Mathematics of Operations Research

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http://pubsonline.informs.org

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MATHEMATICS OF OPERATIONS RESEARCH
Vol. 42, No. 3, August 2017, pp. 745–761
ISSN 0364-765X (print), ISSN 1526-5471 (online)
http://pubsonline.informs.org/journal/moor/

Matroids Are Immune to Braess' Paradox

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Received: April 21, 2015
Revised: March 11, 2016
Accepted: August 8, 2016
Published Online in Articles in Advance: February 14, 2017

MSC2010 Subject Classification: Primary: 91A10; secondary: 91A43, 90B10, 68W01
OR/MS Subject Classification: Primary: games, noncooperative; secondary: mathematics, combinatorics

https://doi.org/10.1287/moor.2016.0825

Abstract. The famous Braess paradox describes the counterintuitive phenomenon in which, in certain settings, an increase of resources, such as a new road built within a congested network, may in fact lead to larger costs for the players in an equilibrium. In this paper, we consider general nonatomic congestion games and give a characterization of the combinatorial property of strategy spaces for which the Braess paradox does not occur. In short, *matroid bases* are precisely the required structure. We prove this characterization by two novel sensitivity results for convex separable optimization problems over polymatroid base polyhedra, which may be of independent interest.

Funding: The work of the first author is supported by JSPS Grant-in-Aid for Scientific Research (B) [Grant 25280004].

Keywords: nonatomic congestion games • Braess' paradox • matroids • polymatroids

1. Introduction

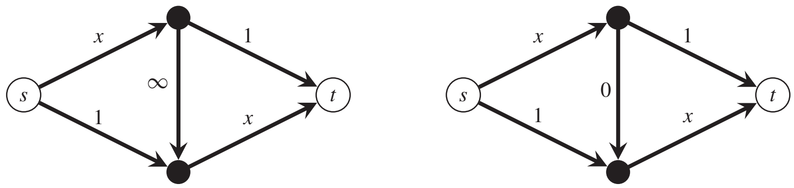
In a congestion game (as introduced by Rosenthal [41, 42]), there is a finite set of players that compete over a finite set of resources. A pure strategy of a player consists of a subset of resources, and the congestion cost of a resource depends only on the number of players choosing the same resource.

Nonatomic congestion games model the interaction of a large number of players with the property that the strategy choice of each player has only a negligible effect on the others. In these kinds of model, it is usually assumed that there is a continuum of players partitioned into *populations* and the strategy space available to a player of a population comprises a population-specific set of allowable subsets of resources. A *pure Nash equilibrium* of a nonatomic congestion game is a strategy distribution from which no player can unilaterally select a different subset of resources with strictly lower cost. Here, the cost of a subset is simply defined as the sum of the resource costs. Nonatomic congestion games have a wide range of applications; for example, they are used to model habitat selection in biology (see Milchtaich [32]), queueing systems (see Korilis et al. [30]), and packet routing in telecommunications (see Qiu et al. [39]). Perhaps the most famous example of a nonatomic congestion game appears in the traffic model of Wardrop [54], where the resources form a (directed) graph and a population corresponds to a continuum of players that want to travel from an origin to some destination in the graph. In this case, the set of allowable subsets corresponds to the set of origin–destination paths and the costs represent travel times. In a *Wardrop* equilibrium (see Wardrop [54]), each player selects a path of minimum cost. The existence of Wardrop equilibria and their characterization via pure Nash equilibria of an associated noncooperative game (assuming continuity of cost functions) has been established since the early 1950s; see Beckmann et al. [4]. In Beckmann et al. [4], the authors show that a strategy distribution is a Wardrop equilibrium if and only if it is a global minimum of an associated separable convex function known as the Beckmann potential. This characterization further implies that for continuous and nondecreasing cost functions, any Wardrop equilibrium has the same cost on every resource (see Correa and Stier-Moses [14] and Remark 2.1).

1.1. Braess' paradox

In this paper we will study a well-known phenomenon originally discovered in the context of the Wardrop routing model: Dietrich Braess, a German mathematician, published in 1968 a paper (Braess [6]; see also Braess et al. [7]) in which he showed that adding a new arc to a transportation network might actually degrade the performance of the resulting Wardrop equilibrium. Here, the performance is measured in terms of the total

Figure 1. Example of the Braess paradox.



travel time experienced by players in a Wardrop equilibrium. Let us briefly recall an example of Braess' paradox. As depicted in Figure 1, there is a single-source single-destination network, and we want to send one unit of flow from s to t. On the arcs, we indicate the travel cost per unit as a function of the congestion; in particular, 1 means that the travel cost per unit is one independent of the congestion, and x signifies that the travel cost per unit is equal to the congestion of the arc. In the left network, the unique Wardrop flow sends evenly one-half units along both the upper and lower paths. This flow is also optimal having total cost of 3/2. Suppose that a new fast road is built (latency function is reduced from ∞ to 0) connecting the two nodes in the middle, as shown in the right-hand-side figure. The new (unique) Wardrop equilibrium sends its flow entirely along the zig-zag path having a total cost of 2, and each player perceives a strictly larger path latency of 2. This example shows the paradoxical situation that a network infrastructure improvement may actually hurt the resulting travel times of the new Wardrop equilibrium.

Let us now consider another type of Braess' paradox that may arise via demand reductions. Note that demand reductions frequently occur in practice, e.g., if commuters switch to the public transport system in case a new railway, tram, or underground line has been built.

Consider the example in Figure 2. There are three populations, $N = \{1,2,3\}$, that want to travel from s_i to t_i, for $i = 1,2,3$. In the original instance, the demands are $d_1 = 1, d_2 = 2$, and $d_3 = M$. The cost function $c(x)$ is defined by $c(x) = 0$ for $0 \le x \le M$ and $c(x) = x - M$ for $M \le x$. The resulting unique Wardrop equilibrium x^* routes the flow of population 1 along the direct edge (s_1, t_1). Thus, the total cost of x^* can be calculated as $C(x^*) = 1 \cdot 2 + 2 \cdot 2 + M \cdot 0 = 6$. Suppose we decrease the demand of population 2 from 2 to $d_2 = 0$. In the new (unique) Wardrop equilibrium $\bar{x}$, the flow of population 1 will be sent entirely on the path (s_1, t_2, t_1) with a total cost of $C(\bar{x}) = M + 2$. It follows that for $M > 4$, the reduction of demand may actually hurt the total cost.

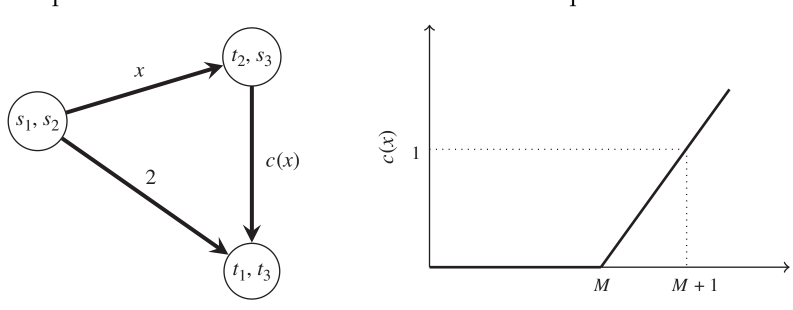
1.2. Our results and techniques

We study nonatomic congestion games and investigate the Braess paradox for arbitrary set systems and for both cost reductions and demand reductions as explained in Figures 1 and 2. Note that there are interesting combinatorial structures of the allowable subsets beyond paths in a graph: tours (as in the traveling salesman problem), spanning trees, or Steiner trees (that frequently occur in telecommunication networks).

We differentiate between a *weak* and a *strong* form of the Braess paradox. A weak Braess paradox occurs if for the new equilibrium (after cost and/or demand reductions) there exists a resource with strictly increased cost. For the strong Braess paradox, there must exist a player with strictly increased private cost. Note that the strong Braess paradox implies the weak, and immunity to the weak Braess paradox implies immunity to the strong Braess paradox (but neither statement holds vice versa in general).

1.2.1. The weak Braess paradox. Our first goal is to characterize the set of allowable subsets of players so that there will be no weak Braess paradox, no matter what kind of continuous and nondecreasing cost functions are associated with the resources. An informal description of our main result is as follows.

Figure 2. Example of Braess' paradox where a demand reduction hurts the equilibrium cost.



A family of set systems is immune to the weak Braess paradox if and only if for every set system of the family, the corresponding clutter (i.e., the set system containing only the inclusion-wise minimal sets) consists of bases of a population-specific *matroid* defined on the ground set of resources.

We note that matroids have a rich combinatorial structure and include, for instance, the class of games, where each player wants to allocate a spanning tree in a graph.

Technically, our first characterization rests on two new results on the sensitivity of optimal solutions minimizing a continuous, differentiable, nondecreasing, and convex separable function (i.e., the Beckmann potential) over a polymatroid base polytope. We show (see Lemma 3.1) that if cost functions are shifted downward, the new global minimum has the property that cost values evaluated at a new optimal solution only decrease. The second sensitivity result considers demand reductions that, as we will argue, can be interpreted in terms of a decomposition of a polymatroid. More precisely, for the second sensitivity result (see Lemma 3.2), we consider a specific polymatroid base polytope that can be decomposed as a Minkowski sum of a finite number of polymatroid base polytopes. We show that by removing one polymatroid base polytope, any new optimal solution of the Beckmann potential has also decreased cost values. The connection of these two results to the Braess paradox is drawn by observing that for games with matroid structure, the problem of computing a Wardrop equilibrium can be reduced to finding a global minimum of the Beckmann potential over a sum of population-specific polymatroid base polytopes. For this, we use the fact that the rank function of a matroid is a submodular function. The two sensitivity results thus imply that for matroid set systems there will be no weak Braess paradox no matter what kind of cost and/or demands reductions occur. We prove the "only if" direction via exploiting the edge-vector characterization of base polytopes due to Tomizawa (see Fujishige [21], Theorem 17.1).

