Asymptotic Behaviour of an Empirical Nearest-Neighbour Distance Function for Stationary Poisson Cluster Processes

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Summary. For stationary Poisson cluster processes (PCP's) $\Phi$ on $R^d$ the limit behaviour, as $n(D) \to \infty$, of the quantity $\left( n(D) \right)^{-1} \sum_{x \in D: \Phi(x) \geq 1} \chi(x, r)$, where $\chi(x, r) = 1$, if $\Phi(b(x, r)) = 1$, and $\chi(x, r) = 0$ otherwise, is studied. A central limit theorem for fixed $r > 0$ and the weak convergence of the normalized and centred empirical process on $[0, R]$ to a continuous Gaussian process are proved. Lower and upper bounds for the nearest neighbour distance function $P_1([\varphi: \varphi(b(0, r)) \geq 1])$ of a stationary PCP are given. Moreover, a representation of higher order Palm distributions of PCP's and a central limit theorem for $m$-dependent random fields with unbounded $m$ are obtained. Both these auxiliary results seems to be of own interest.

1. Introduction

An important quantity for the statistical analysis of stationary point processes is the distribution function $D(r)$ of the distance to the nearest neighbour of a given point of the point process under consideration (see [5, 19]). In the present paper we are concerned throughout with (stationary) Poisson cluster processes. The paper is organized as follows.

Subsequently, in this section we give a series of definitions and sketch the theoretical background. A detailed study of the theory point processes one can find in [16, 19] (in particular for PCP's see [1, 2, 23]) and for the statistical analysis of point processes the reader is referred to [7, 17, 21]. In Sect. 2 we formulate and prove a representation formula for the $k$-th order Palm distribution of PCP's. Note that this result is independent of the remaining part of the paper. Section 3 contains a LINDEBERG-type theorem for $m$-dependent random fields with unboundedly growing $m$ which is a key to prove the main results in Sect. 5. In Sect. 4 we derive lower and upper bounds of the nearest neighbour distance function (NNDF) $D(r)$. Finally, in Sect. 5 we prove a central limit theorem (CLT) and a corresponding functional limit theorem for an unbiased estimator of $1 - D(r)$ constructed from an observation of the PCP on a finite region $W$ (sampling window) whose area tends to infinity.

Let $M$ be the set of all locally finite counting measures $\varphi$ on the $d$-dimensional Euclidean space $R^d$ (equipped with its $\sigma$-algebra of BOREL sets $B^d$) and let $M$ be the
σ-algebra generated by the sets \{φ: φ ∈ M, φ(B) = k\}, k = 0, 1, 2, ..., B ∈ \mathcal{B}_0, where \mathcal{B}_0 denotes the system of bounded sets in \mathbb{R}^d. In what follows the abbreviation ‘x ∈ φ’ means ‘x ∈ \mathbb{R}^d: φ(x) > 0’ and the sum \sum_{x_1, ..., x_k \in \phi} is taken over all k-tuples of pairwise distinct \(x_1, ..., x_k \in \mathbb{R}^d\) with \(φ(x_j) > 0, j = 1, ..., k\). We define a point process on \mathbb{R}^d to be a measurable mapping \(Φ\) from a probability space \([Ω, \mathcal{A}, P]\) into \([M, \mathcal{F}]\). By E and \(D^2\) we designate expectation and variance, respectively, w.r.t. \(P\). \(P = Φ^{-1}\) denotes the probability measure on \([M, \mathcal{F}]\) induced by \(Φ\) and we briefly write \(Φ \sim P\). Provided that \(Φ\) is stationary, simple and \(P = EΦ([0, 1]^2) < ∞\) one can define the (reduced) PALM distribution \(P^1_0\) on \([M, \mathcal{F}]\) by

\[
P^1_0(Y) = \frac{1}{P^1_0(\delta_0)} \int \sum_{x \in \phi} 1_Y(T_xφ - \delta_0) P(dφ), \quad Y ∈ \mathcal{F},
\]

where \((T_xφ)(B) = φ(B + x), \delta_x(B) = 1_B(x)\) (Dirac measure) and \(1_Y(\cdot)\) stands for the indicator of the set \(Y\). Now we are in a position to define the NNDF \(D(\cdot)\) precisely:

\[
D(\cdot) = 1 - P^1_0(\phi(o, r)), \quad r ≥ 0,
\]

where \(Y(x, r) = \{φ: φ ∈ M, φ(b(x, r)) = 0\}\) and \(b(x, r) = \{y: y ∈ \mathbb{R}^d, ||x - y|| ≤ r\}\). The family of PCP’s forms one of the few classes of point processes for which a reasonable theory is known. Well-studied examples of them are compound Poisson, Bartlett-Lewis and Neyman-Scott processes (see [23]) as well as self-exiting, Gaussian-Poisson processes (see [10, 20]) and so-called ‘processes with limited aftereffects’ (see [1, 2]). Besides their theoretical importance they are often used to describe stochastic models in various fields of application (see [7, 21]). We recall the well-known fact that the family of PCP’s coincides with the class of regularly infinite divisible point processes (see [1, 19]), i.e. their structure is determined by the KLM measure \(P\) concentrated on those elements of \(M\) which have finitely many atoms. A more intuitive approach to PCP’s is the following one. A (stationary) PCP \(Φ \sim P\) consists of two components, the process of cluster centres \(Φ_c \sim Q\) forming a (stationary) Poisson process (with intensity 0 < λ_0 < ∞), and the process of cluster members \(Φ_s \sim P_s\), Each point \(x ∈ Φ_c\) triggers a cluster member process \(Φ_s(x) \sim P_s(\cdot)\) that is assumed to be independent of \(Φ_c\) and \(Φ_s(\cdot), y = x,\) and to have the same distribution as \(T_xΦ_s, i.e. P_s(\cdot)(Y) = P_s(T_xY), Y ∈ \mathcal{F}\). The superposition of all clusters yields the PCP \(Φ \sim P\) with \(P(\cdot) = \int \left(\ast \left(D_s(x) \ast P_s(\cdot)\right)\right)(Y) Q(dφ_s),\) where \(P^\ast = P^\ast \ast ... \ast P\) denotes the n-fold convolution of \(P\) with itself. The condition \(\lambda_0 = EN < ∞, where N = Φ_s(\mathbb{R}^d), is sufficient for its existence (see [19, 23]). For convenience we assume in the sequel that both \(Φ_c\) and \(Φ_s\) are simple point processes which implies the simplicity of \(Φ\). Remember the fact that the Poisson process \(Φ_c \sim Q\) is simple iff its intensity measure \(Q_0(\cdot) = EΦ_c(\cdot)\) is diffuse (e.g. in case \(Q_0(\cdot) = λ_0ρ(\cdot), where ρ(\cdot) denotes the Lebesgue measure on \mathbb{R}^d\).

We symbolically write \(X_n \overset{D}{\rightarrow} N(0, σ^2)\) for the convergence in distribution of a sequence of random variables \(X_n\) to a Gaussian random variable with mean zero and variance \(σ^2 > 0\). Further, let \(C_0, C_1, \ldots\) denote positive constants which may be differ from one expression to another.

