Stable Limit Theorems for Sums of Multiply Indexed m-Dependent Random Variables

By LOTHAR HEINRICHS of Freiberg

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1. Introduction

Let \( Z^d, d \equiv 1 \), denote the set \( \{z = (z_1, \ldots, z_d): z_i \in \{0, \pm 1, \ldots \}, i = 1, \ldots, d \} \) equipped with the maximum norm \( \|z\| = \max_{1 \leq i \leq d} |z_i| \). A family \( \{X_z, z \in Z^d\} \) of real-valued random variables (RV’s) defined on a common probability space \( (\Omega, \mathcal{F}, P) \) and indexed by elements of \( Z^d (d \geq 2) \) is called a \( d \)-dimensional random field (RF). For any subset \( V \subset Z^d \) define \( N = |V| = \text{card } V, \partial^s V = \{z: z \in V, \|z - y\| \leq s \text{ for all } y \in Z^d \setminus V\} \) (the \( s \)-boundary of \( V \), \( s \geq 1 \)) and \( S_V = \sum_{z \in V} X_z \). A \( d \)-dimensional RF is said to be \( m \)-dependent if for any finite subsets \( U, V \subset Z^d \) the random vectors \( (X_u)_{u \in U} \) and \( (X_v)_{v \in V} \) are independent whenever \( \|u - v\| > m \) for all \( u \in U \) and \( v \in V \) (see, e.g., [15], [20]). Together with \( Y_z = Y_z(t) = e^{itX_z(t)} - 1, t, u \in \mathbb{R}^1 \), we shall use the following notations and abbreviations, where \( m \) is a fixed positive integer and \( V \) is a fixed subset of \( Z^d \) with \( |V| < \infty \):

\[ g_{V}(t, u) = E \prod_{z \in V} \left[ 1 + u X_z(t) \right], \quad f_{V}(t) = E e^{itS_V}, \]

\[ M_{V} = M_{V}(t) = (\max_{z \in V} E |Y_z(t)|^2)^{1/2}, \quad \Sigma_{V} = \Sigma_{V}(t) = \sum_{z \in V} E |Y_z(t)|^2, \]

\[ V_z^{(k)} = \{y: y \in Z^d, (k - 1) m - \|z - y\| \leq km \}, \quad U_z^{(k)} = \bigcup_{i=1}^{k} V_z^{(k)} \cup \{z\} \]

\[ W_z^{(k)}(u) = \prod_{y \in V_z^{(k)} \cap V} (1 + u Y_y), \quad W_z^{(k)}(u) = (1 + u Y_z) \prod_{i=1}^{k} W_z^{(k)}(u), \]

\[ (1.1) \quad u_k = |U_z^{(k)}| = (2km + 1)^d, \quad v_k = |V_z^{(k)}| = u_k - u_{k-1} \]

\[ e_{z, V}^{(m)}(t, u, a) = \sum_{k=1}^{(m+1)d-1} \frac{u^{k+1}}{(k+1)!} \sum_{u_1, \ldots, u_k \in V_z^{(m)} \cap V \atop \|u_i - u_j\| \geq m} E[e^{iu(X_z - a)} - 1] \]

\[ \times \prod_{j=1}^{k} [e^{iu(X_j - a)} - 1], \quad e^{(m)}(t, u, a) = e_{0, V}^{(m)}(t, u, a) \]

where * indicates that the summation runs over all pairwise different indices \( y_1, \ldots, y_k \in V_z^{(m)} \cap V \).
Further, $G_{n}(x, c)$ denotes the stable distribution function whose characteristic function $g_{n}(t, c) = \int e^{itx}dG_{n}(x, c)$ has the logarithm

\begin{equation}
\log g_{n}(t, c) = -c |t|^\alpha \left( 1 - i\beta \frac{|t|}{t} \omega(t, \alpha) \right),
\end{equation}

where $0 < \alpha \leq 2$, $|\beta| \leq 1$, $c > 0$ and $\omega(t, \alpha) = \frac{2}{\alpha} \log |t|$, if $\alpha = 1 = \tan \frac{\pi \alpha}{2}$, else.

Throughout this paper $C(\cdot)$, $C_k(\cdot)$, $k = 1, 2, \ldots$, denote positive constants (depending on the quantities indicated in the parentheses), which may differ from one expression to another. $(\Theta_k(t, u), k = 1, 2, \ldots)$ stands for a complex-valued continuous function of $u, t \in R^1$ with $\sup \{ |\Theta_k(t, u)| : 0 \leq u \leq 1, t \in R^1 \} \leq C(d) < \infty$.

In the last decade a large number of results concerning rates of convergence in limit theorems for sums of $m$-dependent RV’s (in case $d = 1$) could be proved by using various methods (see [5], [7] – [12], [17], [24], [26], [28]). Stein [17] was the first that obtained the optimal bound in the central limit theorem (CLT) for sums of stationary $m$-dependent RV’s having a finite 8th moment. By using other tools Shergin [28] proved in the non-stationary case that the bound of the error in CLT is given by the third-order Lyapunov ratio multiplied by $C(m + 1)^2$ whenever $E |X_k|^3 < \infty$, $k \geq 1$. In [7] the author gave a general method for proving limit theorems for sums of $m$-dependent RV’s. This method enables us to derive non-uniform estimates and asymptotic expansions in the CLT (see [7], [8], [10]) as well as to prove moderate and large deviation theorems (see [7], [9]), which have the same quality as in the case of independent RV’s. With another technique and under slightly different assumptions Edgeworth expansions in the CLT for $m$-dependent and other types of weakly dependent random vectors are given in [5]. In [11] the author succeeded in obtaining stable limit theorems for sums of $m$-dependent RV’s ($d = 1$) and their rates of convergence. These limit theorems refine earlier ones formulated for more general types of weakly dependent RV’s in [1], [2] and [13]. Furthermore, they extend some bounds obtained for the stable approximation of sums of independent RV’s (see [4], [22]). The main goal of this paper is to provide a representation of the characteristic function $f(t)$ in some neighborhood of $t = 0$ (without any moment conditions) and, basing on it, to derive a CLT, stable limit theorems, and rates of the stable approximation for stationary $m$-dependent RV’s as $N$ tends to infinity. The method presented in Sect. 2 is not so powerful as the factorization method in the case $d = 1$ (see [7]). This is due to the troubles arising from the spatial stochastic dependences although they have only a finite range. The proof of Lemma 2 exploits some ideas which were used in [24], [26] (for $d = 1$) and [6], [21] (for $d \geq 2$) for the derivation and solution of a linear first-order differential equation for $f(t/DS_N)$ in order to derive estimates of the rate in the CLT. We apply these ideas to the more general situation of the function $g(t, u)$. In order to obtain uniform bounds in the CLT for $m$-dependent RV’s both Stein’s and Tikhomirov’s method were employed.
by several authors (see [6], [12], [18], [19], [21], [23]). The uniform estimates
given in [6] under sup \( E |X_x|^2 + \delta \to \infty \), 0 < \( \delta \) < 1, are very close to those for independent RF’s. Under the restrictions \( E |X_x|^3 \equiv C < \infty \) and \( D^2 S_N \cap N \) RIAUBA [23] obtained the optimal bound in the CLT for stationary \( m \)-dependent RF’s, namely, \( O(N^{-1/2}) \), whereas under \( E |X_x|^3 \equiv C < \infty \) the best known bound is \( O(N^{-1/2} (\log N)^{(d-1)/2}) \) (see [6] and [18], [19]). For a special class of stationary \( m \)-dependent RF’s we can show RIAUBA’s bound under \( E |X_x|^3 \equiv C < \infty \) (see Theorem 6) and EDGOWERTH expansions for this special case are given in [12]. Finally, we mention that some authors treated more general cases of RF’s, namely, \( \alpha \)- and \( \varphi \)-mixing and special GIBBS fields (see e.g. [3] (without rates), [18], [19]).

