Non-Uniform Bounds for the Error in the Central Limit Theorem for Random Fields Generated by Functions of Independent Random Variables

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1. Notations and Preliminary Results

Let $\xi_z, z \in \mathbb{Z}_+^d = \{1, 2, \ldots, d\}^d$, be a field of independent random variables (RV’s) on a common probability space $(\Omega, \mathcal{F}, P)$ taking values in an arbitrary measurable space $(K, \mathcal{K})$ and let there be given a family of real-valued BOREL measurable functions $f_z, z \in \mathbb{Z}_+^d$, defined on the measurable product space $(K^d, \mathcal{K}^d)$, see [14].

For every integer $m \geq 1$ and $z = (z_1, \ldots, z_d) \in \mathbb{Z}_+^d$ define:

$$V_z^{(m)} = \{y \in \mathbb{Z}_+^d : y_i \leq z_i + m - 1, \ i = 1, \ldots, d\},$$

$$U_z^{(m)} = \{y \in \mathbb{Z}_+^d : y_i \leq z_i + m, \ i = 1, \ldots, d\},$$

$$\delta_z^{(m)} = \sigma(\xi_y, y \in U_z^{(m)}),$$

$$V_n = \{z \in \mathbb{Z}_+^d : 1 \leq z_i \leq n_i = n_i(n), \ i = 1, \ldots, d\},$$

$$|V_n| = \text{card } V_n = n_1 \cdots n_d,$$

where the sequences $n_i(n), \ i = 1, \ldots, d$, are non-decreasing and satisfy the ordering $n_1(n) \leq \cdots \leq n_d(n)$ for $n = 1, 2, \ldots$ such that $n_d(n) \to \infty$ as $n \to \infty$. Further put

$$X_z = f_z(\xi_y ; y \in U_z^{(\infty)}), \quad z \in \mathbb{Z}_+^d,$$

$$S_n = \sum_{z \in V_n} X_z, \quad B_n^2 = D^2 S_n, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad A_n(x) = |P(S_n - ES_n < xB_n) - \Phi(x)|,$$

and $C, C_0, C_1, \ldots$ stand for positive constants which may be differ from one expression to another.

For $d = 1$ the normal approximation of the distribution of $S_n$ has a rather long history. In his pioneering work [1] already S. N. Bernstein investigated the central limit theorem (CLT) for dependent RV’s of the above type. Later several authors studied bounds for $A_n(x)$, see [3], [9], [12], [13] (for $d = 1$). Most of these papers are motivated

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1) $\sigma(\cdot)$ denotes the smallest $\sigma$-algebra generated by the RV’s in braces.
by the study of

\[ \nu \left( \left\{ t \in (0, 1) : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \sin(2\pi k t) < x \right\} \right) \quad \text{as} \quad n \to \infty, \]

where \( \nu \) denotes the Lebesgue measure on \((\mathbb{R}, \mathcal{B})\) and \( f \mid [0, \infty) \to \mathbb{R} \) has period 1 and \( \frac{1}{0} \int \sin(\omega t) \, dt = 0 \). For details the reader is referred to [9]. We only mention the fact that each irrational number \( t \in (0, 1) \) admits a unique binary expansion

\[ t = \sum_{k=1}^{\infty} \frac{\epsilon_k(t)}{2^k}, \quad \epsilon_k(t) = \frac{1}{2} \left( 1 - r_k(t) \right), \]

where the so-called Rademacher functions \( r_k(t) = \text{sgn} \sin(2\pi k t) \) (and hence the \( \epsilon_k \)'s) prove to be independent RV's on \((\mathbb{S}^1 \cap (0, 1), \nu)\). In statistics of time series (especially of linear processes \( X_t = \sum_{s=-\infty}^{\infty} \xi_{t-s}, \ t = 0, 1, 2, \ldots \)) limit theorems for \( S_n \) can be used to study asymptotic properties of the sample correlation function (see [5]), the integrated periodogram, etc. (see [16, 20, 25]).

The goal of the present paper consists in proving non-uniform rates of convergence in the CLT for the random field (RF) \( X_z, z \in \mathbb{Z}_+^d \). The essential new point of view is that we are able to deal with the case of an arbitrary dimension \( d \geq 1 \). Moreover the obtained bounds of \( A_n(z) \) improve earlier results of LADOKHIN & MOSKVIN [12] and GIL'FANOV [3] for \( d = 1 \). The moment bounds of \( S_n \) derived in Section 3 seems to be of particular interest.

To prove (non-) uniform bounds of \( A_n(x) \) following an idea due to I. A. IBRAIMOV [9] we need

(i) corresponding Berry-Esseen estimates in the CLT for fields of \( m \)-dependent RV's of the form

\[ \bar{X}_z = g_z(\xi_y; y \in U_z^{(m)}), \]

where \( g_z \mid K^{(m+1)^d} \to \mathbb{R} \) is \((\mathbb{S}^{(m+1)^d}, \mathcal{B})\)-measurable, which are as accurate as possible and contain explicitly \( m \).

(ii) bounds of \( E |S_n|^p \) (for some \( p \geq 2 \)) resembling those known for independent RV's provided that \( E |X_z|^p < \infty \) and \( E |X_z - E(X_z | \xi_z^{(m)})|^p \overset{m \to \infty}{\longrightarrow} 0 \) suitably quickly for every \( z \in V_n \).

Another way to establish a CLT for \( S_n \) consists in proving that \( X_z, z \in \mathbb{Z}_+^d \), is \( \alpha \)- or \( \varphi \)-mixing with some mixing rate. Results of that kind are only known for linear \( f_z \)'s and \( d = 1 \) (see [21]). Non-uniform Berry-Esseen bounds and Edgeworth expansions for \( m \)-dependent RF's of the form (1.3) were proved in [7] and [4] (see also [3], [9], [12], [13] for \( d = 1 \)). We note that the non-uniform error estimate in the CLT for general \( m \)-dependent RF's in [5] is unsatisfactory near to \( x = 0 \). For the corresponding uniform bounds for \( m \)-dependent and other types of weakly dependent RF's the reader should compare the recent papers of SUNKLODAS [17], BULINSKII [2], TIKHONIHKOV [18], and further references in [7]. To formulate a modified version of Theorem 5 in [7] we
introduce the following notation:

\[ \bar{Y}_x = \sum_{y \in \mathbb{V}_x} \bar{X}_y, \quad \mathcal{Z}_d = \{1, m + 1, 2m + 1, \ldots\}^d, \]

\[ \bar{S}_n = \sum_{z \in \mathcal{Z}_d} \bar{Y}_x = \sum_{z \in \mathcal{Z}_d} \bar{X}_z, \quad \bar{B}_n^2 = E \bar{S}_n^2, \quad \bar{b}_n^2 = \sum_{z \in \mathcal{Z}_d} E \bar{Y}_z^2, \]

\[ L_{sn}(a) = \sum_{z \in \mathcal{Z}_d} E \left| \bar{Y}_z \right|^s 1_{\{\bar{Y}_z \leq a\}} \bar{B}_n^s, \quad a \geq 0, \quad s \geq 2, \quad \bar{L}_{sn} = \bar{L}_{sn}(\infty), \]

where \(1_A(.)\) denotes the indicator function of the event \(A\).

**Theorem A.** Let the RF \(X_x, z \in \mathbb{V}_n\), defined by (1.3) (where \(m = m(n)\) may depend on \(n\)) satisfy the following conditions: \(E\bar{Y}_z = 0\), \(E \left| \bar{Y}_z \right|^s < \infty\) for some integer \(s \geq 3\), \(z \in \mathcal{Z}_d\), \(n_1(n)/m(n) \to \infty\) as \(n \to \infty\) and further

\[ \bar{L}_{n-1+1}(\bar{b}_n) (\ln |V_n|)^{\frac{d-1}{2}} \xrightarrow{n \to \infty} 0, \quad (m(n))^d (\ln |V_n|)^{d+\frac{1}{2}} |V_n| \xrightarrow{n \to \infty} 0, \]

\[ \bar{B}_n^2 \geq C_1 |V_n|, \quad \max_{z \in \mathcal{V}_n} \frac{E \bar{Y}_z^2}{C_2 (m(n))^d} \quad \text{for} \quad n \geq n_0. \]

Then, for \(n \geq n_0\),

\[ \sup_{x \in \mathcal{Z}^d} (1 + |x|)^s |P(\bar{S}_n < x \bar{B}_n) - \Phi(x)| \leq C_3 (\bar{L}_{3n} + \bar{L}_{sn}), \]

where the constant \(C_3\) only depends on \(d, s, C_1,\) and \(C_2\).