1.2.2. The strong Braess paradox. Our second result gives a characterization of the occurrence of the strong Braess paradox. For this characterization we require that there is no a priori description on how the individual strategy spaces of populations interweave.[1] We say that a set system is *universally* immune to the strong Braess paradox if it is immune to the strong Braess paradox no matter how the strategy spaces of populations interweave. We then obtain the following.

A family of nonempty set systems containing at least two set systems is universally immune to the strong Braess paradox if and only if for every set system of the family, the corresponding clutter consists of bases of a population-specific *matroid* defined on the ground set of resources.

The "if" direction follows directly from our first characterization. For the "only if" direction, we proceed by contradiction. If for a game with at least two populations there exists a population with a nonmatroidal set system, then we derive appropriate cost functions on the resources, the demands, and an embedding of the strategy spaces into resources such that the resulting game admits the strong Braess paradox.

1.3. Related literature

The discovery of the Braess paradox has driven a considerable amount of literature in different fields of science ranging from transportation and traffic networks (see Catoni and Pallotino [8], Dafermos and Nagurney [16], Frank [20], Smith [50]), queueing networks (see Cohen and Kelly [11], Kameda [28], Korilis et al. [30]), electrical and mechanical networks (see Cohen and Horowitz [12]), and computer science (see Correa et al. [15], Fotakis et al. [19], Kameda et al. [29], Lin et al. [31], Roughgarden [43, 44, 45], Roughgarden and Tardos [47], Valiant and Roughgarden [53]) to economics (see Pöppe [38], Samuelson [48]). For an overview of further works, we refer to the website maintained by Braess.[2]

In light of this substantial body of literature, it seems surprising that to date little is known regarding general characterizations of the occurrence of the Braess paradox. Steinberg and Zangwill [51] and later Dafermos and Nagurney [16], Pas and Principio [37], and Hagstrom and Abrams [22] derived *instance-dependent* necessary and sufficient conditions for the Braess paradox to occur. Here, instance-dependent means that these conditions depend on the concrete demand matrix, the cost functions, and the network topology used. Hence, if for a given network topology the used cost functions or demand matrices are not known a priori, these works do not offer any insight on the occurrence of the Braess paradox. This situation occurs naturally whenever a network is built from scratch (as in telecommunications or mechanical networks) or extended (as in traffic networks) and the traffic matrix and realized cost functions are not known precisely. Even if the traffic matrix can be well estimated, the cost functions are subject to changes as street improvements and construction works are continuously ongoing changing the street characteristics. In such cases, it would be valuable to characterize networks that are not vulnerable to Braess' paradox for any instantiation of the demand matrix and the cost functions. Milchtaich [34] derived such a characterization by showing that for undirected single *o-d* networks, series-parallel graphs form the maximal graph class that is immune to the (strong) Braess paradox no matter how many commuters travel and what kind of (continuous and nondecreasing) cost function is used. Note that series-parallel networks are precisely the class of networks that do not contain the network in Figure 1 as a topological minor. He further proved that *any* undirected single *o-d* graph that is not series parallel can be equipped with carefully chosen costs and demands so that the resulting instance admits the strong Braess paradox. His result thus provides a characterization of undirected single *o-d* graph topologies that are immune to the strong Braess paradox. Very recently, Chen et al. [10] and Cenciarelli et al. [9] generalized the characterization of Milchtaich toward directed graphs and allowing for multiple commodities.[3]

Some remarks are in order, to explain how our work differs from that of Milchtaich [34]. As explained above, Milchtaich considered undirected single *o-d* networks and characterizes the maximal network topology that is immune to the strong Braess paradox. In particular, this implies that the resources form an undirected graph and the strategy spaces of players are symmetric because the strategies are the set of *o-d* paths. By contrast, we consider (general) nonatomic congestion games with *asymmetric* strategy spaces, where for a player the allowable set of subsets of resources can have *any* combinatorial structure. Interesting cases beyond paths in graphs include tours, trees, or Steiner trees all in a directed or undirected graph. Additionally, we consider the more general case of cost reductions, demand reductions, or both that might increase the equilibrium cost.

For our characterization of the strong Braess paradox, there is one important additional difference to the result of Milchtaich. In contrast to Milchtaich's characterization, we do not prescribe a priori how the sets of allowable subsets of players are actually embedded in the ground set of resources or, said differently, how the strategy spaces interweave.

It is fair to say that matroids play a special role in the wide area of (integral) congestion games. This connection was first discovered by Ackermann, Röglin, and Vöcking in their important papers (Ackermann et al. [1, 2]). In Ackermann et al. [2], they showed that both weighted and player-specific congestion games admit (pure Nash) equilibria in the case of matroid congestion games, i.e., if the strategy space of each player consists of the bases of a matroid on the set of resources. They also showed that the matroid property is *maximal* in the sense that whenever there are two players both having allowable sets of resources that are not matroidal, then there is a prescribed embedding of the sets into the ground set of resources and cost functions so that the resulting game does not have an equilibrium. It should be noted that our characterization of the weak Braess paradox is direct (relying on a polyhedral combinatorics point of view) and does not rely on the flexibility of embeddings. Also, the "only if" direction of our characterization of the strong Braess paradox exhibits a difference from that used in Ackermann et al. [2]. In Ackermann et al. [2], for obtaining counterexamples, it is required that the strategy space of *all* players is nonmatroidal, whereas we only require that at least one player (or population, in our setting) has a nonmatroid set system, thus allowing for a characterization.

Harks and Peis [24] considered a variant of congestion games—namely, *resource buying games*—in which players jointly design a resource infrastructure and share the congestion-dependent costs of the resources arbitrarily. They showed that for marginally nonincreasing cost functions, such resource buying games always admit an equilibrium as long as the players' strategy spaces form the base set of a matroid, whereas for nonmatroid set systems, there is a two-player game with marginally nonincreasing costs that does not admit an equilibrium. Finally, Harks et al. [25] showed that integral-splittable congestion games with semiconvex cost functions always admit an equilibrium whenever each player's strategy space forms an integral polymatroid.

2. Nonatomic congestion games

A tuple $\mathcal{M} = (N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, (d_i)_{i\in N})$ is called a *nonatomic congestion model* if $N = \{1, \ldots, n\}$ is a nonempty, finite set of populations and $E = \{e_1, \ldots, e_m\}$ is a nonempty, finite set of *resources*. Players are infinitesimally small, and each population i consists of a continuum of players represented by the interval $[0, d_i]$ for some $d_i > 0$. For each population $i \in N$, the set $\mathcal{S}_i$ is a nonempty, finite set of subsets $S \subseteq E$ available to each player of population i. Each player of population $i \in N$ selects a strategy $S \in \mathcal{S}_i$, which leads to a strategy distribution $(x^i_S)_{S\in\mathcal{S}_i}$ satisfying $\sum_{S\in\mathcal{S}_i} x^i_S = d_i$ and $x^i_S \geq 0$ $(\forall\, S \in \mathcal{S}_i)$. We denote by $\mathcal{S}$ the direct sum $\{(i, S) \mid i \in N, S \in \mathcal{S}_i\}$ of $\mathcal{S}_i$ $(i \in N)$, which represents the collection of all strategies of all players. After each player of every population has chosen a strategy, we arrive at the overall strategy distribution $x = (x^i_S)_{(i,S)\in\mathcal{S}}$. The induced load of x on e is denoted by $x_e = \sum_{(i,S)\in\mathcal{S}: e\in S} x^i_S$ (assuming every strategy S of $(i, S) \in \mathcal{S}$ contains each resource at most once and that for every $i \in N$, the rate of consumption of every $S \in \mathcal{S}_i$ on resource $e \in S$ is equal to one). Thus, we can compactly represent the set of feasible strategy distributions by the following polytope

$$P(\mathcal{M}) := \left\{ x \in \mathbb{R}^{\mathcal{S}}_{\geq 0} \;\middle|\; \sum_{S \in \mathcal{S}_i} x^i_S = d_i, \text{ for all } i \in N \right\},$$

where for $x \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$, the value of x on $(i, S) \in \mathcal{S}$ is denoted by x^i_S. We denote by $x_{i,e} = \sum_{S \in \mathcal{S}_i : e \in S} x^i_S$ the load of population i on resource e. Hence, $x_e = \sum_{i \in N} x_{i,e}$. We impose the following assumption on cost functions.

Assumption 2.1. *For every resource $e \in E$, we consider a cost function $c_e: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is nonnegative, continuous, and nondecreasing.*

If in strategy distribution x a player of population i selects $S \in \mathcal{S}_i$, she perceives the disutility, or private cost, of

$$\pi_{i,S}(x) = \sum_{e \in S} c_e(x_e).$$

Since we are often interested in the load on the resources, we define for every polytope $P(\mathcal{M}) \subseteq \mathbb{R}^{\mathcal{S}}_{\geq 0}$ of feasible strategy distributions a corresponding polytope $\tilde{P}(\mathcal{M}) \subseteq \mathbb{R}^{E}_{\geq 0}$ that captures all possible load vectors on the resources obtained by playing a feasible strategy distribution; i.e.,

$$\tilde{P}(\mathcal{M}) := \left\{ \sum_{i \in N} \sum_{S \in \mathcal{S}_i} x^i_S \cdot \chi_S \;\middle|\; x \in P(\mathcal{M}) \right\},$$

where $\chi_S \in \{0,1\}^E$ for $S \subseteq E$ is the characteristic vector of S; hence, $\chi_S(e) = 1$ if $e \in S$ and $\chi_S(e) = 0$ if $e \in E \backslash S$. Note that, defining for each population $i \in N$ a polytope

$$\tilde{P}_i(\mathcal{M}) := \left\{ \sum_{S \in \mathcal{S}_i} x^i_S \cdot \chi_S \;\middle|\; \sum_{S \in \mathcal{S}_i} x^i_S = d_i,\, x^i_S \geq 0 \;(\forall\, S \in \mathcal{S}_i) \right\},$$

we have $\tilde{P}(\mathcal{M}) = \sum_{i \in N} \tilde{P}_i(\mathcal{M})$, the Minkowski sum of polytopes $\tilde{P}_i(\mathcal{M})$ $(i \in N)$.