2. An approximate representation of \( f_v(t) \) in some neighborhood of \( t = 0 \)

In this section we shall represent \( f_v(t) \) in some neighborhood of \( t = 0 \) in such a way that it is easy to recognize the asymptotic behaviour of \( S_y \) as \( N \to \infty \). For this end the first partial derivative of \( g_v(t, u) \) with respect to \( u \) is decomposed and after rearranging the terms an initial-value problem

\[
\frac{\partial}{\partial u} g_v(t, u) = g_v(t, u) a(t, u) + b(t, u), \quad 0 \leq u \leq 1, \quad t \in \mathbb{R}^1
\]

with \( g(t, 0) = 1 \) is established for such \( t \) for which \( M_{v} = M_{v}(t) \) is sufficiently small. The solution of the differential equation (2.1) is given in Lemma 2. Therefore Lemma 2 forms the basis for all further investigations in this paper. For brevity we put

\[
T^{(q, m)}(t, u) = \sum_{l=1}^{q} (-1)^{l-1} \sum_{q_1 + \ldots + q_l = q} \left[ E Y_z \left( W_z^{(1)}(u) - 1 \right) \cdot \ldots \right.
\]

\[
\times \left( W_z^{(q_1)}(u) - 1 \right) - E Y_z E \left( W_z^{(1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_1)}(u) - 1 \right) \]

\[
\times E \left( W_z^{(q_1 + 1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_1 + q_2)}(u) - 1 \right) \cdot \ldots \]

\[
\times E \left( W_z^{(q_1 + \ldots + q_l - 1 + 1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_0)}(u) - 1 \right),
\]

\[
T^{(q, m)}(t, u) = T^{(q, m)}(0, u), \quad q = 1, 2, \ldots
\]

These quantities reflect the “correlation structure” of the RF.

**Lemma 1.** For an arbitrary \( m \)-dependent RF \( \{X, z \in V \subset Z^d\} \) and any integer \( p \geq 2 \) and \( t, u \in \mathbb{R}^1 \) there holds

\[
\frac{\partial}{\partial u} g_v(t, u) = g_v(t, u) \sum_{z \in V} \left[ E Y_z + \sum_{q=1}^{p-1} T^{(q, m)}(t, u) \right]
\]

\[
+ \sum_{z \in V} \left[ (-1)^{l-1} \sum_{l=1}^{p} \sum_{q_1 + \ldots + q_l = p} \left[ E Y_z \left( W_z^{(1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_1)}(u) - 1 \right) \right]
\]

\[
\times \left( W_z^{(q_1)}(u) - 1 \right) - E Y_z E \left( W_z^{(1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_1)}(u) - 1 \right) \]

\[
\times E \left( W_z^{(q_1 + 1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_1 + q_2)}(u) - 1 \right) \cdot \ldots \]

\[
\times E \left( W_z^{(q_1 + \ldots + q_l - 1 + 1)}(u) - 1 \right) \cdot \ldots \cdot \left( W_z^{(q_0)}(u) - 1 \right),
\]

\[
T^{(q, m)}(t, u) = T^{(q, m)}(0, u), \quad q = 1, 2, \ldots
\]

\[
1) \text{The proof of the main result in [21] contains a heavy mistake so that Malevich's result is still unproved.}
\]

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\[ -EY_zE \left( \tilde{W}^{(1)}_z(u) - 1 \right) \cdot \ldots \cdot \left( \tilde{W}^{(p)}_z(u) - 1 \right) E \left( \tilde{W}^{(p_1 + 1)}_z(u) - 1 \right) \]
\[ \times \left( \tilde{W}^{(p_1 + p_2)}_z(u) - 1 \right) \cdot \ldots \cdot E \left( \tilde{W}^{(p_1 + \ldots + p_{l-1} + p_1 + 1)}_z(u) - 1 \right) \cdot \ldots \cdot E \left( \tilde{W}^{(p_2)}_z(u) - 1 \right) \]
\[ \times \prod_{q \geq p + 1} \tilde{W}^{(q)}_z(u). \]

Proof. We verify relation (2.2) by induction (with respect to \( p \)). By using the rule for the differentiation of a product we get
\[
\frac{\partial}{\partial u} g_V(t, u) = \sum_{z \in V} \frac{EY_z \prod_{q \geq 1} \tilde{W}^{(q)}_z(u)}{z \in V} \left( \sum_{z \in V} [EY_z (\tilde{W}^{(1)}_z(u) - 1) \prod_{q \geq 2} \tilde{W}^{(q)}_z(u)} \right) \]
\[ -EY_zE \left( \tilde{W}^{(1)}_z(u) - 1 \right) \prod_{q \geq 2} \tilde{W}^{(q)}_z(u) \]}

Proceeding in this way we arrive at
\[
\frac{\partial}{\partial u} g_V(t, u) = g_V(t, u) \sum_{z \in V} [EY_z + EY_z (\tilde{W}^{(1)}_z(u) - 1) \cdot EY_zE (\tilde{W}^{(1)}_z(u) - 1)] \]
\[ + \sum_{z \in V} EY_z [(\tilde{W}^{(1)}_z(u) - 1) (\tilde{W}^{(2)}_z(u) - 1)] \prod_{q \geq 3} \tilde{W}^{(q)}_z(u) \]
\[ -E (\tilde{W}^{(1)}_z(u) - 1) (\tilde{W}^{(2)}_z(u) - 1) \prod_{q \geq 3} \tilde{W}^{(q)}_z(u) \]
\[ -\sum_{z \in V} EY_z [(\tilde{W}^{(1)}_z(u) - 1) - E (\tilde{W}^{(1)}_z(u) - 1)] \]
\[ \times E (\tilde{W}^{(2)}_z(u) - 1) \prod_{q \geq 3} \tilde{W}^{(q)}_z(u). \]

But even this coincides with (2.2) for \( p = 2 \). Basing on the relation (2.2) for an arbitrary \( p > 2 \) we are now going to prove the corresponding relation for \( p + 1 \).

From the evident decomposition
\[
E (\tilde{W}^{(p')}_z(u) - 1) \cdot \ldots \cdot (\tilde{W}^{(p)}_z(u) - 1) \prod_{q \geq p + 1} \tilde{W}^{(q)}_z(u) \]
\[ =E (\tilde{W}^{(p')}_z(u) - 1) \cdot \ldots \cdot (\tilde{W}^{(p)}_z(u) - 1) \prod_{q \geq p + 2} \tilde{W}^{(q)}_z(u) \]
\[ -E (\tilde{W}^{(p')}_z(u) - 1) \cdot \ldots \cdot (\tilde{W}^{(p)}_z(u) - 1) E (\tilde{W}^{(p+1)}_z(u) - 1) \prod_{q \geq p + 2} \tilde{W}^{(q)}_z(u) \]
\[ +E (\tilde{W}^{(p')}_z(u) - 1) \cdot \ldots \cdot (\tilde{W}^{(p)}_z(u) - 1) g_V(t, u) \],

where \( p' = p_1 + \ldots + p_{l-1} + 1 \), we are led to
\[
\frac{\partial}{\partial u} g_V(t, u) = g_V(t, u) \sum_{z \in V} \left[ EY_z + \sum_{q = 1}^{p} T^{(q, p)}_z(t, u) \right] + \sum_{z \in V} \sum_{l = 1}^{p} (-1)^{l-1} \]
\[ \times \left\{ \sum_{p_1 + \ldots + p_{l-1} + p_1 = p+1} \ldots \sum_{p_1 + \ldots + p_{l-1} = 1, p_1 \geq 2} \right. \]
\[ \times \left( \tilde{W}^{(p+1)}_z(u) - 1 \right) \prod_{q \geq p + 2} \tilde{W}^{(q)}_z(u) \]
\[ + \sum_{p_1 + \ldots + p_{l-1} + p_1 = p} \ldots \sum_{p_1 + \ldots + p_{l-1} = 1} \right. \]
\[ \times \left( \tilde{W}^{(p)}_z(u) - 1 \right) E (\tilde{W}^{(p+1)}_z(u) - 1) \prod_{q \geq p + 2} \tilde{W}^{(q)}_z(u) \right\}. \]
Summarizing the sums in the braces we obtain the desired relation (2.2) for $p + 1$. Thus Lemma 1 is proved.