Here and in the next section let \(Φ \sim P\) be a simple (not necessarily stationary) PCP governed by \(Φ_c\) and \(Φ_s\) and let \(f: (\mathbb{R}^d)^k \times \mathcal{F} → \mathcal{F}\) be a \((\mathcal{F}^d)^k \otimes \mathcal{F}\)-measurable map-
I. Asymptotic Behaviour

1. Representation of higher order Palm distributions

The $k$-th order reduced Palm distribution $P^{1}_{x_{1},...,x_{k}}(Y)$, $Y \in \mathcal{M}$, of a simple point process $\Phi \sim P$ w.r.t. $(x_{1}, \ldots, x_{k}) \in (\mathbb{R}^{d})^{k}$ is defined to be the Radon-Nikodym derivative of the $k$-th order reduced Campbell measure

$$C_{P}^{(k)}(B \times Y) = \int_{\mathcal{M}} \sum^{*}_{x_{1}, \ldots, x_{k} \in \mathcal{Y}} 1_{B}(x_{1}, \ldots, x_{k}) \, 1_{Y}(\varphi - \sum_{l=1}^{k} \delta_{x_{l}}) \, P(d\varphi),$$

$B \in (\mathbb{R}^{d})^{k}$, $Y \in \mathcal{M}$, w.r.t. the $k$-th order factorial moment measure $\alpha_{P}^{(k)}(B) = C_{P}^{(k)}(B \times M)$ provided the latter one is $\sigma$-finite. Hence, for all non-negative $(\mathbb{R}^{d})^{k} \otimes \mathcal{M}$-measurable $f$ on $(\mathbb{R}^{d})^{k} \times M$ the following defining equality holds:

$$\int_{(\mathbb{R}^{d})^{k} \times M} \sum^{*}_{x_{1}, \ldots, x_{k} \in \mathcal{Y}} f(x_{1}, \ldots, x_{k}, \varphi) \, P^{1}_{x_{1},...,x_{k}}(d\varphi) \, \alpha_{P}^{(k)}(d(x_{1}, \ldots, x_{k}))$$

(see [8, 16]).

The following generalization (see [8, 16]) of a well-known theorem due to Slivnyak is needed below. For any Poisson process $\Phi \sim Q$ driven by the intensity measure $\Lambda_{Q}$ we have

$$\alpha_{Q}^{(k)} = \Lambda_{Q} \times \ldots \times \Lambda_{Q} \quad \text{and} \quad Q^{1}_{x_{1},...,x_{k}}(Y) = Q(Y)$$

for all $Y \in \mathcal{M}$ and $(\Lambda_{Q} \times \ldots \times \Lambda_{Q})$-almost every $(x_{1}, \ldots, x_{k}) \in (\mathbb{R}^{d})^{k}$.

**Theorem 1.** Let $\Phi_{c} \sim Q$ and $\Phi_{s} \sim P_{s}$ be simple point processes defining a PCP $\Phi \sim P$ and suppose that $\Lambda_{Q}$ is $\sigma$-finite and $\text{EN}^{k} < \infty$. Then

$$\int_{(\mathbb{R}^{d})^{k} \times M} \int_{M} f(x_{1}, \ldots, x_{k}, \varphi) \, P^{1}_{x_{1},...,x_{k}}(d\varphi) \, \alpha_{P}^{(k)}(d(x_{1}, \ldots, x_{k}))$$

$$= \sum_{l=1}^{k} \sum_{K_{1} \cup \ldots \cup K_{l} = K} \int_{(\mathbb{R}^{d})^{l} \times K_{1} \times \ldots \times K_{l} \times M} \int_{K_{1}} \ldots \int_{K_{l}} \int_{M} f(x_{1}, \ldots, x_{k}, \varphi).$$
For all $(\mathcal{B}^d)^k \otimes \mathcal{B}$-measurable functions $f : (R^d)^k \times \Omega \to R^+$, for which the integrals on both sides make sense. Here $|K_i|$ stands for the cardinality of $K_i$.

Proof of Theorem 1. Using (1.3), (2.1), (2.2) and the definition of PCP's we get

\[
\int_{\Omega} \sum_{x_1, \ldots, x_k \in \Psi} \left( x_1, \ldots, x_k, \varphi - \sum_{j=1}^{k} (\delta_{x_j}) \right) P(d\varphi) = \sum_{i=1}^{k} \sum_{k_i \in K_i} \int \int \sum_{x_1, \ldots, x_k \in \Psi} \left( x_1, \ldots, x_k, \psi - \sum_{j=1}^{k} (\delta_{x_j}) \right) P^{(y_i)}_{\delta} (d\varphi_1) \ldots P_{\delta}^{(y_i)} (d\varphi_k) \frac{P(y_i)(d\varphi_1)}{x_{\in\Psi}} \ldots \frac{P(y_i)(d\varphi_k)}{x_{\in\Psi}}
\]

Finally, we make use of (2.1) and the definition of convolution which immediately lead to the assertion of Theorem 1. 

It should be mentioned that Theorem 1 generalizes in some sense the well-known relation

\[
(P_{\delta}^{(y_i)})_{x_1, ..., x_k \in K_i} \ast \ldots \ast (P_{\delta}^{(y_i)})_{x_1, ..., x_k \in K_i} \ast P (d\varphi) = \sum_{i=1}^{k} \sum_{k_i \in K_i} \int \int \sum_{x_1, \ldots, x_k \in \Psi} \left( x_1, \ldots, x_k, \psi - \sum_{j=1}^{k} (\delta_{x_j}) \right) P_{\delta}^{(y_i)} (d\varphi_1) \ldots P_{\delta}^{(y_i)} (d\varphi_k) \frac{P(y_i)(d\varphi_1)}{x_{\in\Psi}} \ldots \frac{P(y_i)(d\varphi_k)}{x_{\in\Psi}}
\]

(2.3) \[ P_{\delta}^{(y_i)} = P \ast (P_{\delta}^{(y_i)})_{0} \]
which is even valid for an arbitrary stationary infinitely divisible point process characterized by the KLM measure $\tilde{P}$ (see [16, 19]). In the case of a stationary PCP one can show that

$$\left( P_0 \right)_Y = l_\phi^{-1} \sum_{x \in \mathbb{N}} 1_Y (T_x \varphi - \delta_0) P_\phi (d\varphi), \quad Y \in \mathbb{N}.$$  

This can be verified, starting from (1.1) and using (1.3) for $k = 1$, by repeating the steps in the above proof.

3. A Lindeberg-type central limit theorem for $m$-dependent fields with unbounded $m$

The goal of this section is to provide a slight extension of a CLT for multiply indexed $m$-dependent random variables that goes back to B. Rosen [22]. In [13] the author developed a quite general approach for proving limit theorems for sums of random variables forming an $m$-dependent random field on $Z^d = \{ z = (z_1, \ldots, z_d) : z_i = 0, \pm 1, \pm 2, \ldots ; i = 1, \ldots, d \}$. The case $d = 1$ was treated in [11]. With the notation $l(z)$ = max $|z_i|$ an $m$-dependent random field is defined to be a family of real-valued random variables $(X_z)_{z \in Z^d}$ on $(\Omega, \mathcal{F}, P)$ such that for any $A, B \subset Z^d$ with $|A|, |B| < \infty$ and inf $\{l(a - b) : a \in A, b \in B \} > m$ the random vectors $(X_a)_{a \in A}$ and $(X_b)_{b \in B}$ are independent. The conditions which we shall impose on a sequence $(X_z)_{z \in V_n \subset Z^d}$ of $m_n$-dependent random fields ensure the normal convergence of $S_n = \sum_{z \in V_n} X_z$ although $m_n$ is not necessarily bounded. For an arbitrary $\varepsilon > 0$ we introduce the following abbreviations:

$$X_{n2}(e) = X_{n2} I_{\{|x_{n2}| < e\}}, \quad X_{n2}(e) = X_{n2}(e) - EX_{n2}(e),$$