2.1. Nonatomic matroid congestion games

A matroid is a tuple $M = (E, \mathcal{I})$, where E is a finite set, called the *ground set*, and $\mathcal{I} \subseteq 2^E$ is a nonempty family of subsets of E, called *independent sets*, such that (i) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$; and (ii) if $X, Y \in \mathcal{I}$ with $|X| > |Y|$, then $\exists e \in X \backslash Y$ such that $Y \cup \{e\} \in \mathcal{I}$. The inclusion-wise maximal independent sets of $\mathcal{I}$ are called *bases* of matroid M and are usually denoted by $\mathcal{B}$ or $\mathcal{B}(M)$. See Oxley [36], Schrijver [49], and Welsh [55] for more information on matroids.

A nonatomic congestion model $\mathcal{M}$ is called a *matroid congestion model* if for every $i \in N$ there is a matroid $M_i = (E, \mathcal{I}_i)$ such that $\mathcal{S}_i$ equals the set of bases of M_i. In case of nonatomic matroid congestion games, we will write $\mathcal{B}_i$ instead of $\mathcal{S}_i$, B_i instead of S_i, and B instead of S. We give three examples in the areas of queueing, facility location, and minimum spanning tree (MST) games.

Example 2.1 (Queueing Games (see Korilis et al. [30])). There is a set $Q = \{q_1, \ldots, q_m\}$ of $M/M/1$ queues served in a first-come-first-served fashion and a set of $N = \{1, \ldots, n\}$ independent Poisson arrivals of packets, where the arrival rates are denoted by $d_1, \ldots, d_n$. Every queue q has a single server with exponentially distributed service time with mean $1/\mu_q$, $\mu_q > 0$. Each packet is routed to a single queue q out of a set of allowable queues depending on the type. Given a distribution of packets $x \in \mathbb{R}^m_{\geq 0}$, the mean delay of queue q can be computed as $c_q(x_q) = 1/(\mu_q - x_q)$. In this case, the sets $\mathcal{S}_i$, $i \in N$, are uniform rank-1 matroids.

Example 2.2 (Facility Location Games with Supply Functions). Matroid congestion games can also be used as a modeling tool for resource buying games with supply functions. More precisely, whereas so far we interpreted the cost of a resource mostly in terms of a "disutility" such as congestion, one can as well interpret costs as actual renting or buying costs of a resource that depend on the demand; i.e., the more users use a resource, the higher its price. We provide an example phrased in the context of facility location, though the described approach also applies to further settings. Consider a finite set $E = \{e_1, \ldots, e_m\}$ of resources in different locations and a set of populations $N = \{1, \ldots, n\}$. The resources could, for example, correspond to data centers, and the players have to decide which data centers to use to serve their clients. A population groups together players who want to serve clients within the same areas. Each player in population $i \in N$—where, as usual, we assume that there is a total "mass" of d_i players in population i—desires to use some number $k_i \in \mathbb{Z}_{>0}$ of different data centers to cover k_i different areas. Each area j can be served by any data center within a given set $S_j \subseteq E$. The sets S_j may overlap, even for the same player i. However, for reliability reasons, a player cannot use the same data center more than once. Furthermore, to model an offer/demand interplay, the cost c_e for using a particular data center $e \in E$ depends on the total load of players who use data center e. The higher the load on a data center, the larger the cost to use it. In this setting, the strategy space of each population $i \in N$ corresponds to a *transversal matroid* described by the sets S_j for all areas j that population i wants to serve.

Example 2.3 (MST Games). We are given an undirected graph $G = (V, E)$ with nonnegative, continuous, and nondecreasing edge cost functions $c_e(l), e \in E$. In an MST game, every population i is associated with a demand interval $[0, d_i]$ and a subgraph G_i of G. A strategy distribution for population i is to route its demand along the spanning trees of G_i. Formally, the edges correspond to the resources, and the sets $\mathcal{S}_i$, $i \in N$, are the spanning trees of G_i. In this case, M_i is called the *graphic matroid*.

2.2. Wardrop equilibria

A Wardrop equilibrium x for a nonatomic congestion game is a strategy distribution x such that every player of every population uses a strategy with minimum cost. Formally,

$$\pi_i(x) := \sum_{e\in S} c_e(x_e) \le \sum_{e\in S'} c_e(x_e), \quad \text{for any } S, S' \in \mathcal{S}_i \text{ with } x_S^i > 0, \text{ for all } i \in N.$$

We recall the following characterization of Wardrop equilibria that implies their existence.

Theorem 2.1 (Beckmann et al. [4], Section 3.1.2). *A strategy distribution x is a Wardrop equilibrium if and only if it is an optimal solution to*

$$\min_{x\in P(\mathcal{M})} \left\{ \Phi(x) := \sum_{e\in E} \int_0^{x_e} c_e(t)\,dt \right\}. \tag{1}$$

We call Φ the Beckmann potential.

Notice that the problem of finding the minimum value of the Beckmann potential can equivalently be written in terms of $\tilde{P}(\mathcal{M})$ as the following minimization problem:

$$\min_{x\in \tilde{P}(\mathcal{M})} \left\{ \sum_{e\in E} \int_0^{x_e} c_e(t)\,dt \right\}.$$

Here, it should be noted that $x = (x_e)_{e\in E} \in \mathbb{R}^E$, while x appearing in (1) is a strategy distribution in $\mathbb{R}^{\mathcal{S}}$. Later, in our results, we will often refer to this equivalent version of the problem of minimizing the Beckmann potential. For simplicity, we will use $\Phi(x)$ also for the Beckmann potential for load vectors $x \in \tilde{P}(\mathcal{M})$.

Remark 2.1. Using the fact that every Wardrop equilibrium $x \in P(\mathcal{M})$ is a global minimum of (1), we obtain the following well-known properties (see Correa and Stier-Moses [14]). If cost functions $(c_e)_{e\in E}$ are strictly increasing, the Wardrop equilibrium load vector $(x_e)_{e\in E} \in \tilde{P}(\mathcal{M})$ is unique (but there can be different decompositions of the demands among the subsets; that is, there can be $x, y \in P(\mathcal{M}), x \ne y$ with $x_e = y_e$ for all $e \in E$). For the case of nondecreasing costs, the vector of costs $(c_e(x_e))_{e\in E}$ is unique under the possibly nonunique equilibrium load vectors.

2.3. The Braess paradox

Recall the examples of Braess' paradox presented in Figures 1 and 2. In these examples, the equilibrium flows on two (network) congestion models $\mathcal{M}$ and $\bar{\mathcal{M}}$ are compared with each other, where $\bar{\mathcal{M}}$ is related to $\mathcal{M}$ by simply reducing some of the cost functions of the populations (in case of the example in Figure 1, only one cost function is reduced from ∞ to 0), reducing the demands of the populations, or both.

In this work, we allow for general cost reductions of the form $\bar{c}_e(t) \le c_e(t)$ for all $t \ge 0$ and $e \in E$ and general demand reductions $\bar{d}_i \le d_i, i \in N$. We denote the changed model by $\bar{\mathcal{M}}$. Note that for both models $\mathcal{M}$ and $\bar{\mathcal{M}}$, the sets of allowable subsets $(\mathcal{S}_i)_{i\in N}$ remain the same. We define the following notion of the weak and strong Braess paradoxes.

Definition 2.1 (The Weak and Strong Braess Paradoxes). Let $E = \{e_1, \ldots, e_m\}$ be a finite set of resources. A family of set systems $(E, \mathcal{S}_i)_{i\in N}$ with $\mathcal{S}_i \subseteq 2^E$ for all $i \in N$ admits the *weak* Braess paradox (BP) if there are two nonatomic congestion models $\mathcal{M} = (N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, (d_i)_{i\in N})$ and $\bar{\mathcal{M}} = (N, E, (\mathcal{S}_i)_{i\in N}, (\bar{c}_e)_{e\in E}, (\bar{d}_i)_{i\in N})$, with $\bar{c}_e(t) \le c_e(t)$ for all $t \ge 0$ and $\bar{d}_i \le d_i$ for all $i \in N$, and two Wardrop equilibria x and $\bar{x}$ for $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, such that

$$\text{there is } e \in E \text{ with } c_e(x_e) < \bar{c}_e(\bar{x}_e). \tag{weak BP}$$

Then $(E, \mathcal{S}_i)_{i\in N}$ admits the *strong* Braess paradox if there is $i \in N$ with $S, S' \in \mathcal{S}_i$, $x_S^i > 0$, $\bar{x}_{S'}^i > 0$ such that

$$\pi_i(x) = \sum_{e\in S} c_e(x_e) < \sum_{e\in S'} \bar{c}_e(\bar{x}_e) = \pi_i(\bar{x}). \tag{strong BP}$$

We say that $(E, \mathcal{S}_i)_{i\in N}$ is *immune* to the weak and strong Braess paradoxes if no such $\mathcal{M}, \bar{\mathcal{M}}$ exist.

Remark 2.2. The strong Braess paradox implies the weak, but not vice versa. On the other hand, if a family of set systems $(E, \mathcal{S}_i)_{i\in N}$ is immune to the weak Braess paradox, it is also immune to the strong Braess paradox. Moreover, if $(E, \mathcal{S}_i)_{i\in N}$ satisfies that for every resource $e \in E$ there is a player i with $\mathcal{S}_i = \{\{e\}\}$, then the weak and strong Braess paradoxes are equivalent.

The examples in Figures 1 and 2 already show that there are quite simple set systems $(E, \mathcal{S}_i)_{i\in N}$ that admit the strong (and thus the weak) Braess paradox.

The driving question of this paper is the following.

> What is a characterizing property of the set systems $(E, \mathcal{S}_i)_{i\in N}$ so that they are immune to the weak or strong Braess paradox?