The following Lemma 2 gives an estimate for the solution of the differential equation (2.2) at $u = 1$. In other words the characteristic function of $S_Y$ is estimated (for sufficiently small $t$).

**Lemma 2.** Let $\{X_z, z \in V \subset \mathbb{Z}^d\}$ be an $m$-dependent RF and let $s$ and $p$ arbitrary integers with $1 \leq s < p$. Then there exists constants $C_1, \ldots, C_4$ (only depending on $d$) such that, for all $t \in R^1$ with $M_Y = M_Y(t) \leq \frac{1}{4} (2mp + 1)^{-d}$,

$$|f_Y(t) - \exp \left\{ \sum_{z \in V} \left[ E Y_z(t) + \sum_{q=1}^s \int_0^t T^{(q,m)}_{z,v}(t, u) \, du \right] \right\}|$$

$$\leq C_2 2^{s+1} (2ms + 1)^{d(s+1)} M_Y^2 \sum_{v \in V} \exp \left\{ \sum_{z \in V} \left[ \frac{1}{4} \right] \right\}$$

$$\leq C_3 2^{-p+2(s+1)} (2ms + 1)^{d(s+1)} M_Y^2 \sum_{v \in V} \exp \left\{ \sum_{z \in V} \left[ \frac{1}{4} \right] \right\}$$

$$+ C_4 (1 - u)^{2s+1} (2ms + 1)^{d(s+1)} M_Y^2 \sum_{v \in V} \exp \left\{ \sum_{z \in V} \left[ \frac{1}{4} \right] \right\}$$

Proof. First note that the following elementary relations hold:

$$|1 + u Y_z|^2 = 1 - 2u (1 - u) (1 - \cos t X_z) \leq 1,$$

$$E |1 + u Y_z|^2 = 1 - u (1 - u) E |Y_z|^2,$$

$$E |W_{z}^{(q)}(u) - 1|^2 \leq u^2 v_q \sum_{y \in V} E |Y_y|^2$$

and

$$\sum_{l=1}^q \sum_{q_1 + \ldots + q_l = q} 1 = 2^{q-1}.$$

By virtue of (2.4) - (2.6) the 1-dependence of the RV's $Y_z$, $W_{z}^{(1)}(u)$, ..., $W_{z}^{(q)}(u)$, the CAUCHY-SCHWARTZ inequality, and the inequality between the arithmetic and geometric mean yield

$$\begin{align*}
|\sum_{z \in V} T^{(q,m)}_{z,v}(t, u)| &\leq \sum_{l=1}^q \sum_{q_1 + \ldots + q_l = q} \left( E |Y_z|^2 \right)^{1/2} \\
&\times \left[ (E |W_{z}^{(1)}(u) - 1|^2) \right]^{1/2} + \left( E |W_{z}^{(q)}(u) - 1|^2 \right)^{1/2} \\
&\times \left( \prod_{k=2}^{q} E |W_{z}^{(k)}(u) - 1|^2 \right)^{1/2} \\
&\equiv (2u)^q (u_1 u_2)^{1/2} \prod_{k=2}^{q} u_k M_Y^2 \sum_{v \in V} \exp \left\{ \sum_{z \in V} \left[ \frac{1}{4} \right] \right\}.
\end{align*}$$
Since $M \leq 1/4 u_p$ the identity (2.2) turns to the following formal differential equation

(2.8) \[
\frac{\partial}{\partial \nu} g_\nu(t, u) = g_\nu(t, u) \left\{ \sum_{z \in V} \left[ E Y_z + \sum_{q=1}^{\nu} T^{(q, m)}_{z, V}(t, u) \right] 
+ \Theta_1(t, u) (2u_{n+1})^{s+1} M_\nu \Sigma_v \right\} 
+ \Theta_2(t, u) (4u_{n+1})^{s+1} M_\nu \Sigma_v 2^{-p}.
\]

Recall that the solution of the initial-value problem (2.1) with continuous functions $a(t, u)$ and $b(t, u), u \geq 0, t \in R^1$, can be expressed in a closed form, namely, by

\[
g(t, u) = \exp \left\{ \int_0^u a(t, v) \, dv \right\} + \int_0^u b(t, v) \exp \left\{ \int_v^u a(t, w) \, dw \right\} \, dv,
\]
where $u \geq 0$ and $t \in R^1$.

Comparing (2.1) with the differential equation (2.8) and remembering that $g(t, 0) = 1$ we find that, for $M \leq 1/4 u_p$,

(2.9) \[
g_\nu(t, 1) = \exp \left\{ \sum_{z \in V} \left[ E Y_z + \sum_{q=1}^{\nu} T^{(q, m)}_{z, V}(t, u) \right] \right\}
+ \int_0^t u^{s+1} \Theta_1(t, u) \, du \left( 2u_{n+1} \right)^{s+1} M_\nu \Sigma_v
\times \int_0^t \Theta_2(t, u) u^{s+1} \exp \left\{ \sum_{z \in V} \left[ (1-u) E Y_z + \sum_{q=1}^{\nu} T^{(q, m)}_{z, V}(t, v) \right] \right\}
\times \int_0^t \nu^{s+1} \Theta_1(t, v) \, dv \left( 2u_{n+1} \right)^{s+1} M_\nu \Sigma_v \right\} \, dv.
\]

Hence, making use of the inequality $|e^x - 1| \leq |x| e^{|x|}$ we immediately deduce the desired estimate (2.3) and so Lemma 2 is proved.

We proceed with a slightly simplified form of Lemma 2.

**Lemma 3.** Let $\{X_z, z \in V \subset Z^d\}$ be an $m$-dependent RF and let $p \geq 2$. Then there exist constants $C_1, \ldots, C_4$ (only depending on $d$) such that

(2.10) \[
|f_\nu(t) - \exp \left\{ \sum_{z \in V} [E Y_z + e^{(m)}_{z, V}(t, 1, 0)] \right\} |
\leq C_1 3^{(2m+1)d} M_\nu \Sigma_v \exp \left\{ \sum_{z \in V} \text{Re} [E Y_z + e^{(m)}_{z, V}(t, 1, 0)] + C_2 3^{(2m+1)d} M_\nu \Sigma_v \right\}
+ C_3 2^{-p} (m + 1)^{2d} M_\nu \Sigma_v \int_0^t \exp \left\{ \sum_{z \in V} \text{Re} [(1-u) E Y_z + e^{(m)}_{z, V}(t, u, 0)] + C_4 3^{(2m+1)d} (1-u) M_\nu \Sigma_v \right\} \, du
\]
for all $t \in R^1$ with $M_\nu \leq 1/4 u_p$, where $\Sigma_v = \Sigma_v(t) = \sum_{z \in V} |E Y_z(t)|$.

**Proof.** We take $s = 1$ in (2.9) and expand the term $T^{(1, m)}_{z, V}(t, u)$ as follows:

\[
T^{(1, m)}_{z, V}(t, u) = \sum_{k=1}^{M-1} \frac{u^k}{k!} \sum_{y_1, \ldots, y_k \in V^{(1)} \cap V} E Y_{y_k} Y_{y_k+1} \ldots Y_{y_k},
\]
and

\[
\frac{u^M}{M!} \sum_{y_1, \ldots, y_M \in V^{(1)} \cap V} E Y_{y_1} \ldots E Y_{y_M} \prod_{y} (1 + u Y_y) Y_y (W^{(1)}_{z, V}(u) - 1),
\]

where $u \geq 0$ and $t \in R^1$. Thus
where $M = (m+1)^d$ and the product $\Pi'$ is taken over some subset of $V_2^{(1)} \cap V$. Obviously, there are at least two RV's among the $(M+1)$-tupel $(Y_z, Y_{y_1}, ..., Y_{y_M})$ which are stochastically independent of each other.