$$S_{n2}^{(k)}(e) = \sum_{y \in V_n} X_{n2}(e), \quad f_n(t, e) = E \exp \left\{ it \sum_{z \in V_n} X_{n2}(e) \right\},$$

$$W_{n2}^{(k)}(t, e) = \exp \left\{ it S_{n2}^{(k)}(e) \right\},$$

$$W_{n2}^{(k)}(t, e) = \exp \left\{ it \left( X_{n2}(e) + \sum_{j=1}^{k} S_{n2}^{(j)}(e) \right) \right\}.$$
Proof of Theorem 2. In view of (3.2) we find by standard arguments from analysis that there exists a positive non-increasing null sequence \((\epsilon_n)_{n \geq 1}\) satisfying \(\epsilon_n^{-1} L_n(\epsilon_n) \to 0\) as \(n \to \infty\). Because of the inequality
\[
|e^{it\epsilon_n} - f_n(t, \delta_n)| \leq \sum_{z \in V_n} |e^{it\Lambda(z)} - e^{it\Lambda(z)}| \leq 2 |t| \epsilon_n^{-1} L_n(\epsilon_n)
\]
where \(\delta_n = \epsilon \cdot m_n^{-2d}\), it suffices to show that
\[
(f_n(t, \delta_n) \xrightarrow{n \to \infty} e^{-t^2/2}, \quad t \in \mathbb{R}^1.
\]

For doing this we decompose the first derivative of \(f_n(t, \epsilon_n)\) as follows
\[
f'_n(t, \delta_n) = I \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1] f_n(t, \delta_n)
+ \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1] [\overline{W}^{(2)}_{12}(t, \delta_n)]
- E[\overline{W}^{(1)}_{12}(t, \delta_n)] \prod_{k \geq 3} W_k(t, \delta_n).
\]

Making use of the elementary inequalities \(|e^{ix} - 1| \leq |x|, |e^{ix} - 1 - ix| \leq x^2/2\) and \(ESS_n = 1\) the following estimates can be obtained straightforwardly:
\[
\leq |t| \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1] - it\]
\[
\leq |t| \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1 - 2] + (\delta_n) \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1] [\overline{W}^{(2)}_{12}(t, \delta_n) - 1] [\overline{W}^{(3)}_{12}(t, \delta_n)]
\]
\[
\leq 4 |t| L_n(\epsilon_n) + 4t^2(2m_n + 1)2d \delta_n \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1] [\overline{W}^{(2)}_{12}(t, \delta_n) - 1] [\overline{W}^{(3)}_{12}(t, \delta_n)]
\]
\[
\leq 16(4m_n + 1)2d \delta_n \sum_{z \in V_n} \mathbb{E}[\Lambda(z)] [\overline{W}^{(1)}_{12}(t, \delta_n) - 1] [\overline{W}^{(2)}_{12}(t, \delta_n) - 1] [\overline{W}^{(3)}_{12}(t, \delta_n)]
\]

Taking into account \(\epsilon_n = m_n^{2d} \delta_n \to 0\) and the conditions (3.1) and (3.2) we find that
\(D^3S_n(\epsilon_n) \leq 1 + 4^3dL_n(\epsilon_n) \leq C_1\) and
\[
\lim_{n \to \infty} \left[ f_n(t, \delta_n) + t f_n(t, \delta_n) \right] = 0 \quad \text{for all} \quad t \in \mathbb{R}^1.
\]

Therefore, using a lemma in [6] we obtain (3.4) and so Theorem 2 is completely proved. \(\square\)

4. Bounds for the NNDF of PCP's

We first compute explicitly the NNDF of a general (stationary) PCP \(\Phi \sim P\). Adhering to the above notation it is easily seen from (2.3) and (2.4) that
\[
P_{\theta}(Y(0, r)) = P(Y(0, r)) \lambda^{-1} \sum_{\eta \in \mathcal{F}} 1_{Y(0, r)}(T \varphi - \delta) P_{\varphi}(d\varphi).
\]
The definition of PCP's and the shape of the probability generating functional of a Poisson process yield

\[ P(Y(0, r)) = \int_{M} \prod_{x \in \Phi} P_s(Y(x, r)) Q(d\varphi) \]

\[ = \exp \left\{ -\lambda_s \int_{\mathbb{R}^d} \left[ 1 - P_s(Y(x, r)) \right] dx \right\}. \]

Therefore

\[(4.1) \quad D(r) = 1 - p_s(r) \exp \left\{ -\lambda_s \int_{\mathbb{R}^d} \left[ 1 - P_s(Y(x, r)) \right] dx \right\}, \]

where

\[ p_s(r) = \lambda_s^{-1} E \sum_{x \in \Phi_s} \left( \Phi_s - \delta_x \right) = \lambda_s^{-1} \int_{\mathbb{R}^d} (P_s)_{2}^{-1} (Y(x, r)) \alpha_s^{(1)}(dx) \]

and \( \alpha_s^{(k)}(.) \) denotes the \( k \)-th order factorial moment measure of \( \Phi_s \sim P_s \).

In the special case of a Neyman-Scott process, where \( \Phi_s \) consists of points which are independently and identically distributed about the origin according to a density \( f(x) \) and the cluster size \( N \) has the probability generating function \( g(z) = \sum_{n \geq 0} P(N = n) z^n \), we have

\[ P_s(Y(x, r)) = \sum_{n = 0}^{\infty} P(N = \Phi_s(b^c(x, r)) = n) = g \left( F(b^c(x, r)) \right), \]

\[ (P_s)_{2}^{-1} (Y(x, r)) = (EN)^{-1} g' \left( F(b^c(x, r)) \right) \]

and

\[ \alpha_s^{(1)}(B) = ENF(B), \quad B \in \mathcal{B}^d \]

where

\[ b^c(x, r) = R^d \setminus b(x, r) \quad \text{and} \quad F(B) = \int_{B} f(x) \, dx. \]

Thus, we get from (4.1) that

\[ D(r) = 1 - \frac{1}{g'(1)} \int_{\mathbb{R}^d} g' \left( F(b^c(x, r)) \right) f(x) \, dx \]

\[ \exp \left\{ -\lambda_s \int_{\mathbb{R}^d} \left[ 1 - g \left( F(b^c(x, r)) \right) \right] dx \right\}, \]

(see also [4]).

We proceed with some simple estimates of \( p_s(r) \) and the exponent in (4.1). Obviously, we have

\[ P(N = 1) \leq \lambda_s p_s(r) \leq EN. \]

The finiteness of the measure \( \alpha_s^{(1)}(B) = \sum_{n \geq 1} nP(\Phi_s(B) = n) \) implies

\[ \int_{\mathbb{R}^d} \left[ 1 - P_s(Y(x, r)) \right] dx \leq \int_{\mathbb{R}^d} \alpha_s^{(1)}(b(x, r)) \, dx = \nu(b(0, r)) \cdot EN. \]
On the other hand a lower bound can be derived in the following manner:

\[
\int_{\mathbb{R}^d} \left[ 1 - P_s(Y(x, r)) \right] dx = \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} \sum_{\{v : v(R_k \geq 1) \}} \mathbb{1}(T_x \in \mathcal{F}) \mathbb{1}(v(b(0, r)) \geq 1, v(R_k = k)) \, dx \, P_s(dx) \n\]

\[
\geq \int \mathbb{1}(v(b(0, r)) \geq 1, v(R_k = k)) \, P_s(dv) = \mathbb{1}(v(b(0, r)) \geq 1, v(R_k = k)) \, \mathbb{P}(N \geq 1).
\]

Combining all above estimates we arrive at

**Theorem 3.** Let \( \Phi_c \sim Q \) be a stationary Poisson process with \( 0 < \lambda_0 < \infty \) and suppose that \( \Phi_n \sim \Phi \) is simple and \( EN < \infty \). Then the NNDF of the corresponding PCP permits for all \( r \geq 0 \) the inclusion

\[
P(N = 1) = \exp \{-\lambda_0 EN v(b(0, r))\}
\]

\[
1 - D(r) \leq \exp \{-\lambda_0 P(N \geq 1) v(b(0, r))\}.
\]

The bounds of \( 1 - D(r) \) established in Theorem 3 show that \( 1 - D(r) \) decreases exponentially fast as \( r \to \infty \) in analogy to the Poisson case. The above inequalities turn to equalities iff \( P(N = 1) = 1 \), i.e. \( \Phi \) is only an independent pointwise translation of \( \Phi_c \).

