On Moderate Deviations of Sums of $m$-Dependent Random Vectors

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1. Introduction

We consider a sequence $X_1, X_2, \ldots$ of $m$-dependent ($m$ fixed) random vectors (rv's) defined on a probability space $(\Omega, \mathcal{F}, P)$ and taken values in the $k$-dimensional Euclidean space $\mathbb{R}^k$, $k \geq 1$. We suppose that $E\|X_j\|^2 < \infty$ and without loss of generality that $EX_j = 0$, $j = 1, 2, \ldots$, where $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^k$. Let $\Gamma_X$ be the covariance matrix of a rv $X$, $S_n = X_1 + \cdots + X_n$ and let $\lambda_n$ denote the smallest eigenvalue of $\Gamma_s$. It is wellknown (see [3] for $k = 1$) that in the case of a weakly stationary sequence $((X_{ij}, \ldots, X_{ik}))_{i, j = 1, 2, \ldots}$ (i.e. $EX_{is} = 0$ and $EX_{is}X_{it} = EX_{is}X_{it+i-t}$, $s, t = 1, \ldots, k; i, j = 1, 2, \ldots$) of $m$-dependent rv's a symmetric positive semidefinite matrix $\Gamma$ exists such that

$$\frac{1}{n} \Gamma_s = \Gamma + 0 \left( \frac{1}{n} \right) I \quad \text{as } n \to \infty,$$

where $I = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. In this case we additionally assume that

$$\lim_{n \to \infty} \lambda_n/n > 0 \quad \text{(i.e. } \Gamma \text{ is positive definite)}.$$

In Theorem 1 we shall give conditions for weakly stationary sequences to hold

$$P(\|S_n\| \geq x \sqrt{n}) = P(\|N_{0,r}\| \geq x) \left( 1 + o(1) \right) \quad \text{as } n \to \infty$$

uniformly for all $x \in [1, c \sqrt{\log n}, c > 0$.

Here, $N_{0,r}$ denotes a $k$-dimensional Gaussian rv with mean 0 and covariance matrix $\Gamma$. Note that in the case of i.i.d. rv's (see [10]) these conditions are even necessary for (1.3). In particular for $k = 1$ there are some weaker results on moderate deviations for $m$-dependent (and $\Phi$-mixing) sequences (see [1], [5], [13]), whereas the investigation of $m$-dependent rv's seems to be new in this context. The proof of our results is based on the standard methods for moderate deviations of independent random variables and vectors (see [11], [13], [10]) and the methods developed in [6] and [8].

Before we start to formulate the results we mention some facts on the behaviour of the normal law (see [10]):

$$P(\|N_{0,r}\| \geq x) \asymp x^{-k-2e^{-x^2d(x)/2}} \quad \text{as } x \to \infty,$$

1) $a_n \asymp b_n$ means $0 < \lim a_n/b_n \leq \lim a_n/b_n < \infty$.  
$\gamma$
where \( d(.) \) is some function depending on \( ||.|| \) and \( \Gamma \). From its definition (see (4.1), Sect. 4) it is not difficult to find that \( d(x) \geq a := \inf_{||z||=1} z\Gamma^{-1}z^T \) \( (x > 0) \), \( d(x) \to a \) as \( x \to \infty \) and \( |d'(x)| \leq C_0 / x \).

Here and below, \( \Gamma^{-1} \) is the inverse of \( \Gamma \), \( z^T \) is the transpose of \( z \) and \( C_0, C_1 \) stand for positive constants (not depending on \( n, x \) etc.) which may differ from one expression to another. In particular, if either \( ||z|| = (z\Gamma^{-1}z^T/b)^{1/2} \) for some \( b > 0 \) or the set \( \{z : ||z||^2 = z\Gamma^{-1}z^T/b = 1\} \) has a positive \text{LEBESGUE} measure on \( \{z : z\Gamma^{-1}z^T = b\} \), then

\[
P(||N_{0,r}|| \geq x) \asymp x^{k-2}e^{-z^2/2} \quad \text{as} \quad x \to \infty,
\]

whereas for any other choice of \( ||.|| \) (not depending on \( x \)) the asymptotic is

\[
P(||N_{0,r}|| \geq x) = o(1) \quad \text{as} \quad x \to \infty.
\]

By \( ||.||_g \) we denote the usual \text{EUCLIDEAN} norm on \( \mathbb{R}^k \) and we recall the well-known fact that

\[
C_0 ||.||_g \leq ||.|| \leq C_1 ||.||_g, \quad 0 < C_0 \leq C_1 < \infty.
\]

Finally, let \( h = h(y) \) denote the solution of the equation

\[
h \exp \left( h^2/(2c^2) \right) = y, \quad y > 0.
\]

It is easily seen that

\[
h(y) \sim c \sqrt{2 \log y} \quad \text{and} \quad h'(y) \sim \frac{c^2}{yh(y)} \quad \text{as} \quad y \to \infty.
\]

## 2. Results

First let us consider the weakly stationary case.

