Lectures on String Topology

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Introduction

These are notes for a course on string topology given at Augsburg University in Fall 2012.

To a (closed, oriented) $n$-dimensional manifold $M$ with a base point $x_0 \in M$ one can associate three spaces of loops:

- the free loop space
  \[ \Lambda M \coloneqq C^\infty(S^1, M) \simeq C^0(S^1, M); \]

- the based loop space
  \[ \Omega M \coloneqq \{ x \in \Lambda \mid x(0) = x_0 \}; \]

- and the string space (or space of unparametrized loops)
  \[ \Lambda M / \text{Diff}^+(S^1) \simeq \Lambda M / S^1. \]

Here $\text{Diff}^+(S^1)$ denotes the group of orientation preserving diffeomorphisms of $S^1 \coloneqq \mathbb{R}/\mathbb{Z}$, and $\simeq$ means “homotopy equivalent”.

Here are a few motivations for studying these spaces.

**Topology.** Based loop spaces play a central role in algebraic topology. For example, many important spaces such as Eilenberg-McLane spaces and Lie groups are homotopy equivalent to based loop spaces, and an fundamental concept in topology is that of
an $\Omega$-spectrum, i.e., a sequence of spaces $K_1, K_2, \ldots$ with $K_n \simeq \Omega K_{n+1}$ for all $n$.

**Riemannian geometry.** One of the oldest questions in Riemannian geometry concerns the number of closed geodesics on a Riemannian manifold $(M, g)$. Since closed geodesics are critical points of the energy functional $E : \Lambda M \to \mathbb{R}$, $E(x) := \int_0^1 |\dot{x}(t)|^2 dt$, this question translates into one about the dimensions of the homology groups $H_k(\Lambda M)$.

**String theory.** The basic idea of string theory is to replace point particles by little strings, and Feynman graphs by surfaces in spacetime. It was the great insight of Chas and Sullivan, and their motivation for introducing string topology, that many of the algebraic structures in string theory already appear in the topology of loop spaces.

**Symplectic geometry.** My own motivation comes from symplectic geometry, where string topology operations naturally appear in the compactifications of moduli spaces of $J$-holomorphic curves with Lagrangian boundary conditions.

Here is a rough description of the content of these notes.

Chapter 1 is devoted to based loop spaces. Concatenation of loops at the base point makes them $H$-spaces, which in turn implies that their homology carries a canonical product, the *Pontrjagin product*, which together with the cup product on cohomology gives their cohomology the structure of a *Hopf algebra*. We will show that this severely restricts the structure of the cohomology ring and compute some examples.

In the intermediate Chapter 2 we introduce an important tool from algebraic topology, *spectral sequences*, and use it to compute the homology of based loop spaces for more examples.
In Chapter 3 we define and study the loop product on the homology of a free loop space $\Lambda M$, which gives this homology the structure of a Batalin–Vilkovisky algebra as well as a Gerstenhaber algebra. Chapter 4 is another intermediate chapter devoted to another important technique, Sullivan’s minimal models. These provide a very efficient way to compute the homology of free loop spaces. As an application, we obtain the famous result that every closed Riemannian manifold whose real cohomology ring requires at least two generators possesses infinitely many closed geodesics.

In Chapter 5 we define and study the string bracket and cobracket on the $S^1$-equivariant homology $H^*_{S^1}(\Lambda M)$, which gives this homology the structure of a Lie bialgebra. Moreover, we will show that the loop product and string bracket are homotopy invariants of the underlying manifold.

If time permits, we will discuss in Chapter 6 some applications of string topology in symplectic geometry, as well as current work on various extensions of string topology.

**Notation.** $R$ always denotes a commutative ring with unit 1. Unless stated otherwise, all homology and cohomology is taken with coefficients in $R$. 
Chapter 1

Based Loop Spaces and the Pontrjagin Product

This chapter is mostly taken from Hatcher’s book [10].

1.1 H-spaces

Definition. An $H$-space is a topological space $X$ with a continuous map $\mu : X \times X \to X$ (the product) and an element $e \in X$ (the unit) such that the maps $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity through maps $(X, e) \to (X, e)$. It is called associative if the maps $(x, y, z) \mapsto \mu(\mu(x, y), z)$ and $(x, y, z) \mapsto \mu(x, \mu(y, z))$ are homotopic, and commutative if the maps $(x, y) \mapsto \mu(x, y)$ and $(x, y) \mapsto \mu(y, x)$ are homotopic.

A map $f : X \to Y$ between $H$-spaces is an $H$-map if the maps $X \times X \to Y$, $(x, x') \mapsto f\mu(x, x')$ and $(x, x') \mapsto \mu(fx, fx')$ are homotopic, and an $H$-equivalence if there exists an $H$-map $g : Y \to X$ such that $fg$ and $gf$ are homotopic to the identity (everything rel base points).

Example 1.1. Every topological group (e.g., a Lie group) is an associative $H$-space.

\footnote{According to [10], “H” stands for “Hopf”.

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Example 1.2. Multiplication in the real and complex numbers, the quaternions, and the octonions induces $H$-space structures on the unit spheres $S^0$ and $S^1$ (commutative and associative), $S^3$ (associative), and $S^7$ (neither). By a famous theorem of Adams, these are the only spheres that admit $H$-space structures.

Example 1.3. Let $Y$ be a topological space with base point $y_0$. Then its based loop space

$$\Omega Y := \{ y : [0, 1] \to Y \mid y(0) = y(1) = y_0 \}$$

is an $H$-space with product given by concatenation

$$x \cdot y(t) := \begin{cases} x(2t) & t \in [0, 1/2], \\ y(2t - 1) & t \in [1/2, 1], \end{cases}$$

and unit given by the constant loop $e(t) \equiv y_0$. It is associative but in general not commutative, and each path $x$ has an inverse (as always up to homotopy) $x^{-1}(t) := x(1 - t)$.

Example 1.4. Multiplication of polynomials induces $H$-space structures on the spaces $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$ (with the direct limit topologies) which descend to $H$-space structures on the infinite projective spaces $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$.

The following problems establish some basic properties of $H$-spaces.

Problem 1.1. Suppose that $X$ is a CW complex and $e \in X$ a 0-cell. Show that every continuous map $\mu : X \times X \to X$ such that the maps $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity through maps $X \to X$ (not necessarily preserving $e$) is homotopic to a map $\tilde{\mu} : X \times X \to X$ such that $\tilde{\mu}(x, e) = \tilde{\mu}(e, x) = x$ for all $x \in X$ (in particular, $\tilde{\mu}$ defines an $H$-space structure). Conclude that every homotopy equivalence $X \to Y$ between an $H$-space $X$ and a CW complex $Y$ induces an $H$-space structure on $Y$. 
Problem 1.2. Every H-space $X$ is abelian, i.e., $\pi_1 X$ is abelian and acts trivially on each $\pi_k X$, $k \in \mathbb{N}$.

Problem 1.3. The universal cover of an H-space is an H-space, and the product of two H-spaces in an H-space.

Problem 1.4. If $X$ is an H-space, then for any pointed space $K$ the set $\langle K, X \rangle$ of base point preserving homotopy classes of maps $K \to X$ inherits a product with unit. If $X = \Omega Y$, then $\langle K, \Omega Y \rangle$ is a group. What is this group for $K = \text{pt}$ and for $K = S^n$?

Problem 1.5. A map $f : Y \to Z$ induces a natural map $\Omega f : \Omega Y \to \Omega Z$ which intertwines the H-space products. In particular, a homotopy equivalence $f : Y \to Z$ induces a homotopy equivalence $\Omega f : \Omega Y \to \Omega Z$ which intertwines the H-space products, so the H-equivalence class of $\Omega Y$ is a homotopy invariant of $Y$.

Example 1.5. For an abelian group $G$ and a positive integer $n$, an Eilenberg–MacLane space $K(G, n)$ is a topological space whose only nonvanishing homotopy group is $\pi_n K(G, n) = G$. For example, $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$, which follows from the homotopy exact sequences of the universal bundles $\mathbb{Z} \to \mathbb{R} \to S^1$ and $S^1 \to S^\infty \to \mathbb{C}P^\infty$ for the groups $\mathbb{Z}$ and $S^1$. We will see later that such a space $K(G, n)$ exists for all $G, n$ and is unique up to homotopy. Assuming this for now, the homotopy exact sequence of the path fibration $\Omega K(G, n) \to PK(G, n) \to K(G, n)$ shows that

$$\Omega K(G, n) \simeq K(G, n - 1).$$

In view of Problem 1.1, the H-space structure on $\Omega K(G, n + 1)$ thus induces a canonical H-space structure on $K(G, n)$.

Problem 1.6. (a) Show that the two H-space structures on $S^1$ defined in Examples 1.2 and 1.5 are equivalent.

(b) Show that the two H-space structures on $\mathbb{C}P^\infty$ defined in Examples 1.4 and 1.5 are equivalent.
Problem 1.7. (a) Show that for any Serre fibration $F \to E \to B$ with contractible total space $E$ there exists a weak homotopy equivalence $F \to \Omega B$. *Hint:* Use a contraction of $E$ to define a fibre preserving map $E \to PB$ to the path space of $B$.

(b) Conclude that every compact Lie group $G$ is homotopy equivalent to $\Omega BG$, the based loop space of its classifying space $BG$. Is it $H$-equivalent?

1.2 Pontrjagin product

Recall that for any spaces $X, Y$ there is a homology cross product $\times : H_i(X) \otimes H_j(Y) \to H_{i+j}(X \times Y)$. If $R$ is a principal ideal domain and $H_*(X)$ is a free $R$-module, then the *K"unneth formula in homology* asserts that the cross product defines isomorphisms of $R$-modules

$$
\bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \cong H_n(X \times Y),
\bigoplus_{i+j=n} \tilde{H}_i(X) \otimes \tilde{H}_j(Y) \cong \tilde{H}_n(X \wedge Y).
$$

Here $\tilde{H}_*(X) := H_*(X, x_0)$ denotes the reduced homology and

$$
X \wedge Y := (X \times Y) / (x_0 \times Y \cup X \times y_0)
$$

the *smash product* of pointed spaces $(X, x_0)$ and $(Y, y_0)$.

Now consider an $H$-space $X$. The composition

$$
H_i(X) \otimes H_j(X) \xrightarrow{\times} H_{i+j}(X \times X) \xrightarrow{\mu_*} H_{i+j}(X)
$$

induces a product on $H_*(X)$ called the *Pontrjagin product*. It has the unit $[e] \in H_0(X)$ and is in general neither associative nor commutative. For a based loop space $X = \Omega Y$, the Pontrjagin product is associative and thus makes $H_*(\Omega Y)$ an $R$-algebra. By Problem 1.5, a map $f : Y \to Z$ induces an algebra map $(\Omega f)_* :$
1.2. PONTRJAGIN PRODUCT

$H_*(\Omega Y) \rightarrow H_*(\Omega Z)$, which is an isomorphism if $f$ is a homotopy equivalence.

**Example 1.6.** Since the connected components of $\Omega S^1$ are contractible and distinguished by the mapping degrees of the corresponding maps $S^1 \rightarrow S^1$, we have a homotopy equivalence $\Omega S^1 \simeq \mathbb{Z}$. Thus $H_*(\Omega S^1) = \bigoplus_{i \in \mathbb{Z}} R \times \{i\}$ with product $(r, i) \cdot (s, j) = (rs, i + j)$. Similarly, $H_*(\Omega T^n) = \bigoplus_{i \in \mathbb{Z}^n} R \times \{i\}$ with product $(r, i) \cdot (s, j) = (rs, i + j).

**Example 1.7.** If $X, Y$ are $H$-spaces and $R$ is a field, then by the Künneth formula, $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$ with product $(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a||b|} aa' \otimes bb'$.

**Problem 1.8.** Express the Pontrjagin product on $H_*(G)$ for a compact Lie group $G$ in terms of the Lie algebra.

To each pointed space $(X, e)$ we can associate a “free” $H$-space, the James reduced product $JX := \coprod_{k \in \mathbb{N}} X^k / \sim,$

where $X^{k+1} \ni x_1 \cdots x_i e x_{i+1} \cdots x_k \sim x_1 \cdots x_i x_{i+1} \cdots x_k \in X^k$ for all $0 \leq i \leq k$. It is an $H$-space with product $\mu(x_1 \cdots x_i, x_{i+1} \cdots x_{i+j}) := x_1 \cdots x_i x_{i+1} \cdots x_{i+j}$ and unit $e$.

Denote by $J_m X \subset JX$ the image of $X^m$ in $X$. Thus we have inclusions \{e\} = J_0 \subset J_1 \subset J_2 \subset \cdots \subset JX$ and quotient maps $q : X^m \rightarrow J_m X = X^m / \sim,$

where $x_1 \cdots x_{i-1} e x_{i+1} \cdots x_m \sim x_1 \cdots x_{j-1} e x_{j+1} \cdots x_m$ for all $i, j$. If $X$ is a CW complex with $e$ a 0-cell, then $J_m X$ and $JX$ inherit CW structures from $X$.

**Example 1.8.** $JS^n = e_0 \cup e_n \cup e_{2n} \cup e_{3n} \cup \cdots$ is a CW complex with one cell $e_{in}$ of dimension $in$ for each multiple of $n$. Since each
cell is attached to lower dimensional cells, the cellular boundary maps are trivial if \( n \geq 2 \) and \( H_*(JS^n) \) is freely generated by the \( e_{in} \). Since \( \mu(e_{in}, e_{jn}) = e_{(i+j)n} \), this shows that as an \( R \)-algebra,

\[
H_*(JS^n) \cong R[e_n]
\]
is the polynomial algebra on one generator \( e_n \) of degree \( n \).

More generally, we have

**Proposition 1.9.** Let \( R \) be a principal ideal domain, and let \( X \) be a connected CW complex such that \( H_*(X) \) is a free \( R \)-module. Then \( H_*(JX) \) is isomorphic to the tensor algebra

\[
T\tilde{H}_*(X) := R \oplus \bigoplus_{m \in \mathbb{N}} \tilde{H}_*(X)^{\otimes m}
\]
on the reduced homology of \( X \).

**Proof.** The proof is taken from [10]. The compositions

\[
\tilde{H}_*(X)^{\otimes m} \hookrightarrow H_*(X)^{\otimes m} \times H_*(X^m) \xrightarrow{q_*} H_*(J_mX) \rightarrow H_*(JX)
\]
define a map \( \phi : T\tilde{H}_*(X) \rightarrow H_*(JX) \), which is a ring homomorphism because the product on \( JX \) is induced by the natural maps \( X^m \times X^n \rightarrow X^{m+n} \). To show that \( \phi \) is an isomorphism, consider the following commutative diagram of short exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & T_{m-1}\tilde{H}_*(X) & \longrightarrow & T_m\tilde{H}_*(X) & \longrightarrow & \tilde{H}_*(X)^{\otimes m} & \longrightarrow & 0 \\
\downarrow{\phi} & & \downarrow{\phi} & & \times & \cong & \\
0 & \longrightarrow & H_*(J_{m-1}X) & \longrightarrow & H_*(J_mX) & \longrightarrow & \tilde{H}_*(X^{\wedge n}) & \longrightarrow & 0.
\end{array}
\]

Here in the first row, \( T_m\tilde{H}_*(X) := R \oplus \bigoplus_{k \leq m} \tilde{H}_*(X)^{\otimes k} \), so this row is exact. The right-hand vertical map is an isomorphism by the Künneth formula in reduced homology because \( \tilde{H}_*(X) \) is a free \( R \)-module. The second row is the long exact sequence of the pair \((J_mX, J_{m-1}X)\) whose quotient is the \( m \)-fold smash product.
$X^\wedge m := X^m/e \times X^{m-1} \cup \cdots \cup X^1 \times e$; it splits into short exact sequences because of the commutativity of the right-hand square and the fact that the right-hand vertical map is an isomorphism. It follows from this diagram by induction over $m$ and the five-lemma that each $\phi : T_m \tilde{H}_*(X) \to H_*(J_m X)$ is an isomorphism, which as $m \to \infty$ shows that $\phi : T \tilde{H}_*(X) \to H_*(J X)$ is an isomorphism.

1.3 Hopf algebras

As we have seen in the previous section, the homology $H_*(X)$ of an $H$-space carries the Pontrjagin product. On the other hand, the cohomology $H^*(X)$ of any space carries the cup product. To relate these two products, we transfer the Pontrjagin product to a coproduct on cohomology as follows.

Recall that for any spaces $X, Y$ there is a cohomology cross product $\times : H^i(X) \otimes H^j(Y) \to H^{i+j}(X \times Y)$. If $H^k(X)$ is a finitely generated free $R$-module for each $k$, then the Künnett formula in cohomology asserts that the cross product defines isomorphisms of $R$-algebras

$$\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \cong H^n(X \times Y).$$

Now consider an $H$-space $X$. Throughout this section we assume that $X$ is path connected and $H^k(X)$ is a finitely generated free $R$-module for each $k$. The composition

$$H^n(X) \overset{\mu^*}{\longrightarrow} H^n(X \times X) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(X)$$

defines a coproduct

$$\Delta : H^*(X) \to H^*(X) \otimes H^*(X),$$
which is an algebra map with respect to the products induced by the cup product. The existence of the unit \( e \) translates to the formula

\[
\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{0 < i < n} \alpha'_i \otimes \alpha''_{n-i}, \quad \alpha'_i, \alpha''_i \in H^i(X)
\]

for \( \alpha \in H^n(X), \, n > 0 \). To see this, consider the commutative diagram

\[
\begin{array}{c}
H^n(X) \xrightarrow{\mu} H^n(X \times X) \xrightarrow{\times} \bigoplus_{i+j=n} H^i(X) \otimes H^j(X) \\
\downarrow 1 \downarrow \phantom{\bigoplus_{i+j=n}} \downarrow p \\
H^n(X) \end{array}
\]

where \( i : X \to X \times X \) is the map \( x \mapsto (x, e) \) and \( p := i^* \circ \times \). Write \( \Delta \alpha = \alpha'_n \otimes 1 + 1 \otimes \alpha''_n + \sum_{0 < i < n} \alpha'_i \otimes \alpha''_{n-i} \). Then the diagram shows that \( \alpha = p(\Delta \alpha) = p(\alpha'_n \otimes 1) = i^*(\alpha'_n \times 1) = \alpha'_n \), and \( \alpha = \alpha''_n \) follows analogously.

**Definition.** A Hopf algebra is a graded \( R \)-module \( A = \bigoplus_{k \geq 0} A^k \) with a product \( \mu : A \otimes A \to A, \, x \otimes y \mapsto xy \), and a coproduct \( \Delta : A \to A \otimes A \) such that

(i) \( \Delta(xy) = \Delta(x)\Delta(y) \), i.e., the coproduct is an algebra map with respect to the product;

(ii) the product has a unit \( 1 \) and \( A^0 = R \cdot 1 \), i.e., \( A \) is connected;

(iii) \( \Delta(1) = 1 \otimes 1 \), and \( \Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{0 < i < n} x'_i \otimes x''_{n-i} \) with \( x'_i, x''_i \in A^i \) for \( x \in A^n, \, n > 0 \).

Note that in a Hopf algebra \( A \) the product is not required to be associative or commutative (thus \( (A, \mu) \) is not an algebra!), and the product is not required to be coassociative or cocommutative. The preceding discussion shows
Theorem 1.10. Let $X$ be a path connected $H$-space such that $H^k(X)$ is a finitely generated free $R$-module for each $k$. Then the cohomology $H^*(X)$ is a commutative, associative Hopf algebra.

Remark 1.11. The conditions in the definition of a Hopf algebra can be rephrased more formally as follows:

(i) The diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\mu} & A \\
\downarrow{\Delta \otimes \Delta} & & \downarrow{\mu \otimes \mu} \\
A \otimes A \otimes A \otimes A & \xrightarrow{\tau} & A \otimes A \otimes A \otimes A
\end{array}
$$

commutes, where $\tau(x \otimes y \otimes z \otimes w) := (-1)^{|y||z|} x \otimes z \otimes y \otimes w$.

(ii) The compositions

$$
A \cong A \otimes R \xrightarrow{1 \otimes u} A \otimes A \xrightarrow{\mu} A, \quad A \cong R \otimes A \xrightarrow{u \otimes 1} A \otimes A \xrightarrow{\mu} A
$$

are the identity maps, where the unit $u : R \to A$, $r \mapsto r \cdot 1$ induces an isomorphism onto $A^0$.

(iii) The compositions

$$
A \xrightarrow{\Delta} A \otimes A \xrightarrow{1 \otimes c} A \otimes R \cong A, \quad A \xrightarrow{\Delta} A \otimes A \xrightarrow{c \otimes 1} R \otimes A \cong A
$$

are the identity maps, where the counit $c : A \to R$ maps $r \cdot 1 \in A^0$ to $r$ and $A^k$ to 0 for $k > 0$.

In this form, conditions (i)–(iii) are perfectly symmetric in the product and coproduct. This shows that the graded dual

$$
A^* := \bigoplus_{k \geq 0} A_k^*, \quad A_k^* := \text{Hom}(A^k, R)
$$

of a Hopf algebra is again a Hopf algebra with product $\Delta^* : A^* \otimes A^* \to A^*$, coproduct $\mu^* : A^* \to A^* \otimes A^*$, unit $c^* : R \cong R^* \to A^*$, and counit $u^* : A^* \to R^* \cong R$. Need hypothesis that $A$ is free and of finite rank in each degree?
In the following 3 examples we compute the Hopf algebra structure and its dual on a polynomial algebra $R[x]$ and on an exterior algebra $\Lambda[x]$ in one generator of positive degree $n$. Since there are no elements of degree between 0 and $n$, in all these cases we must have

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$ 

In general, an element $x$ in a Hopf algebra with this property is called \textit{primitive}. Since it is an algebra map, the coproduct is completely determined by the product:

$$\Delta(x^k) = (x \otimes 1 + 1 \otimes x)^k.$$ 

\textit{Example 1.12.} Consider $A = \Lambda[x]$ with $|x| = n$ odd. Then $x \otimes 1 \cdot 1 \otimes x = -1 \otimes x \cdot x \otimes 1 = x \otimes x$ and we obtain

$$\Delta(x^2) = x^2 \otimes 1 + 1 \otimes x^2,$$

so both sides vanish because $x^2 = 0$. By degree reasons, the dual $A^*$ is again the exterior algebra $\Lambda[\alpha]$ on a generator of degree $n$. Note that the homology of the $H$-spaces $S^1$, $S^3$ and $S^7$ fits into this picture.

\textit{Example 1.13.} Consider $A = R[x] = \Lambda[x]$ with $|x| = n > 0$ even. Then $x \otimes 1 \cdot 1 \otimes x = 1 \otimes x \cdot x \otimes 1 = x \otimes x$ and we obtain

$$\Delta(x^k) = \sum_{i=0}^{k} \binom{k}{i} x^i \otimes x^{k-i}.$$ 

The dual $A^*$ is again isomorphic to $R[x]$ as an $R$-module. Let $\alpha_k$ be the generator of $A^*_{nk}$ satisfying $\langle \alpha_k, x^k \rangle = 1$, where $\langle , \rangle$ denotes the canonical pairing $A^* \otimes A \rightarrow R$. Since the product on $A^*$ is dual to the coproduct on $A$, we obtain $\langle \alpha_i \cdot \alpha_j, x^k \rangle = \langle \alpha_i \otimes \alpha_j, \Delta(x^k) \rangle = \binom{k}{i}$ for $k = i + j$, and thus

$$\alpha_i \cdot \alpha_j = \binom{i+j}{i} \alpha_{i+j}.$$
It easily follows from this by induction that $\alpha := \alpha_1$ satisfies the relations

$$\alpha^k = k! \alpha_k$$

of a divided polynomial algebra. Note that when $R$ is a field of characteristic zero, then $\alpha^k$ generates $A_{nk}^*$ and $A^*$ is just the polynomial algebra $R[\alpha]$.

**Example 1.14.** Consider $A = R[x]$ with $|x| = n$ odd (thus $A$ is not graded commutative). Then $x \otimes 1 \cdot 1 \otimes x = -1 \otimes x \cdot x \otimes 1 = x \otimes x$ and we obtain

$$\Delta(x^2) = x^2 \otimes 1 + 1 \otimes x^2,$$

Introducing the even generator $y := x^2$, we derive from this as in the previous example the relations

$$\Delta(y^k) = \sum_{i=0}^{k} \binom{k}{i} y^i \otimes y^{k-i} = \sum_{i=0}^{k} \binom{k}{i} x^{2i} \otimes x^{2(k-i)},$$

$$\Delta(xy^k) = \sum_{i=0}^{k} \binom{k}{i} (xy^i \otimes y^{k-i} + y^i \otimes xy^{k-i})$$

$$= \sum_{i=0}^{k} \binom{k}{i} \left( x^{2i+1} \otimes x^{2(k-i)} + x^{2i} \otimes x^{2(k-i)+1} \right).$$

The dual $A^*$ is again isomorphic to $R[x]$ as an $R$-module. Let $\alpha_k$ be the generator of $A_{nk}^*$ satisfying $\langle \alpha_k, x^k \rangle = 1$. Again, the product on $A^*$ is determined by the coproduct on $A$ by duality, $\langle \alpha_i \cdot \alpha_j, x^{i+j} \rangle = \langle \alpha_i \otimes \alpha_j, \Delta(x^{i+j}) \rangle$, and we read off

$$\alpha_{2i} \cdot \alpha_{2j} = \binom{i+j}{i} \alpha_{2(i+j)}, \quad \alpha_{2i+1} \cdot \alpha_{2j+1} = 0,$$

$$\alpha_{2i} \cdot \alpha_{2j+1} = \alpha_{2i+1} \cdot \alpha_{2j} = \binom{i+j}{i} \alpha_{2(i+j)+1}$$
As in the previous example, it follows from this that the generators $\alpha := \alpha_1$ and $\beta := \alpha_2$ satisfy the relations

$$\alpha^2 = 0, \quad \beta^k = k!\alpha_{2k}, \quad \alpha\beta^k = \beta^k\alpha = k!\alpha_{2k+1}.$$ 

So $A^*$ is the tensor product of the exterior algebra $\Lambda[\alpha]$ and the divided polynomial algebra on the even generator $\beta$. Note that when $R$ is a field of characteristic zero, then $A^*$ is just the exterior algebra $\Lambda[\alpha, \beta]$ on two generators of degrees $n$ and $2n$.

According to Example 1.8, the homology of the $H$-space $JS^n$ is of the form described in the previous two examples. So we have determined the cup product structure on $H^*(JS^n)$ purely algebraically from the Pontrjagin product on $H_*(JS^n)$! Let us record this result:

**Proposition 1.15.** Let the coefficient ring $R$ be a principal ideal domain. For $n$ even, $H^*(JS^n)$ is the divided polynomial algebra on one generator $\alpha$ of degree $n$. For $n$ odd, $H^*(JS^n)$ is the tensor product of the exterior algebra on a generator $\alpha$ of degree $n$ and the divided polynomial algebra on a generator $\beta$ of degree $2n$.

**Problem** 1.9. Use the cup product on $(S^n)^k$ and the fact that the quotient map $(S^n)^k \to J_kS^m$ induces homeomorphisms on all cells to directly compute the integral cohomology ring $H^*(JS^n)$.

**Problem** 1.10. Use deep results in topology to prove that any $H$-space which is a closed manifold and whose integral cohomology is an exterior algebra on one generator of odd degree is homeomorphic to $S^1$, $S^3$, or $S^7$. 
1.4 The structure of Hopf algebras

The structure of a Hopf algebra imposes strong restrictions on the underlying algebra. For our purposes, the main result in this direction is

**Theorem 1.16** (Hopf, Leray). Let $A$ be a commutative, associative Hopf algebra over a field $K$ of characteristic 0 such that $A^k$ is finite dimensional in each degree $k$. Then as an algebra, $A$ is isomorphic to an exterior algebra $\Lambda[x_1, x_2, \ldots]$ on generators in positive degrees (at most finitely many in each degree).

**Remark 1.17.** (a) Let us emphasize that Theorem 1.16 does not say anything about the coproduct. For example, on the algebra $\Lambda[x, y]$ over $R$ with $|y| = 2|x|$, for any $c \in R$ we obtain a Hopf algebra structure by setting $\Delta(x) := x \otimes 1 + 1 \otimes x$, $\Delta(y) := y \otimes 1 + 1 \otimes y + cx \otimes x$, and extending $\Delta$ as an algebra map.

(b) Note that if the total space $A$ in Theorem 1.16 is finite dimensional, then only generators of odd degree can occur in $A \cong \Lambda[x_1, x_2, \ldots]$ (this was the special case originally proved by Hopf).

**Proof.** This short and clever proof is taken from [10]. Since $A^k$ is finite dimensional for each $k$, we can choose algebra generators $x_1, x_2, \ldots$ of $A$ with $0 < |x_1| \leq |x_2| \leq \cdots$. Since $\Delta(x_n)$ involves only $x_n$ and generators of lower degrees, the algebra $A_n \subset A$ generated by $x_1, \ldots, x_n$ is a Hopf subalgebra, i.e, $\Delta(A_n) \subset A_n \otimes A_n$. We may assume that $x_n \notin A_{n-1}$ (otherwise we remove the generator $x_n$). Since $A$ is commutative and associative, there is a canonical surjective algebra map $A_{n-1} \otimes \Lambda[x_n] \rightarrow A_n$. By induction on $n$, it suffices to show that this surjection is injective for each $n$. Let $I \subset A_n$ be the ideal generated by $x_n^2$ and the elements in $A_{n-1}$
of positive degree. Since elements of $I$ of degree $|x_n|$ belong to $A_{n-1}$, $x_n \notin A_{n-1}$ implies $x_n \notin I$. Consider the composition

$$A_n \xrightarrow{\Delta} A_n \otimes A_n \xrightarrow{q} A_n \otimes (A_n/I),$$

where $q$ is the natural quotient map. Thus $q\Delta(a) = a \otimes 1$ for $a \in A_{n-1}$ (to see this, distinguish the cases $|a| = 0$ and $|a| > 0$) and $q\Delta(x_n) = x_n \otimes 1 + 1 \otimes \bar{x}_n$, where $\bar{x}_n$ is the image of $x_n$ in $A_n/I$.