Together with (2.5) and $|Y_z| \equiv 2, |V_2^{(1)} \cap V| \equiv u_1 - 1$ the binomial theorem yields

$$\sum_{z \in V} \left[ T_{z, \nu}^{(e, m)}(t, u) - \frac{\partial}{\partial u} e_{z, \nu}^{(e, m)}(t, u) \right] \equiv u_1 M \nu \Sigma + C 3^{u_1-1} M \nu \Sigma.$$  

Therefore after integration of this inequality we may employ Lemma 2. Making use of the elementary estimate $\Sigma \equiv 2 \Sigma' $ and arguing as at the end of the proof of Lemma 2 we obtain (2.10). Thus Lemma 3 is proved.

It should be mentioned that in many situations it suffices to consider only the case $m = 1$. In general, if the RF $\{X_z, z \in Z^d\}$ is $m$-dependent, then the RV's

$$X_z' = \sum_{z_i \leq k_1 \leq z_i + m - 1} X_{(k_1, ..., k_d)}, \quad z \in \{(p_1 m, ..., p_d m): p_i \in \mathbb{Z}, i = 1, ..., d\},$$

form a 1-dependent RF.

Now, let there be given a sequence $\{X_{n_2}, z \in V_n\}, n = 1, 2, ..., \mbox{of } m\text{-dependent RF's } (m \text{ fixed}) \mbox{ on a common probability space } (\Omega, \mathcal{F}, P) \mbox{ satisfying the infinitely smallness condition}$

$$\sum_{t \in V} |E e^{it X_{n_2}} - 1| \leq C |t| \to 0 \quad \forall t > 0$$

(i.e. the RV's $X_{n_2}, z \in V_n$, are asymptotically negligible): \[ \max_{z \in V_n} P (|X_{n_2}| < \epsilon) \to 0 \quad \forall \epsilon > 0 \]

and the additional condition

$$\max_{z \in V_n} \sum_{t \in V} |E e^{it X_{n_2}} - 1| \leq C |t| \to 0 \quad \forall t > 0.$$

Because of

$$\sup_{0 \leq t \leq 1} \sum_{z \in V} \text{Re} [u E Y_z(t) + e_{z, \nu}^{(e, m)}(t, u, 0)] \leq 3^{u_1-1} \Sigma'$$

a trivial application of Lemma 3 yields

**Lemma 4.** For a sequence $\{X_{n_2}, z \in V_n\}, V_n \subset \mathbb{Z}^d, n = 1, 2, ..., \mbox{ of } m\text{-dependent RF's } (m \text{ fixed}) \mbox{ satisfying (2.12) and (2.13) there holds}$

$$|\log E \exp \left( i \sum_{z \in V_n} X_{n_2} - \sum_{z \in V_n} \left[ E (e^{it X_{n_2}} - 1) + e_{z, \nu}^{(e, m)}(t, 1, 0) \right] \right) | \to 0$$

uniformly in every finite $t$-interval.

As a first application Lemma 4 will be utilized to prove a general result on the normal approximation of the sum $S_n = \sum_{z \in V_n} X_{n_2}$. The below Theorem 1 generalizes the first CLT for $m$-dependent RF's obtained in [16]. We replace the assumption on the finiteness of the variances of the summands by the condition (2.12). An analogous result concerning $m$-dependent sequences ($d = 1$) was derived.
We introduce the truncated RV's

\[ X_{n_2}(\varepsilon) = \begin{cases} X_{n_2} & \text{if } |X_{n_2}| < \varepsilon \\ 0 & \text{else} \end{cases} \] and \[ S_n(\varepsilon) = \sum_{\varepsilon \in V_n} X_{n_2}(\varepsilon). \]

**Theorem 1.** Let \{X_{n_2}, z \in V_n \subset Z^d\}, \(n = 1, 2, \ldots\), be a sequence of m-dependent RF's \((m \text{ fixed})\) satisfying the following three conditions (i)-(iii) for every fixed \(\varepsilon > 0\):

(i) \(\sum_{\varepsilon \in V_n} P(|X_{n_2}| \leq \varepsilon) \xrightarrow{n \to \infty} 0\),

(ii) \(\sum_{\varepsilon \in V_n} EX_{n_2}(\varepsilon)^2 \leq C(\varepsilon) < \infty\),

(iii) \(ES_n(\varepsilon) \xrightarrow{n \to \infty} a \in R^1\) and \(D^2S_n(\varepsilon) \xrightarrow{n \to \infty} \sigma^2\).

Then, for \(\sigma > 0\),

\[ \sup_{x \in R^1} \left| P \left( S_n < x \right) - \Phi \left( \frac{x-a}{\sigma} \right) \right| \xrightarrow{n \to \infty} 0 \]

and, for \(\sigma = 0\),

\[ S_n \xrightarrow{n \to \infty} a \quad \text{in probability}, \]

where \(\Phi(x) = G_{20}(x, 1/2)\) denotes the standard normal distribution function.

**Proof.** By virtue of (i) and the standard inequality (see e.g. [9], [14])

\[ |P(S_n < x) - P(S_n(\varepsilon) < x)| \leq \sum_{\varepsilon \in V_n} P(|X_{n_2}| \leq \varepsilon), \]

which is even valid for arbitrarily dependent RV's, it suffices to prove (2.14) for the truncated field \(\{X_{n_2}(\varepsilon), z \in V_n\}\) Since

\[ |E(e^{itX_{n_2}(\varepsilon)} - e^{itX_{n_2}(\varepsilon)}) - 1| \leq \frac{t^2}{2} D^2X_{n_2}(\varepsilon) \]

both condition (2.12) and (2.13) are fulfilled for the RF's \(\{X_{n_2}(\varepsilon) - X_{n_2}(\varepsilon) - EX_{n_2}(\varepsilon), z \in V_n\}\), \(n = 1, 2, \ldots\) Together with (ii) and Lemma 4 the estimates

\[ |E(e^{itX_{n_2}(\varepsilon)} - 1)(e^{itX_{n_2}(\varepsilon)} - 1)| \leq (2\varepsilon |t|)^k - 1 \frac{t^2}{2} (D^2X_{n_2}(\varepsilon) + D^2X_{n_2}(\varepsilon)), \]

yields \(y_1, \ldots, y_k \in V_z^{(l)} \cap V, \ k \equiv 2\),

\[ |E(e^{itX_{n_2}(\varepsilon)} - 1)(e^{itX_{n_2}(\varepsilon)} - 1) + t^2EX_{n_2}(\varepsilon) X_{n_2}(\varepsilon)| \leq \varepsilon |t|^3 (D^2X_{n_2}(\varepsilon) + D^2X_{n_2}(\varepsilon)), \]

and

\[ |E(e^{itX_{n_2}(\varepsilon)} - 1 + \frac{t^2}{2} D^2X_{n_2}(\varepsilon)| \leq \varepsilon |t|^3 D^2X_{n_2}(\varepsilon), \]

which can be derived by a liberal use of the inequalities

\[ |e^{itX_{n_2}(\varepsilon)} - 1| \leq |t| |X_{n_2}(\varepsilon)| \leq 2\varepsilon |t| \quad \text{and} \quad 2 |ab| \leq a^2 + b^2, \]

imply that the difference

\[ \max_{|t| \leq T} \left| \log E \exp \left( it (S_n(\varepsilon) - ES_n(\varepsilon)) \right) + \frac{t^2}{2} D^2S_n(\varepsilon) \right| \]

is
becomes arbitrarily small for sufficiently small $\varepsilon > 0$ and large enough $n$. Hence, by virtue of (iii) and the continuity theorem for characteristic functions the validity of (2.14) and (2.15) is completely shown.