5. Normal convergence of an empirical NNDF

Let \((W_n)_{n \geq 1}\) be a sequence of compact sampling windows in \( \mathbb{R}^d \) such that the eroded sets \( D_n = W_n \ominus b(0, r) := \cap \{W_n + x, n \geq 1\} \), fulfill the following regularity condition:

There exists a sequence of \( d \)-dimensional rectangles

\[ A_n = [0, a_n^{(1)}] \times \ldots \times [0, a_n^{(d)}] \]

and constants \( c_1, c_2 > 0 \) such that

\[
\nu(A_n) \to \infty, \quad \min_{1 \leq i \leq d} a_n^{(i)} \geq c_1
\]

and

\[
D_n \subseteq A_n, \quad \nu(D_n) \geq c_2 \nu(A_n) \quad \text{for every} \ n \geq 1.
\]

In the following we consider an unbiased estimator of the quantity \( S(r) = \lambda v(1 - D(r)) \) which was used and discussed e.g. in [7, 9, 21] and that is defined by

\[
S_n(r) = \frac{1}{\nu(D_n)} \sum_{x \in \Phi} 1_{D_n}(x) 1_{Y(x, r)}(\Phi - \delta_x).
\]

Note that \( S_n(r) \) is completely determined because \( \Phi \) is given in the larger region \( W_n \).

We mention that in order to exhaust more information from a realization observed in \( W_n \) another unbiased estimator for \( S(r) \) was suggested in [9].

However because \( \nu(W_n) \) becomes very large the difference between the estimator (5.3) and that one in [9] is negligible and so for better feasibility of calculations we prove asymptotic normality of \( S_n(r) \) only. Under modified and more restrictive assumptions the weak convergence of a special estimator of \( S(r) \) to a Poisson process was shown in [18]. Asymptotic Gaussianity for estimators of reduced moment measures of Brillinger-mixing point processes and PCP's were proved by Jolivet [15] (see
also [17]) and HEINRICH [14], respectively. A profound study of analogous problems for shot noise processes was carried out in [12]. Limit theorems for a class of empirical distributions, which for example estimate \( P(Y(0, r)) \) for PCP's having clusters with bounded range, were given in [3]. For \( r \geq 0 \) define the centred random variable

\[
Z_n(r) = \left( \nu(D_n) \right)^{1/2} \left( S_n(r) - ES_n(r) \right).
\]

**Theorem 4.** Let \( \Phi_c \sim Q \) be a stationary Poisson process with \( 0 < \lambda \leq \infty \) and suppose that \( \Phi_c \sim P \) is simple and \( EN^2 < \infty \). Further, let \( (D_n)_{n \geq 1} \) satisfy (5.1) and (5.2) such that, for the corresponding PCP \( \Phi \sim P \),

\[
\lim_{n \to \infty} D^2Z_n(r) = \sigma^2(r) > 0.
\]

Then

\[
Z_n(r) \xrightarrow{n \to \infty} N(0, \sigma^2(r)).
\]

**Proof of Theorem 4.** First we introduce the truncated PCP \( \Phi_c \) whose cluster centre process is again \( \Phi_c \sim Q \) but its secondary process \( \Phi_{c_n} \) consists of only those atoms of \( \Phi_c \) which are located in the sphere \( b(0, \varrho) \), i.e. \( \Phi_{c_n}(x) > 0 \) iff \( \Phi_c(x) > 0 \) and \( ||x|| \leq \varrho \). To avoid trivial complications assume that \( \varrho > r \). For \( A \in \mathbb{R}_d^2 \) put

\[
S_n(r, A) = \frac{1}{\nu(D_n)} \sum_{x \in \Phi_c} 1_{A \cap D_n(x)} 1_{Y(x, r)}(\Phi_c - \delta_x)
\]

where the last equality holds \( \mathbb{P} \)-almost surely (see (1.3)). By definition of the PCP \( \Phi_c \), the following lemma is obvious.

**Lemma 1.** The random variables \( S_n(r, A) \) and \( S_n(r, B) \) are independent if the sets \( A \) and \( B \) are separated by a distance greater than \( 2(\varrho + r) \).

Next we will study the asymptotic behaviour of

\[
Z_{n}(r) = \left( \nu(D_n) \right)^{1/2} \left( S_n(r, D_n) - ES_n(r, D_n) \right)
\]

under the assumption that \( \varrho = \varrho_n \) goes to infinity at an appropriately slow rate.

Consequently, according to the definition in Sect. 3 the random variables

\[
X_n(r) = \left( \nu(D_n) \right)^{1/2} \left( S_n(r, E_z) - ES_n(r, E_z) \right),
\]

where \( E_z = [z_1 - 1, z_1] \times \cdots \times [z_d - 1, z_d] \) form an \( m_n \)-dependent random field with \( m_n = \left( 2(\varrho_n + r) \right) + 1 \) and \( V_n = \times_{i=1} \{ 1, 2, \ldots, [a_n^{(i)}] + 1 \} \).

Therefore, to prove

\[
Z_{n}(r)/DZ_{n}(r) \xrightarrow{n \to \infty} N(0, 1),
\]
where \( q_n \rightarrow \infty \) as \( n \rightarrow \infty \) at a suitable rate, we have only to verify the conditions of Theorem 3 with \( m_n \rightarrow \infty \). Later in this section it will be shown that

\[
\text{(5.9)} \quad \sup_{n \geq 1} |DZ_{n^2}(r) - DZ_n(r)| \leq \frac{1}{4} \sigma(r)
\]

for \( r \geq q_0 \).

For any \( \epsilon > 0 \) let \( q_0(\epsilon) \) be chosen such that \( \mathbb{E} \Phi([0, 1]^{d}) \leq \epsilon\left(n(D_n)\right)^{1/2} m_n^{-2d} \sigma(r)/4 \), \( DZ_n(r) \geq 3\sigma(r)/4 \) and \( q_0 \geq q_0 \) for every \( n \geq n_0(\epsilon) \). Using \( n(D_n) S_{n^2}(r, E_2) \leq \Phi(E_2) \), (5.9) and the stationarity of \( \Phi \) we get after some manipulations that, for \( n \geq n_0(\epsilon) \),

\[
L_n(\epsilon) \leq \frac{2m_n^{2d}}{\nu(D_n) D^2 \tau_{n^2}(r)} \sum_{z \in V_n} \left[ 1 + \frac{\epsilon (n(D_n))^{1/2} m_n^{-2d} \sigma(r)}{\mathbb{E} \Phi(E_2)} \right]^2 \times \mathbb{E} \Phi^2(E_2) \left\{ \Phi(E_2) \geq r - m_n^{2d}(n(D_n))^{1/2} DZ_{n^2}(r) - \Phi(E_2) \right\} \\
\leq \frac{16m_n^{2d} \left| V_n \right|}{\nu(D_n) \sigma^2(r)} \mathbb{E} \Phi^2([0, 1]^{d}) \left\{ \Phi([0, 1]^{d}) \geq r - m_n^{2d}(n(D_n))^{1/2} \sigma(r)/4 \right\}.
\]

Hence, (5.1) and (5.2) imply that (3.1) and (3.2) are fulfilled with \( m_n \rightarrow \infty \) at some rate and so, by (3.3), (5.8) holds.