**Theorem 1.** Let \( X_1, X_2, ... \) be a weakly stationary sequence of \( m \)-dependent rv's with \( \mathbb{E}X_1 = 0 \) and let \( \Gamma \) in (1.1) satisfy (1.2). Further, suppose that

\[
P(||X_j|| \geq y) \leq q(y) y^{-2-c_1d(h(y))}(\log y)^{(k+c_2d(h(y)))^2}
\]

for \( j = 1, 2, ... \) where \( q(.) \) is an arbitrary function (not depending on \( j \)) with \( q(y) \to 0 \) as \( y \to \infty \). Then the asymptotic relation (1.3) holds.

For not necessarily stationary sequences of \( m \)-dependent rv's we introduce the following conditions:

\[
\mathbb{E}X_j = 0, \quad \mathbb{E} ||X_j||^2 \leq C_0 < \infty \quad \text{for} \quad j = 1, 2, ...
\]

and

\[
\lambda_n \asymp n \quad \text{as} \quad n \to \infty.
\]

We put \( Z_n = S_{n/m} \lambda_n \) and \( \Gamma_n = \Gamma_{z_n} \).

**Theorem 2.** Let \( X_1, X_2, ... \) be a sequence of \( m \)-dependent rv's satisfying (2.2), (2.3) and

\[
P(||X_j|| \geq y) \leq q(y) y^{-2-c_1d(\log y)^{(k+c_2d)^2}} \quad \text{for} \quad j = 1, 2, ...
\]
where \( c > 0 \) and \( a = \sup_{|z|} |z|^2 \). Then there holds

\[
P(||Z_n|| \geq x) = P(||N_0, r_n|| \geq x) (1 + o(1)) \quad \text{as } n \to \infty
\]

uniformly for all \( x \in [1, c \sqrt{\log n}] \).

**Remark 1.** The assertions of Theorem 1 and 2 remain valid if one replaces the norm \( ||.|| \) in (2.1) and (2.4) by any other norm \( ||.||' \). In the situation of Theorem 2 this is an immediate consequence of (1.6). Applying (1.6) to the condition (2.1) one has to show that

\[
y^p d(h_{(y/C)}) - d(h_y) \text{ is bounded for } 0 < C_1 < \infty.
\]

By (1.7) and the properties of \( d(.) \) one gets

\[
\left| \frac{d}{dz} d'(h(z)) \right| = h'(z) \left| d'(h(z)) \right| \leq \frac{C_0}{z \log z}
\]

for \( z \) between \( y/C_1 \) and \( y \). Together with the mean value theorem one gets (2.6).

**Remark 2.** The following estimate enables us to extent Lemma 2 below to the not necessary stationary case (see Remark B, Sect. 3):

\[
P(||N_0, r_n|| \geq x) = P\left(\sup_{|z|} |z| \leq x \sup_{|z|} |z| \right)
\]

\[
\geq C_0 x^{p-2} \exp \left( -x^2 \sup_{|z|} |z| \right)
\]

where \( ||z|| := (z I^{-1} z^T)^{1/2} \).

Since \( ||N_0, r_n|| \to 0 \) almost sure, we get

\[
\sup_{|z|} |z| \leq \sup_{|z| = 1} |z| = \sup_{|z| = 1} |z| \to 0 \text{ almost sure. This leads to (2.7).}
\]

### 3. A general truncation lemma and its application

The consideration of truncated random variables is a standard step to prove the Central limit theorem and its refinements (see [4], [9]). It is interesting that some truncation results only depend on the tail behaviour of the summands, but not on the dependence structure of the sequence.

Let \( Y_1, Y_2, \ldots \) be an arbitrary sequence of \( \sigma \)-algebras on \( (\Omega, \mathcal{A}, P) \) with values in a topological vector space \( E \) equipped with its \( \sigma \)-algebra \( \mathcal{E} \) of BOREL sets.