Consider first the case $|x_n|$ even. Suppose that the map $A_{n-1} \otimes \Lambda[x_n] \rightarrow A_n$ is not injective and let $\sum_{i=0}^{k} a_i x_n^i = 0$ be a nontrivial relation of smallest degree $k > 0$ in $x_n$, where $a_i \in A_{n-1}$ with $a_k \neq 0$. Applying the algebra map $q\Delta$ yields

$$0 = \sum_i (a_i \otimes 1)(x_n \otimes 1 + 1 \otimes \bar{x}_n)^i$$
$$= (\sum_i a_i x_n^i) \otimes 1 + (\sum_i i a_i x_n^{-1}) \otimes \bar{x}_n$$
$$= (\sum_i i a_i x_n^{-1}) \otimes \bar{x}_n$$

because $\sum_i a_i x_n^i = 0$. Since $x_n \notin I$ and thus $\bar{x}_n \neq 0$, this implies the relation $\sum_{i=1}^{k} i a_i x_n^{-1} = 0$ of lower degree in $x_n$ (here we use that $K$ is a field to conclude $a = 0$ from $a \times b = 0$ and $b \neq 0$). Since $a_k \neq 0$ and $K$ has characteristic 0, we conclude $ka_k \neq 0$, so the relation is nontrivial and we have a contradiction.

In the case $|x_n|$ odd consider a relation $a_0 + a_1 x_n = 0$. Applying $q\Delta$ yields

$$0 = a_0 \otimes 1 + (a_1 \otimes 1)(x_n \otimes 1 + 1 \otimes \bar{x}_n) = (a_0 + a_1 x_n) \otimes 1 + a_1 \otimes \bar{x}_n = a_1 \otimes \bar{x}_n,$$

which implies $a_1 = 0$ and hence $a_0 = 0$. \qed

Theorems 1.10 and 1.16 imply
Theorem 1.18. Let $X$ be a path connected $H$-space whose cohomology $H^k(X)$ with coefficients in a field $K$ of characteristic 0 is finite dimensional for each $k$. Then the cohomology $H^*(X)$ is isomorphic as an algebra to an exterior algebra $\Lambda[x_1, x_2, \ldots]$ on generators in positive degrees.

If the total cohomology $H^*(X)$ is finite dimensional (e.g., if $X$ is a compact Lie group), then only generators of odd degrees occur in $\Lambda[x_1, x_2, \ldots]$.

Remark 1.19 ($\Gamma$-manifolds). For closed manifolds, there is the following weakening of the notion of an $H$-space (this is in fact the notion originally introduced by Hopf in [11]). A $\Gamma$-manifold is a closed connected manifold $M$ with a continuous map $\mu : M \times M \to M$ such that for some (and hence every) point $e \in M$ the maps $M \to M$ given by $\ell_e(x) := \mu(e, x)$ and $r_e(x) := \mu(x, e)$ have nonzero mapping degrees $d_\ell$ and $d_r$. The proof of Theorem 1.10 yields in this case an algebra map $\Delta : H^*(M) \to H^*(M) \otimes H^*(M)$ satisfying $\Delta(1) = 1 \otimes 1$, and $\Delta(x) = x' \otimes 1 + 1 \otimes x'' + \sum_i x'_i \otimes x''_{n-i}$ with $|x'_i|, |x''_i| > 0$ and $x', x'' \neq 0$ for $|x| > 0$. Theorem 1.16 still holds in this situation, and we obtain Hopf’s original result that the cohomology of a $\Gamma$-manifold is isomorphic as an algebra to an exterior algebra $\Lambda[x_1, x_2, \ldots, x_m]$ on finitely many generators of odd degrees.

The following construction of $\Gamma$-manifolds was explained to me by U. Frauenfelder and P. Quast. Let $M$ be a closed Riemannian symmetric space, i.e., a closed connected Riemannian manifold which possesses for each $p \in M$ an isometry $I_p : M \to M$ fixing $p$ whose differential at $p$ is minus the identity (and hence $I^2_p = 1$).

Then one can define a map $\mu : M \times M \to M$ by $\mu(p, q) := I_p(q)$. Since $I^2_p = 1$, the map $q \mapsto I_p(q)$ has mapping degree $d_\ell = \pm 1$.

Whether the degree $d_r$ of the map $p \mapsto I_p(q)$ is nonzero, and thus
\( \mu \) defines a \( \Gamma \)-structure, depends on the space \( M \). For example, for \( M = S^n \) the degree \( d_r \) equals zero if \( n \) is even and 2 if \( n \) is odd, so \( S^n \) is a \( \Gamma \)-manifold if and only if \( n \) is odd (the “only if” follows from the structure of its cohomology ring), while it is an \( H \)-space only for \( n = 0, 1, 3, 7 \). Similarly, this construction yields a \( \Gamma \)-structure on the Lagrangian Grassmannian \( U(n)/O(n) \) for each odd \( n \) (see [1]). It would be interesting to understand in general for which Riemannian symmetric spaces this construction yields a \( \Gamma \)-structure.

**Problem 1.11.** Let \( G \) be a compact Lie group with Lie algebra \( \mathfrak{g} \).

(a) Show that the left-invariant differential forms define a subcomplex \( \Omega^*_{li}(G) \) of the de Rham complex \( (\Omega^*(G), d) \) and the inclusion \( \Omega^*_{li}(G) \hookrightarrow \Omega^*(G) \) induces an isomorphism on cohomology.

(b) Show that under the algebra isomorphism \( \Omega^*_{li}(G) \cong \Lambda \mathfrak{g}^* \) the exterior differential corresponds to \( d : \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^* \),

\[
d\alpha(\xi_0, \ldots, \xi_k) := \sum_{i<j} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_k).
\]

(c) Prove that the differential defined in (b) satisfies \( d^2 = 0 \) for any Lie algebra, and thus gives rise to the Lie algebra cohomology \( H^*(\mathfrak{g}) \). Conclude from (a) that \( H^*(\mathfrak{g}) = H^*(G; \mathbb{R}) \) for any compact Lie group.

**Problem 1.12.** Compute the real cohomology ring of \( SO(4) \)

(a) by finding a homeomorphism \( SO(4) \cong S^3 \times SO(3) \);

(b) by computing the Lie algebra cohomology, using Theorem 1.18.

If this was easy, try \( SO(n) \) for general \( n \).

**Problem 1.13.** Show that each exterior algebra \( A = \Lambda[x_1, \ldots, x_m] \) on finitely many odd generators satisfies Poincaré duality, i.e., \( A^k \cong A^{n-k} \) with \( n := |x_1| + \cdots + |x_m| \).
1.5 The loop space of a suspension

Up to now we have computed the homology $H_*(\Omega Y)$ of a based loop space only in the rather trivial case $Y = T^n$. It turns out that there is a class of spaces for which $H_*(\Omega Y)$ can be computed using our previous techniques: those which are a suspension of another space.

The suspension of a space $X$ is the space

$$S(X) := X \times I / X \times \partial I,$$

where $I := [-1, 1]$. For example, the suspension of $S^n \approx D^n/\partial D^n$ is

$$S(S^n) = S^n \times I / S^n \times \partial I \approx D^n \times I / D^n \times \partial I \cup \partial D^n \times I$$

$$= D^n \times I / \partial(D^n \times I) \approx S^{n+1}.$$

For a pointed space $(X, e)$ it is often more convenient to use the reduced suspension

$$\Sigma X := X \times I / X \times \partial I \cup e \times I,$$

which is homotopy equivalent to $S(X)$ because only the interval $e \times I$ is contracted to a point.

There is a canonical inclusion $\lambda : X \hookrightarrow \Omega \Sigma X$ associating to $x \in X$ the loop $I \ni t \mapsto [x, t] \in \Sigma X$ based at $e$ (reparametrized over the interval $[0, 1]$). See Figure fig:suspension1 for a schematic picture, as well as an illustration of the resulting $S^1$-family of loops in $\Sigma S^1 \approx S^2$.

More generally, to $(x_1, \ldots, x_m) \in X^m$ we can associate the concatenation

$$\lambda(x_1 \cdots x_m) := \lambda(x_1) \cdot \lambda(x_2) \cdots \lambda(x_m) \in \Omega \Sigma X.$$
Here we parametrize the loops $\lambda(x_1 \cdots x_m)$ such that as some $x_i$ approaches $e$, the corresponding loop in the concatenation is parametrized over shorter and shorter intervals and we have

$$\lambda(x_1 \cdots x_i e x_{i+1} \cdots x_m) = \lambda(x_1 \cdots x_i x_{i+1} \cdots x_m).$$

This can be achieved by picking any continuous function $d : X \to [0, 1]$ with $d^{-1}(0) = \{e\}$ and parametrizing the $i$-th loop in $\lambda(x_1 \cdots x_m)$ over an interval of length $d(x_i)/(d(x_1) + \cdots + d(x_m))$. It follows that $\lambda$ descends to a map

$$\lambda : JX \to \Omega \Sigma X$$

on the James reduced product. Now the main result of this section is

**Theorem 1.20.** For every connected CW complex $X$, the map $\lambda : JX \to \Omega \Sigma X$ is a weak homotopy equivalence.

Since by construction $\lambda$ is compatible with the $H$-space products, together with Proposition 1.9 this implies

**Corollary 1.21.** Let $R$ be a principal ideal domain, and let $X$ be a connected CW complex such that $H_*(X)$ is a free $R$-module. Then $\lambda : JX \to \Omega \Sigma X$ induces an algebra isomorphism (with respect to the Pontrjagin products)

$$T \widetilde{H}_*(X) \cong H_*(JX) \xrightarrow{\cong} H_*(\Omega \Sigma X).$$

Since $\Sigma S^n \simeq S^{n+1}$, Theorem 1.20 together with Proposition 1.15 implies

**Corollary 1.22.** Let the coefficient ring $R$ be a principal ideal domain. For $n$ even, $H^*(\Omega S^{n+1})$ is the divided polynomial algebra on one generator $\alpha$ of degree $n$. For $n$ odd, $H^*(\Omega S^{n+1})$ is the tensor product of the exterior algebra on a generator $\alpha$.
of degree $n$ and the divided polynomial algebra on a generator $\beta$ of degree $2n$.

**Proof of Theorem 1.20.** The main step is proving Corollary 1.21 for coefficients in a field $K$. This is done by a nice geometric construction. Let us write

$$Y := \Sigma X = Y_+ \cup_X Y_-$$

as the union of the two reduced cones

$$Y_\pm := C_\pm X := \{ [x, t] \in \Sigma X \mid \pm t \geq 0 \},$$

glued along $X \approx X \times \{ 0 \} \subset \Sigma X$, see Figure fig:suspension2. Consider the path fibration

$$\Omega Y \to PY \xrightarrow{p} Y$$

and write

$$PY = P_+ Y \cup P_- Y, \quad P_\pm Y := p^{-1}(Y_\pm),$$

thus $P_\pm Y$ is the space of paths in $Y$ starting at the base point $[e]$ and ending in $Y_\pm$. Since $P_\pm Y$ and their intersection $P_+ Y \cap P_- Y = p^{-1}(X)$ are deformation retracts of open neighbourhoods in $PY$ (check this!), we can apply the Mayer–Vietoris sequence to the pair $(P_+ Y, P_- Y)$. Since $PY$ is contractible, this yields the first row in the following commutative diagram in reduced homology:

$$\begin{align*}
\tilde{H}_*(P_+ Y \cap P_- Y) & \xrightarrow{i_+ \oplus i_-} \tilde{H}_*(P_+ Y) \oplus \tilde{H}_*(P_- Y) \\
\tilde{H}_*(\Omega Y \times X) & \xrightarrow{\cong} \tilde{H}_*(\Omega Y) \oplus \tilde{H}_*(\Omega Y)
\end{align*}$$

(1.1)
Here $i_{\pm} : P_+Y \cap P_-Y \hookrightarrow P_{\pm}$ are the inclusions, and the isomorphism at the bottom is the Künneth formula (here we use that we have field coefficients). The remaining maps are defined as follows.

Since the cone $Y_+ = C_+X$ is contractible, the restriction $\Omega Y \to P_+Y \to Y_+$ of the path fibration to $Y_+$ splits as a product $P_+Y \simeq \Omega Y \times Y_+ \simeq \Omega Y$. We can define explicit maps

\begin{align*}
 f_+ : P_+Y \to \Omega Y \times Y_+, \quad \gamma &\mapsto \left(\gamma \cdot \gamma_y^+, y = \gamma(1)\right), \\
g_+ : \Omega Y \times Y_+ \to P_+Y, \quad (\gamma, y) &\mapsto \gamma \cdot \bar{\gamma}_y^+.
\end{align*}

Here $\gamma_y^+ : [t, 1] \to Y_+, s \mapsto [x, s]$ is the canonical path in $Y_+$ from $y = [x, t]$ to the base point $[e]$, and $\bar{\gamma}_y^+$ is the inverse path, see Figure \text{fig:suspension2}. Since the loops $\gamma_y^+ \cdot \bar{\gamma}_y^+$ and $\bar{\gamma}_y^+ \cdot \gamma_y^+$ are canonically homotopic with fixed ends to the constant loops at $[e]$ resp. $y$, the formulas

\begin{align*}
 f_+g_+(\gamma, y) = \left(\gamma \cdot \bar{\gamma}_y^+ \cdot \gamma_y^+, y\right) \sim (\gamma, y), \quad g_+f_+(\gamma) = \gamma \cdot \gamma_y^+ \cdot \bar{\gamma}_y^+ \sim \gamma
\end{align*}

show that $f_+$ and $g_+$ are homotopy inverses. The maps $f_-$ and $g_-$ are defined analogously. Note that the maps $f_{\pm}$ and $g_{\pm}$ restrict to homotopy inverse maps

\begin{equation*}
 P_+Y \cap P_-Y \xrightarrow{f_{\pm}} \Omega Y \times X \xrightarrow{g_{\pm}} P_+Y \cap P_-Y.
\end{equation*}

Consider the following commutative diagram:

\begin{equation*}
 P_+Y \cap P_-Y \xrightarrow{(i_+, i_-)} P_+Y \amalg P_-Y \\
 \xrightarrow{g_-} \Omega Y \times X \xrightarrow{(j_+, j_-)} \Omega Y \amalg \Omega Y,
\end{equation*}

where $\pi : \Omega Y \times Y \to \Omega Y$ is the projection onto the first factor and the map $(j_+, j_-)$ is defined by commutativity. Passing to reduced homology yields the remaining maps in the diagram (1.1).
Since \( f_\sim g_\sim \sim 1 \), the map \( j_\sim = \pi f_\sim g_\sim \) is homotopic to the projection \( \pi : \Omega Y \times X \rightarrow X \). On the other hand, since \( f_\sim g_\sim (\gamma, y) = (\gamma \cdot \gamma_y^- \cdot \gamma_y^+, y) \), and \( \gamma_y^- \cdot \gamma_y^+ \) is just the loop \( \lambda(x) \) defined above (where \( y = [x, 0] \)), the map \( j_\sim = \pi f_\sim g_\sim : (\gamma, x) \mapsto \gamma \cdot \lambda(x) \) is just the composition

\[
\Omega Y \times X \xrightarrow{\mathbb{I} \times \lambda} \Omega Y \times \Omega Y \rightarrow \Omega Y,
\]

where the last arrow is the \( H \)-space product. This shows first of all that the map \( j_\sim \) in (1.1) corresponds to the projection

\[
\left( H_*(\Omega Y) \otimes \tilde{H}(X) \right) \oplus \tilde{H}_*(\Omega Y) \rightarrow \tilde{H}_*(\Omega Y)
\]

onto the second factor. Since \( j_\sim \oplus j_\sim \) is an isomorphism, the restriction of \( j_\sim \) to the kernel of \( j_\sim \) must be an isomorphism, which by the preceding discussion is just the composition

\[
H_*(\Omega Y) \otimes \tilde{H}_*(X) \xrightarrow{\mathbb{I} \times \lambda_*} H_*(\Omega Y) \otimes \tilde{H}_*(\Omega Y) \rightarrow \tilde{H}_*(\Omega Y),
\]

where the last arrow is the Pontrjagin product. By the algebraic Lemma 1.23 below (with \( A = H_*(\Omega Y) \), \( V = \tilde{H}_*(X) \), and \( i = \lambda_* \)), this implies that the canonical algebra homomorphism \( T\lambda_* : T\tilde{H}_*(X) \rightarrow H_*(\Omega Y) \) extending \( \lambda_* \) is an isomorphism. Since \( \lambda \) is a map of \( H \)-spaces, \( T\lambda_* \) is given by the composition of maps

\[
T\tilde{H}_*(X) \rightarrow H_*(JX) \xrightarrow{\lambda_*} H_*(\Omega Y),
\]

where the first map is the algebra isomorphism from Proposition 1.9. So we have shown that \( \lambda_* : H_*(JX) \rightarrow H_*(\Omega Y) \) is an isomorphism with field coefficients. Applying the universal coefficient theorem with the fields \( \mathbb{Q} \) and \( \mathbb{Z}_p \), this implies that \( \lambda_* : H_*(JX) \rightarrow H_*(\Omega Y) \) is also an isomorphism with integer coefficients. Since the \( H \)-spaces \( JX \) and \( \Omega Y \) are abelian (Problem 1.2), the Hurewicz theorem for abelian spaces implies that \( \lambda \)
induces isomorphisms on all homotopy groups, so by Whitehead’s theorem it is a homotopy equivalence.

It remains to prove the algebraic lemma used in the proof of Theorem 1.20.

**Lemma 1.23.** Let $A$ be a graded algebra over a field $K$ with $A^0 = K$, and let $V$ be a graded $K$-vector space with $V^0 = 0$. Let $i : V \to \tilde{A} := A/A^0$ be a linear grading preserving map such that the map $\mu : A \otimes V \to \tilde{A}$, $a \otimes v \mapsto a \cdot i(v)$ is an isomorphism. Then the canonical algebra homomorphism $Ti : TV \to A$ extending $i$ is an isomorphism.

**Proof.** We prove by induction on $n$ that each $a \in A^n$ can be uniquely written as $Ti(w)$ with $w \in TV$. For $n = 0$ this follows from $A^0 = K$, so suppose $n > 0$. Since $\mu$ is an isomorphism, we can uniquely write $a = \sum_j a_j i(v_j)$ with $a_j \in A$, $v_j \in V$. Now $V^0 = 0$ implies $|v_j| > 0$, hence $|a_j| < n$, so by induction hypothesis we can uniquely write $a_j = Ti(w_j)$ with $w_j \in TV$. Thus $a = \sum_j Ti(w_j)i(v_j) = Ti(\sum_j w_j \otimes v_j)$ and this representation is unique.

**Problem 1.14.** For pointed spaces $(X, x_0)$ and $(Y, y_0)$ we define their *wedge sum* $X \vee Y := X \amalg Y / x_0 \sim y_0$, and we denote by $\langle X, Y \rangle$ the set of base point preserving homotopy classes of maps $X \to Y$. Show:

(a) The assignment

$$
(x, t) \mapsto \begin{cases} 
(x, 2t) & t \in [0, 1/2], \\
(x, 2t - 1) & t \in [1/2, 1]
\end{cases}
$$
defines a canonical map
\[ \Sigma X \mapsto \Sigma X \vee \Sigma X. \]
Sending \((f, g)\) to the composition \(\Sigma X \longrightarrow \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y\) defines a group structure on the set \(\langle \Sigma X, Y \rangle\). This group is abelian if \(X\) is itself a suspension, and it equals \(\pi_{k+1}(Y)\) for \(X = S^k\).

(b) Associating to \(f : \Sigma X \to Y\) the family of loops \(t \mapsto f[x, t], \ x \in X\), gives a canonical identification (the adjoint relation)
\[ \langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle, \]
which identifies the product on \(\langle \Sigma X, Y \rangle\) defined in (a) with the product on \(\langle X, \Omega Y \rangle\) induced by the \(H\)-space structure on \(\Omega Y\).
Chapter 2

Spectral Sequences and Applications

In this chapter we introduce a more advanced technique from algebraic topology, spectral sequences, and use it to compute the cohomology of based loop spaces in further examples. We follow the discussion in the book by Bott and Tu [2], omitting some details of proofs.

2.1 The spectral sequence of a filtered complex

We begin with an algebraic lemma.

**Lemma 2.1.** Every exact triangle (called an exact couple)

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A \\
\downarrow{k} & & \downarrow{j} \\
E & \xrightarrow{\delta} & E
\end{array}
\]

of \(R\)-modules gives rise to a new exact triangle (called the derived couple)

\[
\begin{array}{ccc}
A' & \xrightarrow{j'} & A' \\
\downarrow{k'} & & \downarrow{j'} \\
E' & \xrightarrow{\delta'} & E'
\end{array}
\]
via the following definitions:

\[ d := jk : E \to E, \quad E' := H(E, d), \quad A' := iA \subset A, \]

\[ i' := i|_{A'}, \quad j'(ia) := [ja], \quad k'[e] := ke. \]

For later usage, let us record these definitions in short-hand notation as

\[ i' = i, \quad j' := ji^{-1}, \quad k' := k, \quad d' := ji^{-1}k. \quad (2.1) \]

**Proof.** Let us check that everything is well-defined. First, \( kj = 0 \) implies \( d^2 = jkjk = 0 \), so the homology \( E' := H(E, d) \) is defined. For the definition of \( j' \), note that \( d(ja) = jkja = 0 \), so \( ja \) represents a homology class \( [ja] \in H(E, d) \). If \( ia = \bar{ia} \), then by exactness \( a - \bar{a} = ke \) and hence \( [ja - j\bar{a}] = [jke] = [de] = 0 \), which shows well-definedness of \( j' \). For well-definedness of \( k' \), note first that \( 0 = de = jke \) implies \( ke = ia \in A' \). Moreover, \( [e] = 0 \) implies \( e = d\bar{e} = jk\bar{e} \), and hence \( ke = kjk\bar{e} = 0 \).

The proof of exactness is left as an exercise. \( \square \)

**Problem 2.1.** Prove exactness of the derived couple in Lemma 2.1.

**Definition.** A filtered complex is a differential complex \((K, D)\) of \( R \)-modules (i.e., \( D : K \to K \) satisfies \( D^2 = 0 \)) together with a decreasing filtration

\[ K = K_0 \supset K_1 \supset K_2 \supset \cdots \]

of submodules satisfying \( D(K_p) \subset K_p \) for all \( p \).

Given a filtered complex, we set \( K_p := K \) for \( p < 0 \). The short exact sequences

\[ 0 \to K_{p+1} \xrightarrow{i} K_p \xrightarrow{j} K_p/K_{p+1} \to 0 \]
induce long exact sequences in homology (with respect to $D$)

$$
\cdots \to H(K_{p+1}) \xrightarrow{i} H(K_p) \xrightarrow{j} H(K_p/K_{p+1}) \xrightarrow{k} H(K_{p+1}) \to \cdots.
$$

All these exact sequences are encoded in the exact couple

$$
A_1 := \bigoplus_{p \in \mathbb{Z}} H(K_p) \xrightarrow{i_1} A_1 \xleftarrow{k_1} E_1 := \bigoplus_{k \in \mathbb{Z}} H(K_p/K_{p+1}).
$$

Iterative application of Lemma 2.1 thus provides a sequence of exact couples

$$
A_r \xrightarrow{i_r} A_r \xleftarrow{k_r} E_r \xleftarrow{j_r} E_r
$$

with $E_{r+1} = H(E_r, d_r = j_r k_r)$.

**Definition.** A spectral sequence is a sequence of differential complexes $(E_r, d_r)$ with $E_{r+1} = H(E_r, d_r)$.

Let us emphasize that, while in the above sequence each exact couple completely determines the next exact couple, in a spectral sequence $(E_r, d_r)$ determines the next $R$-module $E_{r+1}$ but not the differential $d_{r+1}$, which needs to be given as an additional input for each $r$.

It is natural to ask when the spectral sequence above becomes stationary, i.e., after some $r$ all the exact couples are equal. This happens if and only if $i_r$ is injective, or equivalently $k_r = 0$, or equivalently $j_r$ is surjective. Note that these conditions imply $d_r = 0$, but $d_r = 0$ is not sufficient. The simplest case in which this occurs is when the filtration has finite length $\ell$, i.e.,

$$
K = K_0 \supset K_1 \supset \cdots \supset K_\ell \supset K_{\ell+1} = 0.
$$
To see this, note that \( A_1, A_2, A_3, \ldots, A_{\ell+1} \) are the direct sums of the following (non-exact) sequences whose arrows define the maps \( i_r \) (where we write \( i := i_1 \)):

\[
\cdots \leftarrow H(K) \leftarrow H(K_1) \leftarrow iH(K_2) \leftarrow i^2 H(K_3) \leftarrow \cdots \leftarrow i^{\ell} H(K_{\ell}) \leftarrow 0,
\]

\[
\cdots \leftarrow H(K) \supset iH(K_1) \leftarrow iH(K_2) \leftarrow i^2 H(K_3) \leftarrow \cdots \leftarrow i^{\ell} H(K_{\ell}) \leftarrow 0,
\]

\[
\cdots \leftarrow H(K) \supset i^2 H(K_1) \supset i^2 H(K_2) \leftarrow i^2 H(K_3) \leftarrow \cdots \leftarrow i^2 H(K_{\ell}) \supset 0,
\]

\[
\cdots \leftarrow H(K) \supset i^\ell H(K_1) \supset i^\ell H(K_2) \supset i^\ell H(K_3) \supset \cdots \supset i^\ell H(K_{\ell}) \supset 0.
\]

Thus more and more of the maps become inclusions until the map \( i_{\ell+1} \) consists entirely of inclusions, so the sequence becomes stationary at the exact couple

\[
A_{\ell+1} \leftarrow \cdots \leftarrow A_\ell = A_{\ell+1} = \cdots =: A_{\infty}
\]

\[
E_{\ell+1} \leftarrow \cdots \leftarrow E_\ell = E_{\ell+2} = \cdots =: E_{\infty}.
\]

We see that

\[
E_{\infty} \cong A_{\ell+1}/iA_{\ell+1} = \bigoplus_{p=0}^{\ell} i^p H(K_p)/i^{p+1} H(K_{p+1}) = \bigoplus_{p=0}^{\ell} F_p/F_{p+1} =: GH(K).
\]

is the associated graded module of the induced filtration on homology

\[
H(K) = F_0 \supset F_1 \supset F_2 \supset \cdots, \quad F_p := i^p H(K_p).
\]

**Definition.** One says that a spectral sequence \((E_r, d_r)\) converges to a filtered module \( H = F_0 \supset F_1 \supset F_2 \supset \cdots \) if it becomes stationary and \( E_{\infty} = \bigoplus_p F_p/F_{p+1} =: GH \) is the associated graded module of \( H \).
2.1. THE SPECTRAL SEQUENCE OF A FILTERED COMPLEX

The argument above also works if $K = \bigoplus_n K^n$ has an additional grading (referred to as the dimension) such that $D(K^n) \subset K^{n+1}$ and the filtrations $K^n = K^n_0 \supset K^n_1 \supset K^n_2 \supset \cdots$, $K^n_p := K^n \cap K_p$, have finite length in each dimension $n$, where convergence now means that the sequence becomes stationary in each dimension (see [2] for more details). Thus we have shown

**Theorem 2.2.** Let $K = \bigoplus_n K^n = K_0 \supset K_1 \supset K_2 \supset \cdots$ be a graded filtered complex such that the filtration has finite length in each dimension $n$. Then the short exact sequence $0 \to \bigoplus_p K_{p+1} \to \bigoplus_p K_p \to \bigoplus_p K_p/K_{p+1} \to 0$ induces a spectral sequence $(E_r, d_r)$ which converges to the homology $H(K)$. \hfill \Box

**Problem 2.2.** Fill in the details of the proof of Theorem 2.2 in the graded case.

**Remark 2.3.** The terms $A_r$ and $E_r$ in the above exact couples are naturally graded by the filtration degree as follows. For $r = 1$ we set $A^p_1 := H(K_p)$ and $E^p_1 := H(K_p, K_{p+1})$. Then we define inductively $A^p_r := i_{r-1} A^{p+1}_{r-1}$ and $E^p_r := H(E^p_{r-1}, d_{r-1})$. With respect to this grading the maps have degrees $|i_1| = -1$, $|j_1| = 0$, $|k_1| = 1$, so it follows from the formulas (2.1) that $|i_2| = |i_1| = -1$, $|j_2| = |j_1 i_1^{-1}| = 1$, $|k_2| = |k_1| = 1$, next $|i_3| = |i_2| = -1$, $|j_3| = |j_2 i_2^{-1}| = 2$, $|k_3| = |k_2| = 1$, and hence $|i_r| = -1$, $|j_r| = r - 1$, $|k_r| = 1$, $|d_r| = |j_r k_r| = r$.

This gives another way to see that the above spectral sequence converges: If the filtration has finite length $\ell$, then $E^p_1 = H(K_p, K_{p+1}) = 0$ for $p < 0$ and for $p > \ell$. Since $E^p_r$ can only become smaller as $r$ increases, it follows that $E^p_r = 0$ for $p < 0$ and for $p > \ell$, for all $r$. Since $d_r$ increases the filtration degree by $r$, this shows that $d_r = 0$.
for all \( r > \ell + 1 \), so the spectral sequence becomes stationary at \( E_{\ell+1} \).

**Remark 2.4.** Some care must be taken when relating a filtered module \( H = F_0 \supset F_1 \supset F_2 \supset \cdots \) to its associated graded module \( GH = \bigoplus_p F_p/F_{p+1} \) because \( GH \) is in general *not* isomorphic to \( H \). For example, the graded module associated to the filtration \( \mathbb{Z} \supset 2\mathbb{Z} \supset 0 \) of \( \mathbb{Z} \) is \( 2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \mathbb{Z} \). However, we always have \( \text{rank } GH = \text{rank } H \).