To complete the proof of Theorem 4 it remains to show

\[
\text{(5.10)} \quad \lim_{\epsilon \rightarrow \infty} \sup_{n \geq 1} D^2[Z_n(r) - Z_{n^2}(r)] = 0 \quad \text{for } r \geq 0.
\]

Evidently, we have

\[
\text{(5.11)} \quad D^2[Z_n(r) - Z_{n^2}(r)] = EZ_n(r) (Z_n(r) - Z_{n^2}(r)) - EZ_{n^2}(r) (Z_n(r) - Z_{n^2}(r)).
\]

We estimate only the first expression on the right-hand side. In order to shorten the subsequent calculations we will omit some details. The starting point is the following identity

\[
\nu(D_n) EZ_n(r) Z_{n^2}(r)
\]

\[
= \mathbb{E} \sum_{y \in V_n} \sum_{z \in \Phi_{n^2}(y)} 1_{D_n \cap \Phi(y)}(x_2) 1_{Y(x, r)}(\Phi_n(y) - \delta_n) \prod_{z \in \Phi_n} 1_{Y(z, r)}(\Phi_n(z)) + \mathbb{E} \sum_{y \in V_n} \sum_{z \in \Phi_{n^2}(y)} 1_{D_n \cap \Phi(y)}(x_2) 1_{Y(x, r)}(\Phi_n(y) - \delta_n) \times 1_{Y(x, r)}(\Phi_n(y) - \delta_n) \prod_{z \in \Phi_n} 1_{Y(z, r)}(\Phi_n(z)) (\Phi_n(z) - \delta_n) + \mathbb{E} \sum_{y \in V_n} \sum_{z \in \Phi_{n^2}(y)} 1_{D_n \cap \Phi(y)}(x_2) 1_{Y(x, r)}(\Phi_n(y) - \delta_n) \times 1_{Y(x, r)}(\Phi_n(y) - \delta_n) \prod_{z \in \Phi_n} 1_{Y(z, r)}(\Phi_n(z)) (\Phi_n(z) - \delta_n) - 2 \nu(D_n) u_1^{(0)} u_r^{(0)} v_r^{(0)},
\]

\[
= \mathbb{E} \sum_{y \in V_n} \sum_{z \in \Phi_{n^2}(y)} 1_{D_n \cap \Phi(y)}(x_2) 1_{Y(x, r)}(\Phi_n(y) - \delta_n) \prod_{z \in \Phi_n} 1_{Y(z, r)}(\Phi_n(z)) (\Phi_n(z) - \delta_n) - 2 \nu(D_n) u_1^{(0)} u_r^{(0)} v_r^{(0)}.
\]
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where

$$u_t^{(e)} = E \sum_{x \in \Phi_{e}} 1_{b(x,e)}(x) \mathbf{1}_{Y(x,r)}(\Phi_{e} - \delta_{x}),$$

$$v_t^{(e)} = \exp \left\{ -\lambda_{e} \int \left[ 1 - P \left( \Phi_{e} \left( b(z, \epsilon) \cap b(0, \rho) \right) = 0 \right) \right] \, dz \right\},$$

and \( u_r = u_{r}^{(\infty)}, \, v_r = v_{r}^{(\infty)}. \)

Further, we need, for \( 0 \leq \delta \leq t < \infty, \)

$$v_{st}^{(e)}(y) = E \prod_{x \in \Phi_{e}} 1_{Y(x,s)}(\Phi_{e}) \cdot 1_{Y(x+y,t)}(\Phi_{e} - \delta_{x}),$$

$$v_{st}^{(e)}(y) = E \sum_{x \in \Phi_{e}} 1_{Y(x,s)}(\Phi_{e} - \delta_{x}) 1_{Y(x+y,t)}(\Phi_{e} - \delta_{x}),$$

$$v_{st}^{(e)}(y) = v_{st}^{(\infty)}(y) \, , \, u_{st}(y) = u_{st}^{(\infty)}(y) \, \mathrm{and} \, w_{n}(y) = v_{n}(D_{n} \cap (D_{n} + y))/v_{n}(D_{n}).$$

Now, making use of the independence properties of PCP's and applying (2.1) and (2.2) to the stationary Poisson process \( \Phi_{e} \sim Q \) we arrive after some lengthy, but elementary, calculations at

$$\lambda_{e} \mathbb{E}_{U}(r) \left( Z_{n}(r) - Z_{n_{e}}(r) \right)$$

$$= \lambda_{e} \mathbb{E}_{U} \sum_{x \in \Phi_{e}} 1_{b(x,e)}(x) \mathbf{1}_{Y(x,r)}(\Phi_{e} - \delta_{x})$$

$$\quad + \left( \lambda_{e} \mathbb{E}_{U}(D_{n}) \right)^{-1} \int_{\mathbb{R}^{+}} \left[ v_{rr}(y) E \sum_{x_{1}, x_{2} \in \Phi_{e}} 1_{D_{n}}(x_{1} + y) 1_{D_{n}}(x_{2} + y) \mathbf{1}_{Y(x_{1}, r)}(\Phi_{e} - \delta_{x_{1}}) \right.$$

$$\times 1_{Y(x_{2}, r)}(\Phi_{e} - \delta_{x_{2}}) - v_{rr}(y) E \sum_{x_{1}, x_{2} \in \Phi_{e}} 1_{D_{n}}(x_{1} + y) 1_{D_{n}}(x_{2} + y)$$

$$\times 1_{Y(x_{1}, r)}(\Phi_{e} - \delta_{x_{1}}) 1_{Y(x_{2}, r)}(\Phi_{e} - \delta_{x_{2}}) \left. \right] dy$$

$$+ \int_{b(0, t)} w_{n}(y) u_{r} v_{r} (u_{r}^{(e)} v_{r}^{(e)} - u_{r} v_{r}) dy$$

$$+ \int_{b(0, t)} w_{n}(y) \left( v_{r}^{(e)}(y) - v_{rr}(y) \right) u_{rr}(-y) - u_{rr}(y) E \sum_{x \in \Phi_{e}} \left( 1_{Y(x, r)}(\Phi_{e} - \delta_{x}) - 1_{b(0, e)}(x) 1_{Y(x, r)}(\Phi_{e} - \delta_{x}) \right)$$

$$\times 1_{Y(x+y, r)}(\Phi_{e} - \delta_{x}) + (u_{r}^{(e)} - u_{r}) E \sum_{x \in \Phi_{e}}$$

$$\times 1_{Y(x, r)}(\Phi_{e} - \delta_{x}) 1_{Y(x+y, r)}(\Phi_{e} - \delta_{x}) \right] dy$$

$$+ \int_{b(0, t)} w_{n}(y) v_{r}^{(e)}(y) \left( v_{r}^{(e)}(y) - v_{rr}(y) \right) \left( u_{r}^{2} - u_{rr}(y) u_{rr}(-y) \right) dy$$