3. Convergence to stable limit laws

The main objective of this section is the study of the asymptotic behaviour of a suitably standardized sum $Z_N = B_N^{-1} \left( S_V - A_N \right)$ ($A_N$ and $B_N > 0$ are the sequences of real numbers) of RV's connected in a stationary $m$-dependent RF $\{X_z, z \in V \subset \mathbb{Z}^d\}$ as $N = |V| \to \infty$. Under certain restrictions Lemma 4 enables us to find all possible limit distributions of the sum $Z_N$ and to obtain explicit conditions for the convergence to them. It turns out, not surprisingly, that the set of limiting distributions of $Z_N$ coincides with the set of stable distributions. For the sake of brevity we put $F_N(x) = P(Z_N < x)$, $f_N(t) = E e^{itZ_N}$ and

$$
A_N(x, \beta, \gamma) = \sup_{x \in \mathbb{R}^d} |P(Z_N < x) - G_{sp}(x, \gamma)|.
$$

**Theorem 2.** Let $\{X_z, z \in V \subset \mathbb{Z}^d\}$ be a stationary $m$-dependent RF ($m$ fixed) satisfying the condition $|\varrho^{(m)}(V)|/|V| \to 0$. Further, let there exist sequences $a_N$, $A_N$ and $B_N > 0$ with $B_N \to \infty$ such that,

$$
N \max_{|t| = T} |E e^{i(t(x_0 - a_N)/B_N - 1)}| \leq C(T) < \infty \quad \text{for every finite } \quad T > 0
$$

and $F_N(x)$ weakly converges to a non-degenerate distribution function $F(x)$. Then it holds

$$
\log \int e^{itx} dF(x) = \lim_{N \to \infty} \left\{ -\frac{it}{B_N} A_N + N \left[ E e^{i(t(x_0 - a_N)/B_N - 1)} + \frac{it}{B_N} a_N + e^{(m)}(t, 1, a_N) \right] \right\}
$$

uniformly in every finite $t$-interval.

Furthermore, $F(x)$ is necessarily stable and if it has exponent $\alpha$, then $B_N = N^{1/\alpha} h(N)$, where $h(x)$ is a slowly varying function (in the sense of Karamata, see [13]).

**Proof.** As a consequence of the stationarity and the conditions (3.2) and $|\varrho^{(m)}(V)|/|V| \to 0$ it follows from Lemma 4 that

$$
\max_{|t| = T} \left| \log E \exp \left\{ \frac{it}{B_N} (S_V - Na_N) \right\} - N \left[ E e^{i(t(x_0 - a_N)/B_N - 1 + e^{(m)}(t, 1, a_N))} \right] \right| \to 0.
$$

This shows that under the above conditions the limit behaviour of $F_N(x)$ does not depend on the shape of $V$ but only on the number $N = |V|$. Hence, since $f_N(t) = \int e^{itx} dF(x)$ the desired relation (3.3) is proved. The proof that $F(x)$ is stable is quite similar to the proof of an analogous problem for $\alpha$-mixing sequences (see
However a series of modifications are needed. Because of \(B_N \to \infty\) and \(f_N(t) \to \int e^{itx}dF(x)\) one can conclude that \(\lim_{N \to \infty} B_{N+1}/B_N = 1\) (see [13], p. 38) and, therefore, for any positive number \(a_1, a_2\), there exist a sequence \(M = M(N) \to \infty\) such that \(\lim_{N \to \infty} B_{M}/B_N = a_1/a_2\). One can choose a further sequence \(L = L(N) \to \infty\) increasing so slowly that

\[
P = P(N) = [N/L^{d-1}] \to \infty, \quad \text{and} \quad Q = Q(N) = [M(N)/L^{d-1}] \to \infty
\]

and

\[
(3.5) \quad B_{N}^{-1} \sum_{z \in D} X_z \to 0
\]
in probability for every subset \(D\) of

\[
D_u = \{z = (z_1, z_2, \ldots, z_d): u \equiv z_1 \equiv u + m - 1, \quad 0 \leq z_j \leq L - 1, \quad j = 2, \ldots, d\}, \quad u \in \mathbb{Z}
\]

Define the blocks

\[
V_N^{(i)} = \{z = (z_1, \ldots, z_d): 0 \equiv z_1 \equiv P - 1, 0 \leq z_j \leq L - 1, j = 2, \ldots, d\}
\]

\[
V_N^{(2)} = \{z = (z_1, \ldots, z_d): P + m \equiv z_1 \equiv P + Q + m - 1, 0 \leq z_j \leq L - 1, j = 2, \ldots, d\}
\]

and consider the sums

\[
\frac{1}{a_1} \left[ B_{N}^{-1} \left( \sum_{z \in V_N^{(1)}} X_z - A_N \right) - b_1 \right] + \frac{B_M}{a_1 B_N} \left[ B_{M}^{-1} \left( \sum_{z \in V_N^{(2)}} X_z - A_M \right) - b_2 \right] = \frac{1}{a_1 B_N} \sum_{z \in V_N^{(1)} \cup V_N^{(2)} \cup D_P} X_z - C_N - \frac{1}{a_1 B_N} \sum_{z \in D_P} X_z.
\]

By virtue of (3.3), (3.4), \(|e^{it}v_N^{(i)}|/|V_N^{(i)}| \to 0 (i = 1, 2), |V_N^{(i)}|/N \to 1, |V_N^{(2)}|/M(N) \to 1\), and the m-dependence the distribution function on the left side differs from \(F_N(a \mathbf{x} + b_1) * F_M(a \mathbf{x} + b)\) by at most \(o(1)\) as \(N \to \infty\), where \(*\) stands for the convolution. By means of our assumptions it is seen from (3.4) that the limiting distribution function of the right side coincides with \(F(ax + b)\) (see [14], p. 17), where \(a > 0\) and \(b\) are constants. Consequently, \(F(a \mathbf{x} + b) * F(a \mathbf{x} + b_2) = F(ax + b)\) and so \(F(x)\) is stable (see [14], p. 88). Assuming that \(F(x) = G_{\beta}(x, c), 0 < \alpha \equiv 2, |\beta| \equiv 1, c > 0\), we will show that

\[
(3.6) \quad \lim_{N \to \infty} B_{Nk}/B_N = k^{1/2} \quad \text{for every} \quad k = 1, 2, \ldots,
\]

which is necessary and sufficient for \(B_N = N^{1/4}h(N)\) (see [13]). For this purpose we define the blocks \(U_l(N) = \{z = (z_1, \ldots, z_d): (P + m) l \equiv z_1 \equiv (P + m) l + P - 1, 0 \leq z_j \equiv L - 1, j = 2, \ldots, d\}\), and the sums \(S_l(N) = \sum z \in U_l(N) X_z, l = 0, 1, \ldots, k - 1\). In view of our assumptions we get, as \(N \to \infty\),

\[
E \exp \left[ it \left( \sum_{z \in U_l(N)} X_z - A_{Nk} \right)/B_{Nk} \right] = f_{Nl}(t) \left( i + o(1) \right), \quad l = 0, 1, \ldots, k - 1,
\]

\[
E \exp \left[ it \left( \sum_{z \in U(N)} X_z - A_{Nk} \right)/B_{Nk} \right] = f_{Nk}(t) \left( 1 + o(1) \right),
\]
where \( U(N) = \bigcup_{i=0}^{k-1} (U_i(N) \cup D_{(P+\mu)P+P}) \). Furthermore, the latter characteristic function differs from \( E \exp \left\{ i \left( \sum_{i=0}^{k-1} S_i(N) - A_{Nk} \right) / B_{Nk} \right\} \) by at most \( o(1) \) as \( N \to \infty \).

Hence, by the \( m \)-dependence, \( |f_{Nk}(t)| - |f_{N}(tB_N/B_{Nk})| \to 0 \) as \( N \to \infty \) which immediately leads to (3.5). Thus Theorem 2 is completely proved.