3. A characterization of the weak Braess paradox

For our first characterization of the weak Braess paradox, we define for a set system $(E, \mathcal{S}_i)$ the *clutter* as

$$(\mathcal{S}_i)^{\min} := \left\{U \in \mathcal{S}_i \mid \nexists T \in \mathcal{S}_i \text{ with } T \subset U\right\}.$$

The set system $(E, (\mathcal{S}_i)^{\min})$ (for which we also use the term "clutter") contains all inclusion-wise minimal subsets of $\mathcal{S}_i$.

Our first main result gives a complete characterization of the weak Braess paradox.

> **Theorem 3.1.** *Let $(E, \mathcal{S}_i)_{i\in N}$ be a family of set systems; i.e., $\mathcal{S}_i \subseteq 2^E$, for all $i \in N$. Then, the following statements are equivalent:*
> (I) The clutter $(E, (\mathcal{S}_i)^{\min})$ forms the base set of a matroid $M_i = (E, \mathcal{I}_i)$, for all $i \in N$.
> (II) A family of set systems $(E, \mathcal{S}_i)_{i\in N}$ is immune to the weak Braess paradox.

3.1. Proof of Theorem 3.1: (I) ⇒ (II)

The proof of (I) ⇒ (II) consists of a number of steps organized as follows.

In the first step we prove that if every clutter $(E, (\mathcal{S}_i)^{\min})$ corresponds to bases of some matroid $M_i = (E, \mathcal{I}_i)$ for all $i \in N$, then $(E, (\mathcal{S}_i)^{\min})_{i\in N}$ is immune to the weak Braess paradox. To show this, we first model the set of feasible strategy distributions of a nonatomic matroid congestion game via a suitably defined *polymatroid base polytope*. This way, the problem of computing a Wardrop equilibrium of a matroid congestion model can be interpreted as the problem to find a global minimum of a separable convex function (i.e., the Beckmann potential) over a sum of population-specific polymatroid base polytopes that is itself a polymatroid base polytope (see Edmonds [17]).

In the next steps, we prove two sensitivity results for this class of optimization problems stating that whenever (i) cost functions or (ii) demands are decreased, any global minimizer of the Beckmann potential has the property that the new induced cost values component-wise decrease. This implies that the weak Braess paradox does not occur.

In the final step of (I) ⇒ (II), we prove that if the family of clutters $(E, (\mathcal{S}_i)^{\min})_{i\in N}$ is immune to the weak Braess paradox, then so is the family of set systems $(E, \mathcal{S}_i)_{i\in N}$.

3.1.1. Polymatroids. To define polymatroids, we first have to introduce submodular functions. A function $\rho\colon 2^E \to \mathbb{R}$ is called submodular if $\rho(U)+\rho(V) \geq \rho(U \cup V)+\rho(U \cap V)$ for all $U, V \subseteq E$. It is called monotone if $\rho(U) \leq \rho(V)$ for all $U \subseteq V \subseteq E$ and normalized if $\rho(\emptyset)=0$. Given a submodular, monotone, and normalized function ρ, the pair (E, ρ) is called a *polymatroid*. The associated *polymatroid base polytope* is defined as

$$P_\rho := \{x \in \mathbb{R}_+^E \mid x(U) \leq \rho(U),\ \forall\, U \subseteq E,\ x(E)=\rho(E)\},$$

where $x(U) := \sum_{e \in U} x_e$ for all $U \subseteq E$. Given submodular functions ρ_i, $i \in N$ all defined on 2^E and $\rho := \sum_{i \in N} \rho_i$, we know that the Minkowski sum $P_\rho = \sum_{i \in N} P_{\rho_i}$ is also a polymatroid base polytope; see Edmonds [17], Fujishige [21], or Schrijver [49], Theorem 44.6.

3.1.2. From nonatomic matroid congestion games to polymatroids. Consider now a nonatomic matroid congestion model $\mathcal{M}$, where for every $i \in N$ the associated strategy space forms the base set $\mathcal{B}_i$ of a matroid $M_i = (E, \mathcal{I}_i)$. It is well known that the rank function $\mathrm{rk}_i\colon 2^E \to \mathbb{R}$ of matroid M_i satisfies

$$\mathrm{rk}_i(S) := \max\{|U| \mid U \subseteq S \text{ and } U \in \mathcal{I}_i\}, \quad \forall\, S \subseteq E$$

and is submodular, monotone, and normalized. Moreover, the characteristic vectors of the bases in $\mathcal{B}_i$ are exactly the vertices of the polymatroid base polytope P_{rk_i}.

It follows that the polytope

$$\left\{x^i \in \mathbb{R}_+^{\mathcal{B}_i} \,\middle|\, \sum_{B \in \mathcal{B}_i} x_B^i = d_i\right\}$$

corresponds to strategy distributions for population i that lead to load vectors in the following polytope:

$$P_{d_i \cdot \mathrm{rk}_i} = \{x_i \in \mathbb{R}_+^E \mid x_i(U) \leq d_i \cdot \mathrm{rk}_i(U),\ \forall\, U \subseteq E,\ x_i(E) = d_i \cdot \mathrm{rk}_i(E)\}.$$

Thus, the polymatroid base polytope $P := \sum_{i \in N} P_{d_i \cdot \mathrm{rk}_i} = P_{\sum_{i \in N} d_i \cdot \mathrm{rk}_i}$ is equal to $\tilde{P}(\mathcal{M})$. To simplify notation we define the following submodular functions: $\rho_i = d_i \cdot \mathrm{rk}_i$ for $i \in N$ and $\rho = \sum_{i \in N} \rho_i$. Furthermore, let $P_i = P_{\rho_i}$. We thus have $\tilde{P}(\mathcal{M}) = P_\rho = \sum_{i \in N} P_i$.

3.1.3. Two sensitivity results. Consider the following optimization problem:

$$\min_{x \in P_\rho} \left\{\Phi(x) := \sum_{e \in E} \int_0^{x_e} c_e(t)\, dt\right\}, \tag{2}$$

where P_ρ is a polymatroid base polytope with rank function ρ and for all $e \in E$, $c_e\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are nondecreasing and continuous functions. We recall the following necessary and sufficient optimality conditions. Let $\chi_e \in \mathbb{R}_{\geq 0}^E$ for $e \in E$ be the indicator vector with all-zero entries except for the eth coordinate, which is 1. From Fujishige [21], we know that a base $x \in P_\rho$ is optimal for problem (2) if and only if

$$c_e(x_e) \leq c_f(x_f), \quad \text{for any } e, f \in E \text{ such that } x' := x + \epsilon(\chi_f - \chi_e) \in P_\rho \text{ for some } \epsilon > 0. \tag{3}$$

We now prove a result on the sensitivity of optimal solutions minimizing the Beckmann potential over a polymatroid base polytope: we will show that whenever a cost function is shifted downward (i.e., $\bar{c}_e(t) \leq c_e(t)$ for all $t \geq 0$), then any new optimal solution $\bar{x}$ has the property $\bar{c}_e(\bar{x}_e) \leq c_e(x_e)$ for all $e \in E$, where x denotes any optimal solution for the cost functions $c_e, e \in E$, and $\bar{x}$ denotes any optimal solution for the cost functions $\bar{c}_e, e \in E$. This result implies that for matroid set systems, the weak Braess paradox does not occur if only cost reductions are considered.

Lemma 3.1. *Let I and $\bar{I}$ be two instances of problem* (2) *with the only difference that for $\bar{I}$ we use cost functions satisfying $\bar{c}_e(t) \leq c_e(t)$ for all $e \in E$ and $t \geq 0$. Then any optimal solutions x and $\bar{x}$ to the instances I and $\bar{I}$, respectively, satisfy*

$$\bar{c}_e(\bar{x}_e) \leq c_e(x_e), \quad \textit{for all } e \in E.$$

Proof. Assume by contradiction that there is $e \in E$ with $\bar{c}_e(\bar{x}_e) > c_e(x_e)$. Thus, by the monotonicity of c_e, this implies $\bar{x}_e > x_e$. By elementary transformations of bases in polymatroid base polytopes (see Murota [35], Theorem 4.3), there must exist $f \in E \backslash \{e\}$ with $\bar{x}_f < x_f$ and $\epsilon > 0$ such that

$$x + \epsilon(\chi_e - \chi_f) \in P_\rho \qquad \text{and} \qquad \bar{x} + \epsilon(\chi_f - \chi_e) \in P_\rho.$$

As x and $\bar{x}$ are both optimal solutions for their respective optimization problems, and $x_f, \bar{x}_e > 0$, we obtain by the optimality conditions (3) that $c_f(x_f) \leq c_e(x_e)$ and $\bar{c}_e(\bar{x}_e) \leq \bar{c}_f(\bar{x}_f)$. Hence,

$$c_f(x_f) \leq c_e(x_e) < \bar{c}_e(\bar{x}_e) \leq \bar{c}_f(\bar{x}_f) \leq c_f(x_f),$$

a contradiction. □

We now prove a second sensitivity result for minimizers of the Beckmann function over the polymatroid base polytope for the case when the demand is reduced.

Lemma 3.2. *Let I and $\bar{I}$ be two instances of problem (2), with the only difference that for $\bar{I}$ the demand for one population $j \in N$ is decreased from d_j to $\bar{d}_j < d_j$. Hence, the feasible strategy distributions P and $\bar{P}$ of I and $\bar{I}$, respectively, are given by*

$$P = P_\rho = \sum_{i \in N} P_i \qquad \text{and} \qquad \bar{P} = \frac{\bar{d}_j}{d_j} P_j + \sum_{i \in N \setminus \{j\}} P_i.$$

Let $x, \bar{x} \in \mathbb{R}_+^E$ be minimizers of problem (2) over the polytopes P and $\bar{P}$, respectively; then

$$c_e(\bar{x}_e) \le c_e(x_e), \quad \forall\, e \in E.$$

Proof. By contradiction, assume there is $e \in E$ with $c_e(\bar{x}_e) > c_e(x_e)$. Since c_e is nondecreasing, this implies $\bar{x}_e > x_e$. Because $x \in P = \sum_{i \in N} P_i$, we can decompose x as $x = \sum_{i \in N} x_i$, where $x_i \in P_i$ for $i \in N$. Let $x' = \sum_{i \in N \setminus \{j\}} x_i + (\bar{d}_j / d_j) x_j \in \bar{P}$. Clearly, $x' \le x$ component-wise, and we thus have in particular $\bar{x}_e > x_e \ge x'_e$. By exchange properties of base polytopes of polymatroids, there exist $f \in E$ with

$$\bar{x}_f < x'_f \tag{4}$$

and $\epsilon > 0$ such that
 (i) $\bar{x} + \epsilon(\chi_f - \chi_e) \in \bar{P}$, and
 (ii) $x' + \epsilon(\chi_e - \chi_f) \in \bar{P}$.