For \( D \in \mathcal{E} \) put

\[
Y^D_j = \begin{cases} 
Y_j & \text{if } Y_j \in D \\
0 & \text{if } Y_j \in \overline{D} = E \setminus D
\end{cases}
\]

and

\[
S^D_n = Y^D_1 + \cdots + Y^D_n.
\]

**Lemma 1.** For all \( A, B, Y \in \mathcal{E} \) with \( A \subseteq B \) there holds

\[
|P(S^A_n \in Y) - P(S^B_n \in Y)| \leq \max_{D \subseteq \{A, B\}} \left( P\left( S^D_n \in Y, \bigcup_{j=1}^n \{Y_j \in A\} \right) \right)
\]
and
\[(3.2) \quad |P(S_n^A \in Y) - P(S_n^B \in Y) - \Sigma(Y) + \Sigma^B(Y)| \]
\[\leq \max_{D \in \{A, B\}} P(S_n^D \in Y, \bigcup_{i=1}^{n-1} \bigcup_{j=i+1}^n \{Y_i \in \bar{A}, Y_j \in \bar{A}\},\]
where
\[\Sigma^D(Y) = \sum_{j=1}^n P(S_n^D \in Y, Y_j \in A, \bigcap_{i=1}^{j-1} \{Y_i \in A\}, D \in \{A, B\}.\]

Corollary 1. The quantities on the left-hand side of (3.1) and (3.2) can be estimated uniformly for all \(Y \in \mathcal{C}\) by \(\sum_{j=1}^n P(Y_j \in \bar{A})\) and \(\sum_{i=1}^{n-1} \sum_{j=i+1}^n P(Y_i \in \bar{A}, Y_j \in \bar{A})\), respectively.

In case the topology in \(E\) is generated by a norm \(\|\cdot\|\) we put for \(\delta \geq 0\): \(X_j^\delta = X_j^\delta\) and \(S_n^\delta = S_n^\delta\), where \(D = \{x : x \in E, \|x\| < \delta\}\). If \(\delta = \infty\) we omit the index \(\delta\).

Corollary 2. For all real numbers \(\alpha, \beta, \gamma \geq 0\) with \(0 \leq \alpha \leq \beta \leq \infty\) we have
\[(3.3) \quad |P(\|S_n^A\| \geq \gamma) - P(\|S_n^B\| \geq \gamma)| \leq \max_{\delta \in \{\alpha, \beta\}} \sum_{j=1}^n P(\|S_n^\delta\| \geq \gamma, \|Y_j\| \geq \alpha).\]

Proof of Lemma 1. Because of \(\Omega = \bigcup_{j=1}^n \{Y_j \in \bar{A}\} \cup \bigcup_{j=1}^n \{Y_j \in A\}\) we get for \(D \in \{A, B\}:\)
\[P(S_n^D \in Y) = P(S_n^D \in Y, \bigcap_{j=1}^n \{Y_j \in A\}) + P(S_n^D \in Y, \bigcup_{j=1}^n \{Y_j \in \bar{A}\}).\]
From \(A \subseteq B\) it follows
\[(3.4) \quad P(S_n^A \in Y) - P(S_n^B \in Y) = P(S_n^A \in Y, \bigcup_{j=1}^n \{Y_j \in \bar{A}\}) - P(S_n^B \in Y, \bigcup_{j=1}^n \{Y_j \in \bar{A}\}).\]

From this and \(|a - b| \leq \max\{a, b\}, a, b \geq 0\) we obtain (3.1). Substituting the decomposition
\[\bigcup_{j=1}^n \{Y_j \in \bar{A}\} = \bigcup_{j=1}^n \bigcap_{i=1}^{j-1} \{Y_i \in A, Y_j \in \bar{A}\} \cup \bigcup_{i=1}^{n-1} \bigcup_{j=i+1}^n \{Y_i \in \bar{A}, Y_j \in \bar{A}\}\]
into (3.4) we derive the second assertion of Lemma 1.

Let us return to the situation in Sect. 1 and 2. In what follows we set \(\alpha = \sqrt{\Lambda_n/x}\) and \(Z_n^\alpha = S_n^\alpha/\sqrt{\Lambda_n}\), where \(\Lambda_n\) equals either \(n\) or \(\lambda_n\) according to the formulation of Theorem 1 or 2. The next lemma is formulated with respect to Theorem 1.

Lemma 2. Let \(X_1, X_2, \ldots\) be a weakly stationary sequence of \(m\)-dependent rv's satisfying (1.2) and (2.1). Then there holds
\[(3.5) \quad |P(\|Z_n^\alpha\| \geq x) - P(\|Z_n\| \geq x)| = o(1) P(\|N_{0,i}\| \geq x) \quad \text{as } n \to \infty\]
uniformly for all \(x \in \left[1, c \sqrt{\log \Lambda_n}\right]\).
Remark 3. Lemma 2 can be reformulated according to the situation in Theorem 2. If we drop the stationarity assumption and replace (1.2) by (2.2), (2.3); \( \Gamma \) by \( \Gamma_n \); \( \Lambda_n = n \) by \( \Lambda_n = \lambda_n \) and (2.1) by (2.4), then the formulation of Lemma 2 remains valid. This is seen by using relation (2.7) and examining the following proof of Lemma 2.