**Remark.** It should be noted that in case \( d = 1 \) the second assertion of Theorem 2 remains in force without the condition (3.2) (see [13], p. 316). For proving this assertion for \( d \geq 2 \) condition (3.2) is essentially used to ensure that the limit behaviour of \( F_N(x) \) does not depend on the shape of \( V \). However, it might be possible to show this merely by the condition \( |\varepsilon^{(m)}V/\|V\|_N| \to 0 \). In what follows we are interested in imposing conditions on \( Ee^{itX_0} - 1 \) and \( \varepsilon^{(m)}(t, 1, a) \) which ensure the convergence \( A_N(z, \beta, c) \to 0 \). Because of the estimate \( |\varepsilon^{(m)}(1, 1, a)| \leq C(d, m) |Ee^{it(X_0-a)} - 1| \) it is reasonable to suppose that the distribution of \( X_0 \) belongs the domain of attraction of the stable distribution function \( G_{a\beta}(x, 1) \).

The problems arising from the choice of the centering sequence \( A_N \) in the case \( z = 1, 0 < |\beta| \leq 1 \), requires a separate treatment of this case.

**Theorem 3A.** Let \( \{X_z, z \in V \subset \mathbb{Z}^d\} \) be a stationary \( m \)-dependent RF (\( m \) fixed) satisfying the condition \( |\varepsilon^{(m)}V/\|V\|_N| \to 0 \) and suppose that, as \( t \to 0 \),
\[
E e^{it(x_0-a)} - 1 = \log g_{\beta}(t, 1) h_0(t) (1 + o(1))
\]
and
\[
\varepsilon^{(m)}(t, 1, a) = -\log g_{\beta}(t, 1) h^*_1(t) (1 + o(1))
\]
for \( 0 < a \leq 2, \alpha + 1, |\beta| \leq 1 \) and \( \alpha = 1, \beta = 0 \), respectively, where \( a \) stands for some centering constant and \( h_0(t) > 0 \) as well as \( h_1(t) \), \( h_0(t) = \text{const} \), are continuous slowly varying functions (i.e. \( h_i(t)/h_i(t) \to 1 \) for every \( c > 0, i = 1, 2 \)) which fulfill the relation
\[
\limsup_{t \to 0} h^*_i(t)/h_i(t) = b < 1 .
\]

Then, it holds
\[
\lambda_N(a, \beta, 1) \to 0 ,
\]
where \( \lambda_N = Na \) and \( B_N^{-1} = \inf \{t > 0: |t|^\alpha (h_0(t) - h_1(t)) = 1/N\} \).

**Proof.** In order to apply (3.3) with \( A_N = a \) one has to verify condition (3.2). From (3.7) we can conclude (see also [11]) that the function \( h_1(t) - h_0(t) \) is positive, continuous, and slowly varying (as \( t \to 0 \)), i.e. \( h_0(t)/h_1(t) \to 1 \) for every \( c > 0 \). The choice of the normalizing sequence \( B_N \) implies that
\[
N |E \left( e^{it(x_0-a)/B_N} - 1 \right)| \leq C_1(a, \beta) |t|^\alpha \frac{h_0(t/B_N)}{h_0(1/B_N)} \frac{h_0(1/B_N)}{h(1/B_N)} < C_2(a, \beta) |t|^\alpha/(1 - b)
\]
for every finite $t$ and $N$ large enough and so condition (3.2) is fulfilled. Therefore by our assumptions we recognize that (3.3) is only valid for $F(x) = G_{a,\beta}(x, 1)$. Even this is equivalent to (3.8).

As an immediate consequence of (3.3) we obtain

**Theorem 3B.** Let $\{X_z, z \in V \subset Z^d\}$ be a stationary $m$-dependent RF ($m$ fixed) satisfying the condition $|e^{(m)} V|/|V| \xrightarrow[N \to \infty]{} 0$. Suppose that there exist sequences $a_N, B_N \to \infty$ and real constants $c_0 \geq 0, c_1$ with $c_1 < c_0$ such that, for $|\beta| = 1$,

$$N \left( E e^{it x_0 - a_N/B_N - 1} - \log g_{1, \beta}(t, c_0) \right) \xrightarrow[N \to \infty]{} 0$$

and

$$N e^{(m)}(t/B_N, 1, a_N) \xrightarrow[N \to \infty]{} c_1 \log g_{1, \beta}(t, 1)$$

in every finite $t$-interval. Then, with $A_N = N a_N$,

$$A_N (1, \beta, c_0 - c_1) \xrightarrow[N \to \infty]{} 0$$

4. Rates of convergence in stable limit theorems for $m$-dependent RF's

In [4] and [22] estimates of the difference between a given stable distribution and the distribution of a suitably standardized sum of i.i.d. RV's are obtained by using the concept of pseudomoments. Roughly speaking it consists in supposing the smallness of the distance $P(X_0 < x) - G_{a,\beta}(x, c)$ in some sense. In [11] this concept was extended to the case of 1-dependent RV's ($d = 1$) and the case of Markov-dependent RV's was treated in [27]. In view of the obvious fact that the finiteness of a pseudomoment of certain order effects eventually after centering that the distance $E e^{it x_0 - g_{a,\beta}(t, c)}$ decreases with certain power of $t$ as $t \to 0$. Therefore it is reasonable to formulate the conditions needed in the below theorems in terms of characteristic functions and to omit the relatively expensive adaptation of the concept of pseudomoments to $m$-dependent RF's (see also [11]). For the sake of simplicity we do not consider the case $\alpha = 1, 0 \leq |\beta| \leq 1$ and put in all other cases $A_N = 0$. In the first step we prove a fairly general estimate of the difference $f_V(t/B_N) - g_{a,\beta}(t, 1)$. For brevity put

$$R^{(q,m)}(t) = \max_{y_1, \ldots, y_q \in V_0} \left| E Y_0(t) \prod_{j=1}^q Y_{y_j}(t) \right|; \quad q = 1, 2, \ldots$$

and $\varepsilon > 0$ stands for a sufficiently small real number.

**Lemma 5.** Let $\{X_z, z \in V \subset Z^d\}$ be a stationary $m$-dependent RF and let $X_0$ belong to the normal domain of attraction of the distribution function $G_{a,\beta}(x, c_0)$, with $0 < \alpha \leq 2, \alpha \neq 1, |\beta| \leq 1$ and $\alpha = 1, \beta = 0$, respectively, such that, for $|t| \leq \varepsilon$,

$$|E e^{it x_0} - 1 - \log g_{a,\beta}(t, c_0)| \leq K_0 |t|^r$$

where $c_0, K_0, r$ are positive real numbers with $\alpha < r < 1 + \alpha$. Further, let there exist real numbers $c_1, c_2(z \in e^{(m)} V), K_q \geq 0 (q = 1, \ldots, s)$ depending on $m$, where $s = \frac{2r}{\alpha}$, \ldots
such that, for $|t| \equiv \varepsilon$,

$$\begin{align*}
(4.3) & \quad \left| \int_0^1 T^{(1,m)}(t, u) \, du - c_1 \log g_{\alpha \beta}(t, 1) \right| \equiv K_1 |t|^r \\
(4.4) & \quad \left| \int_0^1 T^{(1,m)}(z, u) \, du - c_2 \log g_{\alpha \beta}(t, 1) \right| \equiv K_1 |t|^r, \quad z \in \hat{\delta}^{(m)} V,
\end{align*}$$

and, for $q = 2, \ldots, s$ and $z \in \hat{\delta}^{(m)} V$,

$$\max \left\{ \left| \int_0^1 T^{(q,m)}(t, u) \, du \right|, \left| \int_0^1 T^{(q,m)}(z, u) \, du \right| \right\} \leq K_q |t|^r. \tag{4.5}$$