**Remark 2.5.** Of course, Theorem 2.2 also holds if \( D(\mathcal{K}^n) \subset \mathcal{K}^{n-1} \). Moreover, we also get a spectral sequence from an *increasing* filtration

\[
K_0 \subset K_1 \subset K_2 \subset \cdots \subset K = \bigcup_{p \geq 0} K_p
\]

of submodules satisfying \( D(\mathcal{K}_p) \subset \mathcal{K}_p \) for all \( p \): Setting \( K_p := 0 \) for \( p < 0 \), the short exact sequences

\[
0 \rightarrow K_{p-1} \overset{i}{\rightarrow} K_p \overset{j}{\rightarrow} K_p/K_{p-1} \rightarrow 0
\]

induce long exact sequences in homology (with respect to \( D \))

\[
\cdots \rightarrow H(K_{p-1}) \overset{i}{\rightarrow} H(K_p) \overset{j}{\rightarrow} H(K_p/K_{p-1}) \overset{k}{\rightarrow} H(K_{p-1}) \rightarrow \cdots ,
\]

and thus a sequence of exact couples starting with

\[
A_1 := \bigoplus_{p \in \mathbb{Z}} H(K_p) \xrightarrow{i_1} A_1
\]

\[
E_1 := \bigoplus_{k \in \mathbb{Z}} H(K_p/K_{p-1}).
\]

If the filtration has finite length in each dimension \( n \) (for some additional grading), then the spectral sequence converges to the homology \( H(\mathcal{K}) \) in the sense that

\[
E_{\infty} \cong \bigoplus_{p \geq 0} F_p/F_{p-1} =: GH(\mathcal{K})
\]
is the associated graded module of the induced filtration on homology

\[ F_0 \subset F_1 \subset F_2 \subset \cdots \subset H(K) = \bigcup_{p \geq 0} F_p, \quad F_p := \iota H(K_p), \]

where \( \iota : H(K_p) \to H(K) \) is the map induced by the inclusion \( K_p \hookrightarrow K \).

For later use, let us note the following

**Lemma 2.6.** The differential \( d_1 : H^n(K_p/K_{p+1}) \to H^{n+1}(K_{p+1}/K_{p+2}) \) of the spectral sequence in Theorem 2.2 is the boundary map in the long exact sequence of the triple \( (K_p, K_{p+1}, K_{p+2}) \).

**Proof.** Set \( (A, B, C) := (K_p, K_{p+1}, K_{p+2}) \) and consider the following commutative diagram, where the maps are the obvious inclusions and projections, and the rows and first column are exact:

\[
\begin{array}{cccccc}
0 & \rightarrow & B & \rightarrow & A & \rightarrow & A/B & \rightarrow & 0 \\
\downarrow & & \downarrow j & & \downarrow & & \downarrow = & & \downarrow & \\
0 & \rightarrow & B/C & \rightarrow & A/C & \rightarrow & A/B & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & 0.
\end{array}
\]

It induces the following commutative diagram on homology, where \( j_1 = j_* \) and \( k_1, d_1 \) are the boundary maps in the long exact se-
sequences of the rows:

\[ H^n(A/B) \xrightarrow{k_1} H^{n+1}(B) \]
\[ H^n(A/B) \xrightarrow{d_1} H^{n+1}(B/C). \]

Since \( j_1 k_1 \) is the first differential in the spectral sequence, this proves the lemma.

In topology, spectral sequences usually arise from an increasing sequence

\[ X_0 \subset X_1 \subset X_2 \subset \cdots \subset X = \bigcup_{p \geq 0} X_p \]

of subspaces of a topological space \( X \). If we assume that every compact subset of \( X \) is contained in some \( X_p \), then we get an increasing filtration on the singular chain complex

\[ C_\ast(X_0) \subset C_\ast(X_1) \subset C_\ast(X_2) \subset \cdots \subset C_\ast(X) = \bigcup_{p \geq 0} C_\ast(X_p), \]

and thus a spectral sequence \((E_r, d_r)\) with \( E_1 = \bigoplus_{p \geq 0} H_\ast(X_p, X_{p-1}) \). The spectral sequence converges to \( H_\ast(X) \) provided that for each \( n \) all but finitely many of the maps \( H_n(X_p) \to H_n(X_{p+1}) \) are isomorphisms. By Lemma 2.6, the differential \( d_1 : H_\ast(X_p, X_{p-1}) \to H_{\ast-1}(X_{p-1}, X_{p-2}) \) is the boundary map in the long exact sequence of the triple \((X_p, X_{p-1}, X_{p-2})\).

**Example 2.7** (cellular homology). Suppose that \( X \) is a CW complex and \( X_p \subset X \) is the \( p \)-skeleton for each \( p \). Since \( X_p/X_{p-1} = \bigvee_\alpha S^p_\alpha \) is a wedge sum of \( p \)-spheres, \( H_n(X_p, X_{p-1}) \) equals zero for \( n \neq p \), and the cellular chain group \( \bigoplus_\alpha R \) for \( n = p \). Thus the maps \( H_n(X_p) \to H_n(X_{p+1}) \) are isomorphisms for \( p \neq n, n-1 \) and the above spectral sequence converges to \( H_\ast(X) \). By the preceding
discussion, \( d_1 : H_p(X_p, X_{p-1}) \to H_{p-1}(X_{p-1}, X_{p-2}) \) is the cellular boundary map, and hence

\[
E_2 = \bigoplus_{p \geq 0} H^\text{cell}_p(X)
\]
is the total cellular homology. Note that here the filtration degree coincides with the dimension. Since according to Remark 2.3 the differential \( d_r \) decreases the dimension by 1 and the filtration degree by \( r \), we have \( d_2 = d_3 = \cdots = 0 \) and thus

\[
E_2 = \cdots = E_\infty = \bigoplus_{p \geq 0} F^p/F^{p-1}, \quad F^p := \iota H_*(X_p) \subset H_*(X).
\]

It follows algebraically from this that

\[
H_n(X) \cong H_n^\text{cell}(X)
\]
for all \( n \): Since \( F^p/F^{p-1} \cong H^\text{cell}_p(X) \) lives only in dimension \( n \), the submodules \( F^p_n := F^p \cap H_n(X) \) satisfy \( F^p_n/F^{p-1}_n = 0 \) for \( n \neq p \), i.e.,

\[
0 = F^0_n = F^1_n = \cdots = F^{n-1}_n \subset F^n_n = \cdots = H_n(X),
\]
and hence \( H_n(X) = F^n_n = F^n_n/F^{n-1}_n \cong H_n^\text{cell}(X) \).

**Problem 2.3.** Express the terms \( (R_r, d_r) \) in the spectral sequence See [3] of a filtered complex \( K = K_0 \supset K_1 \supset \cdots \) directly in terms of the relative homology groups \( H(K_p/K_{p+r}, D) \).

### 2.2 The spectral sequence of a double complex

**Definition.** A *double complex* is a bigraded \( R \)-module \( K = \bigoplus_{p,q \geq 0} K^{p,q} \) with two commuting differentials

\[
\delta : K^{p,q} \to K^{p+1,q}, \quad d : K^{p,q} \to K^{p,q+1}, \quad d^2 = \delta^2 = 0, \quad d\delta = \delta d.
\]
To a double complex we can naturally associate a single graded complex \((K = \bigoplus_{n \geq 0} K^n, D)\) via

\[
D := \delta + d' : K^n := \bigoplus_{p+q=n} K^{p,q} \rightarrow K^{n+1}, \quad d' := (-1)^p d \text{ on } K^{p,q}
\]

with the decreasing filtration \(K = K_0 \supseteq K_1 \supseteq \cdots\),

\[
K_p := \bigoplus_{i \geq p, q \geq 0} K^{i,q}.
\]

Since \(i \geq p\) and \(q \geq 0\) implies \(i + q \geq p\), the \(n\)-dimensional part

\[
K^n_p := K_p \cap K^n = \bigoplus_{i \geq p, q \geq 0, i+q=n} K^{i,q}
\]

vanishes for \(p > n\), so the filtration has finite length \(n\) in dimension \(n\). Hence, by Theorem 2.2, the spectral sequence \((E_r, d_r)\) of this filtered complex converges to the total homology \(H(K, D)\).

Let us describe the terms in this spectral sequence more explicitly. We have

\[
K^n_p = K^{p,n-p} \oplus K^{p+1,n-p-1} \oplus \cdots \oplus K^{n,0}
\]

and thus \(K^n_p/K^n_{p+1} = K^{p,q}\), where \(p + q = n\). Since \(D = d' = (-1)^p d\) on this quotient, it follows that

\[
H^n(K_p/K_{p+1}, D) = \frac{\ker(d : K^{p,q} \rightarrow K^{p,q+1})}{\text{im} (d : K^{p,q-1} \rightarrow K^{p,q})} =: H^{p,q}(K, d),
\]

and hence

\[
E_1 = \bigoplus_{p,q \geq 0} E_1^{p,q}, \quad E_1^{p,q} = H^{p,q}(K, d).
\]
2.2. THE SPECTRAL SEQUENCE OF A DOUBLE COMPLEX

The differential \( d_1 = j_1k_1 \) is shown in the following diagram of homology groups with respect to \( D \):

\[
\cdots H^n(K_p/K_{p+1}) \xrightarrow{k_1} H^{n+1}(K_{p+1}) \xrightarrow{i_1} H^{n+1}(K_p) \xrightarrow{j_1} H^{n+1}(K_p/K_{p+1}) \]

Consider \([a]_1 \in E_{1}^{p,q} = H^n(K_p/K_{p+1}, D)\), \( n = p + q \), represented by \( a \in K^{p,q} \) with \( da = 0 \). Since, by Lemma 2.6, \( d_1 \) is the boundary map in the long exact sequence of the triple \((K_p, K_{p+1}, K_{p+2})\), it follows that

\[
d_1[a]_1 = [Da]_1 = [\delta a]_1 \in H^{n+1}(K_{p+1}/K_{p+2}, D) = E_{1}^{p+1,q}.
\]

Since \( d_1 \) is induced by \( \delta \), we have

\[
E_2 = \bigoplus_{p,q \geq 0} E_2^{p,q}, \quad E_2^{p,q} = H^p\left(H^{\cdot,q}(K, d), \delta \right).
\]

To compute \( d_2 \), consider \([a]_2 \in E_2^{p,q} \), represented by \( a \in K^{p,q} \) with \( da = 0 \) and \([\delta a]_1 = 0 \), i.e., \( \delta a = -d'b \) for some \( b \in K^{p+1,q-1} \). For \( n = p + q \), the element \( a + b \in K^n_p \) agrees with \( a \) in \( K^n_p/K^n_{p+1} \) and satisfies

\[
D(a + b) = d'a + (\delta a + d'b) + \delta b = \delta b \in K^{n+1}_{p+1}.
\]

Now note that \( \delta b \) actually lies in \( K^{p+2,q-1} \subset K^{n+1}_{p+2} \), so \( k_1[a]_1 = k_1[a+b]_1 = i_1[\delta b]_1 \in H^{n+1}(K_{p+1}, D) \), where \( i_1 : H^{n+1}(K_{p+2}, D) \rightarrow H^{n+1}(K_{p+1}, D) \) is the map induced by the inclusion \( i : K^{n+1}_{p+2} \hookrightarrow K^{n+1}_{p+1} \). It follows from (2.1) that \( d_2[a]_2 \) is the class of \( j_1[\delta b]_1 \in H^{n+1}(K_{p+2}/K_{p+3}, D) = E_1^{p+2,q-1} \) in \( E_2^{p+2,q-1} \). So we have

\[
d_2 : E_2^{p,q} \mapsto E_2^{p+2,q-1}, \quad [a]_2 \mapsto [\delta b]_2,
\]

where \( a \) and \( b \) form the following zigzag in which horizontal arrows correspond to \( \delta \) (increasing \( p \)), vertical ones to \( d' \) (increasing \( q \)),
and $\delta a = -d'b$:

Continuing inductively, we see that each $E_r$ is bigraded as

$$E_r = \bigoplus_{p,q \geq 0} E_{r}^{p,q},$$

where an element $[a]_r \in E_{r}^{p,q}$ is represented by a zigzag of length $r$ with $a_0 \in K^{p,q}$ and $\delta a_i = -d' a_{i+1}$ for $i = 0, \ldots, r - 2$:

Then $[a_0]_{r-1} = [a_0 + \cdots + a_{r-1}]_{r-1}$ and $D(a_0 + \cdots + a_{r-1}) = \delta a_{r-1} \in K^{p+r,q-r+1}$ imply

$$k_{r-1}[a_0]_{r-1} = [D(a_0 + \cdots + a_{r-1})]_{r-1} = i_{r-1}[\delta a_{r-1}]_{r-1},$$

so we have

$$d_r : E_{r}^{p,q} \mapsto E_{r}^{p+r,q-r+1}, \quad [a_0]_r \mapsto [\delta a_{r-1}]_r.$$
In the limit we find the induced filtration on the total homology
\[ H^n(K, D) = H^n(K_0) \supset iH^n(K_1) \supset \cdots, \]
whose associated graded module consists of the terms
\[ i^p H^n(K_p, D)/i^{p+1} H^n(K_{p+1}, D) \cong E_{p,q}^\infty, \quad p + q = n. \]
Thus we have shown the following refinement of Theorem 2.2 for a double complex:

**Theorem 2.8.** Let \( K = \bigoplus_{p,q \geq 0} K^{p,q} \) be a double complex with differentials \( \delta : K^{p,q} \to K^{p+1,q} \) and \( d : K^{p,q} \to K^{p,q+1} \). Then there exists a spectral sequence \((E_r, d_r)\) converging to the total homology \( H(K, D = \delta + (-1)^p d) \) with the following properties:

\[
E_r = \bigoplus_{p,q \geq 0} E_r^{p,q}, \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},
\]
\[
E_1^{p,q} = H^{p,q}(K, d), \quad E_2^{p,q} = H^p \left( H^{-q}(K, d), \delta \right),
\]
\[
GH^n(K, D) = \bigoplus_{p+q=n} E_{\infty}^{p,q}.
\]

**Problem 2.4.** Let \( (A = \bigoplus_{p \geq 0} A^p, \delta) \) and \( (B = \bigoplus_{q \geq 0} B^q, d) \) be two cochain complexes of \( R \)-modules.

(a) Show that \( A \otimes B \) becomes a double complex with the differentials \( \delta \otimes 1 \) and \( 1 \otimes d \).

(b) If \( R \) is a field, show that in the spectral sequence of this double complex all higher differentials \( d_2, d_3, \ldots \) vanish and deduce the **Eilenberg–Zilber theorem** that the total cohomology of the double complex equals

\[
H^*(A \otimes B, D = \delta \otimes 1 + 1 \otimes d') \cong H^*(A, \delta) \otimes H^*(B, d).
\]
2.3 Products

For many applications one needs to understand the behaviour of spectral sequences with respect to products. Consider a double complex \((K = \bigoplus_{p,q \geq 0} K^{p,q}, \delta, d)\) with a (not necessarily associative or commutative) product

\[
\cdot : K^{p,q} \otimes K^{p',q'} \to K^{p+p',q+q'}
\]

for which \(\delta\) and \(d' = (-1)^p d\) are derivations, i.e.,

\[
\delta(a \cdot b) = \delta a \cdot b + (-1)^{|a|} a \cdot \delta b
\]

and the same for \(d'\), where \(|a| = p + q\) for \(a \in K^{p,q}\) denotes the total degree. It follows that \(D = \delta + d'\) is also a derivation and the product descends to the cohomologies with respect to \(\delta, d,\) and \(D\). In particular, the product descends to \(E_1 = H(K, d)\) and, since \(d_2 = \delta\), to \(E_2 = H(H(K, d), \delta)\). More generally, we have

**Theorem 2.9.** If a double complex \(K = \bigoplus_{p,q \geq 0} K^{p,q}\) carries a product with respect to which \(\delta\) and \(d' = (-1)^p d\) are derivations, then each \(E_r\) in the associated spectral sequence inherits a product with respect to which \(d_r\) is a derivation.

**Proof.** Let us first show that \(d_2\) is a derivation. Consider \([a_0] \in E_2^{p_1,q_1}\) and \([b_0] \in E_2^{p_2,q_2}\) represented by the zigzags

\[
0 \quad 0
\]

\[
\begin{array}{c}
\uparrow \\
da_0 \quad b_0 \\
\uparrow \\
a_1, \quad b_1
\end{array}
\]

Then \(d'a_0 = d'b_0 = 0\), \(\delta a_0 = d'a_1\) and \(\delta b_0 = d'b_1\) imply

\[
\delta(a_0 b_0) = \delta a_0 \cdot b_0 + (-1)^{|a_0|} a_0 \delta b_0 = -(d'a_1 \cdot b_0 + (-1)^{|a_0|} a_0 d'b_1) = -d'(a_0 b_1 +
\]


so \([a_0b_0] \in E^{p_1+p_2,q_1+q_2}_2\) is represented by the zigzag

\[
\begin{array}{c}
0 \\
\uparrow \\
a_0b_0 \\
\uparrow \\
a_0b_1 + a_1b_0.
\end{array}
\]

Since

\[
\delta(a_0b_1 + a_1b_0) = \delta a_1 \cdot b_0 + (-1)^{|a_0|} a_0 \delta b_1 + \left( \delta a_0 \cdot b_1 + (-1)^{|a_1|} a_1 \delta b_0 \right)
\]

and the term in brackets equals \(- (d' a_1 \cdot b_1 + (-1)^{|a_1|} a_1 d' b_1) = -d'(a_1 b_1)\), hence is \(d'\)-exact and vanishes in homology, we conclude

\[
d_2[a_0b_0]_2 = \left[ \delta(a_0b_1 + a_1b_0) \right]_2
= \left[ \delta a_1 \cdot b_0 + (-1)^{|a_0|} a_0 \delta b_1 \right]_2
= d_2[a_0]_2 \cdot [b_0]_2 + (-1)^{|a_0|} [a_0]_2 d_2[b_0]_2.
\]

Thus \(d_2\) is a derivation and the product descends to \(E_3\).

For the general case, assume inductively that the product descends to \(E_r\). To show that \(d_r\) is a derivation, consider \([a_0] \in E_r^{p_1,q_1}\) and \([b_0] \in E_r^{p_2,q_2}\) represented by the zigzags

\[
\begin{array}{c}
0 \\
\uparrow \\
a_0 \\
\uparrow \\
a_1 \\
\uparrow \\
\vdots \\
\uparrow \\
a_{r-1},
\end{array}
\]

\[
\begin{array}{c}
0 \\
\uparrow \\
b_0 \\
\uparrow \\
b_1 \\
\uparrow \\
\vdots \\
\uparrow \\
b_{r-1}.
\end{array}
\]
A simple computation similar to that in the case $r = 2$ shows that $[a_0b_0] \in E^{p_1+p_2,q_1+q_2}_r$ is represented by the zigzag

$$
\begin{array}{c}
0 \\
\uparrow \\
 a_0b_0 \longrightarrow \\
\uparrow \\
 a_0b_1 + a_1b_0 \longrightarrow \\
\uparrow \\
 a_0b_2 + a_1b_1 + a_2b_0 \\
\uparrow \\
 \cdots \\
\uparrow \\
 a_0b_{r-1} + a_1b_{r-2} + \cdots + a_rb_0.
\end{array}
$$

Another short computation gives

$$
\delta(a_0b_{r-1} + a_1b_{r-2} + \cdots + a_{r-1}b_0) = \delta a_{r-1} \cdot b_0 + (-1)^{|a_0|}a_0\delta b_{r-1}
$$

$$
- d'(a_1b_{r-1} + a_2b_{r-2} + \cdots + a_{r-1}b_1),
$$

and hence

$$
d_r[a_0b_0]_r = \left[\delta(a_0b_{r-1} + a_1b_{r-2} + \cdots + a_{r-1}b_0)\right]_r
$$

$$
= \left[\delta a_{r-1} \cdot b_0 + (-1)^{|a_0|}a_0\delta b_{r-1}\right]_r
$$

$$
= d_r[a_0]_r \cdot [b_0]_r + (-1)^{|a_0|}[a_0],d_r[b_0]_r.
$$

Thus $d_r$ is a derivation and the product descends to $E_{r+1}$. \hfill \Box

Problem 2.5. (a) Show that if the cochain complexes $(A, \delta)$ and $(B, d)$ in Problem 2.4 carry products for which $\delta$ resp. $d$ are derivations, then

$$
a \otimes b \cdot a' \otimes b' := (-1)^{|b||a'|}
$$

defines a product on $A \otimes B$ for which $\delta \otimes 1$ and $1 \otimes d'$ are derivations.
(b) Deduce that for $R$ a field the isomorphism

$$H^*(A \otimes B, D = \delta \otimes 1 + 1 \otimes d') \cong H^*(A, \delta) \otimes H^*(B, d).$$

is an isomorphism of $R$-algebras.

2.4 Čech cohomology

**Definition.** A presheaf $\mathcal{F}$ (of $R$-modules) on a topological space $X$ is a contravariant functor

$$\{\text{open subsets of } X, \text{ inclusions}\} \rightarrow \{R\text{-modules, homomorphisms}\},$$

such that $\mathcal{F}(\emptyset) = 0$. Thus $\mathcal{F}$ associates to every open subset $U \subset X$ an $R$-module $\mathcal{F}(U)$, and to every inclusion $V \subset U$ a restriction homomorphism $\rho^U_V : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$, such that

$$\rho^U_U = 1_l, \quad \rho^U_M = \rho^V_W \circ \rho^U_V \text{ for } W \subset V \subset U.$$

One usually writes the restriction $\rho^U_V(\omega)$ of $\omega \in \mathcal{F}(U)$ simply as $\omega|_V$.

**Example 2.10.** For each $R$-module $G$ we can define a sheaf $G$ on $X$ by $G(U) := G$ for all nonempty open sets $U$ and $\rho^U_V = 1_l$ for all $\emptyset \neq V \subset U$.

Consider an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of $X$, where $I$ is an arbitrary index set. For $\alpha_0, \ldots \alpha_p \in I$ we abbreviate

$$U_{\alpha_0 \cdots \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}.$$  

For $p \geq 0$, the $p$-th Čech cochain group is the submodule

$$C^p(\mathcal{U}, \mathcal{F}) \subset \prod_{(\alpha_0, \ldots, \alpha_p) \in I^{p+1}} \mathcal{F}(U_{\alpha_0 \cdots \alpha_p}).$$
consisting of those elements \( (\omega_{\alpha_0\ldots\alpha_p} \in \mathcal{F}(U_{\alpha_0\ldots\alpha_p})) \) satisfying
\[
\omega_{\alpha_0\ldots\alpha_p} = \text{sgn}(\sigma)\omega_{\alpha_{\sigma(0)}\ldots\alpha_{\sigma(p)}}
\]
for every permutation \( \sigma \) of \( \{0, \ldots, p\} \). In particular, \( \omega_{\alpha_0\ldots\alpha_p} = 0 \) whenever two of the indices \( \alpha_0, \ldots, \alpha_p \) are equal. The Čech coboundary operator \( \delta = \delta_p : C^p(U, \mathcal{F}) \to C^{p+1}(U, \mathcal{F}) \) is defined by
\[
(\delta \omega)_{\alpha_0\ldots\alpha_{p+1}} := \sum_{i=1}^{p+1} (-1)^i \omega_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}}|_{U_{\alpha_0\ldots\alpha_{p+1}}},
\]
where \( \hat{\alpha}_i \) means that the index \( \alpha_i \) is omitted. A simple computation shows that \( \delta^2 = 0 \). The quotient
\[
H^p(U, \mathcal{F}) := \ker \delta_p / \operatorname{im} \delta_{p-1}
\]
is called the Čech cohomology of the cover \( U \) with values in \( \mathcal{F} \).

An open cover \( V = (V_\beta)_{\beta \in J} \) is called a refinement of \( U = (U_\alpha)_{\alpha \in I} \) if there exists a refinement map \( \phi : J \to I \) such that \( V_\beta \subset U_{\phi(\beta)} \) for all \( \beta \in J \). The induced map \( \phi^* : C^p(U, \mathcal{F}) \to C^p(V, \mathcal{F}) \),
\[
(\phi^* \omega)_{\beta_0\ldots\beta_p} := \omega_{\phi(\beta_0)\ldots\phi(\beta_p)}|_{V_{\beta_0\ldots\beta_p}},
\]
is easily seen to be a chain map, so it induces a map in cohomology
\[
\phi^* : H^p(U, \mathcal{F}) \to H^p(V, \mathcal{F}).
\]

**Problem 2.6.** If \( \phi, \psi : J \to I \) are two refinement maps, then the map \( K : C^p(U, \mathcal{F}) \to C^{p-1}(V, \mathcal{F}) \) defined by
\[
(K \omega)_{\beta_0\ldots\beta_{p-1}} := \sum_{i=0}^{p-1} \omega_{\phi(\beta_0)\ldots\phi(\beta_i)\psi(\beta_i)\ldots\psi(\beta_{p-1})}|_{V_{\beta_0\ldots\beta_{p-1}}}
\]
defines a chain homotopy between \( \phi^* \) and \( \psi^* \), i.e.,
\[
\psi^* - \phi^* = \delta K + K \delta : C^p(U, \mathcal{F}) \to C^p(V, \mathcal{F}).
\]
Thus the induced map $H^p(U, \mathcal{F}) \to H^p(V, \mathcal{F})$ on cohomology is independent of the refinement map $\phi$. It follows that the cohomology groups $H^p(U, \mathcal{F})$ for all open covers of $X$ form a direct system of $R$-modules with respect to the partial ordering on the open covers defined by $U < V$ iff $V$ is a refinement of $U$ (sic!).

**Definition.** The direct limit

$$H^p(X, \mathcal{F}) := \lim_{\mathcal{U}} H^p(U, \mathcal{F})$$

is called the Čech cohomology of $X$ with values in the presheaf $\mathcal{F}$.

**Problem 2.7.** Show that to every presheaf $\mathcal{F}$ on $X$ we can associate a new presheaf $\mathcal{F}'$ by setting $\mathcal{F}'(U) := \prod_{\alpha} \mathcal{F}(U_{\alpha})$, where $U = \coprod_{\alpha} U_{\alpha}$ is the decomposition of $U$ into its connected components, and that $H^p(X, \mathcal{F}') = H^p(X, \mathcal{F})$.

Applying this construction to the presheaf $G$ in Example 2.10, we obtain the trivial presheaf $G'(\coprod_{\alpha} U_{\alpha}) = \prod_{\alpha} G$, with restriction maps induced by the identity maps for connected open subsets. A presheaf is called *constant* if it is isomorphic to a trivial presheaf.

**Problem 2.8.** Compute the Čech cohomologies with values in the trivial presheaf $G$ of the open covers of the circle $S^1$ by $k \geq 2$ intervals, and show that they stabilize at $k = 3$ to give the Čech cohomology of $S^1$.

A presheaf $\mathcal{F}$ on $X$ is called *locally constant* if $X$ has an open cover $\mathcal{U}$ such that for all $U \in \mathcal{U}$ and $x \in U$ the canonical map $\mathcal{F}(U) \to \lim_V \mathcal{F}(V)$ is an isomorphism, where the direct limit is taken over all open neighbourhoods $V \subset U$ of $x$.

**Problem 2.9.** Show that on a locally contractible space $X$ (e.g., a CW complex), the presheaf that associates to $U \subset X$ its singular cohomology $H^q(U)$ is locally constant, but in general not constant.
Problem 2.10. Define a presheaf $\mathcal{F}$ on $S^1$, with $\mathcal{F}(I) \cong \mathbb{Z}$ for each open interval $I \subset S^1$, which is locally constant but not constant, and compute $H^k(S^1, \mathcal{F})$.

Problem 2.11. Give an appropriate definition of a short exact sequence of presheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ on a space $X$ and show that it induces a long exact sequence in Čech cohomology

$$\cdots \to H^p(X, \mathcal{F}) \to H^p(X, \mathcal{G}) \to H^p(X, \mathcal{H}) \to H^{p+1}(X, \mathcal{F}) \to \cdots.$$

2.5 The Čech–de Rham complex

Now we combine the techniques of the previous sections in the main example of this chapter. Let $M$ be a manifold and $\Omega^q$ the presheaf of $q$-forms on $M$. For an open cover $\mathcal{U}$ of $M$ we then have the Čech–de Rham complex

$$C^p(\mathcal{U}, \Omega^{q+1}) \xrightarrow{d} C^p(\mathcal{U}, \Omega^q) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \Omega^q),$$

where $\delta$ is the Čech differential, and $d$ is induced by the exterior derivative on forms. Since the exterior derivative commutes with restrictions, the two differentials commute and we have a double complex. Hence, by Theorem 2.8, there exists a spectral sequence $(E_r, d_r)$ converging to the total cohomology $H^*(K, D)$ with

$$E_1^{p,q} = C^p(\mathcal{U}, H^q_{dR}), \quad E_2^{p,q} = H^p(\mathcal{U}, H^q_{dR}).$$

Thus $E_1^{p,q}$ and $E_2^{p,q}$ are the Čech cochains resp. cohomologies with values in the locally constant presheaf $H^q_{dR}$ which associates to each $U \subset M$ its de Rham cohomology $H^q_{dR}(U)$. 
Problem 2.12. Prove that every \(n\)-dimensional manifold \(M\) has a good cover, i.e., a locally finite open cover \(\mathcal{U} = (U_\alpha)_{\alpha \in I}\) such that every nonempty intersection \(U_{\alpha_0 \cdots \alpha_p}\) is diffeomorphic to \(\mathbb{R}^n\). Moreover, good covers are cofinal, i.e., every open cover of \(M\) has a good refinement.