Proof of Lemma 2. Applying Corollary 2 for \( \alpha = x, \beta = \infty \) and \( y = x\sqrt{A_n} \) we obtain

\[
P(\|Z_n\| \geq x) - P(\|Z_n\| \geq x | n, \|X_j\| \geq n/x).
\]

We define \( T_j = X_{j-m} + \cdots + X_j + \cdots + X_{j+m}, j = 1, \ldots, n \), and put \( X_i = X_{i+1} = 0 \) for \( i = 0, 1, \ldots \). Then we find that

\[
(3.6) \qquad \left\{ \|S_n\| \geq x \sqrt{n}, \|X_j\| \geq \sqrt{n}/2 \right\} \qquad \cup \quad \left\{ \|S_n\| - \|T_j\| \geq \frac{x}{\sqrt{2}} \sqrt{n}, \|X_j\| \geq \sqrt{n}/x \right\}
\]

Here we have used the inequality \( \|x\| - \|y\| \leq \|x - y\| \) for \( x, y \in \mathbb{R}^k \) and the relation

\[
\left\{ \sum_{i=1}^{p} X_i \right\} \leq \left\{ \sum_{i=1}^{p} Y_i \right\} \leq \frac{a}{p} \quad \text{for} \quad a > 0, \quad p = 1, 2, \ldots \quad \text{Now we shall prove}
\]

\[
(3.7) \quad \sup_{1 \leq x \leq e^{\log n}} \frac{nP(\|X_j\| \geq \gamma \sqrt{n})}{P(\|X_j\| \geq x)} \quad \quad n \to \infty \quad 0
\]

for \( 0 < \gamma < 1, j = 1, 2, \ldots \)

Using (1.4) and (2.1) we see that it suffices to show the uniformly boundedness of

\[
F_n(x) = \frac{(\log n)^{k/2}}{G_1(x) G_2(x) G_3(x)} \quad \text{for} \quad 1 \leq x \leq c \sqrt{\log n},
\]

where \( G_1(x) \equiv (n / \log n)^{c^2 (h'(\sqrt{n}))^2 \sqrt{n}} \), \( G_2(x) = x^{c^2 \sqrt{2}} \sqrt{\log x} \) and \( G_3(x) = \exp \left(-x^2 d(x)/2\right) \). A straightforward differentiation leads to \( G_1'(x) = c^2 \log (n / \log n) f_n(x) G_1(x)/2, G_2'(x) = \left( k + c^2 h'(\sqrt{n}) \right) / x + c^2 f_n(x) \log x \right) G_2(x) \), where \( f_n(x) = \frac{d(\h(x \sqrt{n}))}{h(\h(x \sqrt{n}))} h'(\h(x \sqrt{n})) \Gamma_n \). The derivative of \( G_3(x) \) follows from (4.1) in Sect. 4: \( G_3'(x) = -x^2 (x \sqrt{n} \log x) G_3(x) \), \( 0 < C_0 \leq q(x) \leq C_1 < \infty \). The properties of \( d(.) \) and (1.7) show that

\[
(3.8) \quad f_n(x) = C_0 (x \log n),
\]

and this implies for sufficiently large \( n \) and \( x \geq x_0 \) that

\[
\frac{G_1'(x)}{G_1(x)} + \frac{G_2'(x)}{G_2(x)} + \frac{G_3'(x)}{G_3(x)} < 0.
\]

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Then $F_n(x)$ is non-decreasing and

$$
\sup_{-\sqrt{\log n}} \sup_{x_1 \leq x \leq \sqrt{\log n}} F_n(x) = \exp \left\{ \frac{c^2}{2} \log n \left[ d \left( \frac{\sqrt{\log n}}{\log n} \right) - d \left( \frac{h(c \sqrt{\log n})}{\sqrt{\log n}} \right) \right] \right\}.
$$

Since $c \sqrt{\log n} = h(c \sqrt{\log n})$ we get from the mean value theorem and (3.8) that the exponent on the right hand side is bounded. Thus (3.7) holds.

Condition (2.1) implies $E \|X_j\|^q \leq C_0 < \infty$ for $0 \leq q < c^2 a$. Making use of a Berry-Esseen estimate for sums of $m$-dependent RV's (see [14]) one can find a real number $\varepsilon > 0$ such that $\sup \left\{ P\left( (S_n - T_j) / \sqrt{n} \in K \right) - P\left( N_0, r_0 \in K \right) \right\} \leq C_0 n^\varepsilon$, where $I_{ni} = \frac{1}{n} I_{s_n - r_i}$ and the supremum is taken over all convex Borel sets $K$.