Then there exist positive constants $C_1, \ldots, C_8$ (depending on $d, s, r, \alpha$) such that for $|t| \equiv C_1 B_N q_N(m, p)$ and $1 \equiv s < p$ the estimate

$$\left| f_v \left( \frac{t}{B_N} \right) - g_{\alpha \beta}(t, 1) \right| \equiv K(m) N \left( \frac{|t|}{B_N} \right)^r \left[ C_2 e^{-c_2 |t|^s} + C_4 2^{-p} \left( 1 + R_N(t) \right) \right], \tag{4.6}$$

holds, where

$$q_N(m, p) \equiv C_8 \min \left\{ \left( \frac{K(m) N}{B_N^s} \right)^{-1/(t - \varepsilon)}, (2m + 1)^{-2d/a} \right\},$$

$$R_N(t) = \min \left\{ \exp \left[ -C_5 N \left( \frac{|t|}{B_N} \right)^s + C_6 N \left[ 3^{a_1} \left( \frac{|t|}{B_N} \right)^{3a/2} + \sum_{q = 1}^{(m+1)d-1} \left( \frac{u_1}{q} \right) R^{(q,m)}(t) \right] \right], \quad K(m) = \max \left\{ \sum_{i=0}^{s} K_i, (2m+1)^{d(s+1)} \right\},$$

$$B_N = \left( c_0 |V| + c_1 \sqrt{\delta^{(m)} V} + \sum_{z \in \hat{\delta}^{(m)} V} c_2 \right)^{1/a} \quad \text{and} \quad \tau = \min \left\{ r, \alpha \left( \frac{s}{2} + 1 \right) \right\}.$$
Theorem 4. Let \( \{X, z \in V \subset Z^d\} \) be a stationary \( m \)-dependent RF (\( m \) fixed) satisfying the conditions \( |\tilde{\varphi}^{(m)}(V)/|V| | \to 0 \), and (4.2) - (4.5). Further, suppose that, for some \( \varrho > \varpi \),
\[
(4.7) \quad P(\varphi, m, t) \equiv C_1(m) |t|^q, \quad q = 1, \ldots, (m + 1)^d - 1.
\]

Then there exists a positive constant \( C_2(m) \) such that
\[
A_2(\alpha, \beta, 1) \equiv C_2(m) \left( N^{-\frac{r-s}{s}} + N^{-\frac{1}{s}} (\log N)^d/s \right)
\]
for \( \tau = \min \left\{ r, \frac{\alpha}{2} (s + 2) \right\} \) and \( \alpha < r \equiv 1 + \alpha \).

Proof. The proof is based on the above Lemma 4 and Esseen’s estimate of the distance between two distribution functions (see [14], Chapt. 5). Condition (4.7) implies that \( c_1 = c_2 = 0 (z \in \tilde{\varphi}^{(m)}(V)) \) and that \( R_X(t) \) does not exceed one for sufficiently small \( \varrho \). Since \( B^* = c_0 N \) the estimate (4.6) takes on the form
\[
|f(t/B^*) - g_{ab}(t, 1)| \equiv C_1(m) |t|^{\varrho} (e^{-C_2|t|^s} + 2^{-\varrho})
\]
for
\[
|t| \leq T_N = \frac{C_3(m)}{N^{1/s}} (m + 1)^{-2d/s}
\]
Therefore, putting \( p = \left\lfloor \frac{r \log N}{\alpha \log 2} \right\rfloor \) Esseen’s estimate yields
\[
A_2(\alpha, \beta, 1) \equiv C_4 \int_{|t| \leq T_N} |f(t/B^*) - g_{ab}(t, 1)| \frac{dt}{|t|} + C_5 \frac{T_N^r}{N^{r-s}}
\]
\[
\leq C_6(m) N^{-\frac{r-s}{s}} + C_7(m) N^{-\frac{r-s}{s}} + C_8(m) N^{-1} (\log N)^d/s
\]
This proves the assertion of Theorem 4.

Remark. For \( \varpi = 2 \) Theorem 4 makes only sense if all covariances \( \text{cov} (X_0, X_z, z \in V^{(1)}) \), are equal to zero. In case of a non-normal stable limiting distribution it is easily seen that (4.7) is satisfied if
\[
(4.8) \quad \text{ess sup} \ E(|X_0|^\gamma \mid X_z) < \infty \quad \text{for all} \quad z \in V^{(1)},
\]
where \( \gamma \) is some real number with \( 0 < \gamma < \alpha \), if \( \alpha \equiv 1 \), and \( \gamma = 1 \), if \( 1 < \alpha < 2 \) (see also [27]).

However, there exist numerous examples for which (4.8) does not hold. In the remaining part of this section we shall consider \( m \)-dependent RF’s which are generated by functionals of independent RV’s. Let \( f_x \mid R^{(m+1)d} \to R^1, z \in Z^d \), be a family of BOREL measurable functions and let \( \{Z_x, z \in Z^d\} \) be a field of independent RV’s.

Put \( F_z(x) = P(\xi_z \leq x) \) and \( U_z(n, k) = \{y = (y_1, \ldots, y_d) : y \in Z^d, z_j + km \equiv y_j \equiv z_j + (n-k) m, j = 1, \ldots, d\}, k = 0, 1, \ldots, n = 1, 2, \ldots, U_x = U_z(1, 0) \). The family of RV’s \( X_x = f_x(\xi_y, y \in (4.8)) \).
\( (U_i, z \in \mathbb{Z}^d) \), represents an \( m \)-dependent RF. Some further notations are needed:

\[
Z^d_n = \{ z = (z_1, \ldots, z_d) : z_j = (n m + 1) y_j, y_j \in \mathbb{Z}^1, j = 1, \ldots, d \},
\]

\[
W_{z, v}(n) = \bigcup_{y \in U_z(n, 0) \cap V} U_y \setminus U_z(n + 1, 1), \quad W_0(n) = W_{0, U_z(n, 0)}(n)
\]

(4.10)

\[
\Psi_z(t; x, y) \in W_{z, v}(n) = \int \exp \left\{ i t \sum_{y \in U_z(n, 0) \cap V} f_y(x, u \in U_y) \right\} \prod_{v \in U_{z, (n + 1, 1)}} F_v(dx_e).
\]

Obviously the family of "random" characteristic functions \( \{ \Psi_z(t; x, y) \in W_{z, v}(n) \} \), \( z \in Z^d_n \) forms a \( 1 \)-dependent RF and the characteristic function of \( S_v = \sum_{z \in V} X_z \) takes on the form

\[
f_v(t) = E \prod_{z \in Z^d_n} \Psi_z(t; x, y) \in W_{z, v}(n)).
\]

Hence by Hölder's inequality and the inequality \( x \leq e^{z-1} \) we find that

(4.9)

\[
|f_v(t)| \leq \exp \left\{ -\frac{1}{2d} \sum_{z \in Z^d_n} (1 - E |\Psi_z(t; x, y) \in W_{z, v}(n)|^2) \right\}.
\]

Next we derive a bound for the difference

(4.11)

\[
D_z(t) = E |\Psi_z(t; x, y) \in W_{z, v}(n)|^2 - |E \exp \left\{ i t \sum_{y \in U_z(n, 0) \cap V} X_y \right\}|^2.
\]

Breaking the sum \( \sum y \in U_z(n, 0) \cap V \) in (4.10) into three parts,

\[
\sum_{U_z(n, 2)} + (U_z(n, 1) \cap V) \sum_{U_z(n, 2)} + (U_z(0, 0) \cap V) \sum_{U_z(0, 1) \cap V} = X + Y + Z,
\]

after some elementary calculations we arrive at

\[
|D_z(t)| \leq 2 |E e^{itY} - 1| + 22 |E e^{itZ} - 1|.
\]

Again by a simple but lengthy estimation procedure one can show for any subset \( (X_z)_{z \in U}, U \subset \mathbb{Z}^d \), of an \( m \)-dependent RF that

\[
|E \exp \left\{ i t \sum_{z \in U} X_z \right\} - 1| \leq C_1 (m + 1)^{2d} \sum_{z \in U} |E Y_z| + C_2 (m + 1)^d |U| \sum_{z \in U} |E Y_z|^2,
\]

where \( C_1, C_2 \) are independent of \( m \).