**Remark 1.** The "asymptotic conditions" in Theorem A can be slightly weakened as follows: There exist certain fixed large enough \(N_0\) and certain fixed sufficiently small \(\varepsilon_0 > 0\) such that \(n_1(n)/m(n) \geq N_0\) and the members of the null sequences of (1.4) do not exceed \(\varepsilon_0\) for \(n \geq n_0\). Further note that the conditions of Theorem 5 in [7] (for \(m_1 = \cdots = m_d = m\)) are a little stronger than those of Theorem A. It is readily checked that all conclusions in the proof of Theorem 5 in [7] remain valid if only (1.4) and (1.5) hold. Moreover, the inequality \(2^{-d/2} |\mathcal{Z}_d \cap V_n|^{-d/2} \leq \bar{L}_{sn}\) justifies the elimination of the additional term \(|V_n|^{-d/2}\) on the right-hand side of the estimate (2.20) in [7].

We still formulate a further consequence of Theorem 5 in [7] which is obtained by setting \(m_1 = n_1, \ldots, m_r = n_r\) and \(m_{r+1} = \cdots = m_d = m\) for some \(r \in \{1, \ldots, d - 1\}\). This result is particularly useful if in Theorem A one of the ratios \(n_i(n)/m(n), i = 1, \ldots, d\), remains bounded as \(n \to \infty\), e.g. \(n_r(n)/m(n) < N_0\) for certain fixed and \(n \geq n_0\). In the above notation we replace \(\mathcal{V}^{(m)}_n\) by the index set

\[ \{y \in \mathbb{Z}^d : 1 \leq y_j \leq n_j, j = 1, \ldots, r, z_j \leq y_j \leq z_j + m - 1, j = r + 1, \ldots, d\}, \]

\(\mathcal{Z}_d\) by \(\mathcal{Z}_d-r\) and write \(\bar{Y}^{(r)}_z, \bar{b}^{(r)}_n, \) and \(L^{(r)}_{sn}(a)\) instead of \(\bar{Y}_z, \bar{b}_n, \) and \(\bar{L}_{sn}(a)\), respectively.

**Theorem B.** Let the RF \(X_x, z \in \mathbb{V}_n\), satisfy the following conditions: \(E\bar{Y}_z^{(r)} = 0\), \(E \left| \bar{Y}_z^{(r)} \right|^s < \infty\) for some integer \(s \geq 3\), \(z \in \mathcal{Z}_d-r\), \(n_{r+1}(n)/m(n) \to \infty\) as \(n \to \infty\) for some \(r \in \{1, \ldots, d - 1\}\), and further

\[ \bar{L}^{(r)}_{n-1+1}(\bar{b}^{(r)}_n) (\ln |V_n|)^{\frac{d-r}{2}} \xrightarrow{n \to \infty} 0, \quad \frac{(m(n))^{d-r} (\ln |V_n|)^{d-r+\frac{1}{2}}}{n_{r+1}(n) \cdots n_d(n)} \xrightarrow{n \to \infty} 0, \]
\( E(\bar{Y}_n)^2 \leq C_2 n_1(n) \cdots n_r(n) (m(n))^{d-r} \)

for every \( z = (z_{r+1}, \ldots, z_d) \) with \( 1 \leq z_j \leq n_j(n) \) and \( n \geq n_0 \).

Then, for \( n \geq n_0 \),

\[
\sup_{x \in \mathbb{R}} (1 + |x|)^{s + \delta} |P(\bar{S}_n < x\bar{B}_n) - \Phi(x)| \leq C_3 (\bar{L}_{d+\delta}^{(r)})^{1 - \frac{1}{s + \delta - 2}},
\]

where \( C_3 \) depends on \( d, s, r, C_1, \) and \( C_2 \).

**Remark 2.** The proof of Theorem 5 in [7] reveals that the estimate (1.6) (and (1.9)) remains unchanged if in the whole formulation of Theorem A (and B) the \( \mathbb{RV}'s \) \( \bar{Y}_n^{(r)} \), \( z \in \mathbb{Z}_{d-r} \), for \( r = 0 \) (and \( 1 \leq r \leq d - 1 \)) and the sum \( \bar{S}_n \) are replaced by \( h_z(\bar{Y}_n^{(r)}) \), \( z \in \mathbb{Z}_{d-r} \), and \( \sum h_z(\bar{Y}_n^{(r)}) \), respectively, where \( h_z : \mathbb{R} \rightarrow \mathbb{R}^+, z \in \mathbb{Z}_{d-r} \), are BOREL measurable functions.

In the next step we shall summarize and extend the Theorems A and B in order to include the case when \( s \) is an arbitrary real number greater than 2.

**Theorem C.** Let the RF \( \bar{X}_z, z \in V_n \), satisfy the following conditions: \( E \bar{X}_z = 0, E \bar{X}_z^{1+s+\delta} < \infty \) for some integer \( s \geq 2 \) and \( 0 < \delta \leq 1, n_{r+1}(n)/m(n) \rightarrow \infty \) as \( n \rightarrow \infty \) for some \( r \in \{0, 1, \ldots, d - 1 \} \), and further (1.8) and

\[
E|\bar{X}_z|^{s+\delta} \rightarrow 0, (m(n))^{d-r} (\ln |V_n|)^{d-r+\frac{1}{2}} \rightarrow 0.
\]

Then, for \( n \geq n_0 \),

\[
\sup_{x \in \mathbb{R}} (1 + |x|)^{s + \delta} |P(\bar{S}_n < x\bar{B}_n) - \Phi(x)| \leq C_3 (\bar{L}_{d+\delta}^{(r)})^{1 - \frac{1}{s + \delta - 2}},
\]

\( a \wedge b = \min(a, b) \), where \( C_3 \) depends on \( d, s, r, C_1, \) and \( C_2 \).

**Proof.** The essential point is the following truncation inequality:

\[
|P(\bar{S}_n < x\bar{B}_n) - \Phi(x)| \leq \sum_{z \in \mathbb{Z}_{d-r}} P(\bar{Y}_n^{(r)}) \geq (1 + |x|) \bar{B}_n + \left| P(\bar{S}_n < x\bar{B}_n) - \Phi \left( \frac{x\bar{B}_n - E\bar{S}_n}{\bar{B}_n} \right) \right|
\]

\[
+ \left| \Phi(x) - \Phi \left( \frac{x\bar{B}_n - E\bar{S}_n}{\bar{B}_n} \right) \right| = I_1 + I_2 + I_3,
\]

where

\[
\bar{S}_n = \sum_{z \in \mathbb{Z}_{d-r}} \bar{Y}_n^{(r)}, \quad \bar{B}_n^2 = D^2 \bar{S}_n, \quad \bar{Y}_n^{(r)} = \bar{Y}_n^{(r)} 1_{|\bar{Y}_n^{(r)}| < (1 + |x|)\bar{B}_n}.
\]

It is easily seen that \( I_1 \leq (1 + |x|)^{-(s + \delta)} \bar{L}_{d+\delta}^{(r)} \) and by standard arguments we get

\[
\frac{E\bar{S}_n}{\bar{B}_n} \leq C_4 \frac{\bar{L}_{d+\delta}^{(r)}}{(1 + |x|)^{s + \delta - 1}}, \quad \left| \frac{\bar{B}_n}{\bar{B}_n} - 1 \right| \leq C_5 \frac{\bar{L}_{d+\delta}^{(r)}}{(1 + |x|)^{s + \delta - 2}}.
\]

The mean value theorem applied to \( \Phi(x) \) yields then the estimate \( I_2 \leq C_6 (1 + |x|)^{-(s + \delta)} \times \bar{L}_{d+\delta}^{(r)} \). In order to find a corresponding bound for \( I_3 \) we make use of Theorem A \( (r = 0) \)
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and Theorem B \((1 \leq r \leq d - 1)\), where \(s\) is replaced by \(s + 1\) and the RV's \(\bar{Y}_y^{(r)}\) are substituted by the truncated ones \(h_y(\bar{Y}_y^{(r)}) := \bar{Y}_y^{(r)} - E\bar{Y}_y^{(r)}\) (see Remark 2). By virtue of (1.8) and (1.10),

\[
C_7 |V_n| \leq \bar{B}_n^2 \leq 2(\tilde{B}_n^{(r)})^2 \leq C_8 |V_n|.
\]

This estimate together with (1.9) ensures that the conditions (1.4) and (1.8) are fulfilled with the indicated modifications. We are now in a position to apply the estimates (1.6) and (1.9), respectively. Combining (1.12) with (1.6) and (1.9) we obtain, for \(s = 2, 3, \ldots\),

\[
I_2 \leq C_9 \left(1 + \left|\frac{x\bar{B}_n - ES_n}{\bar{B}_n}\right|^{-\alpha+1}\right) \times \left(\sum_{\epsilon \in \mathbb{N}} E|\bar{Y}_y^{(r)}|^3 + (\bar{B}_n)^{-\alpha+1} \sum_{\epsilon \in \mathbb{N}} E|\bar{Y}_y^{(r)}|^\alpha+1\right) \leq C_{10}(1 + |x|)^{-\alpha+1}(L_{3\lambda(s+\delta)n} + L_{s+3n}).
\]

Finally, the inequalities \(C_{11} \leq \bar{L}_{2n} \leq C_{12}\) implied by (1.7) and

\[
\bar{L}_{3n} \leq \bar{L}_{2n}(\bar{L}_{n+d}/\bar{L}_{2n})^{\delta-2} \quad \text{(see [7])}
\]

complete the proof of Theorem C.