Let \(\mathcal{U}\) be a good cover of \(M\). Then, by the Poincaré lemma, for each nonempty \(U = U_{\alpha_0 \cdots \alpha_p}\) the sequence

\[
0 \longrightarrow \mathbb{R} \xrightarrow{i} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots
\]

is exact, where \(i : \mathbb{R} \hookrightarrow \Omega^0(U)\) denotes the inclusion of the constant functions. Thus we get the following augmented (by a row at \(q = -1\)) double complex with exact columns:

\[
\cdots \xrightarrow{d} C^p(\mathcal{U}, \Omega^1) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \Omega^1) \xrightarrow{d} \cdots
\]

\[
\cdots \xrightarrow{d} C^p(\mathcal{U}, \Omega^0) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \Omega^0) \xrightarrow{d} \cdots
\]

\[
\cdots \xrightarrow{i} C^p(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \mathbb{R}) \xrightarrow{i} \cdots
\]

Taking cohomology with respect to \(d\) yields the augmented \(E_1\) page with only two nontrivial rows at \(q = 0, -1\):

\[
\cdots C^p(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \mathbb{R}) \cdots
\]

\[
= \biguparrow i \quad = \biguparrow i
\]

\[
\cdots C^p(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \mathbb{R}) \cdots
\]
Taking cohomology with respect to $\delta$ yields the augmented $E_2$ page with only two nontrivial rows at $q = 0, -1$:

$$
\cdots H^p(U, \mathbb{R}) \xrightarrow{=} H^{p+1}(U, \mathbb{R}) \cdots
\xrightarrow{\iota_*} \cdots H^p(U, \mathbb{R}) \xrightarrow{=} H^{p+1}(U, \mathbb{R}) \cdots
$$

For degree reasons all the differentials $d_2, d_3, \ldots$ vanish, so the $E_\infty$ page agrees with the $E_2$ page and we have $H^n(U, \mathbb{R}) \cong H^n(K, D)$. More precisely, since $d = 0$ on $C^n(U, \mathbb{R})$, the inclusion $i$ induces a chain map

$$i : \left(C^n(U, \mathbb{R}), \delta\right) \hookrightarrow \left(K^n = \bigoplus_{p+q=n} K^{p,q}, D = \delta + d'\right)$$

and the preceding argument shows that $i_* : H^n(U, \mathbb{R}) \to H^n(K, D)$ is an isomorphism.

Now note that we can interchange the roles of $\delta$ and $d$ to get another spectral sequence, also converging to the total cohomology $H^*(K, D)$, with

$$E_1^{p,q} = H^p(U, \Omega^q),$$

the Čech cohomology with values in the presheaf $\Omega^q$.

**Lemma 2.11.** *For any (not necessarily good) open cover $U = (U_\alpha)$ of a manifold $M$ and any $q \geq 0$ the sequence

$$0 \longrightarrow \Omega^q(M) \xrightarrow{r} C^0(U, \Omega^q) \xrightarrow{\delta} C^1(U, \Omega^q) \xrightarrow{\delta} \cdots
$$

is exact, where $r : \Omega^q(M) \hookrightarrow C^0(U, \Omega^q)$ associates to a global $q$-form $\omega$ the Čech 0-cocycle $(\omega|_{U_\alpha})$.***

**Proof.** The argument uses a partition of unity $(\phi_\alpha)$ subordinate to $(U_\alpha)$, i.e. such that supp $\phi_\alpha \subset U_\alpha$ for all $\alpha$ (but supp $\phi_\alpha$ need
not be compact). For \( q \geq 0, p > 0 \) we define \( K : C^p(\mathcal{U}, \Omega^q) \to C^{p-1}(\mathcal{U}, \Omega^q) \) by

\[
(K\omega)_{\alpha_0 \cdots \alpha_{p-1}} := \sum_\alpha \phi_\alpha \omega_{\alpha_0 \cdots \alpha_{p-1}}
\]

(2.2)

(where we have omitted the obvious restriction maps). We compute

\[
(K\delta\omega)_{\alpha_0 \cdots \alpha_p} = \sum_\alpha \phi_\alpha (\delta\omega)_{\alpha_0 \cdots \alpha_p}
\]

\[
= \sum_\alpha \phi_\alpha \omega_{\alpha_0 \cdots \alpha_p} - \sum_\alpha \sum_{i=0}^p (-1)^i \phi_\alpha \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p},
\]

\[
(\delta K\omega)_{\alpha_0 \cdots \alpha_p} = \sum_{i=0}^p (-1)^i (K\omega)_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}
\]

\[
= \sum_\alpha \sum_{i=0}^p (-1)^i \phi_\alpha \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}.
\]

Adding up the double sums cancel, and since \( \sum_\alpha \phi_\alpha = 1 \) we obtain

\[
K\delta + \delta K = 1.
\]

Hence \( K \) is a chain homotopy between the identity and zero, which implies that all \( \delta \)-homology in degree \( p > 0 \) vanishes. Exactness at \( p = 0 \) holds because Čech 0-cocycles with values in \( \Omega^q \) are precisely the global \( q \)-forms.

\[\square\]

**Remark 2.12.** The only property about the presheaf of \( q \)-forms used in proof of Lemma 2.11 is that it admits partitions of unity. This property can be formalized in the notion of a “fine sheaf”. For example, any sheaf of modules over the sheaf of smooth (or continuous) functions is fine.
By the lemma, we get the following augmented (by a column at \( p = -1 \)) double complex with exact rows:

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
0 \longrightarrow \Omega^{q+1}(M) & \overset{r}{\longrightarrow} & C^0(U, \Omega^{q+1}) \\
\uparrow d & & \uparrow d \\
0 & \overset{r}{\longrightarrow} & C^0(U, \Omega^q) \\
\cdots & \cdots & \cdots \\
\end{array}
\]

Taking cohomology with respect to \( \delta \) yields the augmented \( E_1 \) page with only two nontrivial columns at \( p = 0, -1 \):

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\Omega^{q+1}(M) & \overset{r}{\longrightarrow} & \Omega^{q+1}(M) \\
\uparrow d & & \uparrow d \\
\Omega^q(M) & \overset{r}{\longrightarrow} & \Omega^q(M) \\
\cdots & \cdots & \cdots \\
\end{array}
\]

Taking cohomology with respect to \( d \) yields the augmented \( E_2 \) page with only two nontrivial columns at \( p = 0, -1 \):

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
H_{dR}^{q+1}(M) & \overset{r_*}{\longrightarrow} & H_{dR}^{q+1}(M) \\
\cdots & \cdots & \cdots \\
H_{dR}^q(M) & \overset{r_*}{\longrightarrow} & H_{dR}^q(M) \\
\cdots & \cdots & \cdots \\
\end{array}
\]

For degree reasons all the differentials \( d_2, d_3, \ldots \) vanish, so the \( E_\infty \) page agrees with the \( E_2 \) page and we have \( H_{dR}^n(M) \cong H^n(K, D) \).

More precisely, since \( \delta = 0 \) on \( \Omega^n(M) \), the inclusion \( r \) induces a chain map

\[
r : \left( \Omega^n(M), d' \right) \hookrightarrow \left( K^n = \bigoplus_{p+q=n} K^{p,q}, D = \delta + d' \right)
\]
and the preceding argument shows that $r_* : H^n_{dR}(M) \to H^n(K, D)$ is an isomorphism.

Finally, note that both the de Rham complex $\Omega^*(M)$ and the Čech complex $C^*(\mathcal{U}, \mathbb{R})$ with values in $\mathbb{R}$ carry natural products for which $d$ resp. $\delta$ are derivations: the cup product $\omega \wedge \tau$ on differential forms, and the product $$(fg)_{\alpha_0 \cdots \alpha_p \beta_0 \cdots \beta_p} := f_{\alpha_0 \cdots \alpha_p} g_{\alpha_p \cdots \alpha_{p+p'}}$$ on Čech cochains $f \in C^p(\mathcal{U}, \mathbb{R})$, $g \in C^{p'}(\mathcal{U}, \mathbb{R})$. These induce a product on the Čech–de Rham complex defined on $\omega \in C^p(\mathcal{U}, \Omega^q)$, $\tau \in C^{p'}(\mathcal{U}, \Omega^{q'})$ by $$(\omega \cdot \tau)_{\alpha_0 \cdots \alpha_p \beta_0 \cdots \beta_p} := (-1)^{qp'} \omega_{\alpha_0 \cdots \alpha_p} \wedge \tau_{\alpha_p \cdots \alpha_{p+p'}}.$$ Here the sign $(-1)^{qp'}$ is motivated by the resemblance of the Čech–de Rham complex to the tensor product of the Čech complex with values in $\mathbb{R}$ with the de Rham complex (though it is not a tensor product; see Problem 2.4).

**Problem 2.13.** Show that for the product on the Čech–de Rham complex defined above both differentials $\delta$, $d'$ are derivations, i.e., for $\omega \in C^p(\mathcal{U}, \Omega^q)$, $\tau \in C^{p'}(\mathcal{U}, \Omega^{q'})$ we have $$\delta(\omega \cdot \tau) = \delta \omega \cdot \tau + (-1)^{p+q} \omega \cdot \delta \tau, \quad d'(\omega \cdot \tau) = d' \omega \cdot \tau + (-1)^{p+q} \omega \cdot d' \tau.$$ What happens if we replace the sign $(-1)^{qp'}$ in the definition by $(-1)^{pq'}$?

It is easy to see that the above chain maps $i : C^*(\mathcal{U}, \mathbb{R}) \hookrightarrow K^*$ and $r : \Omega^*(M) \hookrightarrow K^*$ are algebra maps. Since good covers are cofinal, we can summarize the preceding discussion in the following

**Theorem 2.13.** For a cover $\mathcal{U}$ of a manifold $M$ let $(K_*^\mathcal{U}, D_*^\mathcal{U})$ be the associated Čech–de Rham complex. Then: For a good
cover \( U \), the inclusion \( i : \mathbb{R} \hookrightarrow \Omega^0(M) \) induces an isomorphism of \( \mathbb{R} \)-algebras

\[
i_* : H^n(U, \mathbb{R}) \xrightarrow{\cong} H^n(K_U, D_U).
\]

For any (not necessarily good) cover \( U \), the inclusion \( r : \Omega^*(M) \hookrightarrow C^0(U, \Omega^*) \) induces an isomorphism of \( \mathbb{R} \)-algebras

\[
r_* : H^dR_n(M) \xrightarrow{\cong} H^n(K_U, D_U).
\]

In particular, since good covers are cofinal, we get the algebra isomorphism

\[
i_*^{-1}r_* : H^dR_n(M) \xrightarrow{\cong} H^n(M, \mathbb{R}).
\]

between de Rham cohomology and Čech cohomology with values in \( \mathbb{R} \).

\[\square\]

**Problem 2.14.** Prove the assertion in Theorem 2.13 that \( H^n(U, \mathbb{R}) \cong H^n(M, \mathbb{R}) \) for any good cover \( U \).

**Example 2.14 (First Chern class).** Let \( S^1 \to E \to M \) be a circle bundle and \( U = (U_\alpha) \) be a good cover of \( M \). For each \( \alpha \) we pick local trivializations \( \Psi_\alpha : E|_{U_\alpha} \to U_\alpha \times S^1 \). Then

\[
\Psi_\beta \Psi_\alpha^{-1} : U_{\alpha \beta} \times S^1 \to U_{\alpha \beta} \times S^1, \quad (x, \sigma) \mapsto (x, e^{2\pi i \tau_{\alpha \beta}(x)} \sigma)
\]

for transition functions \( \tau_{\alpha \beta} : U_{\alpha \beta} \to \mathbb{R} \) (which are uniquely determined up to adding integer constants). They define a Čech cochain \( \tau = (\tau_{\alpha \beta}) \in C^1(U, \mathbb{R}) \). The definition of \( \tau_{\alpha \beta} \) shows that \( \tau_{\alpha \beta} + \tau_{\beta \gamma} + \tau_{\gamma \alpha} \in \mathbb{Z} \) on \( U_{\alpha \beta \gamma} \), so we get \( \delta \tau \in C^2(U, \mathbb{Z}) \) which is \( \delta \)-closed. Its Čech cohomology class

\[
c_1(E) := [\delta \tau] \in H^2(U, \mathbb{Z}) \cong H^2(M, \mathbb{Z})
\]

is called the *first Chern class* of the circle bundle \( E \).
Problem 2.15. Let $S^1 \to E \to M$ be a circle bundle and $\mathcal{U} = (U_\alpha)$ be a good cover of $M$.

(a) Show that the definition of the first Chern class $c_1(E)$ in Example 2.14 does not depend on the choice of local trivializations and transition functions.

(b) Explain how this definition implicitly uses the boundary map in the long exact sequence in Problem 2.11 associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$.

(c) Find an explicit expression (in terms of transition functions and a partition of unity) for a closed 2-form $\omega$ on $M$ representing the image of $c_1(E)$ under the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \cong H^2_{dR}(M)$.

(d) Compute the first Chern class of the Hopf fibration $S^1 \to S^3 \to S^2$.

Problem 2.16. For a good cover of the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (with coordinates $(\theta, \phi)$) by 9 squares of side length $2/3$, compute Čech cocycles in $C^1(\mathcal{U}, \mathbb{R})$ resp. $C^2(\mathcal{U}, \mathbb{R})$ representing the images of $[d\theta]$, $[d\phi]$ and $[d\theta \wedge d\phi]$ under the isomorphisms $H^i_{dR}(T^2) \cong H^i(T^2, \mathbb{R})$, $i = 1, 2$.

Singular cohomology. The results in this section carry over from the de Rham complex to the singular cochain complex. For this, denote by $C^q$ the presheaf that associates to each open set $U \subset X$ of a topological space $X$ the $R$-module $C^q(U, R)$ of singular cochains on $U$ with values in $R$. The following analogue of Lemma 2.11 is an algebraic consequence of the Mayer–Vietoris sequence for singular cohomology (see [2]):

Lemma 2.15. For any open cover $\mathcal{U} = (U_\alpha)$ of a topological space $X$ and any $q \geq 0$ the sequence

$$0 \to C^q(X) \to r C^0(\mathcal{U}, C^q) \to \delta C^1(\mathcal{U}, C^q) \to \cdots$$
is exact, where \( r : C^q(X) \hookrightarrow C^0(\mathcal{U}, X^q) \) associates to a global singular \( q \)-cochain \( \omega \) the Čech 0-cocycle \( (\omega|_{U_\alpha}) \).

**Problem 2.17.** Deduce Lemma 2.15 from the Mayer–Vietoris sequence.

As above, for an open cover \( \mathcal{U} \) of \( X \) we get a double complex, the Čech–singular complex

\[
\begin{array}{c}
C^p(\mathcal{U}, C^{q+1}) \\
\uparrow d
\end{array}
\begin{array}{c}
C^p(\mathcal{U}, C^q) \\
\xrightarrow{\delta} C^{p+1}(\mathcal{U}, C^q),
\end{array}
\]

where \( \delta \) is the Čech differential, and \( d \) is induced by the singular coboundary operator. The product on Čech cochains and the cup product on singular cochains induce a product on this double complex. Assume now that the space \( X \) possesses a cofinal system of good covers \( \mathcal{U} = (U_\alpha)_{\alpha \in I} \), where “good” now means that each nonempty intersection \( U_{\alpha_0 \cdots \alpha_p} \) is contractible. Then the arguments for the proof of Theorem 2.13 (together with the argument at the end of Example 2.7) yield

**Theorem 2.16.** Let \( X \) be a topological space which possesses a cofinal system of good covers. For a cover \( \mathcal{U} \) of \( X \) let \( (K_\mathcal{U}, D_\mathcal{U}) \) be the associated Čech–singular complex with values in \( R \). Then:

For a good cover \( \mathcal{U} \), the inclusion \( i : R \hookrightarrow C^0(X) \) induces an isomorphism of \( R \)-algebras

\[ i_* : H^n_{\text{Čech}}(\mathcal{U}, R) \cong H^n(K_\mathcal{U}, D_\mathcal{U}). \]

For any (not necessarily good) cover \( \mathcal{U} \), the inclusion \( r : C^*(X) \hookrightarrow C^0(\mathcal{U}, C^*) \) induces an isomorphism of \( R \)-algebras

\[ r_* : H^n_{\text{sing}}(X, R) \cong H^n(K_\mathcal{U}, D_\mathcal{U}). \]
In particular, since good covers are cofinal, we get the algebra isomorphism
\[ i_{\ast}^{-1} r_{\ast} : H_{\text{sing}}^{n}(X, R) \xrightarrow{\simeq} H_{\text{\v{C}ech}}^{n}(X, R). \]

between de Rham cohomology and \v{C}ech cohomology with values in $R$.

\textbf{Problem 2.18.} Prove that simplicial complexes have cofinal systems of good covers. What about CW complexes?

\section{2.6 The Leray–Serre spectral sequence}

A continuous map $\pi : E \to B$ is called a \textit{fibration} if it has the \textit{homotopy lifting property} for each topological space $X$, i.e., any homotopy $h : [0, 1] \times X \to B$ has a lift $H : [0, 1] \times X \to E$ (i.e., $\pi \circ H = h$) extending any given lift $0 \times X \to E$. It is called a \textit{Serre fibration} if it has the homotopy lifting property for all closed $k$-balls $D^k$. We will always assume that $B$ is connected; the homotopy lifting property for a point then implies that $\pi$ is surjective. For $b \in B$ we denote by $F_b := \pi^{-1}(b)$ the \textit{fibre over} $b$. We fix a base point $b_0$ and a \textit{fibre} $F := \pi^{-1}(b_0)$.

\textbf{Problem 2.19.} Prove for a fibration $F \to E \xrightarrow{\pi} B$:

(a) If $B$ is contractible, the inclusion $F \hookrightarrow E$ is a homotopy equivalence.

(b) Every path $\gamma : [0, 1] \to B$ induces a map $f_{\gamma} : F_{\gamma(0)} \to F_{\gamma(1)}$ which is unique up to homotopy. If $\gamma$ and $\delta$ are homotopic with fixed end points, the maps $f_{\gamma}$ and $f_{\delta}$ are homotopic. In particular, each $f_{\gamma}$ is a homotopy equivalence.

(c) The construction in (b) yields a group homomorphism from $\pi_1(B)$ to the group of homotopy equivalences of $F$. In particular,
\( \pi_1(B) \) acts on \( H_q(F) \) and \( H^q(F) \). Under which conditions does \( \pi_1(B) \) act on \( \pi_q(F) \)?

(d) If \( \pi_1(B) \) acts trivially on \( H^q(F) \), then \( U \mapsto H^q(\pi^{-1}(U)) \) defines a constant presheaf on every good cover \( \mathcal{U} = (U_\alpha) \) of \( B \), i.e., there are isomorphisms \( H^q(\pi^{-1}(U_{\alpha_0, \ldots, \alpha_p})) \cong H^q(F) \) under which all restriction maps correspond to the identity map on \( H^q(F) \).

Suppose now that \( F \to E \xrightarrow{\pi} B \) is a fibration and \( \mathcal{U} = (U_\alpha) \) is a good cover of \( B \). Then \( \pi^{-1}\mathcal{U} = (\pi^{-1}(U_\alpha)) \) is an open cover of \( E \) (in general not good), so as in the previous section we have the Čech–singular double complex \( K^{p,q} = C^p(\pi^{-1}\mathcal{U}, C^q) \). By Theorem 2.16, the resulting Leray–Serre spectral sequence \( (E_r, d_r) \) converges to \( H^n(K, D) \cong H^n(E) \), the singular cohomology of the total space \( E \). Its first page

\[
E_1^{p,q} = C^p(\pi^{-1}\mathcal{U}, H^q) = C^p(\mathcal{U}, H^q)
\]

consists of Čech cochains on \( \mathcal{U} \) with values in the presheaf \( H^q(U) := H^q(\pi^{-1}(U)) \). Note that by Problem 2.19 (a) this presheaf is locally constant. The second page

\[
E_2^{p,q} = H^p(\mathcal{U}, H^q)
\]

is the Čech cohomology of \( \mathcal{U} \) with values in the presheaf \( H^q \).

Suppose now that \( \pi_1(B) \) acts trivially on \( H^q(F) \). Then by Problem 2.19 (d) the presheaf \( H^q \) is constant on the good cover \( \mathcal{U} \), and

\[
E_2^{p,q} = H^p(\mathcal{U}, H^q(F))
\]

equals the Čech cohomology of \( \mathcal{U} \) with values in the constant \( R \)-module \( H^q(F) \). Again by Theorem 2.16 (applied with the coefficient ring \( H^*(F) \), provided that \( B \) possesses a cofinal system of good covers), this equals the singular cohomology \( H^p(B, H^q(F)) \) of \( B \) with coefficients in \( H^q(F) \). If \( R \) is a field and \( H^q(F) \) is finite.
2.6. THE LERAY–SERRE SPECTRAL SEQUENCE

dimensional, then this equals $H^p(B) \otimes H^q(F)$ by the universal
coefficient theorem and we have proved

**Theorem 2.17** (Leray, Serre). *Let $F \to E \xrightarrow{\pi} B$ be a fibration whose base $B$ possesses a cofinal system of good covers. Then for every good cover $\mathcal{U}$ of $B$ we get a Leray–Serre spectral sequence $(E_r, d_r)$ which converges to the homology $H^n(E)$. Its second page

$$E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q)$$

is the Čech cohomology of $\mathcal{U}$ with values in the locally constant presheaf $\mathcal{H}^q(\mathcal{U}) = H^q(\pi^{-1}(U))$. If $\pi_1(B)$ acts trivially on $H^*(F)$, $R$ is a field, and each $H^q(F)$ is finite dimensional, then we have an isomorphism of $R$-algebras

$$E_2^{p,q} \cong H^p(B) \otimes H^q(F).$$

\[\square\]

The simplest case of the Leray–Serre spectral sequence occurs when $E \cong B \times F$ is a product:

**Corollary 2.18** (Künneth formula). *Suppose that $B$ possesses a cofinal system of good covers and $H^q(F)$ is a finitely generated free $R$-module for each $q$. Then the cross product on cohomology induces an algebra isomorphism

$$H^*(B \times F) \cong H^*(B) \otimes H^*(F).$$

*Proof.* Pick a good cover $\mathcal{U}$ of $B$ and consider the Čech–singular double complex $K^{p,q} = C^p(\pi^{-1}\mathcal{U}, C^q)$ associated to the product fibration $\pi : B \times F \to B$. Pick singular cocycles $f_j \in C^q(F)$ representing a basis of the free $R$-module $H^q(F)$. Denote by $p_{\alpha_0\ldots\alpha_p} : \pi^{-1}(U_{\alpha_0\ldots\alpha_p}) = U_{\alpha_0\ldots\alpha_p} \times F \to F$ the projections onto the second
factor. Then each class \([\omega] \in E_1^{p,q} = C^p(U, H^q) \cong C^p(U, H^q(F))\) is uniquely represented by \(\omega \in K^{p,q}\) with

\[
\omega_{\alpha_0 \cdots \alpha_p} = \sum_j c_{\alpha_0 \cdots \alpha_p}^j p^*_{\alpha_0 \cdots \alpha_p} f_j, \quad c_{\alpha_0 \cdots \alpha_p} \in R.
\]

Since the projections \(p_{\alpha_0 \cdots \alpha_p}\) commute with restrictions, we have

\[
(\delta \omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i,j} (-1)^i c_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}^j p^*_{\alpha_0 \cdots \alpha_p+1} f_j.
\]

From this we see that \([\delta \omega] = 0 \in E_1^{p,q}\) is equivalent to \(\sum_i (-1)^i c_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}^j\) for all \(j\), and hence to \(\delta \omega = 0 \in K^{p+1,q}\). Thus the element \(\omega\) lives to all pages \(E_r\), so \(d_2\) and all higher differentials vanish and \(E_\infty^{p,q} = H^p(B) \otimes H^q(F)\). By Theorem 2.17, this implies the isomorphism of modules

\[
H^*(B \times F) \cong H^*(B) \otimes H^*(F).
\]

Since it is induced by the canonical algebra map \(C^p(U, R) \otimes C^q(F) \to K^{p,q}\), it is an algebra isomorphism.

**Example 2.19 (Hopf fibration).** As an example with nonvanishing \(d_2\), consider the Hopf fibration \(S^1 \to S^3 \to S^2\). Pick a finite good cover \(U\) of \(S^2\) and consider the Čech–de Rham double complex \(K^{p,q} = C^p(\pi^{-1}U, \Omega^q)\). A generator of \(H^1(S^1)\) is represented by the closed 1-form \(d\theta\) on \(S^1\). Pick trivializations and projections \(p_\alpha : \pi^{-1}(U_\alpha) \cong U_\alpha \times F \to F\). Then the element \(\omega \in K^{0,1}\) with \(\omega_\alpha = p^*_\alpha d\theta\) is \(d\)-closed. Since \((\delta \omega)_{\alpha\beta} = p^*_\alpha d\theta - p^*_\beta d\theta\) is exact, \(\omega\) represents a class \([\omega]_2 \in E_2^{0,1}\). However, in contrast to the situation in Corollary 2.18, \(\delta \omega\) is nonzero. This happens because here the maps \(p_\alpha\) do not commute with restrictions and \(p^*_\alpha d\theta|_{U_\alpha\beta} \neq p^*_\beta d\theta|_{U_\alpha\beta}\). Indeed, writing \(p^*_\alpha d\theta - p^*_\beta d\theta = d\tau_{\alpha\beta}\) for functions \(\tau_{\alpha\beta}\) on \(U_{\alpha\beta}\), we obtain a Čech cocycle \(\delta \tau \in C^2(U, \mathbb{R})\) representing
\[ d_2[\omega]_2 \in E_2^{2,0} \cong H^2(S^2) \otimes H^0(S^1) \cong \mathbb{R}. \]

The nonvanishing of this class reflects the failure of the closed 1-form \( d\theta \) on the fibre to extend to a closed 1-form on the total space \( S^3 \). Note that the higher differentials \( d_3, d_4, \ldots \) vanish for degree reasons and \( E_3 = E_\infty \) gives correctly the cohomology of \( S^3 \), see Figure fig:Hopf.

**Problem 2.20 (Projective space).** Use the Leray–Serre spectral sequence for the fibration \( S^1 \to S^{2n+1} \to \mathbb{C}P^n \) to compute the cohomology ring of \( \mathbb{C}P^2 \) from that of \( S^1 \) and \( S^{2n+1} \).

**Problem 2.21 (Unitary group).** Use the Leray–Serre spectral sequence for the fibration \( U(n-1) \to U(n) \to S^{2n-1} \) and induction on \( n \) to compute the cohomology ring of \( U(n) \).

### 2.7 Applications to based loop spaces

Now we will compute the cohomology ring of some loop spaces by applying the Leray–Serre spectral sequence to the path fibration \( \Omega X \to PX \to X \). Since \( PX \) is contractible, the cohomology of the total space vanishes. This imposes strong restrictions on the terms in the spectral sequence, which sometimes allows us to compute \( H^*(\Omega X) \) from \( H^*(X) \) or vice versa.

**Standing assumption.** Throughout this section, \( X \) is simply connected, \( R \) is a field of characteristic zero, and \( H^q(X) \) is finite dimensional for each \( q \).

**Exterior algebras in one generator.** Suppose that \( H^*(X) = \Lambda[a] \) is an exterior algebra in one generator of even degree \( n \geq 2 \). For degree reasons, the first nontrivial differential \( d_r, r \geq 2 \), in the Leray–Serre spectral sequence can occur on the page \( E_n = E_2 = H^*(X) \otimes H^*(\Omega X) \). Since \( E_\infty^{p,q} = 0 \) for all \( (p,q) \neq (0,0) \), the generator \( a \) must die on the page \( E_n \), so there must be a generator
\( \bar{a} \in H^{\ast}(\Omega X) \) of degree \( n - 1 \) with \( d_n(\bar{a}) = a \). Since \( d_n \) is a derivation and \( d_n(a) = 0 \), it follows that \( d_n(\bar{a}a^k) = a^{k+1} \) for all \( k \geq 1 \), so all the \( a^k \) and \( \bar{a}a^k \) die on the page \( E_n \), see Figure \( \text{fig:ext-even} \). Note that \( \bar{a}^2 = 0 \) because \( n - 1 \) is odd.

Suppose that \( H^{\ast}(\Omega X) \) has some other generator \( b \neq \bar{a} \). Since \( b \) must eventually die in the spectral sequence, we must have \( d_r b \neq 0 \) for some \( r \geq n \). Since the dimension of \( E_r^{p,q} \) can only decrease with increasing \( r \), the page \( E_2 \) must have a nonzero entry at the position of \( d_n b \), and thus \( H^{\ast}(\Omega X) \) must have a generator of degree \( |b| - (r - 1) \). Inductively replacing \( b \) by generators of smaller degrees, we end up at one of the following cases.

- \( r = n \) and \( d_n b = \bar{a}a \): This violates \( d_n^{\ast} = 0 \).
- \( r = n \) and \( b \in H^{n-1}(\Omega X) \) linearly independent of \( \bar{a} \): Then \( d_n b = ta \) for some \( t \in \mathbb{R} \) and the \( d_n \)-cycle \( b - t\bar{a} \) lives to \( E_\infty \), which is impossible.
- \( r = n \) and \( |b| < n - 1 \): Then \( b \) lives to \( E_\infty \), which is impossible.
- \( r > n \): Since the elements \( a^k \) and \( \bar{a}a^k \) have died on the page \( E_n \), we end up with a generator \( b \) of degree \( |b| < r - 1 \). But then \( d_r b = 0 \) and \( b \) lives to \( E_\infty \), which is impossible.

So in all cases we get a contradiction, and it follows that \( H^{\ast}(\Omega X) = \Lambda[\bar{a}] \).

If \( |a| = n \) is odd, there must again be a generator \( \bar{a} \in H^{\ast}(\Omega X) \) of degree \( n - 1 \) with \( d_n(\bar{a}) = a \). Since by Theorem \ref{thm:1.18} the cohomology \( H^{\ast}(\Omega X) \) of the \( H \)-space \( \Omega X \) is an exterior algebra, all powers \( \bar{a}^k \), \( k \geq 1 \), are nonzero in \( H^{\ast}(\Omega X) \). The derivation property of \( d_n \) implies \( d_n(\bar{a}^k) = k\bar{a}^{k-1}a \), so all the elements \( \bar{a}^k \) and \( \bar{a}^ka \) die on the page \( E_n \), see Figure \( \text{fig:ext-odd} \). A similar argument as in the case \( n \) even shows that \( H^{\ast}(\Omega X) \) cannot have any other generators, so we have shown \( H^{\ast}(\Omega X) = \Lambda[\bar{a}] \).
Problem 2.22. For an $H$-space $X$, reverse the preceding arguments to show: If $H^*(\Omega X)$ is an exterior algebra on one generator of degree $n - 1 \geq 1$, then $H^*(X)$ is an exterior algebra on one generator of degree $n$.