Therefore, since \( |U_z(n, 0)| |U_z(n, 2)| = (n m + 1)^d - [(n - 4) m + 1]^d \leq C m^d n^d - 1 \), we get

(4.12)

\[
|E \exp \left\{ i t \sum_{y \in U_z(n, 1) \cap V} X_y \right\} - 1 - \sum_{y \in U_z(n, 1) \cap V} (E Y_y + e^{itY_y}(t, 1, 0))| \leq C m^d - 5(m + 1)^d \max_{z \in V} |E Y_z| + C_2 [n (m + 1)]^{2d} \max_{z \in V} |E Y_z|^{3/2}.
\]

In analogy to (2.11) we continue with the following estimate

(4.13)

\[
\]
For the sake of simplicity we additionally assume that the RV's \( \xi_z, z \in \mathbb{Z}^d \), are i.i.d. and \( f_z = f, z \in \mathbb{Z}^d \). Therefore the RF \( \{f(\xi_y, y \in U_z), z \in \mathbb{Z}^d\} \) is (strictly) stationary.

**Lemma 6.** Let the stationary \( m \)-dependent RF \( \{X_z = f(\xi_y, y \in U_z), z \in V \subset \mathbb{Z}^d\} \) (\( m \) fixed) satisfy the conditions (4.2) and (4.3)' with \( s = 1 \) and let \( B_N \) be asymptotically equivalent to \( (c_0 + c_1) N^{1/2} \) (i.e. \( \lim_{N \to \infty} B_N / N^{1/2} = c_0 + c_1 \)). Further suppose that 
\[
|\varphi^{(n_k)} V|/|V| \xrightarrow{N \to \infty} 0 \quad \text{for } n_k = k 3^{(2m+1)d}, \quad \text{where } k = 1, 2, \ldots \ (\text{not depending on } N \text{ and } m).
\]
Then there exist positive constants \( C_1, C_2(m) \) such that, for \( |t| \equiv C_1 B_N / (n_k m) \), and sufficiently large \( N \) and \( k \),
\[
(4.13) \quad |f_V(t/B_N)| \leq e^{-c_2(m)|t|^a}.
\]

**Proof.** From (4.12) and (4.2), (4.3)' we find that, for \( |t| \equiv C_1 B_N / (n_k m) \),
\[
E \exp \left\{ \frac{it}{B_N} \sum_{y \in U_0(n_k^2)} X_y - 1 \right\} - |U_0(n_k, 1)| (c_0 + c_1) \log g_0 \left( \frac{t}{B_N}, 1 \right)
\]
\[
\leq C_2(n_k m)^{3d} \left( \frac{|t|}{B_N} \right)^a + C_3 n_k^{d-1} (2m+1)^d \left( \frac{|t|}{B_N} \right)^a \leq C_4 \left( \frac{n_k m}{k} \right)^d \left( \frac{|t|}{B_N} \right)^a.
\]
This estimate together with (4.11) leads to
\[
E \left| \varphi_0 \left( \frac{t}{B_N} ; \xi_y, y \in W_0(n_k) \right)^2 - 1 + 2 |U_0(n_k, 1)| (c_0 + c_1) \right( \frac{|t|}{B_N} \right)^a
\]
\[
\leq C_5 \left( \frac{n_k m}{k} \right)^d \left( \frac{|t|}{B_N} \right)^a.
\]
Consequently, for a large enough \( k_0 \), all \( k \geq k_0 \) and \( |t| \equiv C_1 B_N / (n_k m) \),
\[
(4.14) \quad E \left| \varphi_0 \left( \frac{t}{B_N} ; \xi_y, y \in W_0(n_k) \right)^2 - 1 \right. \equiv -C_0(k_0) \left. \left( \frac{|t|}{B_N} \right)^a \right( n_k m \right)^d
\]
where \( C_0(k_0) > 0 \).

In view of the stationarity, (4.9) and (4.14) we have
\[
|f_V \left( \frac{t}{B_N} \right) | \equiv \exp \left\{ -2^{-d}C_1(k) |Z_{n_k}^d \cap (V \setminus \varphi^{(n_k)} V)| (n_k m)^d \left( \frac{|t|}{B_N} \right)^a \right\}, \quad k \geq k_0.
\]
Since, for arbitrary \( k \), \( |\varphi^{(n_k)} V| / |V| \xrightarrow{N \to \infty} 0 \), we obtain
\[
|Z_{n_k}^d \cap V| / |V| \xrightarrow{N \to \infty} |U_0(n_k, 1)|^{-1}
\]
and so (4.13) is completely proved.

**Theorem 5.** Let the stationary \( m \)-dependent RF \( \{X_z = f(\xi_y, y \in U_z), z \in V \subset \mathbb{Z}^d\} \) (\( m \) fixed) satisfy the conditions (4.2) - (4.5) and \( |\varphi^{(n)} V| / |V| \xrightarrow{N \to \infty} 0 \) for every fixed integer \( n \equiv 1 \) (not depending on \( N \)). Then, it holds
\[
(4.15) \quad A_N(\alpha, \beta, 1) = O(N^{-(\tau - e)/a}) \quad \text{as } N \to \infty,
\]
where \( A_N = 0, B_N = (c_0 |V| + c_1 |V| \varphi^{(m)} V| + \sum_{z \in \varphi^{(m)} V} c_z)^{1/2} \) and \( \tau = \min \left\{ r, \alpha \left( \frac{g}{2} + 1 \right) \right\} \).
Proof. Put $T_1 = \varrho_1 B_N$ and $T_2 = \varrho_2 (\log N)^{1/2}$ with suitably chosen $\varrho_1$ and $\varrho_2$. Then by Esseen's estimate (compare with the proof of Th. 4) we can write

$$A_N(\alpha, \beta, 1) \leq C_3 \int_{|t| \leq T_2} \left| f_Y \left( \frac{t}{B_N} \right) - g_{\alpha \beta}(t, 1) \right| \frac{dt}{|t|} + \frac{C_4}{T_1} + C_5 \int_{T_2 \leq |t| \leq T_1} \left( |f_Y(t/B_N)| + |g_{\alpha \beta}(t, 1)| \right) \frac{dt}{|t|} = I_1 + I_2 + I_3.$$ 

From Lemma 6 it is seen that by choosing $\varrho_2$ large enough the integral $I_3$ decreases with the order $O(N^{-(\tau-s)/2})$. Of course, $I_2 = O(N^{-1/2})$. From Lemma 5 we obtain for certain positive constants $C_6, \ldots, C_9$ (depending on $m$) that, for $|t| \leq C_6 B_N^{-2d/2}, |f_Y(t/B_N)| - g_{\alpha \beta}(t, 1)| \leq C_7 N^{-\alpha} |t|^\gamma \left[ e^{-C_8 |t|^2} + 2^{-p} e^{C_9 |t|^2} \right]$. Putting $p = \lceil \log N \rceil^2$ and taking into account that $T_2 = \varrho_2 (\log N)^{1/2}$ we come to the result that $I_1 = O(N^{-(\tau-s)/2})$. This proves the validity of (4.15).

From Theorem 5 we can deduce the rate of convergence in the CLT for this special type of $m$-dependent random fields.

**Theorem 6.** Let the stationary $m$-dependent $RF \{X_z = f(\xi_y, y \in U_z), z \in V \subset Z^d \}$ (m fixed) satisfy the conditions $EX_0 = 0, E |X_0|^3 \leq C < \infty, D^3S_r/N, and |\varrho^{(m)} V|/|V| \to 0$ for every fixed integer $n \geq 1$. Then, it holds

$$\sup_x |P (S_V < x DS_V) - \Phi(x)| = O(N^{-1/2}) \quad \text{as} \quad N \to \infty.$$ 

The conditions imposed on $X_0$ ensure the relations (4.2) and (4.3)' (4.4)' with $s = 1, \alpha = 2, \tau = 3$ and $B_N = DS_V$. Therefore, relation (4.15) yields the assertion of Theorem 6.

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