The connection between (1.11) and the desired non-uniform bound for \(A_n(x)\) is given by the following

Lemma 1. Let \(F\) be an arbitrary distribution function and \(0 \leq \epsilon < 1\). Then for any two RV's \(X\) and \(\bar{X}\) (on \((\Omega, \mathcal{F}, P)\)) and every \(x \in \mathbb{R}^3\) we have

\[
|P(X < x) - F(x)| \leq P(|X - \bar{X}| \geq \epsilon(|x| \lor 1)) + \max \{D_F(x, \epsilon), D_F(x, -\epsilon)\} + \max \{A_F(x, \epsilon), A_F(y, -\epsilon)\},
\]

where \(D_F(x, \pm \epsilon) = |F(x \pm \epsilon(|x| \lor 1)) - F(x)|\) and \(A_F(x, \pm \epsilon) = |F(x \pm \epsilon(|x| \lor 1)) - F(x)|\), \(a \lor b = \max(a, b)\).

Proof. Obviously,

\[
P(\bar{X} < y - \epsilon(|x| \lor 1)) - P(|X - \bar{X}| \geq \epsilon(|x| \lor 1)) 
\leq P(X < x) \leq P(\bar{X} < x + \epsilon(|y| \lor 1)) + P(|X - \bar{X}| \geq \epsilon(|x| \lor 1)),
\]

whence it follows the inclusion

\[
-P(|X - \bar{X}| \geq \epsilon(|x| \lor 1)) - A_F(x, -\epsilon) - D_F(x, -\epsilon) \leq P(X < x) - F(x) \leq P(|X - \bar{X}| \geq \epsilon(|x| \lor 1)) + A_F(x, \epsilon) + D_F(x, \epsilon)
\]

proving the assertion of Lemma 1.

Note that a somewhat cruder version of Lemma 1 in the special case \(F(x) = \Phi(x)\) was stated in [15].
2. Main results

We consider the RF (1.0) on the growing sequence of index sets \((V_n)_{n=1,2,...}\), and suppose that the following basic assumptions are satisfied for some real \(p \geq 2\):

\[(A_p) \quad \sup_{z \in \mathbb{Z}_+^d} \mathbb{E} |X_z|^p \leq A(p, d) < \infty,\]

\[(B_p) \quad \sup_{z \in \mathbb{Z}_+^d} (\mathbb{E} |X_z - E(X_z| \mathcal{F}_m)|^p)^{1/p} \leq A \cdot \tau(m), \quad m = 0, 1, 2, \ldots,\]

where the function \(\tau(.)\) is non-increasing such that

\[\sum_{m=0}^{\infty} m^{d-1} \tau(m) \leq B = B(\tau) < \infty.\]

**Theorem 1.** Suppose the RF (1.0) satisfies the following conditions: \(E X_z = 0, z \in \mathbb{Z}_+^d,\)

\[\beta \geq C |V_n|, C > 0, \text{ for } n \geq n_0, \quad (A_p) \text{ and } (B_p) \text{ with } \tau(m) = m^{-\alpha} (m \geq 1), \quad \tau(0) = 1, \quad \alpha > 0, \text{ for some } p > 2. \]

Then

\[(2.1) \quad A_n(x) \leq \frac{C_1}{(1 + |x|)^p} |V_n|^{-(1+\beta)/(1+\beta)} |V_n|^{-d/2} \]

for all \(x \in \mathbb{R}^d\) and \(n \geq n_0\), where \(C_1\) only depends on \(p, d, \alpha, A, B, C\).

**Theorem 2.** Let the conditions of Theorem 1 be satisfied with \(\tau(m) = e^{-\beta m}\), where \(g(m) \leq 0\) and \(g(m)/\ln m \uparrow \infty\) as \(m \to \infty\). Then

\[(2.2) \quad A_n(x) \leq \frac{C_2}{(1 + |x|)^p} \left( |V_n|^{-(1+\beta)/(1+\beta)} |V_n|^{-d/2} \right),\]

for all \(x \in \mathbb{R}^d\) and \(n \geq n_0\), where \(C_2\) depends on \(p, d, g(.), A, B, C, \) and \(A_n\) is the smallest integer for which \(g(A_n) \geq \left( \frac{3}{2} - \frac{1}{p} \right) \ln |V_n|\).

**Corollary 1.** In the special case \(\tau(m) = e^{-\beta m}, b, c > 0\), (2.2) can be specified to

\[(2.3) \quad A_n(x) \leq \frac{C_3}{(1 + |x|)^p} \left( \ln |V_n|^{1/b} |V_n|^{-d/2} \right),\]

see [8] for \(b = 1\). Under the conditions of Theorem 2 we always have

\[|V_n|^{1/2} \sup_{x \in \mathbb{R}^d} (1 + |x|)^p A_n(x) \xrightarrow{n \to \infty} 0 \text{ for every } \varepsilon > 0.\]

**Remark 3.** I. A. Ibragimov [9] was the first that proved (uniform) rates of convergence of the shape (2.3) for \(d = 1, b = 1, 2 < p \leq 3, f = f, \) and \(\xi_i\) i.i.d. For \(p = 3\) this bound was sharpened to

\[A_n(x) \leq C_4 \frac{\omega_x(n)}{\sqrt{n} (1 + |x|)^3} \quad \text{(see [3], [12])},\]

where

\[\omega_x(n) = \begin{cases} \sqrt{\ln n} & \text{for } |x| \leq e^{\sqrt{n}/\ln n} \\ (\ln n)^3 & \text{otherwise} \end{cases}\]
Obviously, in view of (2.3), \( \omega_x(n) \) can now be replaced by \( \sqrt{\ln n} \). The problem whether the logarithm in \( \omega_x(n) \) can be removed and replaced by a constant is still open. G. A. Misevičius [13] gave a partial answer to this question under much stronger additional restrictions. In some cases Edgeworth expansions of \( A_n(x) \) also provide "optimal" Berry-Esseen bounds for \( A_n(x) \), where the \( X_i \) must fulfill higher moment and certain smoothness conditions (see [24] for \( d = 1 \) and Sect. 5).

Finally we apply (2.3) to the normal approximation of (1.1). Indeed, remembering (1.2) and setting

\[
\epsilon_k(e_k, e_{k+1}, \ldots) := f(2^{k-1}t) = f\left(\sum_{j=k}^{\infty} \frac{e_j(t)}{2^{j-1}}\right)
\]

we put this problem into the frame of Theorem 1 and 2. As it was shown in [9] the smoothness condition

\[
(2.4) \quad \left( \int_0^1 |f(t + h) - f(t)|^p dt \right)^{1/p} \leq C_0 h, \quad h > 0, \quad \alpha > 0, \quad p > 2,
\]

implies

\[
(E |f_k(e_k; e_{k+1}, \ldots) - f_k(e_k, e_{k+1}, \ldots)|^{p})^{1/p} \leq C_1 e^{-c_m}
\]

for some constants \( c \) and \( C_1 \) depending on \( \alpha \) and \( p \). Therefore (2.3) leads to the subsequent strengthening of the CLT's for (1.1) given in [9], [12], and [3].

**Theorem 3.** Let \( f : [0, \infty) \to \mathbb{R}^1 \) be a Borel measurable function with period 1 satisfying

\[
\int_0^1 f(t) dt = 0, \quad \int_0^1 |f(t)|^p dt < \infty \quad \text{for some} \quad p > 2,
\]

\[
\sigma^2 = \int_0^1 f^2(t) dt + 2 \sum_{k=1}^{\infty} \int_0^1 f(t) f(2^k t) dt > 0,
\]

and (2.4). Then

\[
(2.5) \quad \left| \nu \left( \left\{ t \in (0, 1) : \frac{1}{\sigma} \sqrt{n} \sum_{k=0}^{n-1} f(2^k t) < x \right\} \right) - \Phi(x) \right| \leq \frac{C_2}{(1 + |x|)^p} \left( \frac{\ln n}{n} \right)^{1/(1 + p - 1)}
\]

for all \( x \in \mathbb{R}^1 \) and sufficiently large \( n \).