Notice that $\bar{P} + (1 - \bar{d}_j/d_j) P_j = P$, and hence, $x - x' \in (1 - \bar{d}_j/d_j) P_j$. Together with (ii), this implies

$$x + \epsilon(\chi_e - \chi_f) = \underbrace{x' + \epsilon(\chi_e - \chi_f)}_{\in \bar{P}} + \underbrace{(x - x')}_{\in (1 - \bar{d}_j/d_j) P_j} \in P,$$

and therefore

$$c_e(x_e) \ge c_f(x_f), \tag{5}$$

since x is a minimizer of Ψ over P. Similarly, (i) implies

$$c_e(\bar{x}_e) \le c_f(\bar{x}_f), \tag{6}$$

because $\bar{x}$ is a minimizer of Ψ over $\bar{P}$. Putting things together, we obtain

$$\begin{aligned} c_e(x_e) &< c_e(\bar{x}_e) && \text{(by assumption for sake of contradiction)} \\ &\le c_f(\bar{x}_f) && (6) \\ &\le c_f(x'_f) && ((4) \text{ and } c_f \text{ is nondecreasing}) \\ &\le c_f(x_f) && (x' \le x \text{ and } c_f \text{ is nondecreasing}) \\ &\le c_e(x_e) && (5), \end{aligned}$$

thus leading to a contradiction and finishing the proof. □

Furthermore, Lemma 3.1 shows that equilibrium costs only decrease (component-wise) when reducing the cost function. Hence, reducing costs and demands simultaneously can be interpreted as doing first one reduction and then the other. Each of these changes is such that any new Wardrop equilibrium has on any resource a lower cost than before the change. Thus, a family of matroid set systems (as a special case of polymatroids) is immune to the weak Braess paradox.

3.1.4. From clutters to set systems. To complete the direction (I) ⇒ (II) of the proof of Theorem 3.1, it remains to show that if the clutters $(E, (\mathcal{S}_i)^{\min})_{i \in N}$ of some set systems $(E, \mathcal{S}_i)_{i \in N}$ are immune to the weak Braess paradox, then so are the original set systems $(E, \mathcal{S}_i)_{i \in N}$.

For any set $U \in \mathcal{S}_i$, we call $U^* \in S_i$ a *tight subset* of U, if U^* is an inclusion-wise minimal set in $\mathcal{S}_i$ contained in U; i.e, $U^* \subseteq U$ and $U^* \in (\mathcal{S}_i)^{\min}$.

Lemma 3.3. *If $(E, (\mathcal{S}_i)^{\min})_{i \in N}$ is immune to the weak Braess paradox, then so is $(E, \mathcal{S}_i)_{i \in N}$.*

Proof. Let $\mathcal{M} = (N, E, (\mathcal{S}_i)_{i\in N}, (c_e)_{e\in E}, (d_i)_{i\in N})$ and $\bar{\mathcal{M}} = (N, E, (\mathcal{S}_i)_{i\in N}, (\bar{c}_e)_{e\in E}, (\bar{d}_i)_{i\in N})$ be two congestion models defined on the same strategy spaces $(\mathcal{S}_i)_{i\in N}$ and satisfying $\bar{c}_e(t) \le c_e(t)$ for $e \in E, t \ge 0$ and $\bar{d}_i \le d_i$ for $i \in N$. Moreover, let $\mathcal{M}^{\min}$ and $\bar{\mathcal{M}}^{\min}$ be the models obtained from $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, by replacing $(\mathcal{S}_i)_{i\in N}$ with $((\mathcal{S}_i)^{\min})_{i\in N}$.

Let $x \in P(\mathcal{M})$ and $\bar{x} \in P(\bar{\mathcal{M}})$ be two Wardrop equilibria. We have to show $c_e(x_e) \ge \bar{c}_e(\bar{x}_e)$ for $e \in E$. Starting from x, we iteratively pick $i \in N, U \in \mathcal{S}_i \setminus (\mathcal{S}_i)^{\min}$ with $x^i_U > 0$ and a tight subset U^* of U, and set $x^i_{U^*} \leftarrow x^i_{U^*} + x^i_U$, $x^i_U \leftarrow 0$.

We denote by x' the thus obtained point, whose load vector satisfies $(x'_e)_{e\in E} \in \tilde{P}(\mathcal{M}^{\min}) \subseteq \tilde{P}(\mathcal{M})$. We apply the same procedure to $\bar{x}$ to obtain $\bar{x}'$ with $(\bar{x}'_e)_{e\in E} \in \tilde{P}(\bar{\mathcal{M}}^{\min})) \subseteq \tilde{P}(\bar{\mathcal{M}})$. The condition $U^* \subseteq U$ implies that the profiles x' and $\bar{x}'$ satisfy

$$c_e(x'_e) \le c_e(x_e) \qquad \text{and} \qquad \bar{c}_e(\bar{x}'_e) \le \bar{c}_e(\bar{x}_e), \quad \text{for all } e \in E.$$

This implies that x' and $\bar{x}'$ are global minimizers of the Beckmann potential over both polytopes $\tilde{P}(\mathcal{M})$ and $\tilde{P}(\mathcal{M}^{\min})$ and $\tilde{P}(\bar{\mathcal{M}})$ and $\tilde{P}(\bar{\mathcal{M}}^{\min})$, respectively. Thus, x' and $\bar{x}'$ are equilibrium load vectors for both models $\mathcal{M}$ and $\mathcal{M}^{\min}$ and $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}^{\min}$, respectively. Since equilibrium costs per resource are unique (see Remark 2.1), we get

$$c_e(x'_e) = c_e(x_e) \qquad \text{and} \qquad \bar{c}_e(\bar{x}'_e) = \bar{c}_e(\bar{x}_e), \quad \text{for all } e \in E.$$

As, by assumption, $(E, (\mathcal{S}_i)^{\min})_{i\in N}$ is immune to the weak Braess paradox, we have $c_e(x'_e) \ge \bar{c}_e(\bar{x}'_e)$ for $e \in E$, which finally implies

$$c_e(x_e) = c_e(x'_e) \ge \bar{c}_e(\bar{x}'_e) = \bar{c}_e(\bar{x}_e), \quad \text{for all } e \in E,$$

proving the lemma. □

3.2. Proof of Theorem 3.1: (II) ⇒ (I)

Let us recall the definition of a set system being immune to the weak Braess paradox. Let $\mathcal{M}$ and $\bar{\mathcal{M}}$ be any two models of a nonatomic congestion game with the only difference that for $\bar{\mathcal{M}}$ we use cost functions satisfying $\bar{c}_e(t) \le c_e(t)$ for all $e \in E$ and $t \ge 0$ and $\bar{d}_i \le d_i$ for all $i \in N$. Then, if the strategy spaces of $\mathcal{M}, \bar{\mathcal{M}}$ are immune to the weak Braess paradox, we have that any two Wardrop equilibria x and $\bar{x}$ for $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, satisfy

$$\bar{c}_e(\bar{x}_e) \le c_e(x_e), \quad \text{for all } e \in E. \tag{7}$$

We first give an outline of the proof. Let $(\mathcal{S}_i)_{i\in N}$ be strategy spaces that are immune to the weak Braess paradox. We fix the demand of each population to 1; i.e., $d_i = 1$ for $i \in N$. Hence, the possible load vectors that can be obtained by population $i \in N$ by playing a feasible strategy are given by the following polytope:

$$P_i = \text{Convex hull of } \{\chi_S \mid S \in \mathcal{S}_i\}, \quad \forall\, i \in N,$$

where χ_S denotes the characteristic vector of $S \subseteq E$. Thus, the load vectors obtainable by considering all populations combined are described by the Minkowski sums of the polytopes P_i; i.e.,

$$\tilde{P}(\mathcal{M}) = \sum_{i\in N} P_i.$$

For brevity, we set

$$P = \tilde{P}(\mathcal{M}).$$

Analogously, we define the polytopes corresponding to the possible load vectors stemming from the strategy spaces $((\mathcal{S}_i)^{\min})_{i\in N}$; i.e.,

$$P_i^{\min} = \text{Convex hull of } \{\chi_S \mid S \in \mathcal{S}_i^{\min}\}, \quad \forall\, i \in N,$$

and we let

$$P_i^{\uparrow} = P_i + \mathbb{R}^E_{\ge 0} = P_i^{\min} + \mathbb{R}^E_{\ge 0} \quad \forall\, i \in N,$$

be the dominant of P_i for $i \in N$. Observe that

$$P^{\uparrow} = \sum_{i\in N} P_i^{\uparrow}. \tag{8}$$

We will use a polyhedral approach to show (II) ⇒ (I). More precisely, we will show that $(\mathcal{S}_i)_{i\in N}$ being immune to the weak Braess paradox implies that each $P_i^{\min}$ for $i \in N$ is the base polytope of a matroid. To show that $P_i^{\min}$ for $i \in N$ is the base polytope of a matroid, we rely on the following well-known characterization (see, e.g., Fujishige [21], Theorem 17.1), and a corollary thereof.

Lemma 3.4. *Let $Q \subseteq [0,1]^E$ be a $\{0,1\}$-polytope; i.e., all vertices of Q are part of $\{0,1\}^E$. Then Q is the base polytope of a matroid if and only if each edge direction of Q is parallel to some vector $\chi_e - \chi_f$ for distinct $e, f \in E$.*

The above lemma can be rephrased in terms of the dominant $Q^{\uparrow}$ of Q as follows.

