by the sets \(\{\psi \in N : \psi(B) = n\}\), \(n \geq 0\), \(B \in \mathcal{B}_d\)
- \(1_Y(\cdot)\) = indicator function of the set \(Y\)
- \(\delta_x(B) = 1_B(x)\) : Dirac measure

In what follows we put together some basic notions and tools from point process theory needed in the above sections, for details see [2], [13] or [15].

A point process \(\Psi \sim P\) on \(\mathbb{R}^d\) (with distribution \(P = P \circ \Psi^{-1}\)) is defined to be an \((\mathfrak{A}, \mathfrak{M})\)-measurable mapping from \([\Omega, \mathfrak{A}, P]\) into the measurable space \([N, \mathfrak{M}]\), that is, \(\Psi\) is a locally finite random counting measure. Since \(\Psi\) is throughout assumed to be simple, i.e. \(\mathbf{P}(\psi(\{x\} < 1, \forall x \in \mathbb{R}^d) = 1\), we may briefly write \(\Psi = \sum_{i \geq 0} \delta x_i\) expressing the one-to-one correspondence between the measure and its random countable support.

By means of the translation operator \(T_y\) acting on \(N\) such that \((T_y\psi)(\cdot) = \psi(\cdot + y)\) for \(y \in \mathbb{R}^d\), the stationarity of \(\Psi \sim P\) is defined by the invariance property \(T_y\Psi \sim P\) for any \(y \in \mathbb{R}^d\). In this case the intensity measure \(\Lambda(\cdot) = \mathbf{E}\Psi(\cdot)\) coincides with the \(d\)-dimensional Lebesgue measure multiplied by the intensity \(\lambda = \Lambda([0, 1]^d)\) and the second-order factorial moment measure \(\alpha^{(2)}(A \times B) = \mathbf{E} \sum_{i \geq 0} 1_A(X_i) (\Psi - \delta x_i)(B)\) (if it exists) can be decomposed by disintegration with respect to the Lebesgue measure in the following way:

\[
\alpha^{(2)}(A \times B) = \lambda \int_A \alpha^{(2)}_{red}(B - x) \, dx \quad \text{for } A, B \in \mathcal{B}_d
\]
The so-called reduced second moment measure \( \alpha_{\text{red}}(\cdot) \) completely characterizes the second-order behaviour of \( \Psi \sim P \), as does its product density \( \varrho^{(2)}(x) = \lim_{\epsilon \to 0} \alpha_{\text{red}}^{(2)}(b(x, \epsilon))/2\epsilon^{d-1} \) when the latter limit exists almost everywhere. In the isotropic case, \( \varrho^{(2)}(x) \) depends only on \( ||x|| \) so that indeed the pair correlation function

\[
g(r) = \varrho^{(2)}(x)/\lambda = \lim_{\epsilon \to 0} \alpha_{\text{red}}^{(2)}(b(o, r + \epsilon) \setminus b(o, r))/\lambda \omega_d \epsilon^{d-1} \quad \text{for } r = ||x||
\]
suffices to describe the second-order properties. The distribution \( P \) of a stationary simple point process \( \Psi \) is closely connected (by a one-to-one correspondence) with the reduced Palm distribution \( P_{o}^{(1)} \) which can be interpreted as the distribution of \( \Psi - \delta_{o} \) conditional on the null event that an atom of \( \Psi \) is located at the origin, i.e. \( P_{o}^{(1)}(Y) = \mathbb{P}(\Psi - \delta_{o} \in Y | \Psi(\{o\}) > 0) \) for \( Y \in \mathcal{Y} \), see Chapter 12 in [2] for a rigorous definition. It turns out that \( \alpha_{\text{red}}^{(2)}(\cdot) = \int_{N} \psi(\cdot)P_{o}^{(1)}(d\psi) \). The so-called ‘refined’ Campbell theorem links the distributions \( P \) and \( P_{o}^{(1)} \) in the following way:

\[
\int_{N} \int_{\mathbb{R}^{d}} f(x, T_{x}\psi - \delta_{o})\psi(dx)P(d\psi) = \lambda \int_{\mathbb{R}^{d}} \int_{N} f(x, \psi)P_{o}^{(1)}(d\psi)dx
\]

for any non-negative measurable function \( f: \mathbb{R}^{d} \times N \to \mathbb{R}^{1} \).

Furthermore, there are second-order or two-point Palm distributions \( P_{x,y}^{(2)}(Y) \) which are defined as Radon–Nikodym derivatives of the reduced second-order Campbell measure \( C^{(2)} \) defined by

\[
C^{(2)}(A \times B \times Y) = \int_{N} \int_{A} \int_{B} 1_{Y}(\psi - \delta_{x} - \delta_{y})(\psi - \delta_{x})(dy)\psi(dx)P(d\psi), \quad A, B \in \mathcal{B}_{o}, \ Y \in \mathcal{Y},
\]

with respect to the second-order factorial moment measure \( \alpha^{(2)} \).

These regular conditional distributions were systematically studied by Kallenberg [13], see Ambartzumian [1] and Hanisch [5] for earlier contributions. The Radon–Nikodym approach provides the conditional probability interpretation

\[
P_{x,y}^{(2)}(Y) = \mathbb{P}(\Psi - \delta_{x} - \delta_{y} \in Y | \Psi(\{x\}) > 0, \Psi(\{y\}) > 0) \quad \text{for (almost every) } x, y \in \mathbb{R}^{d}.
\]

For the sake of completeness, let us recall that a stationary Poisson process \( \Psi \sim P \) with intensity \( \lambda > 0 \) is determined by the well-known properties that the number \( \Psi(B) \) of atoms in a bounded Borel set \( B \) is Poisson distributed with parameter \( \lambda |B| \); and the numbers \( \Psi(B_{1}), \ldots, \Psi(B_{k}) \) are mutually independent, if \( B_{1}, \ldots, B_{k} \) are pairwise disjoint for any \( k \geq 2 \). These properties imply that

\[
P_{o}^{(1)} = P = P_{x,y}^{(1)} \quad \text{for } x, y \in \mathbb{R}^{d}, \quad \alpha_{\text{red}}^{(2)}(\cdot) = \lambda |\cdot|, \quad g(r) = 1 \quad \text{for } r \geq 0.
\]

For some particular point process models it is indeed possible to derive tractable expressions of the multi-point Palm distributions, see Heinrich [7] for Poisson cluster processes.

Acknowledgements

The author would like to thank Ralf Körner for his assistance in calculating the values of \( L(1(1 + p), p, R) \) in Table 1. The author is also grateful to the referees who helped to improve this paper.
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