$$+ \int_{b(0, t)} w_{n}(y) \left( v_{r}^{(e)}(y) - v_{rr}^{(e)}(y) \right) u_{rr}(u_{r} - u_{r}^{(e)}) dy$$

$$+ \int_{b(0, t)} w_{n}(y) v_{r}^{(e)}(y) \left( u_{r}^{2} - v_{rr}^{(e)}(y) + v_{rr}(y) \right) dy = \sum_{i=1}^{n} I_{n}^{(i)}. \)
Next we give some simple estimates

\[(5.12)\]

\[\nu^\prime_{rr}(y) - v_{rr}(y) \leq \lambda_0 \int_{\mathbb{R}^d} P\left(\Phi_{\delta}(b(z, r) \cap b(0, \varrho) \geq 1\right) \, dz\]

\[\leq \lambda_0 \nu(b(0, r)) \alpha_s^{(1)}(b^c(0, \varrho)),\]

\[\nu_r^{(p)} - v_r \leq \lambda_0 \nu(b(0, r)) \alpha_s^{(1)}(b^c(0, \varrho)),\]

and

\[1_{y(x, r)}(\varphi_{x_0} - \delta_x) - 1_{y(x, r)}(\varphi_{x_0} - \delta_x) \leq 1_{b(0, r^2)}(x)\]

provided \(\varphi_{x_0}(|x|) > 0\). This implies \(|u_r - u_r^{(p)}| \leq \alpha_s^{(1)}(b^c(0, \varrho)) + \alpha_s^{(1)}(b^c(0, \varrho - r))\). Summarizing these estimates we find that

\[|I_n^{(1)}| \leq \lambda_0^{-1} \alpha_s^{(1)}(b^c(0, \varrho)),\]

\[|I_n^{(2)}| \leq \nu(b(0, r)) E N^2 \alpha_s^{(1)}(b^c(0, \varrho)) + 2 \lambda_0^{-1} \alpha_s^{(1)}(R^d, b^c(0, \varrho - r))\]

and

\[|I_n^{(3)}| \leq EN\nu(b(0, r)) \alpha_s^{(1)}(b^c(0, \varrho - r)) \left(\lambda_0 \nu(b(0, r)) + 2\right)\]

In the following we analyse only the integral \(I_n^{(1)}\) in detail because its handling seems to be somewhat more difficult. The remaining terms \(I_n^{(i)}, i = 4, 5, 6\), can be treated step by step by the above inequalities combined with the following estimates:

\[\int_{b(0, r)} (u_r^{(p)}(y) - u_{rr}(y)) \, dy \leq \int E \sum_{b(0, r)} \Phi_{\delta}(b(x + y, r) \cap b^c(0, \varrho)) \, dy\]

\[\leq \nu(b(0, r)) \alpha_s^{(2)}(R^d, b^c(0, \varrho)),\]

\[\int_{b(0, r)} 1_{y(x+ y, r)}(\Phi_{\delta} - \delta_x) \, dy \leq \Phi_{\delta}(R^d, b(0, r)).\]

In this way one can show that

\[\lim_{n \to \infty} \sup_{n \geq 1} |I_n^{(i)}| = 0\]

for \(i = 1, \ldots, 6\) provided \(E N^2 < \infty\).

Obviously, we have

\[|I_n^{(7)}| \leq (EN)^2 \int_{\mathbb{R}^d} |v_{rr}^{(p)}(y) - v_{rr}(y) - v_r(v_{rr}^{(p)} - v_r)| \, dy\]

\[\leq (EN)^2 \int_{\mathbb{R}^d} |v_{rr}^{(p)}(y) - v_{rr}(y)| \left[1 - \frac{v_r v_{rr}^{(p)}}{v_{rr}^{(p)}}(y)\right] \, dy\]

\[+ (EN)^2 \int_{\mathbb{R}^d} \frac{v_r v_{rr}^{(p)} v_r(y)}{v_{rr}^{(p)}(y)} \left[1 - \frac{v_r v_{rr}^{(p)}(y)}{v_{rr}^{(p)}(y)}\right] \, dy.\]

By (5.12), \(1 - e^{-x} \leq x, x \geq 0,\) and the identity

\[v_r v_{rr}^{(p)} = \exp \left\{ -\lambda_0 \int_{\mathbb{R}^d} P\left(\Phi_{\delta}(b(z, r)) \geq 1, \Phi_{\delta}(b(z + y, r) \cap b(0, \varrho)) \geq 1\right) \, dz\right\}\]
we find that the first integral on the right-hand side is bounded by
\[
\lambda_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P \left( \Phi_1(b(z, r)) \leq 1, \phi_2(b(z + y, r) \cap b^c(0, \varrho)) \leq 1 \right) \, dz \, dy
\]
In like manner one can show that the second integral possesses the upper bound
\[
\lambda_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P \left( \Phi_1(b(z, r)) \right) \leq 1, \phi_2(b(z + y, r) \cap b^c(0, \varrho)) \leq 1 \right) \, dz \, dy
\]
\[
= \lambda_0 \varphi^2(b(0, r)) \alpha^{(2)}(R^d, b^c(0, \varrho)).
\]
Hence,
\[
\lim_{n \to \infty} \sup_{n \geq 1} |I_n^{(7)}| = 0
\]
and
\[
\lim_{n \to \infty} \sup_{n \geq 1} |E \mathbb{Z}_n(r) (Z_n(r) - Z_{n\delta}(r))| = 0.
\]
The second term in (5.11) can be handled similarly which shows the validity of (5.10). Thus, the Theorem 4 is completely proved.

Next, we shall give conditions for a functional limit theorem to hold. For this purpose we consider \( Z_n = (Z_n(r), 0 \leq r \leq R), n = 1, 2, \ldots, \) as a sequence of \( D[0, R] \)-valued random elements on \( \Omega, \mathcal{A}, \mathbb{P} \). To show weak convergence of \( Z_n \) we have to prove the corresponding multivariate version of Theorem 4 and, in addition, the tightness of the sequence \( (Z_n)_{n \geq 1} \) which we do by bounding the fourth moments of the increments \( Z_n(t) - Z_n(s), 0 \leq s \leq t \leq R \). Define the \( k \)-th order mixed cumulant \( \Gamma_k(X_1, \ldots, X_k) \) to be equal to
\[
\int \cdots \int \log \mathbb{E} \exp \left( \sum_{j=1}^k t_j X_j \right) \, dt_1 \cdots dt_k
\]
and put \( \Gamma_k(X) = \Gamma_k(X, \ldots, X) \).

Further, we need the abbreviations \( B(x, s, t) = b(x, t) \setminus b(x, s) \) and
\[
\mu_k(s, t) = \mathbb{E} \left[ \frac{1}{k!} \sum_{\pi \in \Phi_k} \prod_{j=1}^k \mathbb{1}_{Y_j(Z_n(t) \cap b_s)} \right], \quad k \geq 1.
\]
Clearly,
\[
\mu_1(s, t) \leq \mu_2(s, t) \leq \ldots .
\]
The relation
\[
(5.13) \quad \mathbb{E} (X - EX)^4 = \Gamma_4(X) + 3(D^2X)^2
\]
makes it possible to express conditions on \( \mathbb{E} (Z_n(t) - Z_n(s))^4 \) by means of the fourth- and second-order cumulants of
\[
\nu(D_n)^{1/2} (S_n(t) - S_n(s)).
\]

**Lemma 2.** In addition to the assumption of Theorem 4 suppose \( \mathbb{E} N^k < \infty \) for some integer \( k \geq 1 \). Then
\[
(5.14) \quad (\nu(D_n))^{k-1} |\Gamma_k(S_n(t) - S_n(s))| \leq C_1 \mu_k(s, t) + C_2 \nu(B(0, s, t)),
\]
where \( C_1, C_2 \) only depend on \( k \) and \( R \).
Proof of Lemma 2. For $k = 1$ one has

$$E(S_n(t) - S_n(s)) = \lambda Q(u_t v_t - u_s v_s).$$

Hence, by $u_s - u_t = \mu(s, t)$ and

$$v_s - v_t \leq \lambda Q \int_{\mathbb{R}^d} \alpha_{s,t}^3(B(z, s, t)) \, dz \leq \lambda Q \lambda_s v(B(0, s, t)), \quad (5.14)$$

(5.14) follows for $k = 1$.