Thus by (1.2) in the stationary case and by (2.2)/(2.4) in the instationary case we get

$$
P\left( \|S_n - T_j\| \geq \frac{x}{2} \sqrt{n} \right) \leq C_0 n^{-\varepsilon} + P\left( \|N_0, r_0\| \geq \frac{x}{2} \right)
$$

$$
\leq C_0 n^{-\varepsilon} + \exp \left\{ -C_1 x^2 \right\}.
$$

Keeping in mind the $m$-dependence, (1.4), (2.1) and the last estimate, one can derive the asymptotic relation

$$
\sum_{j=1}^{n} P\left( \|S_n - T_j\| \geq x \sqrt{n}/2, \|X_j\| \geq \sqrt{n}/x \right) = o(1) P\left( \|N_0, r\| \geq x \right) \text{ as } n \to \infty.
$$

From Lemma 7 in Sect. 5 we deduce

$$
\sum_{j=1}^{n} P\left( \|S_n^a - T_j^a\| \geq x \sqrt{n}/2 \right) \leq C_0 e^{-c x^2}.
$$

Arguing as above we find that

$$
\sum_{j=1}^{n} P\left( \|S_n^a - T_j^a\| \geq x \sqrt{n}/2, \|X_j\| \geq \sqrt{n}/x \right) = o(1) P\left( \|N_0, r\| \geq x \right) \text{ as } n \to \infty.
$$

To complete the proof of Lemma 2 it remains only to combine the relations (3.6), (3.7), (3.9) and (3.10).

4. Asymptotic normality of the conjugated distribution law

We put

$$
\mathcal{B} = \{ w = (w_1, \ldots, w_k) : 0 \leq w_i < \pi, i = 1, \ldots, k - 2, 0 \leq w_{k-1} < 2\pi \}
$$

and the mapping $(w, r) \mapsto z[w, r] = (z_1, \ldots, z_k)$, $w \in \mathcal{B}$, $r \geq 0$, is defined by the usual spherical coordinates

$$
z_1 = r \cdot \cos w_1, \quad z_2 = r \cdot \sin w_1 \cos w_2, \ldots, z_k = r \sin w_1 \cdot \ldots \cdot \sin w_{k-1}.
$$

Further we set $R(w) = 1/\|z[w, 1]\|$ and define the mapping $T: [\mathcal{B} \times [0, \infty) \to R^k$ by

$$
T(w, r) = y(w, r) = R(w) z[w, r].
$$

Note that $\|y(w, r)\| = r$ and $1/C_1 \leq R(w) \leq 1/C_0$ where $C_0, C_1$ are the same constants as in (1.6). Furthermore, $R(w)$ is a continuous,
function and the Jacobian of $T$ equals $r^{k-1}(R(w))^k H(w)$, where $H(w) = 1$ for $k = 1, 2$ and $H(w) = (\sin w_1)_{k-3} \ldots \sin w_{k-3}$ for $k \geq 3$. With the quadratic form $g(w) = y(w, 1) \times \Gamma^{-1} y^T(w, 1), w \in \mathcal{B}$ we define the abovementioned function $d(.)$ by

$$
e^{-n\sigma^2/2} = \int_{\mathcal{B}} \frac{|H(w)| R^k(w)}{g(w)} e^{-n\sigma^2/2} dw = \int_{\mathcal{B}} \frac{|H(w)| R^k(w)}{g(w)} dw.$$

Now, we consider a sequence $X_1, X_2, \ldots$ of $m$-dependent r.v.'s satisfying (2.2) and (2.3). Putting

$$h = h(w, x) = y(w, x) \Gamma^{-1}, \quad w \in \mathcal{B}, \quad 1 \leq x \leq e \sqrt{\log A_n},$$

we define a family of distribution laws $W_n(h, .)$ and a corresponding family of r.v.'s $Z^*_n(h)$ by

$$W_n(h, A) = \mathbb{P}(Z^*_n(h) \in A) = \frac{1}{f_{n,a}(h)} \int_A e^{it^T \mathcal{P}(Z^*_n(h) \in dz)}, A \in \mathcal{B},$$

where $\mathcal{B}^k$ denotes the BOREL-$\sigma$-algebra of $R^k$, $f_{n,a}(u) = \mathbb{E}\exp(uz^T)$, $u \in C^k$, and $\alpha = \sqrt{\lambda_n}/x$. $W_n(h, .)$ is usually called the conjugated distribution of $Z^*_n$. We write

$$w_n(u) = \max_{1 \leq j \leq n} (E|e^{u X_j^T} - 1|^2)^{1/2}, \quad u \in C^2$$

and

$$M^{(q)}_n(v) = \max_{1 \leq j \leq n} E|X_j|_q, e^{[v^T X_j^T], \quad v \in R^k, \quad q = 1, 2, \ldots}$$

In order to investigate the characteristic function

$$E \exp\{it(Z^*_n(h))^T\} = f_{n,a}(it + h)/f_{n,a}(h), \quad t \in R^k,$$

we quote the following lemma from [8]:

Lemma 3. Let $X_1, X_2, \ldots$ be $m$-dependent r.v.'s with mean zero and $\max_{1 \leq j \leq n} |X_j|_E \leq H < \infty$, where $H$ may depend on $n$. Then for all $t, v \in R^k$ satisfying $w_n(it) \leq 1/(24m)$, $w_n(mv) \leq 1/24$ and $|v|_E \leq 1/(24m(\max_{1 \leq j \leq n} E|X_j|_E)^{1/2})$, the moment-generating function $f_{n}(it + v)$ does not disappear and the estimate

$$\left| \frac{\partial \rho_1}{\partial t_p} \log E e^{(i t + v)^T s_{X_j}^2} \right| \leq C_0(|\rho|) \cdot n \cdot m^{|\rho| - 1} \cdot e^{2|mH|v^T s_{X_j}^2} \cdot M^{(p)}_n(mv)$$

holds, where $\frac{\partial \rho_1}{\partial t_p} = \frac{\partial \rho_1}{\partial t_{p_1} \ldots \partial t_{p_k}}$ and $|\rho| = p_1 + \cdots + p_k$.

Corollary 3. For sufficiently large $n$ and

$$|t|_E \leq 2 \sqrt{\lambda_n}/(\max_{1 \leq j \leq n} E|X_j|^2)^{1/2}$$

there holds

$$\left| \frac{\partial \rho_1}{\partial t_p} \log f_{n,a}(it + v) \right| \leq C_1(|\rho|) \cdot n \cdot m^{|\rho| - 1} e^{C_2|mH|v^T s_{X_j}^2} \cdot M^{(p)}_n,$$

where $M^{(q)}_{n,a} = \max_{1 \leq j \leq n} E||X_j||^q$. 

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Proof of Corollary 3. Using Schwarz's inequality and \(|e^{ix} - 1| \leq |x|\) we get
\[
\omega_n(i) \leq |i| \left( \max_{1 \leq j \leq n} E |X_j|^2 \right)^{1/2}.
\]
Obviously, we have
\[
\frac{m}{\sqrt{n}} |h| \left| X_j^* \right| E \leq \frac{mx}{\sqrt{n}} \cdot C_0 \cdot \alpha \leq C_0 \cdot m.
\]
Therefore, using Lemma 3 we obtain (4.4). ■

The next lemma summarizes some auxiliary results.

**Lemma 4.** Let \(X_1, X_2, \ldots\) be \(m\)-dependent \(rv\)'s satisfying (2.2), (2.3) and \(\max_{1 \leq j \leq n} E \|X_j\|^{\alpha + q} \leq C_0 < \infty\) for some \(q > 0\). Then for \(h\) from (4.2) and \(n \to \infty\):

\[
\begin{align*}
EZ_n^* &= o(n^{-\nu}) 1, \\
\begin{array}{c}
M_{n,a}^{(3)} = o(n^{1/2 - \nu}) \\
M_{n,a}^{(4)} = o(n^{-\nu}) I = I_n^a(h),
\end{array} \\
F_n^* &= I_n + o(n^{-\nu}) I, \\
E (h) &= \exp \left( \frac{1}{2} y(w, x) I_n^{-1} y(w, x) \right) [1 + o(n^{-\nu})],
\end{align*}
\]

where \(1 = (1, \ldots, 1) \in \mathbb{R}^k\) and \(0 < \psi < q/2\).

**Proof of Lemma 4.** It is easy to see that

\[
P(||X_j|| \geq \alpha) \leq C_0 \alpha^{-q}, \quad E \|X_j^*\|^3 \leq C_0 \alpha^{1-q}
\]

and

\[
E \|X_j - X_j^*\|^p \leq C_0 \alpha^{p-q-2}, \quad 0 < p < q + 2.
\]

Therefore, (2.2) and (2.3) imply (4.5). Using the abbreviation \(X_j^* = (X_{j1}^*, \ldots, X_{jk}^*)\) the \(m\)-dependence yields

\[
\begin{align*}
I_n^* &= \frac{1}{\sqrt{n}} \left( \sum_{j=1}^{n-1} \text{cov} (X_{j1}^*, X_{j2}^*) + \sum_{j=1}^{n} \sum_{i=1}^{m} \text{cov} (X_{j1}^*, X_{j2}^*) + \text{cov} (X_{j1}^*, X_{j2}^*) \right)_{s,t=1,\ldots,k},
\end{align*}
\]

where we put \(X_j^* = 0\) for \(j > n\), \(s = 1, \ldots, k\).