We summarize this discussion in

**Proposition 2.20.** If $H^*(X) = \Lambda[a]$ is an exterior algebra on one generator of degree $n \geq 2$, then $H^*(\Omega X) = \Lambda[\bar{a}]$ is an exterior algebra on one generator of degree $n - 1$. Conversely, if $X$ is an $H$-space and $H^*(\Omega X) = \Lambda[\bar{a}]$ is an exterior algebra on one generator of degree $n - 1 \geq 1$, then $H^*(X) = \Lambda[a]$ is an exterior algebra on one generator of degree $n$.

Note that this recovers the rational cohomology ring of the based loop space of odd-dimensional spheres in Corollary 1.22. Since $K(\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}, n + 1)$ and $H^*K(\mathbb{Z}, 1) = H^*(S^1) = \Lambda[a_1]$ is an exterior algebra on one generator of degree 1, we obtain by induction the cohomology of the Eilenberg–McLane spaces:

**Corollary 2.21.** $H^*K(\mathbb{Z}, n) = \Lambda[a_n]$ is an exterior algebra on one generator of degree $n$.

**Truncated polynomial algebras in one generator.** Suppose that $H^*(X) = R[a]/a^{n+1}$ is a truncated polynomial algebra in one generator of even degree $|a| = k$. Again, there must be a generator $\bar{a} \in H^*(\Omega X)$ of odd degree $k - 1$ with $d_k(\bar{a}) = a$. Since $d_k(\bar{a}a^\ell) = a^{\ell+1}$, all the elements $a, a^2, \ldots, a^n$ and $\bar{a}, \bar{a}a, \ldots, \bar{a}a^{n-1}$ die on the page $E_k$, see Figure fig:trunc. However, the $d_k$-cycle $\bar{a}a^n$ is so far not a boundary. An argument as in the proof of Proposition 2.20 shows that $H^*(\Omega X)$ cannot have additional generators of degree $< k(n + 1) - 2$. So $\bar{a}a^n$ lives to the page $E_{kn}$ and there must be a generator $\bar{b} \in H^*(\Omega X)$ of even degree $k(n + 1) - 2$.
with \( d_{kn} \bar{b} = \bar{a}a^n \). Since \( d_{kn}(b^\ell) = \ell \bar{b}^{\ell-1} \bar{a}a^n \), all the elements \( \bar{b}^\ell \) and \( \bar{b}^\ell \bar{a}a^n \) die on the page \( E_{kn} \) and \( E_{kn+1}^{p,q} = E_{\infty}^{p,q} = 0 \) for all \( (p, q) \neq (0, 0) \). Now an argument as above shows that \( H^*(\Omega X) \) cannot have any additional generators, so in view of Theorem 1.18 we have shown

**Proposition 2.22.** If \( H^*(X) = R[a]/a^{n+1}, n \geq 1, \) is a truncated polynomial algebra on one generator of even degree \( k \geq 2 \), then \( H^*(\Omega X) = \Lambda[\bar{a}, \bar{b}] \) is an exterior algebra on a generator \( \bar{a} \) of degree \( k-1 \) and a generator \( \bar{b} \) of degree \( k(n+1) - 2 \). □

This recovers the rational cohomology ring of the based loop space of odd-dimensional spheres in Corollary 1.22. Since \( H^*(\mathbb{C}P^n) = R[a]/a^{n+1} \) with \( |a| = 2 \), we also obtain

**Corollary 2.23.** \( H^*(\Omega \mathbb{C}P^n) = \Lambda[\bar{a}, \bar{b}] \) is an exterior algebra on a generator \( \bar{a} \) of degree 1 and a generator \( \bar{b} \) of degree \( 2n \). □

**Problem 2.23.** Deduce from Corollary 2.23 and the Hopf–Leray Theorem 1.16 that for \( n > 1 \) the homology of \( \mathbb{C}P^n \) with the Pontrjagin product is an exterior algebra \( H_*(\Omega \mathbb{C}P^n) = \Lambda[x, y] \) on a generator \( x \) of degree 1 and a generator \( y \) of degree \( 2n \). (Note that this is not true for \( n = 1 \).)

Do this? **Problem 2.24.** Compute \( H^*(\Omega X) \) when \( H^*(X) = \Lambda[a, b]/I \) is generated by two elements \( a, b \) of degree 2 with a finitely generated ideal of relations \( I \).

The last problem shows that the computation of \( H^*(\Omega X) \) from the Leray–Serre spectral sequence of the path fibration becomes increasingly difficult as the number of generators and relations of \( H^*(X) \) increases. Nevertheless, it turns out that there is a simple general answer for \( H^*(\Omega X) \). We will derive this answer in Chapter 4 using minimal models.
2.8 Transgression

The Čech–de Rham complex revisited. Consider again a double complex \((K, \delta, d)\) with exact augmented rows

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots & \\
0 & \longrightarrow & C^{q+1} & \xrightarrow{r} & K^{0,q+1} & \xrightarrow{\delta} & K^{1,q+1} & \cdots \\
\uparrow{d} & & \uparrow{d} & & \uparrow{d} & & \\
0 & \longrightarrow & C^{q} & \xrightarrow{r} & K^{0,q} & \xrightarrow{\delta} & K^{1,q} & \cdots \\
\cdots & \cdots & \cdots & \cdots & 
\end{array}
\]

where \(r\) is the inclusion \(C^{q} := \ker \delta \hookrightarrow K^{0,q}\). Examples of this are the Čech–de Rham complex or the Čech–singular complex for any (not necessarily good) cover. We have seen in Section 2.5 that \(r\) induces isomorphisms \(r_* : H^n(C, d) \xrightarrow{\cong} H^n(K, D)\). The following lemma provides an explicit chain homotopy inverse \(s\) of \(r\). Let \(K : K^{p,q} \rightarrow K^{p-1,q}\) be a chain homotopy such that \(K\delta + \delta K = 1\) on \(K^{p,q}\) for \(p > 0\), and \(K\delta + rK = 1\) of \(K^{0,q}\).

(By [14, Ch. 4 Sec. 2], such a chain homotopy exists whenever the augmented rows are exact and consist of free \(R\)-modules; for the Čech–de Rham complex, \(K\) can be explicitly given by formula 2.2 in terms of a partition of unity.)

**Lemma 2.24.** There exist maps \(s : K^n \rightarrow C^n\) and \(L : K^n \rightarrow K^{n-1}\) satisfying

\[
sr = 1, \quad 1 - rs = DL + LD, \tag{2.3}
\]

which are given on \(K^{p,q}\) by the collating formulas

\[
L := \sum_{i=0}^{p-1} K(-d'K)^i,
\]

\[
s := (-d'K)^p - K(-d'K)^p \delta - K(-d'K)^{p-1}d'. \tag{2.4}
\]
Proof. Let us first motivate the formulas for $s$ and $L$. Note that the map $d'K$ maps $K_{p,q}$ to $K^{p-1,q+1}$. So one can guess that the map $s : K_{p,q} \to K^{0,p+q}$ should be built using $(d'K)^p$, and the chain homotopy $L$ should have on $K_{p,q}$ the form $\sum_{i=0}^{p-1} \pm K(d'K)^i$. Using this ansatz for $L$, one then computes $DL + LD$ to determine the signs.

Let us now check that $L, s$ defined by (2.4) satisfy (2.3). The equation $sr = I$ follows directly from the definition of $s$ for $p = 0$. To show the second equation, we abbreviate $M := -d'K$. Since $\delta$ and $d'$ anti-commute, a simple induction shows

$$\delta M^i = M^i \delta + M^{i-1}d'$$

for all $i \geq 1$. Using this identity, $D = \delta + d'$, and $K\delta + \delta K = I$, we compute on $K_{p,q}$:

$$DL = \sum_{i=0}^{p-1} \left( d'K M^i + \delta K M^i \right)$$

$$= \sum_{i=0}^{p-1} \left( -M^{i+1} + M^i - K\delta M^i \right)$$

$$= I - M^p - \sum_{i=0}^{p-1} K M^i \delta - \sum_{i=1}^{p-1} K M^{i-1}d',$$

$$LD = L\delta + Ld'$$

$$= \sum_{i=0}^{p} K M^i \delta + \sum_{i=0}^{p-1} K M^i d'$$

$$DL + LD = I - M^p + K M^p \delta + K M^{p-1}d'$$

$$= I - s.$$

It remains to verify that $s$ lands in $\ker \delta$. For this, we compute on $K_{p,q}$, using $\delta K + K\delta = I$, equation (2.5), and $\delta^2 = d'^2 = \cdots$.
\[ \delta d' + d'\delta = 0: \]
\[
\delta s = \delta M^p - \delta K M^p \delta - \delta K M^{p-1} d'
\]
\[
= M^p \delta + M^{p-1} d' - (1 - K \delta) M^p \delta - (1 - K \delta) M^{p-1} d'
\]
\[
= K \delta M^p \delta + K \delta M^{p-1} d'
\]
\[
= K M^p \delta^2 + K M^{p-1} d' \delta + K M^{p-1} \delta d' + K M^{p-2} d'^2
\]
\[
= 0.
\]

In particular, restricting \( s \) to \( \ker \delta \cap \ker d \subset K^{n,0} \) we obtain a map
\[ s = (-d' K)^n : K^{n,0} \supset \ker \delta \cap \ker d' \to \ker d \subset C^n. \]

Applied to the Čech–de Rham complex, this map associates to each Čech \( n \)-cocycle a closed \( n \)-form, and for a good cover it induces the isomorphism between Čech and de Rham cohomology in Theorem 2.13. An analogous discussion applies to the Čech–singular complex.

**Transgression in the Leray–Serre spectral sequence.**

Let us now return to a fibration \( F \to E \xrightarrow{\pi} B \) and the Leray–Serre spectral sequence \( C^p(\pi^{-1} U, \Omega^q) \) associated to a good cover \( U \) of \( B \).

We will work with the de Rham complex, but the same discussion applies to singular cochains. An element \([\omega] \in H^n(F) = E_2^{0,n}\) is called transgressive if it lives to \( E_{n+1} \), i.e., the differentials \( d_2, \ldots, d_n \) vanish on \([\omega]\). The following lemma gives a nice geometric interpretation of transgressive elements.

**Lemma 2.25.** An element \([\omega] \in H^n(F)\) is transgressive iff there exists a global \( n \)-form \( \psi \in \Omega^n(E) \) such that \( \psi|_F = \omega \) and \( d\psi = \pi^* \tau \) for a closed \((n + 1)\)-form \( \tau \in \Omega^{n+1}(B) \) on the base.

**Proof.** Suppose first that \([\omega]\) is transgressive. This means that \([\omega]\)
is represented by a zigzag

\[
\begin{array}{ccccccc}
0 & \uparrow & & & & & \\
\uparrow & & & & & & \\
0 & \rightarrow & a_0 & \rightarrow & \ldots & \rightarrow & a_n \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& \uparrow & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
b & \rightarrow & \delta a_n & \rightarrow & \pi^* b & \rightarrow & 0,
\end{array}
\]

where \( a_0 \in C^0(\pi^{-1}\mathcal{U}, \Omega^n) \) restricts to \( \omega \) on \( F \), and \( \delta a_n = \pi^* b \) for a Čech cocycle \( b \in C^{n+1}(\mathcal{U}, \mathbb{R}) \). Applying the map \( s \) from Lemma 2.24 to \( a_0 + \cdots + a_n \in K^n \), we obtain a global \( n \)-form

\[
\psi := \sum_{p=0}^{n} (-d'K)^p a_p - K(-d'K)^n \pi^* b \in \Omega^n(E).
\]

If we choose the partition of unity \( \phi_\alpha \) in the definition of \( K \) such that \( \phi_{\alpha_0}(b_0) = 1 \) for some \( \alpha_0 \), it follows that \( (Kc)_{\alpha_0} = \sum_\alpha \phi_\alpha c_{\alpha_0} \) vanishes at \( b_0 \) for every \( c \in C^1(\pi^{-1}\mathcal{U}, \Omega^q) \), so under restriction to \( F = F_{b_0} \) all terms in \( \psi \) except the first one vanish and we obtain \( \psi|_F = \omega \). Applying \( d \) to \( \psi \), all terms except the last one vanish and we obtain

\[
d\psi = \pi^* \tau, \quad \tau := (-d'K)^{n+1} b \in \Omega^{n+1}(B),
\]

where \( d\tau = d(-d'K)^{n+1} b = 0 \).

Conversely, suppose that \( \omega = \psi|_F \) and \( d\psi = \pi^* \tau \) for \( \psi \in \Omega^n(E) \) and a closed form \( \tau \in \Omega^{n+1}(B) \). Then \( \psi \) restricts to an element \( \psi \in C^0(\pi^{-1}\mathcal{U}, \Omega^n) \) with \( \delta \psi = 0 \) and \( d\phi = \pi^* \tau \). Since the columns in the Čech–de Rham complex \( C^p(\mathcal{U}, \Omega^q) \) are exact, \( \tau \) can be ex-
2.8. TRANSGRESSION

tended to a zigzag

\[ \tau \]

\[ \tau_0 \rightarrow \]

\[ \cdots \rightarrow \]

\[ \tau_n \]

with \( d\tau_0 = -\tau \) and \( d_\tau i = -\delta \tau_{i-1}, i \geq 1 \). Then \( a_0 := \psi + \pi^* \tau_0 \) and \( a_i := \pi^* \tau_i \) defines a zigzag \( a_0, \ldots, a_n \) with \( a_0|_F = \omega \), so \([\omega]\) is transgressive.

The proof shows that the closed form \( \tau \in \Omega^{n+1}(B) \) represents \( d_{n+1}[\omega]_{n+1} \in E_{n+1}^{n+1,0} = H^{n+1}(B) \). The map that assigns to each transgressive element \([\omega] \in H^n(F)\) the element \([\tau] \in H^{n+1}(B)\) is called the transgression map.

**Euler class and Gysin sequence for oriented sphere bundles.** Let us apply the preceding discussion to a sphere bundle \( S^n \to E \xrightarrow{\pi} B \). Suppose that the bundle is oriented, i.e., the action of \( \pi_1(B) \) on \( H_n(S^n) \) is trivial. Let \( \omega \in \Omega^n(S^n) \) be a volume form. Then Lemma 2.25 provides a global angular form \( \psi \in \Omega^n(E) \) which restricts to a volume form on each fibre and satisfies \( d\psi = \pi^* \tau \) for a closed form \( \tau \in \Omega^{n+1}(B) \). The class \([\tau] \in H^{n+1}(B)\) is the Euler class of \( E \).

For degree reasons we have \( E_2 = \cdots = E_{n+1} \) and \( E_{n+2} = \cdots = E_\infty \). So all the information about \( H^*(E) \) is contained in the transgression map \( H^n(S^n) \to H^{n+1}(B), [\omega] \mapsto [\tau] \). Let us use this to compute \( H^*(E) \). The differential

\[ d_{n+1} : E_{n+1}^{p,n} = H^p(B) \otimes H^n(S^n) \to E_{n+1}^{p+n+1,0} = H^{p+n+1}(B) \otimes H^0(S^n) \]
maps $b \otimes [\omega]$ to $b \cup [\tau] \times 1$. Thus

$$H^{p+n}(E) \cong E^{p,n}_\infty \oplus E^{p+n,0}_\infty \cong \ker e_p \oplus H^{p+n}(B)/\text{im } e_{p-1},$$

where $e_p : H^p(B) \to H^{p+n+1}$ denotes the cup product with the Euler class $[\tau]$. Equivalently, this can be expressed as exactness of the Gysin sequence

$$\cdots H^{p-1}(B) \xrightarrow{e_{p-1}} H^{p+n}(B) \xrightarrow{\pi^*} H^{p+n}(E) \xrightarrow{\delta} H^p(B) \xrightarrow{e_p} H^{p+n+1}(B) \cdots$$

where $\pi^* : H^{p+n}(B) \to H^{p+n}(E) \cong \ker e_p \oplus H^{p+n}(B)/\text{im } e_{p-1}$ is the quotient map to the second factor and $\delta : H^{p+n}(E) \cong \ker e_p \oplus H^{p+n}(B)/\text{im } e_{p-1} \to H^p(B)$ is the projection onto $\ker e_p \subset H^p(B)$.

**Hirsch lemma for principal fibrations.** A similar situation arises for a fibration $K(V,n) \to E \xrightarrow{\pi} B$ whose fibre is an Eilenberg–McLane space $K(V,n)$. Suppose that the fibration is principal, i.e., the action of $\pi_1(B)$ on $H^*K(V,n)$ is trivial. Let us take coefficients in $\mathbb{R}$ and assume that $V$ is a finite dimensional $\mathbb{R}$-vector space. Again, we have $E_2 = \cdots = E_{n+1}$ and $E_{n+2} = \cdots = E_{\infty}$, so all the information about $H^*(E)$ is contained in the transgression map $d_{n+1} : V \cong H^nK(V,n) \to H^{n+1}(B)$. To compute $H^*(E)$, note that

$$E_{n+1}^* \cong H^*(B) \otimes \Lambda[V,n],$$

where $\Lambda[V,n]$ denotes the exterior algebra on the vector sapce $V$ in degree $n$ (i.e., the polynomial algebra for $n$ even and the exterior algebra for $n$ odd). The differential $d_{n+1} : E_{n+1}^* \to E_{n+1}^*$ vanishes on $H^*(B)$, and is the derivation induced by the transgression map on $\Lambda[V,n]$. Thus $(E_{n+1}^*, d_{n+1})$ is a commutative differential graded algebra (dga), i.e., a graded commutative algebra with a differential which is a derivation, and we get an algebra isomorphism.
\[ H^*(E) \cong H^*(E_{n+1}, d_{n+1}). \]

If \( E \) is a smooth fibre bundle (which actually never happens for \( n > 1 \)), we can use Lemma 2.25 to obtain \( H^*(E) \) as the cohomology of a commutative dga which is free as an algebra: Equip \( \Omega^*(B) \otimes \Lambda[V, n] \) with the derivation \( d \) which is induced by the exterior derivative on \( \Omega^*(B) \) and by a linear map \( d : V \to \Omega^{n+1}(B) \) that associates to each \( v \in V \) a closed \((n+1)\)-form \( dv \) representing \( d_{n+1}v \).

**Lemma 2.26** (Hirsch lemma). There exists a map of commutative differential graded algebras

\[ \rho : \left( \Omega^*(B) \otimes \Lambda[V, n], d \right) \to \left( \Omega^*(E), d \right) \]

which induces an isomorphism in cohomology and satisfies

\[ \rho|_{\Omega^*(B)} = \pi^* : \Omega^*(B) \to \Omega^*(E). \]

**Proof.** Since \( \Omega^*(B) \otimes \Lambda[V, n] \) is free as an algebra, we can define \( \rho \) on generators and then extend it as an algebra map. On \( \Omega^*(B) \) we set \( \rho := \pi^* \). The linear map \( \rho : V \to \Omega^n(E) \) must satisfy

\[ d\rho(v) = \rho(dv) = \pi^*(dv) \]

so \( \rho \) must associate to each \( v \in V \cong H^n(V) \) a global \( n \)-form \( \rho(v) \) on \( E \) such that \( d\rho(v) = \pi^*(dv) \) for the closed \((n+1)\)-form \( dv \) on \( B \) representing \( d_{n+1}v \). But such a map \( \rho \) is precisely provided by Lemma 2.25 with \( \rho(v) = \psi \) and \( dv = \tau \) (defined on a basis of \( V \) and then extended linearly)!

It remains to show that the induced map \( \rho^* \) on cohomology is an isomorphism. For this, we use the Čech–de Rham complex for a
good cover $\mathcal{U}$ of $B$. Consider the following diagram of dfg maps

$$
\begin{array}{ccc}
\left(\Omega^* (B) \otimes \Lambda[V, n], d\right) & \xrightarrow{\rho} & \left(\Omega^* (E), d\right) \\
\uparrow s & & \downarrow r \\
\left( C^* (\mathcal{U}, \mathbb{R}) \otimes \Lambda[V, n], \delta\right) & \xrightarrow{\sigma} & \left( C^* (\pi^{-1} \mathcal{U}, \Omega^*), D\right),
\end{array}
$$

where

- $s$ is induced by the map $s : C^*(\mathcal{U}, \mathbb{R}) \to \Omega^*(B)$ in Lemma 2.24;
- $r$ is the inclusion $\Omega^*(E) \hookrightarrow C^0(\pi^{-1} \mathcal{U}, \Omega^*)$;
- $\delta|_{C^*(\mathcal{U}, \mathbb{R})}$ is the Čech differential;
- $\delta : V \to C^{m+1}(\mathcal{U}, \mathbb{R})$ sends $v \in V$ to a Čech cocycle $\delta v$ representing $d_{n+1}v \in H^{n+1}(\mathcal{U}, \mathbb{R}) \cong H^{n+1}(B)$;
- $\sigma|_{C^*(\mathcal{U}, \mathbb{R})}$ is the inclusion $C^*(\mathcal{U}, \mathbb{R}) \hookrightarrow C^*(\pi^{-1} \mathcal{U}, \Omega^0)$;
- $\sigma : V \to \bigoplus_{p+q=n} C^p(\pi^{-1} \mathcal{U}, \Omega^q)$ satisfies $D(\sigma v) = \pi^* \delta v$.

By Theorem 2.13 the vertical maps $s$ and $r$ induce isomorphisms in cohomology, and by the proof of Lemma 2.25 the diagram commutes on cohomology. Hence it suffices to prove that $\sigma$ induces an isomorphism on cohomology.

For this, note that $\sigma$ preserves the filtrations on both complexes defined by

$$
\bigoplus_{i \geq p} C^i(\mathcal{U}, \mathbb{R}) \otimes \Lambda[V, n] \text{ resp. } \bigoplus_{i \geq p} C^i(\pi^{-1} \mathcal{U}, \Omega^*),
$$

so it induces a map between the associated spectral sequences. The map between the $E_2 = E_{n+1}$ pages

$$
\sigma_* : H^* (B) \otimes \Lambda[V, n] \to H^* (B) \otimes H^* K(V, n)
$$
is induced by the identity on $B$ and the isomorphism $V \cong H^nK(V, n)$, hence an isomorphism. Moreover, by definition, the differentials $d_{n+1}$ on both sides vanish on $H^*(B)$ and are induced by the transgression map $d_{n+1} : V \to H^{n+1}(B)$. Hence $\sigma_*$ continues as an isomorphism to the pages $E_{n+2} = E_\infty$ and the lemma is proved. \hfill \Box

The Hirsch lemma also holds for arbitrary fibrations, with the de Rham complex replaced by the singular cochain complex. However, in contrast to de Rham complex, the singular cochain complex is highly non-commutative. In Chapter 4 we will construct a commutative cochain complex which is defined on simplicial complexes, computes the singular cohomology, and satisfies an analogue of the Hirsch lemma.
Chapter 3

Free Loop Spaces and the Chas–Sullivan operations

3.1 Orientations

Orientations and the resulting signs are very important for the string topology operations. In this section, we fix the orientation conventions and derive some consequences. The discussion follows Chapter 8 in [7].

(1) The boundary $\partial X$ of an oriented manifold $X$ is oriented by an outward normal vector field $\nu$ by

$$T_x X = \nu_x \times T_x (\partial X).$$

(2) Let $E \to U$ be an oriented vector bundle (i.e., both $U$ and the fibres are oriented), and $s : U \to E$ be a section which is transverse to the zero section. Then the manifold $s^{-1}(0)$ is oriented by

$$T_x U = E_x \times T_x s^{-1}(0).$$

Note that the zero sets of the sections $\mathbb{1} \times s$ of $\mathbb{R}^k \oplus E \to \mathbb{R}^k \times U$ and $s \times \mathbb{1}$ of $E \times \mathbb{R}^\ell$ coincide as sets and their orientations are related by

$$(\mathbb{1} \times s)^{-1}(0) = s^{-1}(0) = (-1)^{\ell(\dim U - \text{rank } E)}(s \times \mathbb{1})^{-1}(0).$$
Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be submersions between oriented manifolds. Then their fibre product

$$X_1 \times_Y X_2 := \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}$$

is a manifold which we orient for $f_1(x_1) = f_2(x_2) = y$ by

$$T_{(x_1,x_2)}(X_1 \times_Y X_2) = \ker d_{x_1}f_1 \times T_yY \times \ker d_{x_2}f_2,$$

where the orientations of $\ker d_{x_i}f_i$ are induced from those of $\text{im} d_{x_i}f_i = T_yY$ via (note the different orders!)

$$T_{x_1}X_1 = \ker d_{x_1}f_1 \times \text{im} d_{x_1}f_1,$$

$$T_{x_2}X_2 = \text{im} d_{x_2}f_2 \times \ker d_{x_2}f_2.$$

More generally, let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be transverse smooth maps between oriented manifolds. Then their fibre product $X_1 \times_Y X_2$ is oriented as follows. Pick oriented vector bundles $E_i \to U_i$ with sections $s_i : U_i \to E_i$ transverse to the zero section such that $X_i = s_i^{-1}(0)$ and the maps $f_i$ extend to submersions $f_i : U_i \to Y$. (For example, for $Y = \mathbb{R}^n$ we can take $U_i = X_i \times \mathbb{R}^n$ and $E_i = U_i \times \mathbb{R}^n$ with the section $s_i(x_i, y) = (x_i, y, y, \ldots)$ and the submersion $U_i \to Y$, $(x_i, y) \mapsto f_i(x_i) + y.$) We orient the fibre product $U_1 \times_Y U_2$ as in (3). Restriction of the bundle $E_1 \oplus E_2 \to U_1 \times U_2$ to $U_1 \times_Y U_2$ gives an oriented vector bundle $E_1 \oplus E_2 \to U_1 \times_Y U_2$ with a section $s_1 \oplus s_2$ that is transverse to the zero section and whose zero set coincides as a set with $X_1 \times_Y X_2$. We orient $(s_1 \oplus s_2)^{-1}(0)$ as in (2) and define the orientation of the fibre product by

$$X_1 \times_Y X_2 := (-1)^{\text{rank} E_2(\dim X_1 - \dim Y)} (s_1 \oplus s_2)^{-1}(0).$$

Important special cases of the fibre product are:

- the product $X_1 \times X_2 = X_1 \times_{pt} X_2$,
3.1. ORIENTATIONS

- the intersection $X_1 \cap X_2 = X_1 \cap_Y X_2$ of two transverse submanifolds $X_1, X_2 \subset Y$.

**Problem 3.1.** Show:

(a) The orientation of the fibre product $X_1 \times_Y X_2$ does not depend on the choice of the bundles $E_i \to U_i$ in its definition.

(b) The orientation of $X_1 \times X_2$ as the fibre product over a point coincides with the product orientation.

(c) Fibre products of any smooth map $f : X \to Y$ with $\mathbb{1} : Y \to Y$ give orientation preserving identifications

$$X \times_Y Y = X = Y \times_Y X.$$ 

The following lemma is proved in [7], where maps are assumed transverse so that the fibre products are defined.

**Lemma 3.1.** (a) *(Boundary)* If $X_1, X_2$ have boundary and $Y$ has no boundary, then for $X_1 \to Y$ and $X_2 \to Y$ we have

$$\partial(X_1 \times_Y X_2) = \partial X_1 \times_Y X_2 \amalg (-1)^{\dim X_1 + \dim Y} X_1 \times_Y \partial X_2.$$

(b) *(Associativity)* For $X_1 \to Y_1$, $X_1 \to Y_1 \times Y_2$ and $X_3 \to Y_2$ we have

$$(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 = X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3).$$

(c) *(Iteration)* For $X_1 \to Y_1 \times Y_2$, $X_2 \to Y_1$ and $X_3 \to Y_2$ we have

$$X_1 \times (Y_1 \times Y_2) (X_2 \times X_3) = (-1)^{\dim Y_2 (\dim Y_1 + \dim X_2)} (X_1 \times_{Y_1} X_2) \times_{Y_2} X_3.$$

(d) *(Diffeomorphism)* If diffeomorphisms $f_i : X_i \to X_i'$ and $g : Y \to Y'$ change orientations by $\varepsilon(f_i)$ resp. $\varepsilon(g)$, then the induced diffeomorphism

$$f_1 \times g f_2 : X_1 \times_Y X_2 \to X_1' \times_Y X_2'$$

changes orientations by $\varepsilon(f_1) \varepsilon(f_2) \varepsilon(g)$. 
As an application of this lemma, we obtain

**Corollary 3.2.** For transverse maps \( f_i : X_i \rightarrow Y \), the canonical map \( X_1 \times X_2 \rightarrow X_2 \times X_1 \) induces an orientation preserving diffeomorphism

\[
X_1 \times_Y X_2 \xrightarrow{\cong} (-1)^{(\dim X_1 + \dim Y)(\dim X_2 + \dim Y)} X_2 \times_Y X_1.
\]

**Proof.** Consider the commuting diagram

\[
\begin{array}{ccc}
\Delta & \rightarrow & Y \times Y \\
\tau \downarrow & & \tau \downarrow \\
\Delta & \rightarrow & Y \times Y
\end{array}
\]

\[
\begin{array}{ccc}
X_1 \times X_2 & \xleftarrow{f_1 \times f_2} & X_1 \times X_2 \\
\tau \downarrow & & \tau \downarrow \\
\Delta & \rightarrow & Y \times Y
\end{array}
\]

in which \( \Delta \rightarrow Y \times Y \) is the inclusion of the diagonal and the maps \( \tau \) interchange the factors. By Lemma 3.1 (d), the maps \( \tau \) and \( \mathbb{I} \) induce an orientation preserving diffeomorphism

\[
\Delta \times_{(Y \times Y)} (X_1 \times X_2) \xrightarrow{\cong} (-1)^{x_1x_2+y} \Delta \times_{(Y \times Y)} (X_2 \times X_1),
\]

where \( x_i = \dim X_i \) and \( y = \dim Y \). On the other hand, Lemma 3.1 (c) and \( \Delta \times_Y X_i = Y \times_Y X_i = X_i \) implies

\[
\Delta \times_{(Y \times Y)} (X_1 \times X_2) = (-1)^{y(y+x_1)} (\Delta \times_Y X_1) \times_Y X_2 = (-1)^{y(y+x_1)} X_1 \times_Y X_2,
\]

\[
\Delta \times_{(Y \times Y)} (X_2 \times X_1) = (-1)^{y(y+x_2)} (\Delta \times_Y X_2) \times_Y X_1 = (-1)^{y(y+x_2)} X_2 \times_Y X_1.
\]

Thus the map \( \tau : X_1 \times X_2 \rightarrow X_2 \times X_1 \) induces an orientation preserving diffeomorphism \( X_1 \times_Y X_2 \xrightarrow{\cong} (-1)^s X_2 \times_Y X_1 \), where the total sign is given by \( s = x_1x_2 + y + y(y + x_1) + y(y + x_2) \equiv (x_1 + y)(x_2 + y) \mod 2 \).