In case of $k = 2$ the left-hand side of (5.14) equals

$$E Z_n(s) (Z_n(s) - Z_n(t)) - E(Z_n(s) - Z_n(t)) Z_n(t).$$

Again (see (5.10), (5.11)) it suffices to treat the first term which can be expressed by means of the notation introduced in the preceding proof as follows:

$$\lambda Q(u_s v_s - u_t v_t) + \lambda Q^2 E \sum_{x_1, x_2 \in \Phi_s} w_n(x_1 - x_2) \left[1_{Y(x_1, s)}(\Phi_s - \delta_{x_1}) \times 1_{Y(x_2, s)}(\Phi_s - \delta_{x_2}) - 1_{Y(x_1, t)}(\Phi_s - \delta_{x_1}) 1_{Y(x_2, t)}(\Phi_s - \delta_{x_2}) \times v_{ts}(x_2 - x_1) \right] + \lambda Q^2 \int_{\mathbb{R}^d} v_n(-y) \times \left[\alpha_{s,t}^3(y) - \alpha_{s,t}^3(-y) \right] v_{ts}(y) - \left(u_s v_s - u_t v_t\right) u_s v_s \, dy.$$

It is easily seen that the absolute value of the expression in the first brackets does not exceed

$$1_{Y(x_1, s)} 1_{Y(x_1, t)}(\Phi_s - \delta_{x_1}) + v_{ts}(y) - v_{ts}(y).$$

After rearranging the terms in the second brackets and taking into account $u_{st}(y) \leq u_s \leq \lambda_s$ we get the following upper bound for its absolute value:

$$\lambda_s \left|u_{st}(y) - u_{ts}(y) - u_s + u_t\right| + \lambda_s \left|u_{st}(-y) - u_{ts}(-y)\right|$$

$$+ (u_s - u_t) \left[u_s - u_{st}(-y) + \lambda_s (v_{st}(y) - v_{st}^3)\right]$$

$$+ \lambda_s^2 \left|v_{st}(y) - v_{ts}(y) - v_s - v_t\right|$$

$$+ \lambda_s \left(v_s - v_t\right) \left(u_t - u_{ts}(y) + u_s - u_{st}(-y)\right).$$

Simple manipulations show that

$$\int_{\mathbb{R}^d} (u_s - u_t - u_{st}(y) + u_{ts}(y)) \, dy$$

$$\leq \int_{\mathbb{R}^d} E \sum_{x \in \Phi_s} 1_{Y(x, s)} 1_{Y(x, t)}(\Phi_s - \delta_{x}) \Phi_s(b(x + y, s)) \, dy \leq v(b(0, s)) \mu_2(s, t),$$

$$\int_{\mathbb{R}^d} \left|v_{st}(y) - v_{ts}(y) - v_s - v_t\right| \, dy$$

$$\leq \int_{\mathbb{R}^d} \left(v_{st}(y) - v_{ts}(y)\right) \left(1 - \frac{v_{ts}(y)}{v_{ts}(y)}\right) \, dy + \int_{\mathbb{R}^d} \left(1 - \frac{v_{st}(y)}{v_{st}(y)}\right) v_s^2 \, dy$$

$$\leq \lambda Q \lambda \lambda_s v(b(0, s)) \left[\lambda Q \lambda_s v(b(0, s)) + 1\right].$$
and for the remaining terms we have
\[ 0 \leq v_{y_s}(y) - v_{y_t}(y) \leq \lambda_0 \lambda_0 (B(0, s, t)), \]
\[ \int_{\mathbb{R}^2} (v_{y_s}(y) - v_{y_t}(y)) dy \leq \lambda_0 \lambda_0 (B(0, s, t)), \]
\[ \int_{\mathbb{R}^2} (u_{y_s}(y) - u_{y_t}(y)) dy \leq \lambda_0 \lambda_0 (b(0, s, t)), \]
\[ \int_{\mathbb{R}^2} (u_s - u_{y_t}(y)) dy \leq \lambda_0 \lambda_0 (b(0, s, t)). \]

Summarising all above estimates we obtain
\[ |E_Z(n) [Z_n(s) - Z_n(t)]| \leq C_{1,2}(s, t) + C_{2,2}(B(0, s, t)) \]
which in fact implies (5.14) for \( k = 2 \).

The validity of (5.14) for \( k = 3 \) can be verified by the same technique as it was just employed in the cases \( k = 1 \) and \( k = 2 \). However, the length of all these elementary calculations increases enormously. Therefore, to avoid a detailed estimation procedure in the general case we outline the proof of Lemma 2 in the simpler situation when \( \Phi_1(b_r(0, b)) = 0 \) \( \text{P-a.s. for some fixed } \eta > 0, \text{i.e. } S_n(r) = S_n(\eta, D_n), 0 \leq r \leq R. \)

Obviously, we need only to estimate mixed cumulants of the form
\[ \Gamma_k(Y_{y_r}(s) - Y_{y_r}(t), Y_{y_r}(s), \ldots, Y_{y_r}(s), Y_{y_r}(s), \ldots, Y_{y_r}(s)), \]
which, in view of the dependence of the random variables \( Y_{y_r}(r) = S_{y_r}(\eta, E_r), 0 \leq r \leq R, z \in V_n, \text{with } m = [2(\theta + R)] + 1, \text{equals} \)
\[ \sum_{i,j \leq m, i=1,...,k-1} \Gamma_k(Y_{y_r}(s) - Y_{y_r}(t), Y_{y_r}(s), \ldots, Y_{y_r}(s), Y_{y_r}(s), \ldots, Y_{y_r}(s)). \]

Next we use the representation of the mixed cumulants by mixed moments (see e.g. [12]) and apply the inequality \( |x_1 \ldots x_l| \leq \frac{1}{l} (|x_1|^l + \cdots + |x_l|^l) \) to the \( Y_{y_r}(s) \) and \( Y_{y_r}(t) \). Then, the stationarity of \( \Phi \) leads us to the following bound of the quantities (5.15):
\[ C(k) (2km + 1)^{k-1} \sum_{y \in V_n} \sum_{i+j=k} \sum_{|z-y| \leq km} \| \mathbb{E} \Phi(E_y) \mathbb{E} \Phi^i([0, 1]_d). \]

Finally, (5.1), (5.2), (1.3) for \( k = 1 \), and the properties of PCP's provide the estimate (5.14) for all \( k \geq 1 \) with constants \( C_1, C_2 \) depending on \( \eta. \)

**Theorem 5.** If the assumptions of Theorem 3 are satisfied, the limit
\[ \lim_{n \to \infty} E_Z(n) Z_n(t) = K(s, t) \]
exists for \( 0 \leq s \leq t \leq R, \) and if, in addition, \( EN^4 < \infty \) and
\[ \mu_4(s, t) \leq C(t - s) \]

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hold, then the random processes $Z_n = (Z_n(r), 0 \leq r \leq R)$ converge weakly in $D[0, R]$ as $n \to \infty$ to an a.s. continuous centred Gaussian process $Z = (Z(r), 0 \leq r \leq R)$ with covariance function $E(Z(s)Z(t)) = K(s, t)$.