Corollary 3.1. *Let $Q \subseteq [0,1]^E$ be a $\{0,1\}$-polytope such that no two distinct vertices u, v of Q satisfy $uleqv$ (component-wise). Then Q is the base polytope of a matroid if and only if each edge direction of $Q^{\uparrow}$ is parallel to some vector $\chi_e - \chi_f$ for distinct $e, f \in E$.*

Proof. If Q is a matroid polytope then, by Lemma 3.4, each edge direction of Q is parallel to $\chi_e - \chi_f$ for distinct $e, f \in E$. Since each edge direction of the dominant of a polytope is also an edge direction of the original polytope, we have that the edge directions of $Q^{\uparrow}$ are also of this form.

Conversely, assume that each edge direction of $Q^{\uparrow}$ is parallel to $\chi_e - \chi_f$ for some distinct $e, f \in E$. First observe that Q and $Q^{\uparrow}$ have the same set of vertices. This follows by the fact that no two distinct vertices u, v of Q satisfy $u \leq v$ (component-wise). Moreover, each edge direction of $Q^{\uparrow}$ is parallel to some vector of the form $\chi_e - \chi_f$ for distinct $e, f \in E$. Since the set of all the vertices of $Q^{\uparrow}$ is linked (connected) by the edges of $Q^{\uparrow}$ (see, e.g., Balinski [3]), this implies that all vertices of $Q^{\uparrow}$ have the same l_1-norm and thus lie on a hyperplane of the form $\langle -\mathbf{1}, x\rangle = b$, where $\mathbf{1} \in \mathbb{R}^E$ is the all-ones vector. Hence, $\langle -\mathbf{1}, x\rangle = b$ is a supporting hyperplane of $Q^{\uparrow}$ and the face of $Q^{\uparrow}$ defined by this hyperplane is equal to Q. Since Q is a face of $Q^{\uparrow}$, each edge of Q is also an edge of $Q^{\uparrow}$, and is thus parallel to $\chi_e - \chi_f$ for distinct $e, f \in E$. Lemma 3.4 now implies that Q is indeed a matroid base polytope. □

Hence, it suffices to show that any edge direction of $P_i^{\uparrow} = (P_i^{\min})^{\uparrow}$ for $i \in N$ is of type $\chi_e - \chi_f$ for distinct $e, f \in E$. Notice that Corollary 3.1 can indeed be applied to the polytopes $(P_i^{\min})^{\uparrow}$, because any two distinct vertices u, v of $P_i^{\min}$ correspond to characteristic vectors of two distinct sets in the clutter $(E, (S_i)^{\min})$, which implies $u \not\leq v$ as required by Corollary 3.1.

The following well-know property links edge directions of a Minkowski sum of polytopes to the edge directions of the summands (see, e.g., Thomas [52, Lemma 6.14]).

Lemma 3.5. *Let $Q_1, \ldots, Q_k \subseteq \mathbb{R}^E$ be polytopes and $Q = \sum_{i=1}^k Q_i$ be their Minkowski sum. Then the set of edge directions of Q is the union of the sets of edge directions of the Q_i for $i \in [k]$.*

However, we are interested in the edge directions of $P_i^{\uparrow}$ for $i \in N$, which are unbounded polyhedra. Lemma 3.5 can easily be generalized to dominants of polytopes as shown below. This allows us to derive properties on the edge directions of $P_i^{\uparrow}$ for $i \in N$ from properties of edge directionis of their Minkowski sum $P^{\uparrow}$.

Lemma 3.6. *Let $Q_1, \ldots, Q_k \subseteq \mathbb{R}^E$ be polytopes and $Q = \sum_{i=1}^k Q_i$ be their Minkowski sum. As usual, let $Q_i^{\uparrow}, \ldots, Q_k^{\uparrow}$ denote the dominants of $Q_i, \ldots, Q_k$ and Q. Then the set of edge directions of $Q^{\uparrow}$ is the union of the sets of edge directions of the $Q_i^{\uparrow}$ for $i \in [k]$.*

Proof. Let $j \in [k]$ and consider an edge $\overline{uv}$ of $Q_j^{\uparrow}$. We first show that $Q^{\uparrow}$ contains an edge parallel to $\overline{uv}$. There is a non-zero vector $h \in \mathbb{R}^E$ such that $\overline{uv}$ are the set of all maximizers of $\langle h, x\rangle$ for $x \in Q_j^{\uparrow}$. Because $Q_j^{\uparrow}$ is a dominant, we must have $h < 0$. For $i \in [k]$, let $Q_i^h \subseteq Q_i^{\uparrow}$ be the set of all maximizers in $Q_i^{\uparrow}$ of the linear function $\langle h, x\rangle$. analogously, let $Q^h \subseteq Q^{\uparrow}$ be all points in $Q^{\uparrow}$ maximizing $\langle h, x\rangle$. Because the maximizers of a Minkowski sum are the sum of the maximizers of the summands, we have $Q^h = \sum_{i=1}^k Q_i^h$. Moreover, because $Q_1^{\uparrow}, \ldots, Q_k^{\uparrow}, Q^{\uparrow}$ are all dominants of polytopes and $h < 0$, we have that $Q_1^h, \ldots, Q_k^h$, and Q^h are also polytopes. Applying Lemma 3.5 to $Q^h = \sum_{i=1}^k Q_i^h$, we obtain that Q^h contains an edge parallel to $\overline{uv}$. Finally, since Q^h is a face of $Q^{\uparrow}$, every edge of Q^h is also an edge of $Q^{\uparrow}$, implying that $Q^{\uparrow}$ contains an edge parallel to $\overline{uv}$, as desired.

Conversely, we have to show that for any edge $\overline{uv}$ of $Q^{\uparrow}$ there is an index $j \in [k]$ such that $Q_j^{\uparrow}$ contains an edge parallel to $\overline{uv}$. This can be proven analogously to the previous case, by considering a vector $h < 0$ such that $\overline{uv}$ are all maximizers of $\langle h, x\rangle$ for $x \in Q^{\uparrow}$, and considering for each $Q_1^{\uparrow}, \ldots, Q_k^{\uparrow}$, and $Q^{\uparrow}$, the face corresponding to all maximizers of $\langle h, x\rangle$. The result then again follows by Lemma 3.5. □

Hence, Corollary 3.1 and Lemma 3.6 imply that to prove the direction (II) ⇒ (I) of Theorem 3.1, it suffices to show the following

Proposition 3.1. *Each edge direction of $P^{\uparrow}$ is parallel to some vector $\chi_e - \chi_f$ with distinct $e, f \in E$.*

Indeed, proving Proposition 3.1 implies by Lemma 3.6 that each edge direction of $P_i^{\uparrow}$ for $i \in N$ is parallel to $\chi_e - \chi_f$ for some distinct $e, f \in E$, which, by applying Corollary 3.1 to $Q = P_i^{\min}$ (notice that $(P_i^{\min})^{\uparrow} = P_i^{\uparrow}$), implies that $P_i^{\min}$ for $i \in N$ are matroid base polytopes as desired. Hence, it remains to prove Proposition 3.1.

We start with a simple observation.

Lemma 3.7. *Let $\overline{uv}$ be an edge of $P^{\uparrow}$; then $\overline{uv}$ is parallel to some vector in $\{-1,0,1\}^E$ with at least two nonzero values.*

Proof. Notice that every edge of $P^{\uparrow}$ must also be an edge of P. Hence, by Lemma 3.5 the edge $\overline{uv}$ (of P) must be parallel to some edge direction of one of the polytopes P_i for some $i \in N$. The statement now follows by observing that all edge directions of P_i are within $\{-1,0,1\}^E$ because P_i is a $\{0,1\}$-polytope; moreover, the edge direction cannot be of the form χ_e since $P^{\uparrow}$ is up-closed, implying that any edge direction of $P^{\uparrow}$ must have at least two nonzero values. □

The following lemma shows the existence of particular types of cost functions that will help us to derive further properties on edge directions by using the fact that no weak Braess paradox exists for the strategy spaces $(\mathcal{S}_i)_{i \in N}$.

Lemma 3.8. *Let x^0 be the midpoint of an edge $\overline{uv}$ of $P^{\uparrow}$, and let $w \in \{-1,0,1\}^E$ be a vector parallel to $\overline{uv}$. Then there exist strictly positive, continuous, and strictly increasing cost functions c_e for $e \in E$ such that the following holds:*

(i) *The Beckmann potential $\Phi(x)$ that corresponds to $(c_e)_{e\in E}$ has x^0 as a unique minimizer over $P^{\uparrow}$.*

(ii) *For any $f \in E$ with $w_f = 1$ and $\alpha > 0$, let $c_e^{(\alpha,f)}$ for $e \in E$ be the following cost function:*

$$c_e^{(\alpha,f)}(t) = \begin{cases} c_e(t) & \text{if } e \in E\backslash\{f\}, \\ c_e(t) - \alpha & \text{if } e = f. \end{cases}$$

Then for a sufficiently small $\alpha > 0$, the costs $(c_e^{(\alpha,f)})_{e\in E}$ are nonnegative, continuous, and strictly increasing, and the corresponding Beckmann potential $\Phi^{(\alpha,f)}(x)$ has a unique minimizer over $P^{\uparrow}$, which is of the form $x^0 + \beta w$ for some $\beta > 0$.

Proof. Let $\langle h, x\rangle = b$ be a supporting hyperplane of $P^{\uparrow}$ defining the edge $\overline{uv}$. Hence, $\overline{uv}$ are the set of all maximizers of $\langle h, x\rangle$ for $x \in P^{\uparrow}$. Notice that $h < 0$ (component-wise) because $P^{\uparrow}$ is a dominant. Indeed, $h_e > 0$ for some $e \in E$ is not possible because in this case $\overline{uv}$ cannot be maximizers with respect to h as increasing the component corresponding to e leads to better points. Moreover, there is also no $e \in E$ with $h_e = 0$, because this would again imply that, given any point y on $\overline{uv}$, one can increase the e-component of y arbitrarily and remain a maximizer with respect to h. However, as $\overline{uv}$ is an edge, it is bounded and it is therefore not possible to increasing any component arbitrarily and remain on the set of maximizers $\overline{uv}$. Thus, $h < 0$.