**Problem 3.2.** Let \( X, Y \subset Z \) be two transversely intersecting submanifolds of an oriented manifold \( Z \).
(a) Show that the orientation of $X \cap Y$ as the fibre product $X \times_Z Y$ agrees with the orientation given in terms of coorientations by (note the reversed order!)

$$\text{(coorientation of } X \cap Y) = (\text{coorientation of } Y) \times (\text{coorientation of } X),$$

where the orientation and coorientation of a submanifold $X \subset Z$ are related by

$$\text{(orientation of } Z) = (\text{coorientation of } X) \times (\text{orientation of } X).$$

(b) Derive from part (a) the rules

$$\partial(X \cap Y) = \partial X \cap Y + (-1)^{\operatorname{codim} X} X \cap \partial Y,$$

$$X \cap Y = (-1)^{\operatorname{codim} X \operatorname{codim} Y} Y \cap X.$$  

### 3.2 The intersection product

As preparation for the loop product, let us first discuss the intersection product

$$\cap : H_i(M) \otimes H_j(M) \to H_{i+j-n}(M),$$

$i + j \geq n$, on an $n$-dimensional oriented manifold $M$. This is usually defined via the cup product $\cup : H^i_c(M) \otimes H^j_c(M) \to H^{i+j}(M)$ and Poincaré duality $\text{PD} : H^i_c(M) \to H_{n-i}(M)$ as

$$a \cap b := \text{PD}\left(\text{PD}^{-1}(a) \cup \text{PD}^{-1}(b)\right)$$

for $a \in H_i(M), b \in H_j(M)$, where $H^i_c(M)$ denotes cohomology with compact support. However, this approach does not extend to define the loop product.

With coefficients in a field $\mathbb{R}$ of characteristic zero, there is another approach based on the following result of Thom [16]: Any integer
homology class \( a \in H_i(M, \mathbb{Z}) \) has a multiple \( ka, \ k \in \mathbb{N} \), which can be represented by a smooth map \( f : P \to M \) from a closed oriented \( i \)-manifold \( P \). Similarly represent \( \ell b, \ \ell \in \mathbb{N} \), by \( g : Q \to M \) for a closed oriented \( j \)-manifold \( Q \). After perturbing \( f \), we may assume that \( f \times g : P \times Q \to M \times M \) is transverse to the diagonal \( \Delta \subset M \times M \). Then

\[
S := (f \times g)^{-1}(\Delta) \subset P \times Q
\]

is a closed oriented manifold of dimension \( i + j - n \) and \( f : S \to M \) represents a homology class \( c \in H_{i+j-n}(M, \mathbb{Z}) \). The intersection product of \( a \) and \( b \) is then

\[
a \cap b := \frac{1}{k\ell} c \in H_{i+j-n}(M, R),
\]

and it is easy to see that this definition is independent of all choices and induces (via the universal coefficient theorem) a product \( \cap : H_i(M, R) \otimes H_j(M, R) \to H_{i+j-n}(M, R) \). This approach can be used to define the loop product, but it does not work for integer coefficients.

The idea for the third approach, which works for arbitrary coefficients and extends to define the loop product, is to use the same construction on the chain level: Represent \( a \in H_i(M) \) and \( b \in H_j(M) \) (with arbitrary coefficients \( R \)) by singular cycles \( \sum_k a_k f_k \) and \( \sum_\ell b_\ell g_\ell \), where \( f_k : \Delta^i \to M \) and \( g_\ell : \Delta^j \to M \) are smooth maps from the standard simplices. Again, we can perturb the \( f_k \) to make each \( f_k \times g_\ell : \Delta^i \times \Delta^j \to M \times M \) transverse to the diagonal \( \Delta \subset M \times M \). Then \( S_{k\ell} := (f_k \times g_\ell)^{-1}(\Delta) \subset P_k \times Q_\ell \) will be a submanifold with corners, and after triangulating each \( S_{k\ell} \), the sum \( \sum_{k,\ell} a_k b_\ell f_k|_{S_{k\ell}} \) will represent the cohomology class \( a \cap b \in H_{i+j-n}(M) \). The goal of this section is to make this approach rigorous.
Manifolds with corners. We begin by collecting some facts about manifolds with corners. An \( n \)-dimensional manifold with corners is a second countable Hausdorff space \( P \) with a maximal atlas \( (U_i, \phi_i)_{i \in I} \), where

- \( \phi_i : U_i \to V_i \) is a homeomorphism from an open subset \( U_i \subset P \) onto an open subset \( V_i \subset \mathbb{R}^n_+ := [0, \infty)^n \);
- the transition maps \( \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \) are smooth.

Here a map \( A \to \mathbb{R}^m \) defined on a subset \( A \subset \mathbb{R}^n \) is called smooth if it extends to a smooth map on an open neighbourhood of \( A \) in \( \mathbb{R}^n \). It follows that

\[
P = \bigsqcup_{k=0}^{n} \partial^k P,
\]

where the codimension \( k \) stratum \( \partial^k P \) consists of those points \( p \in P \) whose image \( x \in \mathbb{R}^n_+ \) under some (and hence every) coordinate map has exactly \( k \) components equal to zero. In particular, \( \partial^0 P \) is the interior of \( P \). The closure of a connected component of \( \partial^k P \) is called a codimension \( k \) face of \( P \). Note that \( \partial^k P \) is an \( (n-k) \)-dimensional manifold (without boundary and corners). A neighbourhood in \( P \) of a point \( p \in \partial^k P \) corresponds under some coordinate chart to a neighbourhood of the origin in \( \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \).

The following consequence of the implicit function theorem gives a more intrinsic characterization of these sets.

**Problem 3.3.** Let \( F : \mathbb{R}^n \to \mathbb{R}^k \) be smooth with \( F(0) = 0 \) and \( dF(0) \) surjective. Then there exists a diffeomorphism \( \Phi : U \to \Phi(U) \) on a neighbourhood \( U \) of 0 with \( \Phi(0) = 0 \) such that

\[
\Phi^{-1}(F^{-1}(\mathbb{R}^k_+)) = (F \circ \Phi)^{-1}(\mathbb{R}^k_+) = (\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) \cap U.
\]
A map $f : P \to Q$ between manifolds with corners is called smooth if $\psi \circ f \circ \phi^{-1}$ is smooth for all coordinate maps $\phi, \psi$ for $P, Q$. The notions of diffeomorphisms etc are defined in the obvious way.

Recall that two smooth maps $f : P \to M$ and $g : Q \to M$ between manifolds (without boundary and corners) are called transverse if $T_pf + T_qg : T_pP \oplus T_qQ \to T_mM$ is surjective for all $p, q, m$ with $f(p) = g(q) = m$. If $P, Q$ are manifolds with corners and $M$ is a manifold, we call two smooth maps $f : P \to M$ and $g : Q \to M$ transverse if $f|_{\partial^k P}$ is transverse to $g|_{\partial^\ell Q}$ for all $k, \ell$.

**Problem 3.4.** Let $P, Q$ be manifolds with corners and $M$ be a manifold. Show:

(a) Any continuous map $f : P \to M$ can be uniformly approximated by a smooth map, which we can choose equal to $f$ on a neighbourhood of each closed subset $A \subset P$ near which $f$ is already smooth.

(b) Two smooth maps $f : P \to M$ and $g : Q \to M$ can be made transverse by a $C^k$-small perturbation of $f$ (for any $k \geq 0$). This perturbation can be chosen fixed on a neighbourhood of each closed subset $A \subset P$ over which $f$ is already transverse to $g$.

(c) If $f$ and $g$ are transverse, then $\{f = g\} \subset P \times Q$ is a submanifold with corners.

We define the boundary $\partial P = \coprod_j \partial_j P$ of a compact oriented manifold with corners as the disjoint union of the closures of the connected components of $\partial^1 P$, equipped with the boundary orientation (see Section 3.1). Then each $\partial_j P$ is again a compact oriented manifold with corners whose boundary equals $\partial\partial_j P = \coprod_i \partial_i \partial_j P$, where $\partial_i \partial_j P = \partial_i P \cap \partial_j P$ is oriented as boundary of $\partial_j P$. Note
that
\[ \partial_i \partial_j P = - \partial_j \partial_i P, \]
where \(-\) denotes the opposite orientation.

The product \( P \times Q \) of two manifolds with corners is again a manifold with corners, whose boundary equals (with the orientation conventions in Section 3.1)
\[ \partial(P \times Q) = \partial P \times Q \coprod (-1)^{\dim P} P \times \partial Q. \quad (3.1) \]

For our purposes, the most important example of a manifold with corners is the standard \( n \)-simplex
\[ \Delta^n := \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, t_0 + \cdots + t_n = 1 \}. \]

A simplicial complex is a space \( S = \bigsqcup_j \Delta^n_j / \sim \) obtained by gluing a collection of standard simplices \( \Delta^n_j \) along common faces via linear isomorphisms. A triangulation of a compact \( n \)-dimensional manifold with corners \( P \) is a homeomorphism \( \phi : S \to P \) from a simplicial complex \( S \) such that

- \( \phi \) is piecewise smooth, i.e., smooth on each simplex \( \Delta^n_i \);
- each \( \phi(\Delta^n_i) \cap \partial^j P \) is either empty, or the image of a face of \( \Delta^n_i \).

For the proof of the following result, see e.g. [12] (where it is proved for manifolds with boundary, but the proof can be adapted to manifolds with corners).

**Theorem 3.3.** Every given triangulation on a (possibly empty) closed subset \( A \subset P \) of a compact manifold with corners \( P \) can be extended to a triangulation of \( P \).

**Homology via manifolds with corners.** Consider now a topological space \( X \). Let \( \Sigma^n(X) \) be the set of equivalence classes of
pairs \((P, f)\), where \(P\) is a compact oriented \(n\)-dimensional manifolds with corners and \((P, f) \sim (P', f')\) iff there exists an orientation preserving diffeomorphism \(\phi: P \to P'\) such that \(f = f' \circ \phi\). That \(\Sigma^n(X)\) is indeed a set follows from the fact that every \(n\)-dimensional manifold with corners can be embedded in \(\mathbb{R}^{2n+1}\), so the equivalence classes of \(n\)-dimensional manifolds with corners can be viewed as a quotient of the set of all \(n\)-dimensional submanifolds with corners in \(\mathbb{R}^{2n+1}\), and similarly for pairs \((P, f)\). Denote by \(P_n(X)\) the free \(R\)-module generated by \(\Sigma^n(X)\) modulo the linear subspace generated by the following relations:

(i) \([P, f] + [−P, f]\), where \(-\) denotes the opposite orientation;

(ii) \([P, f]\) whenever \(f\) is \textit{degenerate}, i.e., it factors through an \((n−1)\)-dimensional manifold with corners;

(iii) \([P, f] − \sum_i [P_i, f|P_i]\) for any decomposition \(P = \bigcup_i P_i\).

Here a \textit{decomposition} of a compact \(n\)-dimensional manifold with corners \(P\) is a union \(P = \bigcup_{i=1}^N P_i\) such that

- \(P_i \subset P\) are smoothly embedded \(n\)-dimensional manifolds with corners;
- each \(P_i \cap P_j\) is either empty, or a codimension \(k\) face of both \(P_i\) and \(P_j\) for some \(k\);
- each \(P_i \cap \partial^j P\) is either empty, or a codimension \(k\) face of \(P_i\) for some \(k\).

For a space pair \((X, A)\) we define \(P_n(X, A) := P_n(X)/P_n(A)\).

Define the \(R\)-linear map \(\partial: P_n(X, A) \to P_{n-1}(X, A)\) by

\[
\partial[P, f] := \sum_j [\partial_j P, f|\partial_j P],
\]
where the sum runs over the codimension 1 boundary components. One easily checks that this is well-defined. Now $\partial_i \partial_j P = -\partial_j \partial_i P$ and condition (i) imply

$$\partial^2 [P, f] = \sum_{i,j} [\partial_i \partial_j P, f|_{\partial_i \partial_j P}] = 0,$$

so we can define the homology groups

$$HP_n(X, A) := \ker \partial / \text{im} \partial.$$  

Associating to each singular simplex $f : \Delta^n \to X$ the element $[\Delta^n, f]$ defines a canonical map

$$\Gamma : C_n(X, A) \to P_n(X, A)$$

from singular chains to $P_n(X, A)$. Recall that the singular boundary operator is given on a generator $f : \Delta^n \to X$ by

$$\partial f := \sum_{i=0}^n (-1)^i f \circ \lambda_i,$$

where $\lambda_i : \Delta^{n-1} \to \partial_i \Delta^n := \Delta^n \cap \{ t_i = 0 \}$ is the unique linear homeomorphism preserving the ordering of the vertices. Since the boundary orientation on $\partial_i \Delta^n$ differs from that induced by $\lambda_i$ by $(-1)^i$, it follows that $\Gamma$ is a chain map:

$$\Gamma(\partial f) = \sum_{i=0}^n (-1)^i [\Delta^{n-1}, f \circ \lambda_i] = \sum_{i=0}^n [\partial_i \Delta^n, f|_{\partial_i \Delta^n}] = \partial(\Gamma f).$$

**Proposition 3.4.** For all CW pairs $(X, A)$, the map $\Gamma : C_n(X, A) \to P_n(X, A)$ induces an isomorphism on homology

$$\Gamma_* : H_n(X, A) \xrightarrow{\cong} HP_n(X, A).$$

**Proof.** It suffices to show that the covariant functor $(X, A) \mapsto HP_n(X, A)$ satisfies the Eilenberg–Steenrod axioms (see e.g. [10]).
Homotopy invariance holds because if $P$ is a manifold with corners, then so is $P \times [0,1]$. Additivity for disjoint unions is obvious, and the long exact sequence of a pair $(X,A)$ is derived as in the case of singular homology.

The excision property $HP_n(X \setminus U, A \setminus U) \cong HP_n(X,A)$ for triples $(X, A, U)$ with $\bar{U} \subset \text{int} A$ can be proved in two steps. First, define a homology $HQ_n(X,A)$ in the same way as $HP_n(X,A)$, but allowing instead of manifolds with corners only standard simplices as domains. Condition (iii) and the fact that manifolds with corners can be triangulated imply that the canonical map $HQ_n(X,A) \to HP_n(X,A)$ is an isomorphism. But for $HQ_n(X,A)$, excision can be proved just as for singular homology, using barycentric subdivision.

The dimension axiom $HP_n(\text{pt}) = 0$ for $n \neq 0$, and $R$ for $n = 0$, follows from condition (ii), according to which $P_n(\text{pt}) = 0$ for $n \neq 0$, and $R$ for $n = 0$.

**Problem** 3.5. Does Proposition 3.4 still hold if we drop condition (iii) in the definition of $HP_n(X,A)$? (I don’t know the answer to this.)

**Cross product.** The definition of the cross product

$$H_i(X) \otimes H_j(Y) \to H_{i+j}(X \times Y)$$

using simplicial chains requires the choice of canonical triangulations of the products $\Delta^i \times \Delta^j$ of two simplices. For chains built on manifolds with corners this is unnecessary because the product of two manifolds with corners is again a manifolds with corners. So we have a natural $R$-bilinear product

$$\times : P_i(X) \otimes P_j(Y) \to P_{i+j}(X \times Y), \quad [P, f] \times [Q, g] := [P \times Q, f \times g].$$
Equation (3.1) translates into
\[ \partial(a \times b) = \partial a \times b + (-1)^i a \times \partial b \]
for \( a \in P_i(X) \) and \( b \in P_j(Y) \). Hence the product descends to a cross product on homology.

**Intersection product.** Let now \( M \) be an oriented manifold of dimension \( d \). Consider first two transverse smooth maps \( f : P \to M, g : Q \to M \) from manifolds with corners of dimensions \( i, j \) with \( i + j \geq d \). Then \( \{f = g\} \subset P \times Q \) is an \((i + j - d)\)-dimensional submanifold with corners, which we orient as the fibre product \( P \times_M Q \) according to the conventions in Section 3.1. So it defines an element

\[ [P, f] \cap [Q, g] := \{f = g\}, f|_{\{f=g\}} \in P_{i+j-d}(M), \]

whose boundary is given by Lemma 3.1 (a) to be

\[ \partial([P, f] \cap [Q, g]) = \partial[P, f] \cap [Q, g] + (-1)^{d-i}[P, f] \cap \partial[Q, g]. \] (3.2)

Consider now two cycles \( a = \sum_k a_k[P_k, f_k] \in P_i(M) \) and \( b = \sum_\ell b_\ell[Q_\ell, g_\ell] \in P_j(M) \). Using Problem 3.4 inductively, we can uniformly approximate the maps \( f_k, g_\ell \) by smooth maps \( \tilde{f}_k, \tilde{g}_\ell \) such that each \( \tilde{f}_k \) is transverse to each \( \tilde{g}_\ell \). Moreover, we can arrange that for each relation \( \partial_\nu f_k = \partial_\mu_\ell f_\ell \) in \( \partial f = 0 \) the corresponding relation \( \partial_\nu \tilde{f}_k = \partial_\mu_\ell \tilde{f}_\ell \) holds, hence \( \tilde{f} = \sum_k a_k \tilde{f}_k \) is again a cycle, and similarly for \( \tilde{g} = \sum_\ell b_\ell \tilde{g}_\ell \). The uniform closeness implies that the cycles \( f \) and \( \tilde{f} \) are homotopic (e.g., following minimal geodesics of some Riemannian metric), and thus homologous. By equation (3.2), the chain

\[ \tilde{a} \cap \tilde{b} := \sum_{k, \ell} a_k[P_k, \tilde{f}_k] \cap [Q_\ell, \tilde{g}_\ell] \in P_{i+j-d}(M) \]

is closed, and a similar argument shows that its homology class

\[ [a] \cap [b] := [\tilde{a} \cap \tilde{b}] \in H P_{i+j-d}(M) \]
does not depend on the chosen approximations \( \tilde{f}_k \) and \( \tilde{g}_\ell \). So we have defined the intersection product

\[
\cap : H_i(M) \otimes H_j(M) \to H_{i+j-d}(M).
\]

Lemma 3.1 (a) and (b) and Corollary 3.2 imply

**Proposition 3.5.** The intersection product on transversal chains is associative and satisfies

\[
\partial (a \cap b) = \partial a \cap b + (-1)^{\text{codim} a} a \cap \partial b, \tag{3.3}
\]

\[
a \cap b = (-1)^{\text{codim} a \text{ codim} b} b \cap a. \tag{3.4}
\]

**Problem 3.6 (linking numbers).** Let \( M \) be an oriented connected \( n \)-dimensional manifold, and \( K, L \subset M \) be closed oriented submanifolds of dimensions \( k = \dim K, \ell = \dim L \) with \( k + \ell = n - 1 \). Suppose that \( K \) and \( L \) are null-homologous and define their linking number

\[
\text{lk}(K, L) := C \cap L \in H_0(M; \mathbb{Z}) \cong \mathbb{Z},
\]

where \( C \) is a \((k + 1)\)-chain transverse to \( L \) with \( \partial C = K \). Show:

(a) The definition does not depend on the choice of \( C \).

(b) The linking number satisfies

\[
\text{lk}(K, L) = (-1)^{\text{codim} K \text{ codim} L} \text{lk}(L, K).
\]

### 3.3 The loop product

Now we will begin discussing the operations introduced by Chas and Sullivan in [5]. For the rest of this chapter we will use the notation from [5]. In particular:

- \( M \) is an orientable connected manifold of dimension \( d \);
• $\mathbb{L} = \mathbb{L}M$ is the \textit{(free) loop space} of $M$, i.e., the space of piecewise smooth maps $S^1 \to M$ from the circle $S^1 = \mathbb{R}/\mathbb{Z}$;

• $K$, $K_x$ etc will denote compact oriented manifolds with corners.

Note that maps $K \to \mathbb{L}$ canonically correspond to maps $K \times S^1 \to M$. We call a map $K \to \mathbb{L}$ from a manifold with corners \textit{smooth} if the corresponding map $K \times S^1 \to M$ is smooth. Thus every continuous map $K \to \mathbb{L}$ can be uniformly approximated by a smooth one.

\textbf{Problem 3.7.} (a) Show that the \textit{evaluation map}

$$\text{ev} : \mathbb{L}M \to M, \quad \gamma \mapsto \gamma(0)$$

defines a locally trivial fibre bundle whose fibre is the based loop space $\Omega M$.

(b) Use the Leray–Serre spectral sequence of this fibration to compute the cohomology ring of the free loop space of an odd-dimensional sphere.

The \textit{loop product}

$$\bullet : H_i(\mathbb{L}) \otimes H_j(\mathbb{L}) \to H_{i+j-d}(\mathbb{L})$$

is defined as follows. Consider two smooth maps $x : K_x \to \mathbb{L}$, $y : K_y \to \mathbb{L}$ from compact oriented manifolds with corners of dimensions $i$, $j$ the are \textit{transverse} in the sense that the evaluation map

$$\text{ev}_x \times \text{ev}_y : K_x \times K_y \to M \times M, \quad (k_x, k_y) \mapsto (x(k_x)(0), y(k_y)(0))$$

is transverse to the diagonal $\Delta \subset M \times M$. Then

$$K_x \bullet y := (\text{ev}_x \times \text{ev}_y)^{-1}(\Delta) \subset K_x \times K_y$$
is a compact oriented manifold with corners of dimension \( i + j - n \), and concatenation of loops yields a map \( x \bullet y : K_{x \bullet y} \to \mathbb{L} \),

\[
(x \bullet y)(k_x, k_y)(t) := \begin{cases} 
  x(k_x)(2t) & t \in [0, 1/2], \\
  y(k_y)(2t - 1) & t \in [1/2, 1]. 
\end{cases}
\]

Note that the evaluation map \( \text{ev}_{x \bullet y} : K_{x \bullet y} \to M \) is just the intersection product \( \text{ev}_x \cap \text{ev}_y \). Moreover, if we denote by \( \partial x \) the restriction of \( x \) to \( \partial K_x \), then as in Section 3.2 we have

\[
\partial K_{x \bullet y} = K_{\partial x \bullet y} \amalg (-1)^{i-d}K_{x \bullet \partial y},
\]

and hence

\[
\partial(x \bullet y) = \partial x \bullet y + (-1)^{i-d}x \bullet \partial y. \tag{3.5}
\]

Thus we have an operation \( \bullet \) that is transversely defined, i.e., defined on transverse pairs \((x, y) \in P_i(\mathbb{L}) \times P_j(\mathbb{L})\). Since by Problem 3.7 every pair \((x, y)\) of continuous maps can be uniformly approximated by a transverse pair, the same arguments as in the case of the intersection product in Section 3.2, using (3.5), yield a well-defined loop product on homology. Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
H_i(\mathbb{L}M) \otimes H_j(\mathbb{L}M) & \overset{\bullet}{\longrightarrow} & H_{i+j-d}(\mathbb{L}M) \\
\downarrow \quad \text{ev}_* \otimes \text{ev}_* & & \downarrow \quad \text{ev}_*
\end{array}
\]

\[
\begin{array}{ccc}
H_i(M) \otimes H_j(M) & \longrightarrow & H_{i+j-d}(M) \\
\downarrow \quad \cap & & \downarrow \quad \\
\end{array}
\]

It will be convenient in the following to shift all gradings down by \( d \) and define

\[
\mathbb{H}_i := H_{i+d}(\mathbb{L}), \quad i = -d, -d + 1, \ldots.
\]

The original grading will be denoted by \( \dim x = \dim K_x \), and the new grading in \( \mathbb{H} \) by

\[
|x| := \dim x - d.
\]
The reason for this shift is that the loop product preserves the new grading:

\[ \bullet : \mathbb{H}_i \otimes \mathbb{H}_j \to \mathbb{H}_{i+j}. \]

Moreover, with the shifted grading we have

**Proposition 3.6.** The loop product

\[ \bullet : \mathbb{H}_i \otimes \mathbb{H}_j \to \mathbb{H}_{i+j} \]

is associative and (graded) commutative.

**Proof.** For associativity, consider three transverse chains \( x, y, z \). Since \( K_{(x \bullet y) \bullet z} \) is the domain underlying the intersection product \( (\bar{x} \cap \bar{y}) \cap \bar{z} \) and the intersection product is associative, it agrees together with its orientation with \( K_{c \bullet (y \bullet z)} \). Now associativity of the loop product on homology follows by homotoping the parametrization of the loops in \((x \bullet y) \bullet z\) to that in \((x \bullet (y \bullet z))\) as in the proof of associativity of the Pontrjagin product on the based loop space.

For commutativity, we introduce a new operation on suitably transverse chains \( x, y \) as follows. Let \( \bar{x} = \text{ev} \circ x \) and define the chain

\[ \hat{y} : [0, 1] \times K_y \to M, \quad (s, k_y) \mapsto y(k_y)(s). \]

Suppose that the map

\[ \bar{x} \times \hat{y} : K_x \times [0, 1] \times K_y \to M \times M, \quad (k_x, s, k_y) \mapsto \left( x(k_x)(0), y(k_y)(s) \right) \]

is transverse to the diagonal \( \Delta \subset M \times M \) and define the * operator

\[ x \ast y : K_{x+y} := (\bar{x} \times \hat{y})^{-1}(\Delta) = K_x \times_M ([0, 1] \times K_y) \to M \]

by

\[
x \ast y(k_x, s, k_y)(t) :=
\begin{cases}
  y(k_y)(2t) & t \in [0, \frac{s}{2}], \\
  x(k_x)(2t - s) & t \in [\frac{s}{2}], \\
  y(k_y)(2t - 1) & t \in [\frac{s+1}{2}, 1].
\end{cases}
\]
Thus the loop \( x \circ y(k_x, s, k_y) \) follows the loop \( y \) from \( y(0) \) to \( y(s) \), then traverses the loop \( x \), and then follows \( y \) from \( y(s) \) to \( y(1) \); see Figure \textbf{fig:star}. Note that at \( s = 0 \) this equals \( x \circ y(k_x, k_y) \) and at \( s = 1 \) it equals \( y \circ x(k_x, k_y) \). The following identity shows that \( x \circ y \) yields a chain homotopy between \( x \circ y \) and \( y \circ x \), and hence implies commutativity on homology:\footnote{This formula differs from the one in \cite{5} by a sign.}

\[
\partial(x \circ y) = \partial x \circ y + (-1)^{|x|} x \circ \partial y + (-1)^{|x|} |x||y| y \circ x - x \circ y.
\]

To see this, note that

\[
K_{\partial \hat{y}} = \partial([0, 1] \times K_y) = 1 \times K_0 - 0 \times K_y - [0, 1] \times \partial K_y,
\]

which by equation \((3.4)\) implies

\[
K_{\bar{x} \cap \partial \hat{y}} = K_{\bar{x} \cap \hat{y}_1} - K_{\bar{x} \cap \hat{y}_0} - K_{\bar{x} \cap \partial \hat{y}} = (-1)^{|x||y|} K_{\hat{y}} + K_{x \circ y} + K_{x \circ y} - K_{x \circ y}.
\]

Here \( \hat{y}_0 \) and \( \hat{y}_1 \) denotes the map \( \hat{y} \) restricted to \( s = 0 \) resp. \( s = 1 \). Now equation \((3.3)\) yields

\[
K_{\partial (x \circ y)} = K_{\partial (\bar{x} \cap \hat{y})} = K_{\partial \bar{x} \cap \hat{y}} + (-1)^{|x|} K_{\bar{x} \cap \partial \hat{y}} = K_{\partial (x \circ y)} + (-1)^{|x|} (-1)^{|x||y|} K_{y \circ x} - K_{x \circ y} + K_{x \circ y}.
\]

Restricting the map \( x \circ y \) to these subsets yields formula \((3.6)\). \( \square \)

**Relation to intersection product and Pontrjagin product.** Recall the loop fibration

\[
\Omega \longrightarrow \mathbb{L} \longrightarrow \mathbb{L}
\]

with \( \Omega = \Omega M \) and the section

\[
\varepsilon : M \longrightarrow \mathbb{L}
\]
given by the constant loops. Transverse intersection with the fibre $\Omega$ yields a map

$$\cap\Omega : \mathbb{H}_* = H_{*+d}(\mathbb{L}) \to H_*(\Omega)$$

of degree $-d$. Indeed, for a chain $x : K_x \to \mathbb{L}$ whose evaluation $\bar{x} : K_x \to M$ is transverse to the base point $b \in M$, the chain $x \cap \Omega$ is given by simply restricting $x$ to $\bar{x}^{-1}(b)$. The following proposition is now obvious, where $\cdot$ denotes the Pontrjagin product.