Note that, by simple calculations, one can check the relation

\[
\sup_{n \geq 1} \left| \int_0^1 \left( \sum_{k=0}^{n-1} f(2^k t) \right) dt - n\sigma^2 \right| \leq 2 \sum_{k=1}^\infty \left| \int_0^1 f(t) f(2^k t) dt \right| < \infty
\]

giving the possibility to substitute \( n\sigma^2 \) by the integral over the squared sum \( f(t) + \cdots + f(2^{n-1}t) \) without changing the bound on the right-hand side of (2.5).
3. Moment bounds for $S_n$ under the mixing condition ($B_*$)

For the sake of generality we study moment bounds for sums of the form

$$S_{nk} = \sum_{z \in V_{nk}} X_{z+y}, \quad 1 \leq k \leq d,$$

where $V_{nk} = \{z \in \mathbb{Z}^d : 1 \leq z_i \leq n_i \ (j = i_1, \ldots, i_r) , z_j = 1 (j + i_1, \ldots, i_r), 1 \leq i_1 < \ldots < i_r \leq d\}$, and $y \in \mathbb{Z}^d = \{0, 1, 2, \ldots\}^d$. In Lemma 3 we establish a bound for $E |S_{nk}|^p, p \geq 2$, which, in fact, does not depend on $y \in \mathbb{Z}^d$. To prove this result we employ a modified version of a technique which is due to Doob (see [6], pp. 225-227). We mention that in the relatively simple case $d = 1$ and $\tau(m) = e^{-cm}$ the proof of Lemma 3 was briefly sketched in [9]. Doob's idea was also used to obtain moment bounds for sums of stationary $\alpha$-mixing and $\varphi$-mixing RV's (see [10], [11], [23]). Other methods for obtaining similar results are discussed in [19] and [23]. In what follows we shall show how Doob's technique works in the more difficult case of RF's (multidimensional indices), where in addition the stationarity assumption is replaced by the uniform boundedness of the $p$-th moments of $X_z, z \in \mathbb{Z}^d$. To begin with we state a simple lemma concerning the behaviour of the variance of (3.1).

**Lemma 2.** If the RF (1.0) satisfies (A2) and (B2), then

$$D^2 S_{nk} \leq A \sup_{z \in \mathbb{Z}^d} (EX_z^2)^{1/2} \left(1 + 2k^3 \sum_{m=0}^\infty (m+1)^{k-1} \tau(m)\right),$$

where $|V_{nk}| = n_{i_1} \cdots n_{i_r}$ and $1 \leq k \leq d$.

**Proof.** We have

$$D^2 S_{nk} = \sum_{z \in V_{nk}+y} D^2 X_z + \sum_{z \in V_{nk}+y} \sum_{m=0}^\infty \sum_{z' \in V_{nk}+y} \text{Cov}(X_z, X_{z'}) \text{Cov}(X_{z'}, X_{z'}),$$

where $||z|| = \max |z_i|$. Since, for $||z - z'|| \geq m + 1$, $U_z^{(m)} \cap U_{z'}^{(m)} = \emptyset$, this is, $E(X_z \mid \mathcal{F}_z^{(m)})$ and $E(X_{z'} \mid \mathcal{F}_{z'}^{(m)})$ are independent RV's, we may write

$$\text{Cov}(X_z, X_{z'}) = E(X_z - E(X_z \mid \mathcal{F}_z^{(m)})) X_{z'},$$

$$+ E(X_{z'} - E(X_{z'} \mid \mathcal{F}_{z'}^{(m)})) E(X_z \mid \mathcal{F}_z^{(m)}).$$

Thus, the assumptions (A2) and (B2), Schwarz's inequality and the fact that card $\{z' \in V_{nk} : ||z' - z|| = m + 1\} \leq (2m + 3)^k - (2m + 1)^k$ lead to

$$D^2 S_{nk} \leq A \sup_{z \in \mathbb{Z}^d} (EX_z^2)^{1/2} |V_{nk}| \left(1 + 2 \sum_{m=0}^\infty ((2m + 3)^k - (2m + 1)^k) \tau(m)\right).$$

Finally, the inequality $(2m + 3)^k - (2m + 1)^k \leq k3^k(m + 1)^{k-1}$ completes the proof of Lemma 2.

We now turn to the main result of this section.

**Lemma 3.** Let the RF (1.0) fulfill (Ap) and (Bp) for some $p > 2$ and without loss of generality let $EX_z = 0, z \in \mathbb{Z}^d$. Then

$$E |S_{nk}|^p \leq C_0 |V_{nk}|^{p/2}, \quad 1 \leq k \leq d,$$

for some constant $C_0$ only depending on $p, d, k, A,$ and $B$. 

As a consequence we have

**Corollary 2.** Under the assumptions of Lemma 3 the estimate

\[
\sup_{x \in x_0} \mathbb{E} \left| \sum_{z \in \mathbb{Z}^{d+n}} X_z \right|^p \leq C_1 |V_n|^p/2
\]

is valid, where \( C_1 \) depends on \( p, d, A, \) and \( B \).

The conspicuous feature of the estimates (3.4) and (3.5) is their analogy to corresponding moment bounds for sums of mutually independent RV's. This fact shows that (3.4) and (3.5) cannot be improved concerning the order of \( V_{nk} \) and \( V_n \), respectively.

**Proof of Lemma 3.** In view of Lemma 2 it suffices to assume that (3.4) is already true if \( p \) is an integer \( s \geq 2 \) and to prove that it is then true if \( p = s + \delta, 0 < \delta \leq 1 \), whenever \((A_{s+\delta})\) and \((B_{s+\delta})\) hold. This will be shown by induction on the subdimensions \( k \in \{1, \ldots, d\} \). To carry out this proof in detail we distinguish two essential cases. We first regard the case when all \( n_i \)'s take on the same value and in the second step this restriction will be omitted.

Step 1. For \( k \in \{0, 1, \ldots, d - 1\} \) let a further integer \( i_{k+1} \) be taken such that \( 1 \leq i_1 < \cdots < i_k < i_{k+1} \leq d \), and for simplicity assume that, in (3.1) and (3.4), \( n_i = \cdots = n_k = n, i_{k+1} = n \geq 1 \). With these modifications our induction hypothesis may be formulated as follows: Suppose

(i) (3.4) is valid for \( p = s \) \((\geq 2)\) and \( V_{nk+1} \) instead of \( V_{nk} \),

(ii) (3.4) is valid for \( p = s + \delta \) \((0 < \delta \leq 1)\) and each of the index subsets \( \{z \in \mathbb{Z}^d : 1 \leq z_{i_1}, \ldots, z_{i_k} \leq n, z_j = 1 \} (j = i_1, \ldots, i_k) \leq V_{nk+1}, 1 \leq j_1 < \cdots < j_k \leq k + 1, 1 \leq l \leq k, \) instead of \( V_{nk} \). (If \( k = 0 \) then (ii) coincides with \((A_{s+\delta})\)).

Then we are to prove the validity of (3.4) for \( p = s + \delta \) and \( V_{nk+1} \) instead of \( V_{nk} \), i.e.

\[
E |S_{nk+1}|^{s+\delta} \leq C_1 n^{(k+1)(s+\delta)/2}
\]

with an appropriate choice of \( C_1 \). First verify that

\[
E \left| \sum_J S_{nk+1}^{(J)} \right|^{s+\delta} \leq (2k+1 + \varepsilon_0) \max_J E |S_{nk+1}^{(J)}|^{s+\delta} + C_2 n^{(k+1)(s+\delta)/2}
\]