For any two \(rv\)'s \(X = (X_s)_{s=1,\ldots,k}\) and \(Y = (Y_t)_{t=1,\ldots,k}\) with mean zero one can get

\[
|EX_s Y_t - \text{cov} (X_s, Y_t)| \leq 2C_0^2 (E \|X - X^*\|^2 + E \|Y - Y^*\|^2).
\]

As a consequence of (2.3) and (4.9) we have

\[
I_n^* = I_n + o(n^{-\nu}) I \quad \text{for} \quad 0 < \psi < q/2.
\]

In the next step we calculate the first and second order moments of \(Z_n^* (h)\). Because of

\[
\begin{align*}
EZ_n^*(h) &= \left( \frac{\partial}{\partial h_s} \log f_{n,a}(h) \right)_{s=1,\ldots,k}
\end{align*}
\]
after using Taylor's expansion formula we obtain
\[ EZ_n^a(h) = EZ_n^a + h\Gamma_{z_n} + R_1 \]
and
\[ \Gamma_{z_n} = \Gamma_{z_n}^a + R_2, \]
where
\[ R_1 = \frac{1}{2} \left( \sum_{s=1}^k h_s h_t \frac{\partial^3}{\partial h_s \partial h_t \partial h_r} \log f_{n,a}(h)|_{h=0,h} \right)_{r=1,...,k} \|	heta_r\| \leq 1 \]
and
\[ R_2 = \left( \sum_{r=1}^k h_r \frac{\partial^3}{\partial h_r \partial h_s \partial h_t} \log f_{n,a}(h)|_{h=0,h} \right)_{s,t=1,...,k} \|	heta_{st}\| \leq 1. \]
From (2.3), (4.4) and the above estimates we find that
\[ R_1 = 0(n \|h\|^{3/2} \lambda_n^{-3/2} M_{n,a}^{(3)}) I = o(n^{-\nu}) I \quad \text{as } n \to \infty \]
and
\[ R_2 = 0(n \|h\| \lambda_n^{-3/2} M_{n,a}^{(3)}) I = o(n^{-\nu}) I \quad \text{as } n \to \infty. \]
The last two estimates together with the above relations imply (4.6) and (4.7). In the same manner one can prove (4.8).

**Lemma 5.** Let \( X_1, X_2, \ldots \) be \( m \)-dependent rv's satisfying the assumptions of Lemma 4. Then there exist positive constants \( C_0 \) and \( C_1 \) (depending on \( m \)) such that
\[ \int_{\|\| \leq C_0 L_n} \left( |E \exp (it(\tilde{Z}_n^a(h) - EZ_n^a(h)))| - \exp (-|t\Gamma_{z_n}^a T/2|) \right) dt \leq C_1 L_n, \]
where \( n \) sufficiently large and \( L_n = n M_{n,a}^{(3)} \lambda_n^{-3/2} \).

Note that the proof of Lemma 5 is quite similar to the proof of a corresponding lemma for Markov-depending rv's (see [7], p. 191). That's why we only sketch the proof of Lemma 5.

**Proof of Lemma 5.** Using (4.3), Corollary 3, (1.6) and Taylor's expansion formula (see e.g. [2], [7]) we get
\[ \| \log E \exp (it(\tilde{Z}_n^a(h) - EZ_n^a(h))) \| + t|\Gamma_{z_n}^a|t T/2| \leq R_3, \]
where
\[ R_3 = \max |t|, \sum_{1 \leq s \leq k} \max |p| \left\| \frac{\partial |p|}{\partial \theta} \log f_{n,a}(\tilde{h} + h) \right\|_{\theta=0} \leq C_1(m) |t|_{L_n}^2 |t| \Lambda_n. \]
Taking into account the estimates \( \lambda_n \leq \sup E(zS_n^2) \lambda_n \leq E |S_n|_{L_n}^2, E |S_n|_{L_n}^2 \leq (m + 1) \times \sum_{|l|=1}^n E |X|_{L_n}^2 = (m + 1) \sum_{j=1}^n (E |X|_{L_n}^2 + E |X - X_j|_{L_n}^2) \leq C_0 m \cdot n \cdot M_{n,a}^{(2)} \) and \( (M_{n,a}^{(3)})^2 \leq M_{n,a}^{(3)} \) we find that (4.11) holds for all \( t \in R^k \) with \( \|t\| \leq C_0(m)/\Lambda_n \). Because of (4.7)
we have $t \Gamma_{2n(h)} \geq |t|^2 (1 - C_0 n^{-v})$. Applying (4.11) and the last estimate we obtain

$$
|\log E \exp \left( i \tilde{h}(Z_n^m(h) - E Z_n^m(h)) \right) + \frac{t \Gamma_{2n(h)} |t|^2}{2} \right| \leq - \frac{|t|^2}{4}
$$

for $||t|| \leq \min \left( C_0(m), \frac{C_1(m)}{L_n} \right)$ and large enough $n$. Finally, our assertion follows without any difficulties from

$$
|e^z - e^y| \leq |z - y| e^{\text{max}|z| + y}, \quad z \in C^1, \quad y \in R^1. \quad \blacksquare
$$