Furthermore, we define

$$\gamma = \min\left\{\frac{-h_e}{4x_e^0} \,\middle|\, e \in E, x_e^0 > 0\right\},$$

and for $t \geq 0$, we let

$$c_e(t) = -h_e + 2\gamma(t - x_e^0),$$

which leads to the Beckmann potential

$$\Phi(x) = -\langle h, x\rangle + \gamma\|x - x^0\|_2^2 - \gamma\|x^0\|_2^2.$$

Clearly, the cost functions c_e are strictly positive, continuous, and strictly increasing (recall that $h_e < 0$ for all $e \in E$). Moreover, x^0 is indeed the unique minimizer of $\Phi(x)$ on $P^{\uparrow}$ since it is the unique minimizer of $\gamma\|x - x^0\|_2^2$ over $\mathbb{R}^E$ and x^0 is a maximizer of $\langle h, x\rangle$ over $P^{\uparrow}$. This shows (i).

To show (ii) we show that for a sufficiently small $\alpha > 0$, the Beckmann potential $\Phi^{(\alpha,f)}(x)$ has as a unique minimizer the point $y = x^0 + \beta_\alpha w$, where

$$\beta_\alpha = \frac{\alpha}{2\gamma\|w\|_2^2}.$$

We will choose $\alpha > 0$ small enough such that y is still in the relative interior of the edge $\overline{uv}$.

Notice that by definition of the Beckmann potential, its gradient at a point x is given by the cost functions; hence,

$$\nabla\Phi^{(\alpha,f)}(x) = (c_e^{(\alpha,f)}(x))_{e\in E}.$$

Since $\Phi^{(\alpha,f)}(x)$ (like $\Phi(x)$) is a strictly convex function, it has a unique minimizer over $P^{\uparrow}$. Moreover, by the Karush-Kuhn-Tucker conditions, we have that y is a minimizer of $\Phi^{(\alpha,f)}(x)$ over $P^{\uparrow}$ if and only if $-\nabla\Phi^{(\alpha,f)}$ is spanned by the cone $\mathscr{C}(y) \subseteq \mathbb{R}^E$ of the normal vectors of the linear constraints of $P^\uparrow$ that are tight at y. To be precise, for the above statement to hold, we consider an inequality description of $P^\uparrow$ where, in particular, a linear equality constraint is represented by two opposing inequality constraints. Notice that $\mathscr{C}(x)$ is the same cone for any point x in the relative interior of the edge $\overline{uv}$, since all points in the relative interior of $\overline{uv}$ have the same set of tight constraints of $P^\uparrow$.

Furthermore, h is in the relative interior of $\mathscr{C}(y)$ because the supporting hyperplane $\langle h, x\rangle = b$ defines the edge $\overline{uv}$. This implies that there exists an $\epsilon > 0$ such that any vector $h' \in \mathbb{R}^E$ with $\|h - h'\|_2 \le \epsilon$ and $h' \perp w$ satisfies $h' \in \mathscr{C}(y)$. We show that for a small enough α, the negative gradient $-\nabla\Phi^{(\alpha,f)}(y)$ can be chosen as such an h'.

We indeed have $-\nabla\Phi^{(\alpha,f)}(y) \perp w$ since

$$\begin{aligned}\langle \nabla\Phi^{(\alpha,f)}(y), w\rangle &= -\langle h, w\rangle + 2\gamma\beta_\alpha \|w\|_2^2 - \alpha\\ &= 0 \quad (\text{using } h \perp w \text{ and the definition of } \beta_\alpha).\end{aligned}$$

Furthermore, since $\nabla\Phi^{(\alpha,f)}(y)$ is continuous in α, and $-\nabla\Phi^{(0,f)}(y) = -\nabla\Phi(x^0) = h$, we have for a small enough $\alpha > 0$ that $\|h - (-\nabla\Phi^{(\alpha,f)}(y))\|_2 \le \epsilon$, as desired. □

We finally complete the proof of (II) ⇒ (I) by showing Proposition 3.1.

Proof of Proposition 3.1 Assume for the sake of contradiction that the proposition does not hold. Hence, there is an edge $\overline{uv}$ of $P^\uparrow$ that is parallel to some vector $w \in \{-1, 0, 1\}^E$ with at least two nonzero entries by Lemma 3.7, and such that w has either two 1 entries or two -1 entries. Without loss of generality, we can assume that w has two 1 entries (for otherwise, consider $-w$). Let $f, g \in E$ be two such entries; i.e., $w_f = w_g = 1$. We now invoke Lemma 3.8 with respect to the edge $\overline{uv}$ and the element $f \in E$ to obtain the following. There are cost functions $(c_e)_{e\in E}$ such that the corresponding Beckmann potential $\Phi(x)$ has its unique minimum at the midpoint of the edge $\overline{uv}$, denoted by x^0. Moreover, it suffices to reduce the cost function for element f to obtain new cost functions $(\bar{c}_e)_{e\in E}$ with a new corresponding potential ($\Phi^{(\alpha,f)}$ in Lemma 3.8) whose unique minimum over $P^\uparrow$ is equal to $y = x^0 + \beta w$ for some $\beta > 0$. However, since $w_g = 1$, this implies that $y_g > x_g^0$, and hence

$$\bar{c}_g(y_g) = c_g(y_g) > c_g(x_g^0),$$

where the second equality follows from the fact that c_g is strictly increasing by Lemma 3.8. Hence, this shows that we have the weak Braess paradox if the combined strategy spaces are described by $P^\uparrow$.

However, because the unique minimizers of the two considered Beckmann potentials above over $P^\uparrow$ are both part of P—since they lie on $\overline{uv}$, which is an edge of P—they are also the unique minimizers of the same Beckmann potentials over the smaller set P. This implies that we have the weak Braess paradox over the strategy spaces $(\mathscr{S}_i)_{i\in N}$, which is a contradiction. □

4. The strong Braess paradox for nonmatroid set systems

We now investigate a combinatorial property of the set systems so that the strong Braess paradox does not occur. In contrast to the weak Braess paradox, the matroid property is not necessary for immunity against the strong Braess paradox. Milchtaich [34], for example, shows that if the strategy space of every player is symmetric and corresponds to the paths of a series-parallel s-t graph, then there will be no strong Braess paradox. Note that in this case the resulting set systems need not be bases of matroids.

In this section, we derive a characterization of the strong Braess paradox that does not take into account the global structure of the game. Specifically, we show that the matroid property is the maximal condition on the players' strategy spaces that guarantees that the strong Braess paradox does not occur *without* taking into account how the strategy spaces of different players interweave (see Ackermann et al. [1, 2], who introduced the notion of interweaving of strategy spaces). To state this property mathematically precisely, we introduce the notion of *embeddings*. Let $\tilde{E} = \{e_1, \dots, e_p\}$ be a set consisting of $p = \sum_{i\in N} |E_i|$ elements, where, as before, $E_i = \bigcup_{S\in\mathscr{S}_i}, i \in N$. Formally, an embedding is a map $\tau := (\tau_i)_{i\in N}$, where every $\tau_i\colon E_i \to \tilde{E}$ is an injective map from E_i to $\tilde{E}$. The embedding of $\mathscr{S}_i$ in $\tilde{E}$ according to τ is then defined by identifying every $S = \{e_1, \dots, e_k\} \in \mathscr{S}_i$, with $\tau_i(S) := \{\tau_i(e_1), \dots, \tau_i(e_k)\}$ and $\tau(\mathscr{S}_i) := \{\tau_i(S) \mid S \in \mathscr{S}_i\}$. Given $(\mathscr{S}_i)_{i\in N}$ and τ, the new combined strategy space is then denoted by $(\tau_i(\mathscr{S}_i))_{i\in N}$.

Definition 4.1. A family of set systems $(E, \mathscr{S}_i)_{i\in N}$ with $\mathscr{S}_i \subseteq 2^E, i \in N$ is said to be *universally* immune to the strong Braess paradox if for all embeddings τ in $\tilde{E}$, the set systems $(\tilde{E}, \tau_i(\mathscr{S}_i))_{i\in N}$ do not admit the strong Braess paradox (in the sense of Definition 2.1).

Since any embedding of the set of bases of a matroid into a ground set of resources is a set of bases of a matroid again, we obtain the following immediate consequence of Theorem 3.1.

Corollary 4.1. *If for each $i \in N$ the clutter $(E, (\mathcal{S}_i)^{\min})$ forms the base set of a matroid $M_i = (E, \mathcal{I}_i)$, then the family of set systems $(E, \mathcal{S}_i)_{i\in N}$ is universally immune to the strong Braess paradox.*

Our second result now gives a complete characterization of set systems that are *universally* immune to the strong Braess paradox.

Theorem 4.1. *Let $|N| \geq 2$ and $(E, \mathcal{S}_i)_{i\in N}$, with $\mathcal{S}_i \subseteq 2^E \setminus \{\varnothing\}$ for each $i \in N$. Then, the following three statements are equivalent*:

(I) $(E, (\mathcal{S}_i)^{\min})$ forms the base set of a matroid $M_i = (E, \mathcal{I}_i)$ for each $i \in N$.

(II) $(E, \mathcal{S}_i)_{i\in N}$ is immune to the weak Braess paradox.

(III) $(E, \mathcal{S}_i)_{i\in N}$ is universally immune to the strong Braess paradox.

4.1. Proof of Theorem 4.1

(I) $\Leftrightarrow$ (II): See Theorem 3.1.

(I) $\Rightarrow$ (III): See Corollary 4.1.