for \( n = 1, 2, \ldots \), where \( S_{nk+1}^{(J)} = \sum_{z \in V_{nk+1}} X_z \) with \( V_{nk+1}^{(J)} = \{z \in \mathbb{Z}^d : j_a(n + q[n^c]) < z_{i_a} \leq n + j_a(n + q[n^c]), a = 1, \ldots, k + 1, z_j = 1 \} (j = i_1, \ldots, j_{k+1}) \), for some \( q \in \mathbb{Z}^d_+ \) and \( c = \frac{k}{2(k+1)} \) \([x] \) denotes the integer part of \( x \). Here \( \sum_J \left( \max_J \right) \) means summing (maximizing) over all \((k+1)\)-tuples \( J = (j_1, \ldots, j_{k+1}) \in \{0, 1\}^{k+1} \). The constant \( C_2 \) on the right side of (3.7) does not involve \( q \) and \( n \), however, \( \varepsilon_0 = \varepsilon_0(q[n^c]) \) can be chosen sufficiently small in dependence of the magnitude of \( q[n^c] \), in other words \( \varepsilon_0(u) \) tends to zero as \( u \to \infty \).
It is readily seen that
\[
E \left| \sum_j S_{nk+1}^{(J)} \right|^{s + \delta} \leq E \left( \sum_j S_{nk+1}^{(J)} \right)^{s + \delta} \leq \sum_j E |S_{nk+1}^{(J)}|^{s + \delta} + \sum_{r=1}^{s} c_r \sum_{j+J'} E |S_{nk+1}^{(J)}|^{s - r + \delta},
\]
where \( J = J' \) indicates that at least one component of \( J \) differs from the corresponding one of \( J' \), e.g. \( j_b = j_b' \) for some \( b \in \{1, \ldots, k + 1\} \). For brevity put
\[
\tilde{S}^{(J)} = \sigma(\xi + y : j_b(n + q[n^e]) < z_{i_b}) \leq (j_a + 1)(n + q[n^e]) + n(a \in \{1, \ldots, k + 1\} \setminus \{b\}),
\]
\[
j_b(n + q[n^e]) < z_{i_b} \leq (j_b + 1)(n + q[n^e]),
\]
\[
1 \leq z_j \leq n + q[n^e] (j = \hat{j}_1, \ldots, \hat{j}_{k+1})
\]
and
\[
\tilde{S}_{nk+1}^{(J)} = E(S_{nk+1}^{(J)} | \tilde{S}^{(J)}).
\]
Clearly, \( \tilde{S}_{nk+1}^{(J)} \) and \( \tilde{S}_{nk+1}^{(J')} \) are independent RV's, and hence, by using Hölder's inequality, we get
\[
E |S_{nk+1}^{(J)}|^{s + \delta} \leq 2^{-1} E |S_{nk+1}^{(J)} - \tilde{S}_{nk+1}^{(J)}|^s |S_{nk+1}^{(J')}|^{s - r + \delta} + 2^{s + 2 - \delta} E |\tilde{S}_{nk+1}^{(J)}|^s |S_{nk+1}^{(J')} - \tilde{S}_{nk+1}^{(J')}|^s - r + \delta
\]
\[
+ 2^{s + 2 - \delta} E |\tilde{S}_{nk+1}^{(J)}|^s E |\tilde{S}_{nk+1}^{(J')}|^s - r + \delta
\]
\[
= 2^{-1}(E |S_{nk+1}^{(J)} - \tilde{S}_{nk+1}^{(J)}|^{s + \delta})^{(s + \delta)/(s + \delta)} (E |S_{nk+1}^{(J')}|^{s + \delta/(s + \delta)})^{(s + \delta)/(s + \delta)}
\]
\[
+ 2^{s + 2 - \delta} (E |\tilde{S}_{nk+1}^{(J)}|^{s + \delta})^{(s + \delta)/(s + \delta)} (E |\tilde{S}_{nk+1}^{(J')}|^{s + \delta})^{(s - r + \delta)/(s + \delta)}
\]
\[
+ 2^{s + 2 - \delta} (E |\tilde{S}_{nk+1}^{(J)}|^{s + \delta})^{(s - r + \delta)/(s + \delta)} E |\tilde{S}_{nk+1}^{(J')}|^{s - r + \delta}
\]
Now, for \( z \in V_{nk+1}^{(J)} + y \),
\[
|E [X_z - E(X_z | \tilde{S}^{(J)})]|^{s + \delta})^{1/(s + \delta)}
\]
\[
\leq (E [X_z - E(X_z | \tilde{S}_{nk+1}^{(J)}(j_b+1)(n + q[n^e]) + y_{i_b} - z_{i_b})]|^{s + \delta})^{1/(s + \delta)}
\]
\[
+ (E [E(X_z - E(X_z | \tilde{S}_{nk+1}^{(J)}(j_b+1)(n + q[n^e]) + y_{i_b} - z_{i_b}) | \tilde{S}^{(J)})]|^{s + \delta})^{1/(s + \delta)}
\]
\[
\leq 2A\tau((j_b + 1)(n + q[n^e]) + y_{i_b} - z_{i_b})
\]
by the inclusion \( \tilde{S}_{nk+1}^{(J)}{(j_b+1)(n + q[n^e]) + y_{i_b} - z_{i_b}) \subseteq \tilde{S}^{(J)}, \) Minkowski's inequality and \((B_{s+\delta})\).

Therefore, again by using Minkowski's inequality,
\[
(E |S_{nk+1}^{(J)} - \tilde{S}_{nk+1}^{(J)}|^{s + \delta})^{1/(s + \delta)}
\]
\[
\leq \sum_{z \in V_{nk+1}^{(J)} + y} (E |X_z - E(X_z | \tilde{S}^{(J)})|^{s + \delta})^{1/(s + \delta)} \leq 2A n^k \sum_{l=0}^{n-1} \rho(q[n^e] + l).
Since \( k + 1 \leq d \) the properties of \( \tau(\cdot) \) imply
\[
\sum_{l=0}^{n-1} \tau(q[n^e] + l) \leq (n/[n^e]) \sum_{l=0}^{n-1} (q[n^e] + l)^k \tau(q[n^e] + l) \leq n^{(k+1)/2} \epsilon_1(q[n^e]),
\]
where \( \epsilon_1(u) \) goes to zero as \( u \to \infty \).

Combining the latter and the above estimates and taking into account that, by (ii),
\[
E \left| S_{nk+1}(J) \right|^s \leq E \left| S_{nk+1}(J') \right|^s \leq C_1 n^{s(k+1)/2}
\]
we obtain
\[
(3.9) \quad E \left| S_{nk+1}^{(J)} \right|^r \left| S_{nk+1}^{(J')} \right|^{s-r+\delta} \leq C_3 \epsilon_1(q[n^e]) n^{s(k+1)/2} \left( \max_j E \left| S_{nk+1}^{(J)} \right|^{s+\delta} \right) + C_4 n^{(s-\delta)(k+1)/2}
\]
for \( r = 1, \ldots, s \).

Finally, the inequality \( x^{\delta(s+\delta)} y^{s(s+\delta)} \leq \frac{\delta}{s+\delta} x + \frac{s}{s+\delta} y \), \( x, y \geq 0 \), and (3.8) show the validity of (3.7).

Next for every integer \( n \geq 1 \) (with \( n_{i_1} = \cdots = n_{i_{k+1}} = n \)) define
\[
a_{n,k+1} = \sup_{y \in Z_0^q} \sum_{z \in E_{nk+1}^k + y} X_z^{s+\delta}.
\]
We now prove that for every \( \epsilon > 0 \) there is a value of \( q = q(\epsilon) \) and a constant \( C \) depending on \( q, k, s \), and \( \delta \) such that, for \( n = 1, 2, \ldots, \)
\[
(3.10) \quad a_{2n,k+1} \leq (2^{k+1} + \epsilon) a_{n,k+1} + C n^{(s+\delta)(k+1)/2}.
\]

Minkowski’s inequality and the induction hypothesis (ii) provide
\[
\sup_{y \in Z_0^q} \sum_{z \in E_{nk+1}^k(q_1, \ldots, q_r) + y} X_z^{s+\delta} \leq C_1 (q[n^e])^{(s+\delta)(k+1-r)} n^{(s+\delta)/2} \leq C_1 \left( \frac{q}{n^l/(k+1)} \right)^{(s+\delta)(k+1-r)} n^{(s+\delta)(k+1)/2}
\]
for \( r = 0, 1, \ldots, k \) and every subset \( \{q_1, \ldots, q_r\} \subset \{i_1, \ldots, i_{k+1}\} \), where
\[
V_{nk+1}(q_1, \ldots, q_r) = \{z \in Z_+^k : 1 \leq z_{q_1}, \ldots, z_{q_r} \leq n, 1 \leq z_j \leq q[n^e] \in \{i_1, \ldots, i_{k+1}\} \setminus \{q_1, \ldots, q_r\}, z_j = 1 \in i_1, \ldots, i_{k+1} \}
\]
The preceding estimate and (3.7) yield
\[
(3.11) \quad \left( E \left| S_{2nk+1}^{(J)} \right|^{s+\delta} \right) \left( E \left| S_{2nk+1}^{(J')} \right|^{s+\delta} \right) \leq \left( E \left| S_{nk+1}^{(J)} \right|^{s+\delta} \right) \left( E \left| S_{nk+1}^{(J')} \right|^{s+\delta} \right) \leq \left( 2^{k+1} + \epsilon_0 \right) a_{n,k+1} + C_2 n^{(s+\delta)(k+1)/2} \left( \frac{q^{k+1}}{n^{l/2}}, \frac{q}{n^{l/2}(k+1)} \right).
\]
Since the right side does not depend on \( y \in \mathbb{Z}_t \) we may take the supremum over all \( y \in \mathbb{Z}_t \) on the left side. Hence, for \( n \geq q^{4(k+1)} \) we get
\[
a_{2n,k+1} \leq \left( (2^{k+1} + \varepsilon_2) a_{n,k+1} + C_2 q^{(s+\delta)(k+1)/2} \right) \left( 1 + \frac{C_5}{q^{C_2^{(s+\delta)}}} \right)^{s+\delta} = (2^{k+1} + \varepsilon) a_{n,k+1} + C_2 q^{(s+\delta)(k+1)/2},
\]
where
\[
\varepsilon = \varepsilon_2 q^{(n^c(t)])} + 2^{k+1} \left( 1 + C_5 q^{C_2^{(s+\delta)}} \right)
\]
and
\[
C = C_2 + C_2 q^{(s+\delta)/2} q.
\]
On the other hand, the condition \((A_{s+\delta})\) and a repeated use of Minkowski\'s inequality ensure that
\[
a_{2n,k+1} \leq 2^{s+\delta}(k+1) a_{n,k+1} \leq 2^{k+1} a_{n,k+1} + 2^{k+1} \left( 2s+\delta - 1 \right) A^{s+\delta} q^{(s+\delta)(k+1)} \leq 2^{k+1} a_{n,k+1} + C_2 q^{(s+\delta)(k+1)/2}
\]
for \( n = 1, 2, \ldots, q^{4(k+1)} - 1 \), where
\[
C = 2^{k+1}(2s+\delta - 1) A^{s+\delta} q^{(s+\delta)(k+1)^r}.
\]
Thus the relation \((3.10)\) is true for all \( n = 1, 2, \ldots \)