To formulate the next lemma we need some further notations. Let $\mu(\cdot)$ denote the LEBESGUE measure on $(R^k, \mathcal{B})$,

$$(B)_\eta = \{ y \in R^k : y = u + v, u \in B, ||v|| < \eta \}$$

and

$$(B)_{\eta} = R^k \setminus (R^k \setminus B)_\eta \quad \text{for} \quad B \in \mathcal{B} \quad \text{and} \quad \eta > 0.$$ 

**Lemma 6.** Let $X_1, X_2, \ldots$ be $m$-dependent rv's satisfying the assumptions of Lemma 4. Then there holds

$$
\left| P \left( Z_n^m(h) \in B \right) - P \left( N_0, r Z_n^m(h) \in B \right) \right| \leq C_0 L_n \mu ((B)_\eta) + C_1 \mu ((B)^{2k} \setminus (B)^{-2k}),
$$

where $B \in \mathcal{B}$ and $\epsilon = C_0 L_n$.

Starting from Lemma 5 and using a general estimate of the discrepancy between an arbitrary distribution law and the normal law on $R^k$ (see [12], [7] or [2], p. 95-97, Lemma 11.4 and Corollary 11.5) one can verify Lemma 6 in an analogous way as in [12], [7]. That's why we omit this proof.

**5. Proof of Theorem 1 and 2.** Let us begin with some notations:

$$
\omega_{j_1, \ldots, j_{k-1}} = \frac{1}{N} \left( \pi j_1, \ldots, \pi j_{k-2}, 2 \pi j_{k-1} \right), \quad j_1, \ldots, j_{k-1} \in \{0, 1, \ldots, N - 1\},
$$

(for brevity we put $\bar{\omega} = \omega_{j_1, \ldots, j_{k-1}}$ if $j_1, \ldots, j_{k-1}, N$ are fixed),

$$
\mathcal{B}(\bar{\omega}) = \left[ \frac{\pi j_1}{N}, \frac{\pi}{N} (j_1 + 1) \right] \times \ldots \times \left[ \frac{\pi j_{k-2}}{N}, \frac{\pi}{N} (j_{k-2} + 1) \right]
$$

and

$$
V_{n,a}(A) = P(Z^a_n \in A), \quad A \in \mathcal{B}.
$$

Clearly, we have

$$
V_{n,a}(T[\mathcal{B} \times [x, \infty]]) = \sum_{j_1, \ldots, j_{k-1} = 0}^{N-1} V_{n,a}(T[\mathcal{B}(\omega_{j_1, \ldots, j_{k-1}}) \times [x, \infty]])
$$

for every $N \geq 1$ and so this identity holds also as $N \to \infty$. Carrying out the coordinate transformation $T$ and using (4.8) we obtain

$$
V_{n,a}(T[\mathcal{B}(\bar{\omega}) \times [x, \infty]])
$$

$$
= \int_{\mathcal{B}(\bar{\omega})} e^{-h(\bar{\omega}, x)} W_n(h(\bar{\omega}, x), d(r, u)) d(\bar{\omega})
$$

$$
= e^{y(\bar{\omega}, x) t^\gamma} \int_{\mathcal{B}(\bar{\omega})} e^{-h(\bar{\omega}, x)} W_n(h(\bar{\omega}, x), d(r, u)) \left( 1 + o(n^{-v}) \right).
$$
Using the latter transformation and Lemmas 4 and 6 one can directly follow the proof of Lemma 2 in [10] and then we achieve at:

**Lemma 7.** Let $X_1, X_2, \ldots$ be $m$-dependent r.v.'s satisfying the assumptions of Lemma 4. Then there holds

$$V_n, a(T[\mathcal{B} \times [x, \infty)}) = (1 + o(1)) P(\|N_{0,R_n}\| \geq x) \quad \text{as } n \to \infty$$

uniformly for all $x \in [1, c \sqrt{\log \lambda_n}]$.

If we consider a weakly stationary sequence then we have to replace $\lambda_n$ by $n$ and $R_n$ by $\frac{1}{n} R_S$. Making these changes in Sect. 4 and 5 we obtain

$$P(\|S_n\| \geq x \sqrt{n}) = (1 + o(1)) P(\|N_0, R_{S_n}\| \geq x).$$

By (1.1) and (1.2) it is easy to show that the right-hand side of (5.1) equals $(1 + o(1)) \times P(\|N_0, r\| \geq x)$. Combining Lemma 2, Lemma 7, Remark 3 and (5.1) we obtain the assertions of Theorems 1 and 2.

**References**


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