We prove (III) $\Rightarrow$ (I) by contradiction. Consider a family of nonempty set systems $(E, \mathcal{S}_i)_{i\in N}$ with $n := |N| \geq 2$, and assume that at least one of the induced clutters $(E, (\mathcal{S}_i)^{\min})_{i\in N}$, say, $(E, (\mathcal{S}_1)^{\min})$, is *not* the base set of a matroid. As above, let $E_i := \bigcup_{S\in\mathcal{S}_i} S$ denote the set of those resources that occur in at least one set in $\mathcal{S}_i$. We will show that the family of set systems $(E, \mathcal{S}_i)_{i\in N}$ admits embeddings $\tau_i\colon E_i \to \tilde{E}$, $i \in N$, such that $\tau(\mathcal{S}) = (\tau_1(\mathcal{S}_1), \ldots, \tau_n(\mathcal{S}_n))$ admits the strong Braess paradox.

Let us call, in general, a nonempty clutter $(E, \mathcal{F})$ a *nonmatroid* if the set system $(E, \{X \subseteq S\colon S \in \mathcal{F}\})$ is not a matroid.

Our proof relies on a certain property of nonmatroids stated in the following lemma. Its proof can also be derived from the proof of Lemma 5.1 in Harks and Peis [24] or the proof of Lemma 16 in Ackermann et al. [2].

Lemma 4.1. *If clutter $(E, \mathcal{F})$ with $\mathcal{F} \neq \varnothing$ is a nonmatroid, then there exist $X, Y \in \mathcal{F}$ and $\{a, b, c\} \subseteq X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$ such that for each set $Z \in \mathcal{F}$ with $Z \subseteq X \cup Y$, either $a \in Z$ or $\{b, c\} \subseteq Z$.*

Proof. Recall the *basis exchange property* for matroids: a clutter $(E, \mathcal{F})$ is the family of bases of some matroid if and only if for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$ there exists some $f \in Y \setminus X$ such that $X - e + f \in \mathcal{F}$. Thus, if the clutter $(E, \mathcal{F})$ is a nonmatroid, there must exist $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$ such that *for all* $f \in Y \setminus X$ the set $X - e + f$ does *not* belong to $\mathcal{F}$. We choose such X, Y and $e \in X \setminus Y$ with $|Y \setminus X|$ minimal (among all $Y' \in \mathcal{F}$ with $X - e + f' \notin \mathcal{F}$ for all $y' \in Y' \setminus X$). Note that $|Y \setminus X| \geq 1$ since $\mathcal{F}$ is a clutter. We distinguish the two cases $|Y \setminus X| = 1$ and $|Y \setminus X| > 1$: In case $|Y \setminus X| = 1$, set $\{a\} = Y \setminus X$ and choose any two distinct elements $\{b, c\} \in X \setminus Y$. Note that $|X \setminus Y| \geq 2$ because otherwise, if $X \setminus Y = \{e\}$, then $Y = X - e + a$, in contradiction to our assumption. Now, for any set $Z \subseteq (X \cup Y) - a$ with $Z \in \mathcal{F}$, the clutter property implies $Z = X$, and therefore $\{b, c\} \subseteq Z$, as desired.

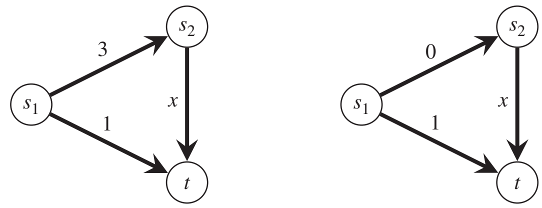
In the latter case $|Y \setminus X| > 1$, we choose any two distinct elements $\{b, c\} \in Y \setminus X$ and set $a = e$. Consider any $Z \in \mathcal{F}$ with $Z \subseteq (X \cup Y) - a$ and suppose, for the sake of contradiction, that $\{b, c\} \not\subseteq Z$. Since $Z \setminus X \subseteq Y \setminus X$, there cannot exist some $g \in Z \setminus X$ with $X - a + g \in \mathcal{F}$. However, $|Z \setminus X| < |Y \setminus X|$, in contradiction to our choice of Y. $\square$

Using this property of nonmatroids, we now define embeddings $\tau(\mathcal{S}) = (\tau_1(\mathcal{S}_1), \ldots, \tau_n(\mathcal{S}_n))$ that admit the strong Braess paradox. The rough idea can be described as follows: we choose the embeddings, demands, and cost functions in such a way that the first two populations are independent of the remaining populations and such that the game of the first two populations is isomorphic to the routing game illustrated in Figure 3 that admits the strong Braess paradox.

Let us set the demands of all populations d_i with $i \in N \setminus \{1, 2\}$ to zero. This way, the game is basically determined by the players in populations 1 and 2. We set the demands $d_1 = d_2 = 1/2$.

Let us choose two sets X, Y in $(\mathcal{S}_1)^{\min}$ and $\{a, b, c\} \subseteq X \cup Y$ as described in Lemma 4.1. Let $e := \tau_1(a)$, $f := \tau_1(b)$, and $g := \tau_1(c)$, with load-dependent costs $c_f(t) = t$, $c_g(t) = 3$, and $c_e(t) = 1$ for any $t \in \mathbb{R}_{\geq 0}$. We set the costs of all resources in $\tau_1(E_1) \setminus (\tau_1(X) \cup \tau_1(Y))$ to some very large cost M (large enough so that no player of population 1 would ever use any of these resources). The cost on all resources in $(\tau_1(X) \cup \tau_1(Y)) \setminus \{e, f, g\}$ is set to zero. This way, each player of population 1 always chooses a strategy $\tau_1(Z) \subseteq \tau_1(X) \cup \tau_1(Y)$, which, by Lemma 4.1, either contains e or contains both f and g.

Figure 3. There are two populations 1 and 2 that want to send 1/2 units of demand, each from s_1 and s_2, respectively, to t. In the left network, there is a unique Wardrop equilibrium, where each population uses its direct edge leading to a cost of 1 for every agent of population 1 and a cost of 1/2 for the agents of population 2. Decreasing the cost from 3 to 0 for the arc (s_1, s_2) induces now the unique Wardrop equilibrium, where agents of population 1 now choose the path (s_1, s_2, t). As a consequence, the private cost of the players in population 2 increase from 1/2 to 1. Also, the social cost increases from 3/4 to 1.



To guarantee that each player of population 2 always selects a strategy containing f, we select a set $S \in (\mathcal{S}_2)^{\min}$ of minimal cardinality, and some arbitrary resource $k \in S$, and define the embedding τ_2 such that $\tau_2(k) = f$ and $\tau_1(E_1) \cap \tau_2(E_2) = \{f\}$. We set the resource costs such that $c_r(x) = 2$ for all resources $r \in \tau_2(E_2) \backslash \{f\}$.

Note that the game of populations 1 and 2 can be represented by the routing game illustrated by the left network in Figure 3 if we interpret resource e as arc (s_1, t), resource f as arc (s_2, t), and resource g as arc (s_1, s_2). By the choice of our cost functions, each player of population 2 always selects the "direct connection," i.e., a strategy containing f, but neither e nor g. As long as the cost on $g := (s_1, s_2)$ is 3 (such as in the left network), each player of population 1 selects the "direct connection," i.e., a strategy containing e, but neither f nor g. However, if the cost on g is reduced from 3 down to 0 (such as in the right network in Figure 3), each player selects the seemingly cheaper strategy containing both f and g, but not e (the "indirect connection"), resulting in a Wardrop equilibrium in which each player of population 2 pays twice as much as in the Wardrop equilibrium for the left network, i.e., before the costs have been reduced. Not only the private cost of the players of population 2 but also the total cost of the new Wardrop equilibrium increased after the cost on resource g has been decreased. Note that the entire construction only involved cost reductions.

Remark 4.1. We can also characterize the universally strong Braess paradox for the case where only demand reductions are considered. For this, we can use Lemma 4.1 to construct a game that is isomorphic to the instance presented in Figure 2. Note that in this instance we need two "trivial" players having one resource each; thus, we need at least three populations.

Remark 4.2. Let us finally remark that the proof of (III) ⇔ (I) of Theorem 4.1 also works for characterizing set systems being universally immune to a "global" version of the Braess paradox, where instead of population specific costs, the total (social) cost is considered. Indeed, the counterexample derived in the proof of (III) ⇒ (I) shows that for any nonmatroid set system, there exists an embedding admitting this global form of Braess' paradox.

5. Conclusions

Our results give a characterization of the weak Braess paradox for arbitrary set systems: for any set system that is immune to the weak Braess paradox, the corresponding clutters must correspond to bases of some matroid. For the strong Braess paradox, we only derived a weaker characterization requiring the flexibility of arbitrary embeddings of set systems into the ground set of resources. Characterizing the strong Braess paradox without the use of embeddings remains an open question.

Our characterization can be transferred to (discrete) congestion games of Rosenthal [41] as follows: if we compare two special pure Nash equilibria (namely, the global potential minima) of two congestion models (for which only cost functions or demands are decreased), then our characterizations still hold. For atomic splittable congestion games (see Bhaskar et al. [5], Cominetti et al. [13], Harks [23], Roughgarden and Schoppmann [46]), it seems unclear whether or not similar results hold true.

Acknowledgments

The authors thank two anonymous reviewers for their very valuable feedback. In particular, the authors are grateful to one reviewer who pointed out the special role of clutters in their characterization. The authors also thank Christian Fersch and Andreas S. Schulz for helpful comments of an earlier draft of the manuscript.

Endnotes

[1] A formal definition of embeddings of strategy spaces into the resources is given in Section 4; see Definition 4.1.

[2] See http://homepage.ruhr-uni-bochum.de/Dietrich.Braess/; accessed October 29, 2014.

[3] There are related characterizations of series-parallel graphs in different contexts, e.g., uniqueness and Pareto efficiency of Nash equilibria (see Milchtaich [33], Richman and Shimkin [40]) and strong equilibria in congestion games (see Epstein et al. [18], Holzman and Law-Yone [26, 27]).

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