According to \((3.10)\)
\[
a_{2r,k+1} \leq (2^{k+1} + \varepsilon)^r a_{n,k+1} + C \sum_{j=1}^{r} (2^{k+1} + \varepsilon)^r \frac{2(l-1)(s+\delta)(k+1)/2}{2^{(k+1)(s+\delta)/2} - (2^{k+1} + \varepsilon)}
\]
whenever \( 2^{k+1} + \varepsilon < 2^{(k+1)(s+\delta)/2} \).

If \( \varepsilon \) is selected in this way we have
\[
a_{2r,k+1} \leq C_6 2^{(s+\delta)(k+1)/2} \quad \text{for} \quad r = 0, 1, 2, \ldots,
\]
where
\[
C_6 = A^{s+\delta} + C \left( 2^{(k+1)(s+\delta)/2} - (2^{k+1} + \varepsilon) \right).
\]
Now let \( n \) be an arbitrary integer such that \( 2^{r} < n \leq 2^{r+1} \). We argue as follows: For an arbitrarily chosen \( y \in \mathbb{Z}_t \) define the RF \( X'_z, z \in \mathbb{Z}^d_+ \), by
\[
X'_z = \begin{cases} 0 & \text{for} \ z \in (V_{2^{r+1}} \setminus V_{2^{r+1} + 1}) + y \\ X_z & \text{for all other} \ z \in \mathbb{Z}^d_+ \end{cases}
\]
Obviously, the RV\'s \( X'_z, z \in \mathbb{Z}^d_+ \), satisfy \((A_{s+\delta})\) and \((B_{s+\delta})\) and therefore, by virtue of \((3.11)\), we conclude that
\[
E \left| \sum_{z \in V_{2^{r+1}+1}} X_z \right|^{s+\delta} = E \left| \sum_{z \in V_{2^{r+1}+1}+y} X'_z \right|^{s+\delta} \leq C_6 2^{(r+1)(s+\delta)(k+1)/2} \leq C_6 (2n)(k+1)(s+\delta)/2
\]
where \( C_3 \) is independent of the choice of \( y \in \mathbb{Z}_d^k \). This provides the desired estimate (3.6) and, hence, (3.4) is completely proved in the special case \( n_{i_1} = \cdots = n_{i_k} \).

In order to accomplish the proof of Lemma 3 for a general rectangle \( V_{n_k} (k \geq 2) \) we proceed as follows: Based on Step 1 we next show (3.4) for \( n_{i_1} \leq \cdots \leq n_{i_k} \) and then, using this result, we may turn to prove (3.4) for \( n_{i_1} \leq \cdots \leq n_{i_k} \) and so on until we arrive at the most general case \( n_{i_1} \leq \cdots \leq n_{i_k} \). Of course this programme will be realized by induction.

For some \( h \in \{1, \ldots, k-1\} \) assume that \( n_{i_1} \leq \cdots \leq n_{i_h} = \cdots = n_{i_h} \) and define

\[
V_{u_1, \ldots, u_{k-h}} = \{z \in \mathbb{Z}_d^k : 1 \leq z_k \leq n_{i_h} (q = 1, \ldots, h), (u_{p-h} - 1) n_{i_h} < z_i \}
\]

and

\[
Y_{u_1, \ldots, u_{k-h}} = \sum_{z \in V_{u_1, \ldots, u_{k-h}}} X_z \quad \text{for} \quad y \in \mathbb{Z}_d^k, \ (u_1, \ldots, u_{k-h}) \in \mathbb{Z}_d^{k-h}.
\]

**Step 2.** For some \( l \in \{0, 1, \ldots, k - h - 1\} \) consider a fixed \((l + 1)\)-tuple \((k_1, \ldots, k_{l+1})\) such that \( 1 \leq k_1 < \cdots < k_{l+1} \leq k - h \) holds. Suppose that for some integer \( s \geq 2 \), \( 0 < \delta \leq 1 \) and every \( N \geq 1 \) the following induction hypotheses are satisfied:

(iii) \[
\sup_{y \in \mathbb{Z}_d} \left| \sum_{i_{k_1}, \ldots, i_{k_{l+1}}} Y_{u_1, \ldots, u_{k-h}} \right|^{s+\delta} \leq C_1 (n_{i_1} \cdots n_{i_h} n_{i_h}^{k-h} N^l)^{s/2},
\]

(iv) \[
\sup_{y \in \mathbb{Z}_d} \left| Y_{u_1, \ldots, u_{k-h}} \right|^{s+\delta} \leq C_2 (n_{i_1} \cdots n_{i_h} n_{i_h}^{k-h})^{(s+\delta)/2},
\]

and

(v) \[
\sup_{y \in \mathbb{Z}_d} \left| \sum_{i_{k_1}, \ldots, i_{k_{l+1}}} Y_{u_1, \ldots, u_{k-h}} \right|^{s+\delta} \leq C_2 (n_{i_1} \cdots n_{i_h} n_{i_h}^{k-h} N_m)^{(s+\delta)/2}
\]

for each subset \( \{q_1, \ldots, q_m\} \subset \{k_1, \ldots, k_{l+1}\}, 1 \leq m \leq l \).

Under these assumptions we have to show that the estimate

\[
(3.13) \quad \sup_{y \in \mathbb{Z}_d} \left| \sum_{i_{k_1}, \ldots, i_{k_{l+1}}} Y_{u_1, \ldots, u_{k-h}} \right|^{s+\delta} \leq C_3 (n_{i_1} \cdots n_{i_h} n_{i_h}^{k-h} N^{l+1})^{(s+\delta)/2}
\]

is true for every \( N \in \mathbb{Z}_d^l \), where \( C_3 \) does not depend on \( n_{i_1}, \ldots, n_{i_h} \) and \( N \). The proof of (3.13) closely resembles that of (3.6) of Step 1.

Therefore (though some technical modifications are necessary) the details of this proof are left to the reader. As a consequence of Step 2 we have

\[
\sup_{y \in \mathbb{Z}_d} \left| \sum_{i_{k_1}, \ldots, i_{k_{l+1}}} Y_{u_1, \ldots, u_{k-h}} \right|^{s+\delta} \leq C_3 (n_{i_1} \cdots n_{i_h} n_{i_h}^{k-h} N)^{(s+\delta)/2}
\]

for every \( N \in \mathbb{Z}_d^l \). Finally, setting \( N = [n_{i_h} / n_{i_h}] + 1 \) and applying the trick used at the end of the proof of Step 1 again we recognize that (3.4) is also true in the case of \( n_{i_1} \leq \cdots \leq n_{i_k} \leq n_{i_k} \). By these final remarks the rather technical proof of Lemma 3 is terminated.
Remark 4. The preceding proof reveals that in order to hold (3.4) for some fixed \( k \in \{1, \ldots, d\} \) the following condition on \( \tau(\cdot) \) is sufficient:

\[
\sum_{m=0}^{\infty} (m+1)^{k-1} \tau(m) \leq B < \infty.
\]

4. Proofs of the Theorems 1 and 2

Before passing to the proofs of Theorem 1 or 2 we first derive a general bound for \( \Delta_n(x) \) which is based on Lemma 1. For doing this we specify Lemma 1 as follows:

\[
F(x) = \Phi(x), \quad X = S_n/B_n, \quad \overline{X} = \overline{S}_n/B_n, \quad \overline{B}_n = E\overline{S}_n^2,
\]

where

\[
\overline{S}_n = \sum_{z \in V_n} \overline{X}_z \quad \text{and} \quad \overline{X}_z = E(X_z | \overline{S}_z^{(m)}) = g_z(\xi_y; y \in U_z^{(m)}).
\]

So we have

\[
\Delta_n(x) \leq P(|S_n - \overline{S}_n| \geq \varepsilon(|x| \vee 1) B_n) + \max_{\delta \in (-\varepsilon, \varepsilon)} \left| \Phi \left( \frac{B_n}{\overline{B}_n} (x + \delta(|x| \vee 1)) \right) - \Phi(x) \right| \geq \frac{\varepsilon(|x| \vee 1) B_n}{\overline{B}_n},
\]