Proof of Theorem 5. By repeating the truncation technique and the other arguments used for proving Theorem 4 we obtain

$$\sum_{j=1}^{k} t_j Z_n(r_j) \to N\left(0, \sum_{i,j=1}^{k} \int K(r_i, r_j) \right)$$

for each collection $t_1, \ldots, t_k \in R^1$ and $r_1, \ldots, r_k \in [0, R]$. Hence, the application of the method of Cramér and Wold (see [5]) shows that $(Z_n(r_j))_{j=1}^{k}$ converges in distribution (as $n \to \infty$) to a centred Gaussian vector with covariance matrix $(K(r_i, r_j))_{i,j=1}^{k}$.

It remains to prove tightness of $(Z_n)_{n \geq 1}$ in $D[0, R]$. The succeeding arguments are quite similar to those employed in the proof of Theorem 22.1 of [5] (see also [3], pp. 454–455).

From Lemma 2, (5.13) and (5.17) it follows that

$$E(Z_n(t) - Z_n(s)) \leq C_1(t - s)^{1/2} \sigma(D_n) + (t - s)^2.$$ 

For given $\varepsilon \in (0, 1)$ we find that for all $n \geq n_0(\varepsilon)$ satisfying $\varepsilon^2/\sigma(D_n) \leq t - s$, the latter estimate becomes

$$E(Z_n(t) - Z_n(s)) \leq 2C_1(t - s)^{1/2} \varepsilon.$$ 

Further, choose $n_1(\varepsilon) \geq n_0(\varepsilon)$ so that $\sigma(D_n) \leq \sigma(D_{n_1(\varepsilon)})$ for all $n \geq n_1(\varepsilon)$. Assume that $h = h(\varepsilon)$ satisfies $\varepsilon^2/\sigma(D_{n_1(\varepsilon)}) \leq h \leq \varepsilon/(\sigma(D_{n_1(\varepsilon)}))^{1/2}$ and, for given $\eta > 0$, define the integer $p = [\eta \varepsilon^2/h]$.

Consider now the increments

$$Z_n(s + (i - 1) h) - Z_n(s + ih), \quad i = 1, \ldots, p.$$ 

By (5.19) and Theorem 12.2 of [5],

$$E[Z_n(s + i h) - Z_n(s)] \leq C_2 \frac{(ph)^2}{\varepsilon h^4}.$$ 

for $n \geq n_1(\varepsilon)$.

Because each sample path of $S_n(r)$ is non-increasing in $r$ and $E(S_n(s) - S_n(s + h)) \leq C_3 h$ we find for $s \leq t \leq s + h$ that

$$|Z_n(s) - Z_n(t)| \leq |Z_n(s) - Z_n(s + h)| + C_3(\sigma(D_n))^{1/2} h.$$ 

Consequently,

$$\sup_{s \leq t \leq s + ph} |Z_n(s) - Z_n(t)| \leq 3 \max_{1 \leq i \leq p} |Z_n(s) - Z_n(s + ih)| + C_3 h(\sigma(D_n))^{1/2}.$$ 

Choosing $\delta = \delta(\varepsilon, \eta) = ph$ and taking into consideration the choice of $p$ and $h$ it follows from (5.19) that

$$\frac{1}{\delta} P\left\{ \sup_{s \leq t \leq s + \delta} |Z_n(s) - Z_n(t)| \geq (3 + C_3) \varepsilon \right\} < C_3 \eta$$ 

for every $n \geq n_1(\varepsilon)$ and $0 \leq s \leq R - \delta$. 

This implies that, for given \( \epsilon, \eta > 0 \), there exists some \( \delta = \delta(\epsilon, \eta) > 0 \) such that for the modulus of continuity \( w(Z_n, \cdot) \)

\[
P(w(Z_n, \delta) \geq 3(3 + C_3) \epsilon) \leq C_4 \eta
\]

holds provided \( n \) is taken large enough. Thus, by Theorem 15.5 of \([5]\) the sequence \( (Z_n)_{n \geq 1} \) is tight and \( P(Z \in C[0, R]) = 1 \).

Once more we return to the NEYMAN-SCOTT process governed by \( \lambda_q, f(x) \) and \( g(x) \) (see Sect. 4).

**Lemma 3.** If, for some \( k \geq 1, g^{(k+1)}(1) < \infty \) and \( \sup_{x \in R^d} \int f(x+y) f(y) \, dy = L < \infty \),

\[
\lim_{x, y \to 0} \frac{1}{x^d} \left( \int_{R^d} \left( \sum_{j=1}^{k} \sigma_{j,k}(h_j(s) - h_j(t)) \right) \, dx \right)
\]

where \( \sigma_{j,k}, j = 1, \ldots, k \), denote the STIRLING numbers of second kind. Using (2.1) for \( k = 1 \) w.r.t. \( P \), we get

\[
h_j(r) = \sum_{n=j-1}^{\infty} n(n - 1) \cdots (n - j + 2) \times \left( \int_{R^d} \left( \sum_{j=1}^{k} \sigma_{j,k}(h_j(s) - h_j(t)) \right) \, dx \right).
\]

Theorem 12.8 of \([16]\), p. 113, shows that the integrand is equal to

\[
\lim_{\epsilon \to 0} \frac{P(\Phi(s) = n, \Phi(B(x, \epsilon, r)) = 0, \Phi(B(x, \epsilon)) = 1)}{P(\Phi(s) = n) = 1)}
\]

for \( j = 1, 2, \ldots \). The inequalities \( |z_1^j - z_2^j| \leq j |z_1 - z_2| \), \( |g^{(j)}(z_1) - g^{(j)}(z_2)| \leq E^{j+1} |z_1 - z_2| \), and \( g^{(j)}(z) \leq E^{j} \), \( \Phi(s) - \Phi(t) \leq \mu(E^{j+1}) \int F(B(x, s, t)) \, dx \)

\[
\leq jEN^{j+1} + EN^j + EN^j \int F(B(x, s, t)) \, dx
\]

Thus, together with (5.20) the proof of Lemma 3 is completed. 

**Concluding Remarks.** The technical result stated in Lemma 3 shows that condition (5.17) of Theorem 5 is valid for a NEYMAN-SCOTT process satisfying \( E^{N^5} < \infty \) and \( \sup_{x \in R^d} \int f(x+y) f(y) \, dy \leq L < \infty \). In particular, in the case of a stationary Poisson process with intensity \( \lambda \), \( \mu \), \( \eta \), \( \epsilon \), and \( \delta \) all the assumptions of Theorem 5 are fulfilled with

\[
K(s, t) = \lambda e^{-\lambda(x_0, t)} + \sum_{n=1}^{\infty} e^{-\lambda(x_0, t)} \left( \int_{B(x_0, t)} \left( \sum_{n=1}^{\infty} e^{-\lambda(x_0, t)} \, dx - \mu(B(0, t+s)) \right) \right),
\]

whenever, in addition, \( \nu(D_n \ominus B(0, R)) \to 1 \).
References

[22] Rosén, B., A note on asymptotic normality of sums of higher dimensionally indexed random variables. Arkiv for Mat. 8 (1969) 33—43

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