**Proposition 3.7.** All maps in the following diagram preserve the products:

$$
\begin{array}{ccc}
(H_{*+d}(M), \cap) & \xrightarrow{\varepsilon} & (\mathbb{H}_*, \cdot) \\
\downarrow & & \downarrow \text{ev} \\
(H_*(\Omega), \cdot) & \xrightarrow{\cap\Omega} & (H_{*+d}(M), \cap).
\end{array}
$$

### 3.4 The loop bracket

The chains $x$ and $y$ play very different roles in $x \ast y$. In fact, one should think of this as an operation of $x$ on $y$ rather than a product. However, it gives rise to a symmetric operation as follows. Note that the term in brackets on the right hand side of

$$\partial(y \ast x) = \partial y \ast x + (-1)^{|y|+1} y \ast \partial x + (-1)^{|y|} ((-1)^{|x||y|} x \cdot y - y \cdot x)$$

differs from the corresponding term in $\partial(x \ast y)$ in (3.6) only by the sign $(-1)^{(|x|+1)(|y|+1)}$. Since all other terms are boundary terms, this shows that the loop bracket

$$\{x, y\} := x \ast y - (-1)^{(|x|+1)(|y|+1)} y \ast x$$

descends to homology. Moreover, this leads to a familiar algebraic structure:
Proposition 3.8. The loop bracket induces a graded Lie bracket of degree $+1$ on homology, i.e., for $x, y, z \in \mathbb{H}_*$ it satisfies

$$\{x, y\} = -(-1)^{(|x|+1)(|y|+1)}\{y, x\},$$
$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{(|x|+1)(|y|+1)}\{y, \{x, z\}\}.$$

Let us first explain the signs. Note that the usual Jacobi identity on an ungraded Lie algebra,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

can be equivalently written as a derivation property

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

In the graded case this becomes

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]].$$

Now the loop bracket $\{\}$ : $\mathbb{L} \otimes \mathbb{L} \to \mathbb{L}$ is an operation of degree $+1$ (i.e., it increases degrees by 1), so it is natural that it should satisfy graded commutativity and graded Jacobi with all degrees shifted by 1, which is exactly the content of Proposition 3.8.

The proof will be a purely algebraic consequence of

Lemma 3.9. The associator of $*$ on suitably transverse $x, y, z$ is symmetric in the first two variables up to chain homotopy,

$$x*(y*z) - (x*y)*z \simeq (-1)^{(|x|+1)(|y|+1)}(y*(x*z) - (y*x)*z).$$

Proof. Up to signs, this is evident from Figure fig:associator. The chain homotopy is simply connecting different parametrizations of the triple concatenations of loops. For the sign we compute, setting $I_s = I_t = [0, 1]$ and using the orientation rules in
Lemma 3.1 and Corollary 3.2

\[ K_{x(y*z)} - (x*y)*z \]

\[ = \{(k_x, s, k_y, t, k_z) \in K_x \times I_s \times K_y \times I_t \times K_z \mid \]

\[ x(k_x)(0) = z(k_z)(s), \quad y(k_y)(0) = z(k_z)(t) \} \]

\[ = K_x \times_M \left( I_s \times K_y \times_M (I_t \times K_z) \right) \]

\[ \cong (-1)^{|y|(|z|+1)} K_x \times_M (I_s \times I_t \times K_z) \times_M K_y \]

\[ \cong (-1)^{|y|(|z|+1)+|y|(|x|+|z|)+|x||z|} K_y \times_M (I_s \times I_t \times K_z) \times_M K_x \]

\[ \cong (-1)^{|y|(|z|+1)+|y|(|x|+|z|)+|x||z|+1+|x||z|+1} K_y \times_M (I_t \times K_x \times_M (I_s \times K_z)) \]

\[ = (-1)^{|x|+1} |y|+1) K_y \times_M \left( I_t \times K_x \times_M (I_s \times K_z) \right) \]

\[ \cong (-1)^{|x|+1} |y|+1) \{(k_y, t, k_x, s, k_z) \in K_y \times_I t \times K_x \times I_s \times K_z \mid \]

\[ y(k_y)(0) = z(k_z)(t), \quad x(k_x)(0) = z(k_z)(s) \} \]

\[ = (-1)^{|x|+1} |y|+1) K_y*(x*z) - (y*x)*z. \]

\[ \square \]

**Problem 3.8.** Deduce Proposition 3.8 from Lemma 3.9.

**Remark 3.10.** The definition of the loop product can be generalized to the following setting. Consider a Serre fibration \( F \to X \xrightarrow{\pi} M \) over an oriented \( d \)-dimensional manifold \( M \). Suppose that there exists a continuous map \( \mu : X \times_M X \to X \). By the Serre fibration property, any pair of continuous maps \( x : K_x \to X \) and \( y : K_y \to X \) can be perturbed to \( \tilde{x}, \tilde{y} \) such that \( \pi \circ \tilde{x} : K_x \to M \) and \( \pi \circ \tilde{y} : K_y \to M \) are smooth and transverse. So we obtain a map

\[ \tilde{x} \bullet \tilde{y} : K_x \times_M K_y \xrightarrow{\tilde{x} \times_M \tilde{y}} X \times_M X \xrightarrow{\mu} X. \]

Arguing as for the loop product, this induces a product

\[ \bullet : H_i(X) \otimes H_j(Y) \to H_{i+j-d}(X). \]
One may think of $X \to M$ as an $H$-fibration. An example is provided by the space $LM$ of free (piecewise smooth or continuous) loops. The product $\bullet$ will be associative resp. commutative if the map $\mu$ has these properties up to homotopy. However, the commutativity of the loop product despite the lack of commutativity of the Pontrjagin product, as well as the further operations discussed below, are specific to the free loop space do not seem to generalize.

Compatibility of loop bracket and loop product.

Lemma 3.11. On suitably transverse chains $x, y, z \in P_*(\mathbb{L})$, the star products $x*$ and $*z$ are left resp. right derivations of the loop product up to chain homotopy:

(a) $x \ast (y \bullet z) \simeq (x \ast y) \bullet z + (-1)^{(|x|+1)|y|} y \bullet (x \ast z)$

(b) $(x \bullet y) \ast z \simeq x \bullet (y \ast z) + (-1)^{|y|(|z|+1)} (x \ast z) \bullet y$.

Proof. Up to signs, part (a) is evident from Figure fig:loop-a. The chain homotopy is simply connecting different parametrizations of the triple concatenations of loops. For the sign we compute, canonically identifying $[0, 1/2] \cong [1/2, 1] \cong [0, 1] = I$ and using the orientation rules in Lemma 3.1 and Corollary 3.2,

$$K_{x *(y \bullet z)} = K_1 \cup K_2$$
with

\[ K_1 = \{(k_x, s, k_y, k_z) \in K_x \times [0, \frac{1}{2}] \times K_y \times K_z \mid \]
\[ x(k_x)(0) = y(k_y)(2s), \quad y(k_y)(0) = z(k_z)(0) \}\]
\[ \cong \{(k_x, s, k_y, k_z) \in K_x \times [0, 1] \times K_y \times K_z \mid \]
\[ x(k_x)(0) = y(k_y)(s), \quad y(k_y)(0) = z(k_z)(0) \}\]
\[ = K_{(x*y)\bullet z}, \]

\[ K_2 = \{(k_x, s, k_y, k_z) \in K_x \times [\frac{1}{2}, 1] \times K_y \times K_z \mid \]
\[ x(k_x)(0) = z(k_z)(2s - 1), \quad y(k_y)(0) = z(k_z)(0) \}\]
\[ = K_x \times M \left( \left[ \frac{1}{2}, 1 \right] \times (K_y \times_M K_z) \right) \]
\[ \cong (-1)^{|y||z|} K_x \times_M \left( \left[ \frac{1}{2}, 1 \right] \times (K_z \times_M K_y) \right) \]
\[ \cong (-1)^{|y||z|+|y||x|+|y||z|+1)} K_y \times_M \left( K_x \times_M \left( [0, 1] \times K_z \right) \right) \]
\[ = (-1)^{|x|+1})y|\{(k_y, k_x, s, k_z) \in K_y \times K_x \times [0, 1] \times K_z \mid \]
\[ x(k_x)(0) = z(k_z)(s), \quad y(k_y)(0) = z(k_z)(0) \}\]
\[ = K_{y\bullet (x*z)}. \]

Part (b) is more interesting because the domains do not agree on the nose. A chain homotopy is provided by a 2-parameter family parametrized by a 2-simplex whose boundary components corresponds to the three terms in (b) as shown in Figure fig:loop-b. More precisely, we orient the 2-simplex

\[ \Delta := \{(t, s) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq 1 \} \]

as a subset of \( \mathbb{R}^2 \) and define a degree 2 chain

\[ \phi : P_*(\mathbb{L}) \otimes P_*(\mathbb{L}) \otimes P_*(\mathbb{L}) \to P_*(\mathbb{L}), \]
\[ \phi(x, y, z) : K = K_{\phi(x,y,z)} \to \mathbb{L} \]
by
\[ K := \{(k_x, k_y, t, s, k_z) \in K_x \times K_y \times \Delta \times K_z | x(k_x)(0) = z(k_z)(s), y(k_y)(0) = z(k_z)(t)\} = K_x \times_M (K_y \times_M (\Delta \times K_z)), \]

\[ \phi(x, y, z)(k_x, k_y, t, s, k_z)(u) := \begin{cases} 
  z(k_z)(3u) : & u \in [0, \frac{s}{3}], \\
  x(k_x)(3u - s) : & u \in [\frac{s}{3}, \frac{s+1}{3}], \\
  z(k_z)(3u - 1) : & u \in [\frac{s+1}{3}, \frac{t+1}{3}], \\
  y(k_y)(3u - t - 1) : & u \in [\frac{t+1}{3}, \frac{t+2}{3}], \\
  z(k_z)(3u - 2) : & u \in [\frac{t+2}{3}, 1]. 
\end{cases} \]

We claim that
\[ (-1)^{|x|+|y|} \partial \phi(x, y, z) = x \bullet (y*z) + (-1)^{|y|(|z|+1)}(x*z) \bullet y - (x \bullet y) \ast z + T, \]

where \( T \) consists of terms involving \( \partial x, \partial y, \) or \( \partial z \), so \( \phi \) is the desired chain homotopy. Up to signs, this identity is evident from Figure fig:loop-b. For the signs, note that

\[ \partial \Delta = \partial_1 \Delta \amalg \partial_2 \Delta \amalg \partial_3 \Delta, \]
\[ \partial_1 \Delta = \{(t, 0) | t \in I\} \cong I, \]
\[ \partial_2 \Delta = \{(1, s) | s \in I\} \cong I, \]
\[ \partial_3 \Delta = -\{(s, s) | s \in I\} \cong -I, \]

and thus
\[ \partial K = \partial_1 K \amalg \partial_2 K \amalg \partial_3 K \amalg B, \]

where \( \partial_i K \) is the boundary component corresponding to \( \partial_i \Delta \) and \( B \) denotes boundary components involving boundaries of \( K_x, K_y, \)
or $K_z$. Using Lemma 3.1 and Corollary 3.2, we compute

$$\partial_1 K = \{(k_x, k_y, t, k_z) \in K_x \times K_y \times I \times K_z \mid x(k_x)(0) = z(k_z)(t), y(k_y)(0) = z(k_z)(t)\}$$

$$= K_x \cdot (y \ast z),$$

$$\partial_2 K = \{(k_x, k_y, s, k_z) \in K_x \times K_y \times I \times K_z \mid x(k_x)(0) = z(k_z)(s), y(k_y)(0) = z(k_z)(1)\}$$

$$= K_x \times_M \left( K_y \times_M (I \times K_z) \right)$$

$$= (-1)^{|y|(|z|+1)} K_x \times_M (I \times K_z) \times_M K_y$$

$$= (-1)^{|y|(|z|+1)} \{(k_x, s, k_z, k_y) \in K_x \times I \times K_z \times K_y \mid x(k_x)(0) = z(k_z)(s), y(k_y)(0) = z(k_z)(0)\}$$

$$= (-1)^{|y|(|z|+1)} K_{(x \ast z) \cdot y},$$

$$\partial_3 K = -\{(k_x, k_y, s, k_z) \in K_x \times K_y \times I \times K_z \mid x(k_x)(0) = z(k_z)(s) = y(k_y)(0)\}$$

$$= -K_{(x \cdot y) \ast z},$$

where the signs $(-1)^{|x|+|y|}$ come from moving $\partial$ past $K_x \times K_y$ in $K$. This proves Lemma 3.11. □

Combining Lemma 3.11(a) and (b), we obtain the derivation property for the string bracket on homology:

$$\{a, b \bullet c\} = a \ast (b \bullet c) - (-1)^{|a|+1|b|+|c|+1}(b \bullet c) \ast a$$

$$= (a \ast b) \bullet c + (-1)^{|a|+1|b|} b \bullet (a \ast c)$$

$$- (-1)^{|a|+1|b|+|c|+1} b \bullet (c \ast a) - (-1)^{|a|+1|b|+1} (b \ast a) \bullet c$$

$$= \{a, b\} \bullet c + (-1)^{|a|+1|b|} b \bullet \{a, c\}.$$

So we have shown

**Theorem 3.12.** The loop product $\bullet$ and the loop bracket $\{,\}$ give the loop homology $\mathbb{H}_*(\mathbb{L})$ the structure of a Gerstenhaber algebra, i.e.,
(i) • is an associative and graded commutative product (of degree 0);

(ii) \{ , \} is a Lie bracket of degree 1;

(iii) \{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{(|a|+1)|b|} b \bullet \{a, c\}.

3.5 The operator $\Delta$

The circle action on the free loop space induces a degree +1 operator

$$\Delta : P_i(\mathbb{L}) \to P_{i+1}(\mathbb{L})$$

that associates to an $i$-chain $x : K_x \to \mathbb{L}$ the $(i + 1)$-chain

$$\Delta x : S^1 \times K_x \to \mathbb{L}, \quad \Delta x(s, k_x)(u) := x(k_x)(s + u).$$

Since $\partial \Delta x = -\Delta \partial x$, the operator $\Delta$ descends to homology. Since

$$\Delta \circ \Delta x(s, t, k_x)(u) = x(k_x)(s + t + u) = \Delta x(s + t, k_x)(u),$$

the $(i + 2)$-chain $\Delta \circ \Delta x$ factors through the $(i + 1)$-chain $\Delta x$ and is therefore degenerate, hence zero in $P_i(\mathbb{L})$. So the operator $\Delta : H_\ast \to H_\ast$ satisfies $\Delta \circ \Delta = 0$.

To study the compatibility of $\Delta$ with the loop product and bracket, we decompose $\Delta = \Delta_1 + \Delta_2$ into two degree +1 operators $\Delta_j : P_i(\mathbb{L}) \to P_{i+1}(\mathbb{L})$ that are defined on $x : K_x \to \mathbb{L}$ by

$$\Delta_1 x : [0, \frac{1}{2}] \times K_x \to \mathbb{L}, \quad \Delta_1 x(s, k_x)(u) := x(k_x)(s + u),$$

$$\Delta_2 x : [\frac{1}{2}, 1] \times K_x \to \mathbb{L}, \quad \Delta_2 x(s, k_x)(u) := x(k_x)(s + u).$$

Note that $\Delta_1$ and $\Delta_2$ do not descend to homology. On the other hand, the map

$$P_i(\mathbb{L}) \otimes P_j(\mathbb{L}) \to P_{i+j+1}(\mathbb{L}), \quad x \otimes y \mapsto x \bullet \Delta y,$$
is the composition of the two chain maps $\mathbb{1} \times \Delta$ and $\bullet$, and hence descends to homology.

**Lemma 3.13.** On suitably transverse chains $x, y \in P_*(\mathbb{L})$ we have the chain homotopy\(^2\)

$$x \ast y + (-1)^{|x|} \Delta_2(x \bullet y) \simeq x \bullet \Delta y. \quad (3.7)$$

**Proof.** The proof is similar to that of Lemma 3.11. Again, we orient the 2-simplex

$$\Delta := \{(t, s) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq 1\}$$

as a subset of $\mathbb{R}^2$ and define a degree 2 chain

$$\psi : P_*(\mathbb{L}) \otimes P_*(\mathbb{L}) \to P_*(\mathbb{L}),$$

$$\psi(x, y) : K = K_{\psi(x,y)} \to \mathbb{L}$$

by

$$K := \{(k_x, t, s, k_y) \in K_x \times \Delta \times K_y \mid x(k_x)(0) = y(k_y)(t)\} = K_x \times_M (\Delta \times K_y),$$

$$\psi(x, y)(k_x, t, s, k_y)(u) := \begin{cases} y(k_y)(2u + s) : & u \in [0, \frac{t-s}{2}], \\ x(k_x)(2u - t + s) : & u \in [\frac{t-s}{2}, \frac{t-s+1}{2}], \\ y(k_y)(2u + s - 1) : & u \in [\frac{t-s+1}{2}, 1]. \end{cases}$$

We claim that

$$(-1)^{|x|} \partial \psi(x, y) = x \ast y + (-1)^{|x||y|+1} \Delta_1(y \bullet x) - x \bullet \Delta y + T,$$

where $T$ consists of terms involving $\partial x$ or $\partial y$. Since $\Delta_1(y \bullet x) \simeq (-1)^{|x||y|} \Delta_2(x \bullet y)$, this will prove the lemma. Up to signs, this

\(^2\)Here one sign differs from \([5]\).
identity is evident from Figure 3.1. For the signs, we again write
\[
\partial \Delta = \partial_1 \Delta \amalg \partial_2 \Delta \amalg \partial_3 \Delta,
\]
\[
\partial_1 \Delta = \{ (t, 0) \mid t \in I \} \cong I,
\]
\[
\partial_2 \Delta = \{ (1, s) \mid s \in I \} \cong I,
\]
\[
\partial_3 \Delta = -\{ (s, s) \mid s \in I \} \cong -I,
\]
and thus
\[
\partial K = \partial_1 K \amalg \partial_2 K \amalg \partial_3 K \amalg B,
\]
where \( \partial_i K \) is the boundary component corresponding to \( \partial_i \Delta \) and \( B \) denotes boundary components involving boundaries of \( K_x, K_y, \) or \( K_z \). Using Lemma 3.1 and Corollary 3.2, we compute
\[
(-1)^{|x|} \partial_1 K = \{ (k_x, t, k_y) \in K_x \times I \times K_y \mid x(k_x)(0) = y(k_y)(t) \} = K_{x \ast y},
\]
\[
(-1)^{|x|} \partial_2 K = \{ (k_x, s, k_y) \in K_x \times I \times K_y \mid x(k_x)(0) = y(k_y)(1) \} = K_x \times_M (I \times K_y)
\]
\[
= (-1)^{|x|(|y|+1)} I \times K_y \times_M K_x
\]
\[
= (-1)^{|x|(|y|+1)} I \times K_{y \bullet x}
\]
\[
= (-1)^{|x|(|y|+1)} K_{\Delta_1(y \bullet x)},
\]
\[
(-1)^{|x|} \partial_3 K = -\{ (k_x, t, k_y) \in K_x \times I \times K_y \mid x(k_x)(0) = y(k_y)(t) \} = -K_{x \bullet \Delta_y},
\]
where the signs \((-1)^{|x|}\) come from moving \( \partial \) past \( K_x \) in \( K \). This proves Lemma 3.13. \qed

Remark 3.14. Note that \((-1)^{|x||y|} \Delta_1(y \bullet x)\) in Figure 3.1 provides another chain homotopy between \( x \bullet y \) and \((-1)^{|x||y|} y \bullet x\).

Exchanging \( x \) and \( y \) in equation (3.7) and using \( \Delta_2(y \bullet x) \cong (-1)^{|x||y|} \Delta_1(x \bullet y) \), we obtain
\[
y \ast x + (-1)^{|x|+1} |y| \Delta_1(x \bullet y) \cong y \bullet \Delta x.
\]
Subtracting this with the sign \((-1)^{|x|+1}|y|+1\) from (3.7), we obtain
\[
0 \simeq x \ast y + (-1)^{|x|}\Delta_2(x \bullet y) \simeq x \bullet \Delta y
- (-1)^{|x|+1}|y|\Delta_1(x \bullet y) \simeq y \bullet \Delta x
\]
\[
= \{x, y\} + (-1)^{|x|}(\Delta_1 + \Delta_2)(x \bullet y) - x \bullet \Delta y + (-1)^{|x|+1}|y|y \bullet \Delta x
\]
\[
\simeq \{x, y\} + (-1)^{|x|}\Delta(x \bullet y) - x \bullet \Delta y - (-1)^{|x|}\Delta x \bullet y.
\]
Passing to homology, we have thus proved

**Corollary 3.15.** The loop bracket on homology is the deviation of \(\Delta\) from being a derivation of the loop product, i.e., for \(a, b \in \mathbb{H}_*\) we have
\[
\{a, b\} = (-1)^{|a|}\left(\Delta(a \bullet b) - \Delta a \bullet b - (-1)^{|a|}a \bullet \Delta b\right).
\]

Combining this with Theorem 3.12 and \(\Delta \circ \Delta = 0\), we obtain

**Theorem 3.16.** The loop product \(\ast\) and the operator \(\Delta\) give the loop homology \(\mathbb{H}_*(L)\) the structure of a Batalin-Vilkovisky (BV) algebra, i.e.,

(i) \(\ast\) is an associative and graded commutative product (of degree 0);

(ii) \(\Delta\) is an operator degree 1 satisfying \(\Delta \circ \Delta = 0\);

(iii) \((-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b\) is a derivation in both variables. \(\square\)

**Problem 3.9.** Let \((A, \cdot)\) be a graded commutative algebra with a degree +1 operator \(\Delta : A \to A\) satisfying \(\Delta \circ \Delta = 0\). Show:

(a) \((A, \cdot, \Delta)\) is a Batalin-Vilkovisky algebra if and only if the 7-term relation holds for all \(a, b, c \in A:\)
\[
\Delta(a \cdot b \cdot c) = \Delta(a \cdot b) \cdot c + (-1)^{|a|}a \cdot \Delta(b \cdot c) + (-1)^{|a|+1}|b|b \cdot \Delta(a \cdot c)
- \Delta a \cdot b \cdot c - (-1)^{|a|}a \cdot \Delta b \cdot c - (-1)^{|a|+|b|}a \cdot b \cdot \Delta c.
\]
(b) If $(A, \cdot, \Delta)$ is a Batalin-Vilkovisky algebra, then $\{a, b\} := (-1)^{|a|} \Delta(a \cdot b) - (-1)^{|a|} \Delta a \cdot b - a \cdot \Delta b$ is a Lie bracket of degree $+1$.

### 3.6 The string bracket

In this section we study induced operations on the $S^1$-equivariant homology of the free loop space for the circle action $\gamma \mapsto \gamma(\cdot + s)$ that rotates the base point.

**Equivariant homology.** We first recall some facts about group actions and equivariant homology. Throughout this section, $G$ is a compact Lie group.

1. A $G$-space is a Hausdorff space $X$ with a continuous left $G$-action $G \times X \to X$. A principal $G$-bundle is a (locally trivial) fibre bundle $G \to P \xrightarrow{\pi} B$ with a continuous left $G$-action which preserves the fibres and is free and transitive on each fibre. Thus $B$ is homeomorphic to $P/G$. Principal bundles are characterized by the following theorem of Gleason (see [3, Theorem II 5.4]): *If a $G$-space $X$ is completely regular (i.e., points and closed sets can be separated by continuous functions) and the $G$-action is free, then the quotient map $X \to X/G$ is a principal $G$-bundle.* We will assume from now on that all $G$-spaces are completely regular.

2. Given a principal $G$-bundle $\pi : P \to B$ and a $G$-space $Y$, one can form the associated fibre bundle with fibre $Y$ 

$$
\pi : P \times_G Y := (P \times Y)/G \to B, \quad [p, y] \mapsto \pi(p),
$$

where $G$ acts diagonally on $P \times Y$. By Gleason’s theorem, if $X, Y$ are $G$-spaces and the action on $X$ is free, then the projection onto the first factor

$$
\pi : (X \times Y)/G \to X/G
$$
defines a fibre bundle with fibre \( Y \).

(3) A *universal \( G \)-bundle* is a principal \( G \)-bundle \( G \to EG \to BG \) whose total space \( EG \) is contractible. The base \( BG \) is called a *classifying space* for \( G \).

**Problem 3.10.** For a compact Lie group \( G \), prove:

(a) There exists a universal \( G \)-bundle \( EG \to BG \) over a CW complex \( BG \).

(b) Every principal \( G \)-bundle \( P \to B \) over a CW complex \( B \) is the pullback of \( EG \to BG \) under a map \( f : P \to BG \).

(c) The universal \( G \)-bundle is unique up to \( G \)-equivariant homotopy equivalence, so in particular \( BG \) is unique up to homotopy equivalence.

(4) The *Borel construction* associates to a \( G \)-space \( X \) the space

\[
X_G := EG \times_G X.
\]

It fits into the following diagram with the obvious projections and inclusions:

\[
\begin{array}{ccc}
G & \longrightarrow & EG \times X \\
\downarrow \pi & & \downarrow \\
X & \overset{\iota}{\longrightarrow} & X_G \\
\downarrow \sigma & & \downarrow \tau \\
& X/G.
\end{array}
\]  

(3.8)

Here \( \tau : X_G \to BG \) is the projection of the associated fibre bundle with fibre \( X \), and the quotient map \( \pi : EG \times X \to X_G \) defines a principal \( G \)-bundle. The map \( \tau \) is in general not a fibration. By the preceding discussion, if the action on \( X \) is free, then \( \tau \) defines a fibre bundle with contractible fibre \( EG \), and hence a homotopy equivalence.
(5) The \(G\)-equivariant (co)homology of a \(G\)-space \(X\) is defined as
\[
H^G_i(X) := H_i(X_G), \quad H^i_G(X) := H^i(X_G).
\]
The diagram (3.8) induces a diagram
\[
\begin{array}{ccc}
H^i(X) & \xrightarrow{\iota_*} & H^G_i(X) & \xrightarrow{\tau_*} & H^i(BG) \\
& & \downarrow{\sigma_*} & & \\
& & H^i(X/G) & &
\end{array}
\]
By (4), if \(X\) is completely regular and \(G\) acts freely on \(X\), then \(\sigma_* : H^G_i(X) \to H^i(X/G)\) is an isomorphism. At the other extreme, if the action is trivial and the coefficient ring is a field, then \(X_G \cong BG \times X\) and thus \(H^G_* (X) \cong H_* (BG) \otimes H_*(X)\). In particular, \(H_*(pt) = H_*(BG)\). Note that the map \(\tau^* : H^*(BG) \to H^*_G(X)\) makes equivariant cohomology an \(H^*(BG)\)-module.

(6) To better understand the relation between equivariant and non-equivariant homology, consider more generally a fibre bundle \(F \to E \xrightarrow{\pi} B\) whose fibre \(F\) is a closed manifold of positive dimension \(n\). Then we can associate to an \(i\)-chain \(x : K_x \to B\) it “preimage” under \(\pi\), i.e., the \((i + n)\)-chain given by the obvious projection \(x^*E := K_x \times_B E \to E\). Note that \(x^*E\) is a bundle over \(K_x\) with fibre \(F\) and thus a manifold with corners whose boundary equals \(\partial K_x \times_B E\). So this operation defines a chain map \(\pi^* : P_i(B) \to P_{i+n}(E)\), which induces on homology the pullback map
\[
\pi^* : H_i(B) \to H_{i+n}(E).
\]
Since the \((i + n)\)-chain \(\pi_*(\pi^*x)\) factors through the chain \(x\), it is degenerate, so on homology we have \(\pi_* \circ \pi^* = 0\). On the other hand, the composition
\[
\Delta := \pi^* \circ \pi_* : H_i(E) \to H_{i+n}(E), \quad \Delta \circ \Delta = 0,
\]
can be nontrivial.
Problem 3.11. Show that for a sphere bundle $S^n \to E \xrightarrow{\pi} B$, the maps $\pi_*$ and $\pi^*$ fit into the exact homological Gysin sequence
\[ \cdots \to H_{i+n+1}(E) \xrightarrow{\pi_*} H_{i+n+1}(B) \xrightarrow{\cap c} H_i(B) \xrightarrow{\pi^*} H_{i+n}(E) \to \cdots, \]
where $c \in H^{n+1}_*(B)$ is the Euler class of the circle bundle.

For a principal bundle $G \to E \xrightarrow{\pi} B$, the map $\Delta : H_i(E) \to H_{i+n}(E)$ can be described more explicitly: It is induced by the chain map that associates to an $i$-chain $x : K_x \to E$ the $(i+n)$-chain
\[ \Delta x : G \times K_x \to E, \quad \Delta x(g, k_x) = g \cdot x(k_x). \]

(7) For a $G$-space $X$ with $n = \dim G > 0$, we apply the preceding discussion to the principal bundle $G \to EG \times X \xrightarrow{\pi} X_G$. Since $EG$ is contractible, we obtain maps
\[
E := \pi_* : H_i(X) \to H_i(X_G), \quad M := \pi^* : H_i(X_G) \to H_{i+n}(X),
\]
\[ \Delta := M \circ E : H_i(X) \to H_{i+n}(X) \]
satisfying
\[ E \circ M = 0, \quad \Delta \circ \Delta = 0. \]

Moreover, in the case $G = S^1$, these maps fit into the Gysin sequence
\[ \cdots \to H_i(X) \xrightarrow{E} H_i(X_{S^1}) \xrightarrow{\cap c} H_{i-2}(X_{S^1}) \xrightarrow{M} H_{i-1}(X) \to \cdots, \]
where $c \in H^2(X_{S^1})$ is the Euler class of the circle bundle.

String homology. Consider now the free loop space $\mathbb{L}$ of an oriented $d$-manifold $M$ with the circle action $\gamma \mapsto \gamma(\cdot + s)$ that rotates the base point. The string homology of $M$ is the equivariant homology
\[ H_i^{S^1}(\mathbb{L}) := H_i(\mathbb{L}_{S^1}). \]
By the preceding discussion, the circle bundle
\[ S^1 \to ES^1 \times \mathbb{L} \to \mathbb{L}_{S^1}. \]
induces a Gysin sequence
\[ \cdots \to H^i(\mathbb{L}) \xrightarrow{E} H^i(\mathbb{L}_{S^1}) \xrightarrow{\cap c} H^i-2(\mathbb{L}_{S^1}) \xrightarrow{M} H^i-1(\mathbb{L}) \to \cdots, \]
where \( c \in H^2(\mathbb{L}_{S^1}) \) is the Euler class of the circle bundle and \( E \circ M = 0 \).