(4.1)

By standard inequalities we get

\[
I_1(x) \leq \frac{\varepsilon(|x| \vee 1) B_n}{\overline{B}_n} \sup_{z \in V_n} E |X_z - \overline{X}_z|^p.
\]

so that, if \((B_p)\) and \(B_n^p \geq C |V_n|\) hold,

(4.2)

Now regard the discrepancy \( B_n - \overline{B}_n \). SCHWARZ's and MINKOWSKI's inequality yield

\[
|B_n^2 - \overline{B}_n^2| = |E(S_n - \overline{S}_n) (S_n + \overline{S}_n)|
\]

\[
\leq (E(S_n - \overline{S}_n)^2 E(S_n + \overline{S}_n)^2)^{1/2}
\]

\[
\leq (B_n + \overline{B}_n) (E(S_n - \overline{S}_n)^2)^{1/2}.
\]

Since for the RF \( X_z^* = X_z - \overline{X}_z, z \in Z_+^d \),

\[
E(X_z^* - E(X_z^* | \overline{S}_z^{(m)}))^2 = E(X_z - E(X_z | \overline{S}_z^{(m)}))^2 \leq (A \tau(k \vee m))^2
\]

for \( k = 0, 1, 2, \ldots \), we deduce from Lemma 2 and \((B_p)\) that

\[
(B_n - \overline{B}_n)^2 \leq C_1 \tau(m) \left( (m+1)^d \tau(m) + \sum_{k \geq m+1} (k + 1)^{d-1} \tau(k) \right) |V_n|
\]
whence, by $B_n^2 \geq C |V_n|$, 

\[(4.3) \quad \left| \frac{B_n}{\overline{B}_n} - 1 \right| \leq \frac{|B_n - \overline{B}_n|}{B_n - |B_n - \overline{B}_n|} \leq C_2 \left( (m+1)^4 \tau^4(m) + \tau(m) \sum_{k=m+1}^\infty (k+1)^{d-1} \tau(k) \right)^{1/2} \]

for sufficiently large $m$.

Bounds for $I_2(x)$ and $I_3(x)$ can now be computed by applying the mean value theorem:

\[
I_2(x) \leq \frac{C_3 \varepsilon}{(1 + |x|)^p (1 - \varepsilon)^p} \exp \left\{ -\frac{1}{2} (|x| - \varepsilon(|x| + 1))^2 \right\} \leq \frac{C_3 \varepsilon}{(1 + |x|)^p (1 - \varepsilon)^p}
\]

and

\[
I_3(x) \leq \frac{C_4}{(1 + |x|)^p (1 - \varepsilon)^p} \left| \frac{B_n}{\overline{B}_n} - 1 \right|
\]

with an appropriate choice of $C_3$ and $C_4$.

It remains to study $I_4(x)$ by the help of Theorem C. For the moment set $p = s + \delta$ for some integer $s \geq 2$ and $0 < \delta \leq 1$ and suppose that $m = m(n)$ is monotonically non-decreasing such that

\[(4.4) \quad \frac{m^d (\ln |V_n|)^{d+1/2}}{|V_n|} \to 0 \quad \text{as} \quad n \to \infty
\]

implying $n_d(n)/m(n) \to \infty$ as $n \to \infty$. Keeping in mind Remark 1 after Theorem A which is correspondingly applicable to Theorem C we find some sufficiently large $N_0$ such that, for $n \geq n_0 = n_0(N_0)$,

\[
\frac{n_1(n)}{m(n)} \leq \cdots \leq \frac{n_r(n)}{m(n)} < N_0 \leq \frac{n_{r+1}(n)}{m(n)} \leq \cdots \leq \frac{n_d(n)}{m(n)}
\]

where $r \in \{0, 1, \ldots, d-1\}$ may depend on $n$. Further, it is immediately seen from Lemma 3 that

\[
E(\overline{Y}_1^{(d)})^2 \leq C_5 n_1 \cdots n_r m^{d-r}
\]

and

\[
\overline{L}_{s+\delta n}^{(r)} \leq \frac{m^{d-r}}{n_{r+1} \cdots n_d} \leq C_6 N_0^{(s+\delta-2)/2} \left( \frac{m^d}{|V_n|} \right)^{(s+\delta-2)/2}.
\]

Therefore from (4.3) and (4.4) together with $B_n^2 \geq C |V_n|$ it follows that condition (1.10) of Theorem C are satisfied. Hence

\[(4.5) \quad I_4(x) \leq C_7 \left( 1 + \frac{B_n}{\overline{B}_n} (|x| - \varepsilon(|x| + 1)) \right)^{-p} \left( \overline{L}_{\overline{B}_n}^{(r)} \right)^{1/2} \frac{1}{p-2}
\]

\[
\leq C_8 \left( (1 + |x|) (1 - \varepsilon) \right)^{-p} \frac{m^d}{|V_n|} \left( \frac{1}{2} (1\Lambda(p-2)) \right).
\]

We are now in a position to prove the theorems.
First specify $m = m(n)$ and $\epsilon = \epsilon(n)$ as follows:

$$m = |V_n|^\alpha, \quad 0 < \alpha < \frac{1}{d}, \quad \text{and} \quad \epsilon = |V_n|^{-\beta}, \quad \beta > 0.$$ 

In case $t(m) = m^{-(d+\alpha)}$ we conclude from (4.1) - (4.3) and (4.5) that

\[
\Delta_n(x) \leq \frac{C_9}{(1 + |x|)^p} \left( |V_n|^{-p(1+n)\alpha - \frac{1}{2}} + |V_n|^{-\beta} + |V_n|^{-\frac{a}{2}(d+2\alpha)} 
+ |V_n|^{-\frac{1}{2}(1+\alpha)(1-2\alpha)} \right)
\]

for all $x \in R^d$.

After some simple calculations it turns out that the minimum of the bracket on the right side of (4.6) is attained for

$$\alpha = \frac{p + (p + 1)(1 \wedge (p - 2))}{2p + (p + 1)(1 \wedge (p - 2))d + 2p}$$

and

$$\beta = \frac{1}{2}(1 \wedge (p - 2)) \frac{p(d + 2\alpha)}{2p(d + \alpha) + d(p + 1)(1 \wedge (p - 2))}.$$ 

This gives $\Delta_n(x) \leq 4C_9(1 + |x|)^{-p} |V_n|^{-\beta}$ proving the assertion of Theorem 1.

On the other hand, on setting $m = A_n$ and $\epsilon = |V_n|^{-1/2}$ and using (4.1) - (4.3) and (4.5) as well as the fact that $t(m) = e^{-g(m)}$ we recognize that

$$\Delta_n(x) \leq \frac{C_{10}}{(1 + |x|)^p} \left( |V_n|^{-p(1+n)\alpha} + |V_n|^{-1/2} + e^{-g(A_n)/2} + \frac{A_n^4}{V_n} \right).$$

Since, by definition of $A_n$ in Theorem 2, $g(A_n) \geq \left(\frac{3}{2} - \frac{1}{p}\right) \ln |V_n|$ the latter estimate yields the desired relation (2.2).

Thus both Theorem 1 and 2 are completely proved.

5. Some Remarks on Asymptotic Expansions

In this final section we briefly consider one of the possible refinements of the error estimates of $\Delta_n(x)$ obtained in the previous sections. Starting from an Edgeworth expansion of $P(S_n < xB_n)$ — the same notation as in Sect. 4 is used — proved and discussed in detail in [7] (Theorem 4, p. 88) we approximate the probability $P(S_n < xB_n)$ by the asymptotic expansion

\[
\Psi_{pn}(x) = \Phi(x) + \sum_{k=1}^{p-2} \sum_{l=1}^{k} \frac{(-1)^{k+2l}}{l! B_{n}^{k+2l}} \sum_{k_1, \ldots, k_l \geq 1} \frac{l! \Gamma_{k_1+2}(S_n)}{(k_1 + 2)!} \frac{d^{k+2l}}{dx^{k+2l}} \Phi(x),
\]

$p = 3, 4, \ldots,$
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where

\[ I_k(X) = i^{-k} \frac{d^k}{dt^k} \ln E e^{itX} |_{t=0} \]  

the k-th order cumulant of X,

and give a non-uniform little-o rate of the remainder term. The succeedingly formulated Theorem 4 improves the main result of G"orze and H"ipp [24] for the special case of — as called there — weakly dependent shifts \( I_z(\xi_y; y \in U_z^{(\infty)}) \). Note that in [24] the \( I_z \)'s are vector-valued but \( z \in Z_+^d \). In [25] such a higher order approximation of \( S_n/B_n \) was recently applied to a sophisticated statistical analysis of linear processes. As already mentioned in Sect. 2 asymptotic expansions also yield optimal Berry-Esseen bounds provided that \( \tau(m) \) decreases quickly enough and \( |I_k(S_n)| \leq C_k |V_n|, k = 3, \ldots, p \). However, the conditions needed for establishing asymptotic expansions are more restrictive than those required in Theorem 1 and Theorem 2. In particular, certain smoothness condition expressed by a weighted conditional Cram\'er-condition has to be imposed on the \( X_z \)'s, more precisely, we need:

\begin{enumerate}
  \item[(C_p)] There exist \( \delta, \varepsilon > 0 \) such that, for \( n \geq n_0 \) and all \( t \in R^d \) with \( |t| \geq \delta \),
  \begin{equation}
  \frac{1}{|V_n|} \sum_{z \in V_n} g_z(t) \leq 1 - \varepsilon,
  \end{equation}

where

\[ g_z(t) = E \left[ E \left[ \exp \left( it \sum_{y \in V_n \setminus \{I_z^{(m)}\}} E(X_y \mid S_z^{(m)}) \right) \right] \right]^{1/t_z}
\]

and \( m = m(n) \) stands for the smallest positive integer for which \( \tau(m) |V_n|^{p/2} \leq 1 \).

Furthermore we introduce a slight strengthening of \( (A_p) \), namely:

\begin{enumerate}
  \item[(A_p^*)] \sup \limits_{z \in Z_+^d \setminus \{z \mid |z| \geq N \}} E |X_z|^p \to 0 \quad \text{as} \quad N \to \infty,
\end{enumerate}

i.e. the RF (1.0) is uniformly \( p \)-integrable (see [14]).

**Theorem 4.** Suppose the RV's \( X_z, z \in Z_+^d \), defined by (1.0) have mean zero, \( B_n^z \geq C |V_n| \) with \( C > 0 \) for \( n \geq n_0 \), and \( |V_n| \) increases such that \( n_1(n)/m(n) \to \infty \) as \( n \to \infty \). Further, assume for some integer \( p \geq 3 \) that the conditions \( (A_p^*) \), \( (B_p) \), and \( (C_p) \) are fulfilled with a mixing rate \( \tau(m) = O(m^{-d(p^2 + 0)(p-2)}) \), as \( m \to \infty \), with \( a > 0 \). Then

\begin{equation}
(1 + |x|)^p |P(S_n < x B_n) - \Psi_{pn}(x)| = o \left( \frac{m(n)^{d_p}}{|V_n|^{(p-2)/2}} \right) \quad \text{as} \quad n \to \infty
\end{equation}

uniformly in \( x \in R^d \), where \( m(n) \) is defined in \( (C_p) \).

We will confine ourselves to some hints to the proof of Theorem 4 which in principle follows the line of Sect. 4. Let \( \overline{\Psi}_{pn}(x) \) denote the right side of (5.1) when \( S_n \) and \( B_n \) are replaced by \( \overline{S}_n \) and \( \overline{B}_n \), respectively. After short calculations utilizing our assumptions, (4.2), (4.3), and Lemma 3 we find that

\[ |E S_n^k - E \overline{S}_n^k| \leq \sum_{z \in V_n} |E (X_z - X_z) (S_n^{k-1} + S_n^{k-2} \overline{S}_n + \cdots + \overline{S}_n^{k-1})| \]
\[ \leq C_1 \tau(m) |V_n|^{(k+1)/2}
\]

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implying that
\[ \left| \Gamma_k \left( \frac{S_n}{B_n} \right) - \Gamma_k \left( \frac{\bar{S}_n}{B_n} \right) \right| \leq C_2 \tau(m) |V_n|^{1/2}, \quad k = 3, 4, \ldots, \]
and, hence,
\[
(5.4) \quad \max_{\delta \in (-\varepsilon, \varepsilon)} \left| \Psi_{pn} \left( \frac{B_n}{\bar{B}_n} \right) \left( x + \delta(|x| \lor 1) \right) - \Psi_{pn}(x) \right|
\leq \frac{C_2 \varepsilon}{(1 + |x|)^p} + \frac{C_4 \tau(m) |V_n|^{1/2}}{(1 + |x|)^p}.
\]
for \(0 \leq \varepsilon \leq 1/2\).

In analogy to (4.1) we have
\[
(5.5) \quad |P(S_n < xB_n) - \Psi_{pn}(x)|
\leq P\left(|S_n - \bar{S}_n| \geq \varepsilon B_n(|x| \lor 1)\right)
+ \max_{\delta \in (-\varepsilon, \varepsilon)} \left| \Psi_{pn} \left( \frac{B_n}{\bar{B}_n} \right) \left( x + \delta(|x| \lor 1) \right) - \Psi_{pn}(x) \right|
+ \max_{\delta \in (-\varepsilon, \varepsilon)} \left| P\left( \bar{S}_n < B_n(x + \delta(|x| \lor 1)) - \Psi_{pn} \left( \frac{B_n}{\bar{B}_n} \right) \left( x + \delta(|x| \lor 1) \right) \right) \right|.
\]

Put \(\varepsilon = m(n)^{d_p}/|V_n|^{(p-2+\eta_1)/2}\) with \(\eta_1 = \frac{1}{p}\). By definition of \(m = m(n)\), (4.2), (4.3) and
since the rate of decay of \(\tau(m)\) is less or equal than a multiple of \(m^{-d(p^2+\alpha)/(p-2)}\) one can easily verify that the first two terms on the right side of (5.5) possess the order \((1 + |x|)^{-p} \times o(m(n)^{d_p}/|V_n|^{(p-2)/2})\) as \(n \to \infty\) and moreover, \(m(n)^{d(p^2+\alpha)}/|V_n|^{(p-2)/2}\) converges, as \(n \to \infty\), to zero for \(0 < \eta_2 < \frac{1}{p}\). The proof of Theorem 4 is finished whenever the conditions (2.14)–(2.17) of Theorem 4 in [7], p. 88, applied to the RF \(\bar{X}_t \equiv E(\bar{X}_t | \bar{S}_z^{(m)})\), \(z \in Z^d_t\), hold. Whereas, by (4.1) and (2.18), the validity of (2.4)–(2.7) is rapidly seen we will discuss the verification of the smoothness condition (2.18) in some detail. First, note that (2.18) is implied by the following somewhat stronger assumption: There exist fixed \(\varepsilon_1, \varepsilon_2 > 0\) such that
\[
(5.6) \quad \sum_{z \in V_n} (1 - g_z(t, b_n)) \geq \varepsilon_1 |V_n| m(n)^{-d}, \quad n \geq n_0,
\]
uniformly for all \(t \in R^1\) with \(|t| \geq \varepsilon_2 \bar{B}_n \bar{L}_{pn}^{1/(p-2)}\), where
\[
b_n^2 = \sum_{z \in V_n} E \bar{X}_n^2, \quad \bar{L}_{pn} = \frac{1}{B_n^2} \sum_{z \in \delta_p} E |\bar{Y}_z|^p \quad \text{(see Sect. 1)}
\]
and
\[
g_z(t, b_n) = E \left[ E \left| \exp \left( it \sum_{z \in V_n \cap U_z^{(m)}} \bar{X}_n_{t|z| \in \delta_u} \right) \right|^p \right]^{1/p} \times \left| \sigma(\bar{z}_y; y \in U_z^{(2m)} \setminus \{z + (m, \ldots, m)\}) \right|^{2d}.\]
Since, by standard manipulations,

$$|g_z(t, b_n) - g_z(t)| \leq 2^d \sum_{y \in V_n \cup V^{(m)}} P(|\bar{X}_y| \geq b_n)$$

and

$$\tilde{L}_{pn} \leq C_\delta (m^d |V_n|)(p-2)/2$$

condition (5.6) is immediately derived from (C) in the domain $|t| \geq \delta$ (for large enough $n_0$ and perhaps smaller $\epsilon_1$); it remains to show

$$\sum_{z \in V_n} (1 - g_z(t)) \geq \epsilon_3 |V_n| m(n)^{-d}$$

for $\epsilon_4 m(n)^{-d/2} \leq |t| \leq \delta$ ($\epsilon_3, \epsilon_4 > 0$), which is done by relying on the proof of a technical lemma in [24], p. 223: It is a well-known fact that for any real-valued characteristic function $g(t)

$$4^s(1 - g(t)) \geq 1 - g(2^st), \quad s = 0, 1, 2, \ldots .$$

For an arbitrary $t \in \mathbb{R}^1$ with $|t| \leq \delta$ choose $s \geq 0$ such that

$$2^{s+1} |t| > \delta \geq 2^s |t| .$$

Then, in view of (5.2),

$$\sum_{z \in V_n} (1 - g_z(t)) \geq \frac{1}{4^{s+1}} \sum_{z \in V_n} (1 - g_z(2^{s+1}t)) \geq \frac{\epsilon_3^2}{\delta^2} |V_n| ,$$

and hence, (5.7) follows. Thus the proof of Theorem 4 is completed.

References