Moreover, the description in (6) shows that
\[ \Delta := M \circ E : H^i(\mathbb{L}) \to H^{i+1}(\mathbb{L}), \quad \Delta \circ \Delta = 0, \]
is the operator introduced in Section 3.5.

**Remark 3.17.** Here is an alternative description of the string homology \( H^*_{S^1}(\mathbb{L}) \). Let \( \widetilde{\mathbb{L}} \subset \mathbb{L} \) be the open dense subset on which the circle action is free, i.e., the set of loops that are neither constant nor multiply covered. Since \( \mathbb{L} \setminus \widetilde{\mathbb{L}} \) has infinite codimension, the inclusion \( ES^1 \times_{S^1} \widetilde{\mathbb{L}} \hookrightarrow \mathbb{L}_{S^1} \) is a weak homotopy equivalence (see Problem 3.12 below). Since the circle action on \( \widetilde{\mathbb{L}} \) is free, the projection \( ES^1 \times_{S^1} \widetilde{\mathbb{L}} \to \widetilde{\mathbb{L}}/S^1 =: \Sigma \) is also a weak homotopy equivalence. Combining these, we see that \( H_*(\mathbb{L}_{S^1}) \cong H_*(\Sigma) \), so we can use the space \( \Sigma \) to compute the string homology. In this model, the maps \( E \) and \( M \) have very nice chain level descriptions: the degree 0 map \( E \) ("erase") forgets the base points on the loops in a chain \( K_x \to \widetilde{\mathbb{L}} \), and the degree +1 map \( M \) ("mark") places base points on the loops in a chain \( K_x \to \Sigma \) in all possible positions.

**Problem 3.12.** Show that the inclusion \( ES^1 \times_{S^1} \widetilde{\mathbb{L}} \hookrightarrow \mathbb{L}_{S^1} \) is a weak homotopy equivalence.

**The string bracket.** The maps \( M \) and \( E \) can be used to transfer operations from \( H_*(\mathbb{L}) \) to \( H^*_{S^1}(\mathbb{L}) \) and vice versa: An
operation $\sigma : H_*^1(\mathbb{L})^\otimes k \to H_*^1(\mathbb{L})$ of degree $\ell$ yields an operation $E \circ \sigma \circ M^\otimes k : H_*^1(\mathbb{L})^\otimes k \to H_*^1(\mathbb{L})$ of degree $\ell + k$, and vice versa. For example, the identity on $H_*^1(\mathbb{L})$ induces the degree +1 operation $\Delta = M \circ E$ on $H_*^1(\mathbb{L})$.

**Problem 3.13.** Show that the operations on string homology induced by the operation $\Delta$ and by the loop bracket are trivial.

The *string bracket*

$$[\ , \ ] : H_*^1(\mathbb{L}) \otimes H_*^1(\mathbb{L}) \to H_*^{i+j+2-d}(\mathbb{L})$$

is (up to a sign) the degree $(2 - d)$ operation on string homology induced by the loop product $\bullet$ on $H_*^1(\mathbb{L})$,

$$[a, b] := (-1)^{|a|} E(Ma \bullet Mb).$$

**Theorem 3.18.** The string bracket $[\ , \ ]$ on string homology $H_*^1(\mathbb{L})$ is a Lie bracket of degree $(2 - d)$.

**Proof.** Skew-symmetry follows from graded symmetry of the loop product:

$$[b, a] = (-1)^{|b|} E(Mb \bullet Ma)$$

$$= (-1)^{|b|+|a|+1+|b|+1} E(Ma \bullet Mb)$$

$$= -(-1)^{|a||b|}[a, b].$$

The Jacobi identity follows from the 7-term relation applied to $Ma, Mb, Mc$. Since $\Delta M = 0$, the last 3 terms vanish and we obtain

$$\Delta(Ma \bullet Mb \bullet Mc)$$

$$= \Delta(Ma \bullet Mb) \bullet Mc + (-1)^{|a|+1} Ma \bullet \Delta(Mb \bullet Mc)$$

$$+ (-1)^{|a|(|b|+1)} Mb \bullet \Delta(Ma \bullet Mc)$$

$$= (-1)^{|a|} M[a, b] \bullet Mc + (-1)^{|a|+|b|+1} Ma \bullet M[b, c]$$

$$+ (-1)^{|a||b|} Mb \bullet M[a, c],$$
where for the last equality we have replaced \( \Delta \) by \( ME \) and inserted the definition of \([\ , \ ]\). Applying \( E \) to this equation, the left hand side vanishes since \( E\Delta = 0 \) and we obtain, inserting again the definition of \([\ , \ ]\),

\[
0 = (-1)^{|b|}[a, b, c] + (-1)^{|b|+1}[a, [b, c]] + (-1)^{|a||b|+|b|}[b, [a, c]],
\]

which is the Jacobi identity

\[
[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]].
\]

\[\square\]

### 3.7 The loop product on Lie groups

As an example, let us compute the loop product on a Lie group \( G \). Note first that we have a homeomorphism

\[
\phi : \mathbb{L} \to G \times \Omega, \quad \gamma \mapsto (\gamma(0), \gamma(0)^{-1}\gamma)
\]

with inverse

\[
\phi^{-1} : G \times \Omega \to \mathbb{L}, \quad (g, \gamma) \mapsto g\gamma.
\]

**Reference.** By a result of Bott, \( \Omega \) has the homotopy type of a CW complex with only even-dimensional cells. Thus its homology \( H_*(\Omega, \mathbb{Z}) \) is free and vanishes in odd dimensions, and the Künneth formula yields an isomorphism

\[
H_*(G \times \Omega) \cong H_*(G) \otimes H_*(\Omega).
\]

**Proposition 3.19.** For any Lie group \( G \), the map \( \phi \) induces via the Künneth formula an isomorphism of \( R \)-algebras

\[
\phi_* : \left( \mathbb{H}_*(\mathbb{L}), \bullet \right) \xrightarrow{\cong} \left( \mathbb{H}_*(G), \cap \right) \otimes \left( H_*(\Omega), * \right),
\]

where \( \bullet, \cap \) and \( * \) denote the loop, intersection and Pontrjagin product, respectively.
**Proof.** Consider chains $x_1 : K_{x_1} \rightarrow G$, $x_2 : K_{x_2} \rightarrow \Omega$, and the resulting product chain

$$x : K_{x_1} \times K_{x_2} \rightarrow L, \quad (k_{x_1}, k_{x_2}) \mapsto x_1(k_{x_1})x_2(k_{x_2}).$$

Let $y_1, y_2, y$ be defined analogously. Using Lemma 3.1 and Corollary 3.2, we compute

$$K_{x \bullet y} = (K_{x_1} \times K_{x_2}) \times_G (K_{y_1} \times K_{y_2})$$

$$= (-1)^{\dim x_1 \dim x_2} (K_{x_2} \times K_{x_1}) \times_G (K_{y_1} \times K_{y_2})$$

$$= (-1)^{\dim x_1 \dim x_2} K_{x_2} \times (K_{x_1} \times_G K_{y_1}) \times K_{y_2}$$

$$= (-1)^{\dim x_1 \dim x_2 + (\dim x_1 + \dim y_1 - d)} \dim x_2 (K_{x_1} \times_G K_{y_1}) \times K_{x_2} \times K_{y_2}$$

$$= (-1)^{|y_1| \dim x_2} K_{x_1 \cap y_1} \times K_{x_2} \times K_{y_2}.$$

With $g := x_1(k_{x_1}) = y_1(k_{y_1})$, the loop product is given on this domain by

$$(x \bullet y)(k_{x_1}, k_{y_1}, k_{x_2}, k_{y_2}) = \begin{cases} gx_2(k_{x_2})(2t) & : t \in [0, 1/2], \\ gy_2(k_{y_2})(2t - 1) & : t \in [1/2, 1] \end{cases}$$

$$= g(x_2(k_{x_2}) \ast y_2(k_{y_2}))$$

$$= \phi^{-1}(g, x_2(k_{x_2}) \ast y_2(k_{y_2})).$$

Since $g = (x_1 \cap y_1)(k_{x_1}, k_{y_1})$, we have derived the commutative diagram

$$\begin{array}{ccc}
K_{x \bullet y} & \xrightarrow{\sim} & (-1)^{|y_1| \dim x_2} K_{x_1 \cap y_1} \times (K_{x_2} \times K_{y_2}) \\
\xrightarrow{x \bullet y} & & \downarrow \phi \\
\xrightarrow{\sim} & & G \times \Omega.
\end{array}$$

This shows that under the isomorphisms

$$\mathbb{H}_n(L) \xrightarrow{\phi} \mathbb{H}_n(G \times \Omega) \cong \sum_{i+j=n} \mathbb{H}_i(G) \otimes H_j(\Omega)$$
the loop product corresponds to the composition
\[
\mathbb{H}_i(G) \otimes H_j(\Omega) \otimes \mathbb{H}_k(G) \otimes H_\ell(\Omega)
\cong (-1)^{jk}
\mathbb{H}_i(G) \otimes \mathbb{H}_k(G) \otimes H_j(\Omega) \otimes H_\ell(\Omega)
\rightarrow \bigcap \otimes \ast
\mathbb{H}_{i+k}(G) \otimes H_{j+\ell}(\Omega),
\]
which is the tensor product of the operations $\bigcap$ and $\ast$.

\begin{example}
For the Lie group $G = S^3$ we have
\[
\left( \mathbb{H}_*(S^3), \bigcap \right) \cong \Lambda[a], \quad |a| = -3,
\]
\[
\left( H_*(\Omega S^3), \ast \right) \cong \Lambda[b], \quad |b| = 2,
\]
where $a$ is the class of a point, and hence
\[
\left( \mathbb{H}_*(\mathbb{L} S^3), \bullet \right) \cong \Lambda[a, b], \quad |a| = -3, \ |b| = 2.
\]
By a theorem of Cohen, Jones, and Yan [6], this formula remains true for all odd-dimensional spheres $S^n$:
\[
\left( \mathbb{H}_*(\mathbb{L} S^n), \bullet \right) \cong \Lambda[a, b], \quad |a| = -n, \ |b| = n-1, \quad n > 1 \text{ odd.}
\]
\end{example}

In the same paper, they also compute the loop product for even-dimensional spheres and complex projective spaces.

Let us try to understand the loop product on a sphere $S^n$ more geometrically. To every pair $x, y \in S^n$ with $\langle x, y \rangle = 0$ (i.e., $y$ is a unit tangent vector to $S^n$ at $x$) we canonically associate a map from an $(n-1)$-dimensional sphere to the loop space based at $x$,

\[
\lambda_{x,y} : \{ z \in S^n \mid \langle y, z \rangle = 0 \} \cong S^{n-1} \rightarrow \Omega_x S^n,
\]
\[
\lambda_{x,y}(z)(t) := \frac{x + z}{2} + \cos \frac{t}{2} (x - z) + \sin \frac{t}{2} \sqrt{1 - \langle x, z \rangle^2} y.
\]
One easily checks that $|\lambda_{x,y}(z)(t)| = 1$. Geometrically, $\lambda_{x,y}(z)$ is the circle obtained by intersecting $S^n$ with the affine plane through $x$ in the directions $z-x$ and $y$; see Figure fig:suspension1. Note that each $\lambda_{x,y}$ is equivalent to the map $\lambda : S^{n-1} \to \Omega \Sigma S^{n-1} \cong \Omega S^n$ used in Section 1.5. It follows from Theorem 1.20 that $\lambda_{x,y}$ represents a generator of $H_{n-1}(\Omega_x S^n)$.

Now suppose that $n$ is odd. Then there exists a unit vector field $v$ on $S^n$, and we obtain a map

$$\lambda : P := \{(x, z) \in S^n \mid \langle v(x), z \rangle = 0 \} \to \mathbb{L}S^n,$$

$$(x, z) \mapsto \lambda_{x,v(x)}(z).$$

Note that the projection $p : (x, z) \mapsto x$ defines a sphere bundle $S^{n-1} \to P \xrightarrow{p} S^n$ which fits into the commuting diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\lambda} & \mathbb{L}S^n \\
p \downarrow & & \downarrow \text{ev} \\
S^n & \xrightarrow{\mathbb{I}} & S^n.
\end{array}
$$

**Problem 3.14.** Show that for $n$ odd, the map $\lambda : P \to \mathbb{L}S^n$ represents a generator $b$ of $\mathbb{H}_{n-1}(\mathbb{L}S^n)$ and $b \cap [\Omega S^n]$ is a generator of $H_{n-1}(\Omega S^n)$.

Now we can compute the loop product on $\mathbb{H}_*(\mathbb{L}S^n)$ for $n$ odd. We know that as a vector space, $\mathbb{H}_*(\mathbb{L}) \cong \Lambda[a, b]$, where $a$ with $|a| = -n$ is the class of a point, and $b$ with $|b| = n-1$ is represented by the map $\lambda$ above. Since the map $\cap [\Omega]$ intertwines the loop and Pontrjagin product, it follows that the $k$-fold loop product $b^k$ gets mapped to a generator $a \bullet b^k$ of $H_{k(n-1)}(\Omega)$, so $b^k$ is a generator of $\mathbb{H}_{k(n-1)}(\mathbb{L})$. Since we can represent classes $ab^k$ and $ab^\ell$ by disjoint cycles lying over different points in $S^n$, we have $ab^k \bullet ab^\ell = 0$. This shows that $(\mathbb{H}_*(\mathbb{L}S^n), \bullet) \cong \Lambda[a, b]$ as an algebra.
For $n$ even the situation is more difficult. According to Cohen, Jones, and Yan [6], the loop product on even-dimensional spheres $S^n$ is given by

$$\left( H_\ast (\mathbb{L}S^n), \bullet \right) \cong \Lambda [a, b, v], \quad |a| = -n, \ |b| = -1, \ |v| = 2n-2$$

(3.10)

Let us at least describe the generators in the case $S^2$. The generator $a$ of degree $-2$ is represented by a point. Recall that $(H_\ast (\Omega S^2), \ast) \cong \Lambda [x]$ with $|x| = 1$. The generator $b$ of degree $-1$ is the image of $x$ under the canonical map $H_1(\Omega) \to H_1(\mathbb{L}) = \mathbb{H}_{-1}(\mathbb{L})$. In contrast to the odd-dimensional case, the generator $x$ now does not give rise to a generator of $H_1(\mathbb{L})$ that maps onto the fundamental cycle under evaluation. But $x^2$ gives rise to a generator $v \in \mathbb{H}_2(\mathbb{L})$ that can be described as follows. Consider the commutative diagram

$$
\begin{array}{cccc}
S^2 & \longrightarrow & V_{4,2} & \longrightarrow & L\mathbb{S}^3 & \longrightarrow & L\mathbb{S}^2 \\
& p & \downarrow & \mu & \downarrow & \pi & \rightarrow \\
& S^3 & \longrightarrow & S^3 & \longrightarrow & S^2,
\end{array}
$$

where $V_{4,2}$ denotes the Stiefel manifold of orthonormal frames in $\mathbb{R}^4$, $p(v, w) = w$ defines a fibre bundle $S^2 \to V_{4,2} \to S^3$,

$$
\mu(v, w)(t) := \cos(t)v + \sin(t)w,
$$

and $\pi : S^3 \to \mathbb{C}P^1 \cong S^2$ is the Hopf fibration associating to $v \in S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ the complex line $\mathbb{C}v$. Complex multiplication $e^{i\theta} (v, w) = (e^{i\theta} v, e^{i\theta} w)$ defines a free circle action on $V_{4,2}$ leaving the composition $L\pi \circ \mu$ invariant, so we get an induced diagram

$$
\begin{array}{cccc}
S^2 & \longrightarrow & V_{4,2}/S^1 & \longrightarrow & L\mathbb{S}^2 \\
& p & \downarrow & ev & \rightarrow \\
S^3/S^1 & \longrightarrow & S^2,
\end{array}
$$

where $V_{4,2}/S^1$ is the total space of the principal bundle $V_{4,2} \to V_{4,2}/S^1 \to S^3/S^1$.
Hence the map $L\pi \circ \mu$ represents a generator $v \in H_2(\mathbb{L})$ such that $ev \circ L\pi \circ \mu = p$ defines the bundle $S^2 \to V_{4,2}/S^1 \to S^2$.

**Problem 3.15.** Show that the generator $v \in H_2(\mathbb{L}S^2)$ defined above satisfies $v \cap \Omega S^2 = x^2 \in H_2(\Omega S^2)$, and try to explain the other relations for the loop product on $S^2$ given in (3.10).

**Problem 3.16.** Show that the BV operator on $H^*_s(\mathbb{L}S^3) \cong \Lambda[a, b]$ is given by

$$\Delta(b^\ell) = 0, \quad \Delta(ab^\ell) = \ell b^{\ell-1},$$

and compute the loop and string bracket. **Hint:** For the first relation, show that $\Delta(b^k)$ is represented by a degenerate chain. For the second relation, show that the chain $S^1 \times (S^2)^\ell \times S^1 \to S^3$ representing $ab^k$ is homotopic to the map $(s, k_1, \cdots, k_\ell, t) \mapsto b(k_1)(\ell s + t)b(k_2)(t) \cdots b(k_\ell)(t)$.

### 3.8 The Goldman bracket on surfaces

As another example, let us consider the string bracket on a closed connected oriented surface $M$ of genus $g \geq 2$.

**Problem 3.17.** Let $\mathbb{L}$ be the free loop space of a closed connected oriented surface $M$ of genus $g \geq 2$. Show:

(a) The connected component $\mathbb{L}^0 \subset \mathbb{L}$ consisting of contractible loops deformation retracts $S^1$-equivariantly onto the space $M \subset \mathbb{L}$ of constant loops.

(b) Each connected component $\mathbb{L}^\alpha \subset \mathbb{L}$ consisting of noncontractible loops deformation retracts $S^1$-equivariantly onto the circle given by a single loop and its reparametrizations.

So we have $S^1$-equivariant homotopy equivalences

$$\mathbb{L} = \coprod_{\alpha \in \pi_1} \mathbb{L}^\alpha, \quad \mathbb{L}^0 \simeq M, \quad \mathbb{L}^\alpha \simeq S^1 \text{ for } \alpha \neq 0,$$
Here, $\tilde{\pi}_1$ denotes the set of free homotopy classes of loops on $M$. Since the $S^1$-action is trivial on $L^0 \simeq M$ and free on $L^\alpha \simeq S^1$ for $\alpha \neq 0$, we obtain

$$L_{S^1} = \bigsqcup_{\alpha \in \tilde{\pi}_1} L^\alpha_{S^1}, \quad L^0_{S^1} \simeq M \times \mathbb{C}P^\infty, \quad L^\alpha \simeq pt \text{ for } \alpha \neq 0.$$ 

The string bracket has degree $2 - d = 0$ and hence induces a Lie bracket on

$$H_0(L_{S^1}) \cong \bigoplus_{\alpha \in \tilde{\pi}_1} R$$

which can be geometrically described as follows. Represent $\alpha, \beta \in \tilde{\pi}_1$ by transverse smooth loops $a, b$. For each intersection point $p \in a \cap b$, let $a\#_p b$ be the concatenation of $a$ and $b$ at $p$, i.e., the loop starting at $p$ and running through $a$ and $b$. Let $\varepsilon_p = +1$ if the tangent vectors to $a$ and $b$ at $p$ form a positive basis of $T_p M$, and $-1$ otherwise. Then

$$[\alpha, \beta] = \sum_{p \in a \cap b} \varepsilon_p \{a\#_p b\},$$

where $\{\gamma\}$ denotes the free homotopy class of a loop $\gamma$. This Goldman bracket, discovered by Goldman [9], was the main motivation for the work of Chas and Sullivan.

**Problem 3.18.** Let $L$ be the free loop space of a closed connected oriented surface $M$ of genus $g \geq 2$.

(a) Show that the geometric description of the Goldman bracket on $H_0(L_{S^1})$ agrees with the general definition of the string bracket (including signs!).

(b) Prove directly from the geometric description that the Goldman bracket on $H_0(L_{S^1})$ is a Lie bracket.

Since a contractible loop can be made disjoint from any other loop, the space $H_0(L^0_{S^1})$ corresponding to contractible loops is contained
3.8. THE GOLDMAN BRACKET ON SURFACES

in the center of the Lie algebra. Moreover, the complementary space

$$\mathcal{V} := H_0(\mathbb{L}_{S^1}^\neq) = \bigoplus_{0 \neq \alpha \in \tilde{\pi}_1} H_0(\mathbb{L}_{S^1}^\alpha)$$

corresponding to noncontractible loops is a Lie subalgebra. To see this, suppose that for some \( p \in a \cap b \) the loop \( a\#_p b \) is contractible. Since a contraction can be modified to fix the base point \( p \), this is equivalent to \( ab = 1 \) in \( \pi_1(M, p) \), so \( b \) is freely homotopic to \( a^{-1} \). Perturb \( a \) so that it has transverse self-intersections and push the loop \( a^{-1} \) slightly off \( a \) so that it runs parallel to \( a \) in opposite direction. According to Figure \textbf{fig:Goldman}, intersection points of \( a \) and \( a^{-1} \) appear in pairs \( p, q \) with opposite signs and \( a\#_p a^{-1} \sim (a\#_q a^{-1})^{-1} \). Now either \( a\#_p a^{-1} \) and \( a\#_q a^{-1} \) are both contractible, in which case the corresponding terms in \([\{a\}, \{a^{-1}\}]\) cancel, or they are both noncontractible (in which case they may or may not cancel). Thus the algebraic count of terms in \([\{a\}, \{a^{-1}\}]\) corresponding to contractible loops is zero.

The Goldman bracket on the space \( \mathcal{V} \) is highly nontrivial. Here are some of its properties. Let us call a free homotopy class \( \alpha \) simple if it has a simple (i.e., embedded) representative.

1. If \([\alpha, \beta] = 0\) and one of them is simple, then \( \alpha \) and \( \beta \) have disjoint representatives (Goldman \cite{Goldman}). This fails without the simplicity hypothesis: there exist classes \( \alpha \neq \beta \) which are not multiples of simple classes, satisfy \([\alpha, \beta] = 0\), and do not have disjoint representatives (Chas \cite{Chas}).

2. More generally, if \( \alpha \) or \( \beta \) is simple, then the number of terms in \([\alpha, \beta]\) equals the minimal number of intersection points (counted positively but with multiplicities) of representatives of \( \alpha \) and \( \beta \) (Chas \cite{Chas}).

3. Numerical evidence led Chas to the following conjectures: A
class $\alpha$ is a multiple of a simple class iff $[\alpha, \alpha^{-1}] = 0$. More generally, for a primitive (i.e., not a multiple of another class) class $\alpha$, the number of terms in $[\alpha, \alpha^{-1}]$ equals the minimal number of self-intersection points of representatives of $\alpha$.

(4) It also seems to be an open question whether the Lie algebra $\mathcal{V}$ has trivial center.

(4) Wolpert [17] and Goldman [9] have discovered deep connections between the Goldman bracket on curves on a surface and the Poisson bracket for the Weil-Petersson symplectic form on Teichmüller space (or more general representation varieties).
Chapter 4

Minimal Models and Applications

4.1 Rational de Rham theory

In introductory algebraic topology one encounters the surprising fact that the highly non-commutative cup product on singular cochains becomes commutative on cohomology. The problem of commutative cochains asks for a commutative cochain model for singular cohomology, or more precisely for a contravariant functor that assigns to each space $X$ a commutative DGA $A^*(X)$ such that

(i) the cohomology of $A^*(X)$ is isomorphic to the singular cohomology ring $H^*(X)$, and

(ii) for each subspace $Y \subset X$ the restriction map $A^*(X) \to A^*(Y)$ is surjective.

On the category of differentiable manifolds, the de Rham complex $\Omega^*(M)$ solves the problem of commutative cochains with coefficients in $\mathbb{R}$. On the other hand, cohomology operations such as Steenrod squares provide obstructions to the solution of this prob-
lem with coefficients in $\mathbb{Z}$, even on the category of smooth manifolds. On the category of simplicial complexes and with coefficients in $\mathbb{Q}$, the problem of commutative cochains was first solved abstractly by Quillen [13], and then more concretely by Sullivan [15] using his rational de Rham theory which we now describe.

To an $n$-simplex $\sigma$ we associate the space $A^*(\sigma)$ of differential forms on $\sigma$ (of all degrees) whose coefficients in some (and hence any) affine coordinates are polynomials with $\mathbb{Q}$-coefficients. To a simplicial complex $K$ (i.e., a set $K$ of simplices with linear identifications of faces) we then associate the $\mathbb{Q}$-vector space of rational forms on $K$

$$A^*(K) := \left\{ (\omega_\sigma) \in \prod_{\sigma \in K} A^*(\sigma) \left| \omega_\sigma|_\tau = \omega_\tau \text{ whenever } \tau \text{ is a face of } \sigma \right. \right\}.$$ 

The wedge product and exterior derivative $d$ on each $A^*(\sigma)$ are compatible with restrictions are thus make $A^*(K)$ a commutative DGA over $\mathbb{Q}$. Simplicial maps $f : K \to L$ (i.e., maps mapping simplices linearly onto simplices) induce pullbacks $f^* : A^*(L) \to A^*(K)$, so $A^*$ is a contravariant functor on the category of simplicial complexes. Moreover, integration over simplices defines a natural transformation to simplicial cochains

$$\rho : A^*(K) \to C^*(K; \mathbb{Q}), \quad \langle \rho \omega, \sigma \rangle := \int_\sigma \omega_\sigma.$$ 

**Theorem 4.1** (Rational de Rham theorem). *Integration over simplices induces an algebra isomorphism $\rho^*$ from the cohomology of $A^*(K)$ to simplicial cohomology $H^*(K; \mathbb{Q})$.*

**Lemmas about rational forms.** The proof of Theorem 4.1 will be based on 3 lemmas about rational forms.
**Lemma 4.2** (Rational Poincaré lemma). Let $CK$ be the cone of a finite simplicial complex $K$. Then every closed $\alpha \in A^k(CK)$, $k > 0$, equals $\alpha = d\beta$ for some $\beta \in A^{k-1}(CK)$.

**Proof.** Recall the proof of the smooth Poincaré lemma. It associates to each homotopy $h_t : M \to M$ with $h_0 \equiv p$ and $h_1 = 1$ an operator

$$H : \Omega^k(M) \to \Omega^{k-1}(M), \quad H\alpha := \int_0^1 h_t^*(i_{X_t}\alpha)dt,$$

where $X_t$ is the vector field on $M$ defined by $\dot{h}_t(x) = X_t(h_t(x))$. Then Cartan’s formula yields

$$dH\alpha = \int_0^1 h_t^*(di_{X_t}\alpha)dt$$

$$= \int_0^1 h_t^*(L_X\alpha)dt - \int_0^1 h_t^*(i_{X_t}d\alpha)dt$$

$$= \int_0^1 \frac{d}{dt}(h_t^*\alpha)dt - Hd\alpha$$

$$= h_t^*\alpha - h_0^*\alpha - Hd\alpha.$$

Since $h_1^*\alpha = \alpha$, and $h_0^*\alpha = 0$ for $k > 0$, this yields the homotopy formula

$$dH + Hd = 1 : \Omega^k(M) \to \Omega^k(M), \quad k > 0.$$

In particular, if a $k$-form $\alpha$ with $k > 0$ is closed, then $\beta := H\alpha$ satisfies $d\beta = \alpha$. Note that the integrand in the formula for $H\alpha$ is the second term in the pullback under the map $h : I \times M \to M$, $(t, x) \mapsto h_t(x)$,

$$h^*\alpha = h_t^*\alpha + dt \wedge h_t^*(i_{X_t}\alpha) \in \Omega^{k+1}(I \times M).$$

Now we write the cone over a simplicial complex $K$ as

$$CK = \{sk + (1 - s)p \mid k \in K, \ s \in [0, 1]\},$$
where we have linearly embedded $K$ into some $\mathbb{R}^N$ with rational
vertices and $p \in \mathbb{Q}^{N+1} \setminus \mathbb{Q}^N$. We pick the linear contraction
\[
h_t : I \times CK \to CK, \quad h_t(sk + (1 - s)p) := tsk + (1 - ts)p.
\]
Then for $\alpha \in A^k(CK)$ closed the pullback $h^*\alpha$ is a differential form
in coordinates $(k, s, t)$ whose coefficients are rational polynomials,
hence the associated homotopy operator yields a rational form $\beta := H\alpha \in A^{k-1}(CK)$ with $d\beta = \alpha$.

For the next two lemmas, we consider the standard $n$-simplex
\[
\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1\}.
\]
Rational $k$-forms on $\Delta^n$ can be uniquely written as
\[
\alpha = \sum_{I = (i_1, \ldots, i_k)} a_I(t)dt_{i_1} \wedge \cdots \wedge dt_{i_k}
\]
with rational polynomials $a_I$, subject to the relations $t_0 + \cdots + t_n = 1$ and $dt_0 + \cdots + dt_n = 0$.

**Lemma 4.3** (Extension lemma). Each $\alpha \in A^k(\partial \Delta^n)$ can be
(non-uniquely) extended to a form $\tilde{\alpha} \in A^k(\Delta^n)$.

**Proof.** Consider the vertex $v := (1, 0, \ldots, 0)$ and the face $\sigma := \{(t_0, \ldots, t_n) \in \Delta^n \mid t_0 = 0\}$. Then we have the stereographic projection
\[
\pi : \Delta^n - v \to \sigma, \quad (t_0, \ldots, t_n) \mapsto \left(\frac{t_1}{1 - t_0}, \ldots, \frac{t_n}{1 - t_0}\right).
\]
For $\beta \in A^k(\sigma)$ the pullback $\pi^*\beta$ on $\Delta^n - v$ is a rational form in
the variables $t_1, \ldots, t_n, (1 - t_0)^{-1}, dt_1, \ldots, dt_n$ and $d(1 - t_0)^{-1} = (1 - t_0)^{-2}dt_0$. Thus for sufficiently large $N \in \mathbb{N}$, the form $\tilde{\beta} := (1 - t_0)^N\pi^*\beta$ defines a rational form on $\Delta^n$ extending $\beta$. Note that
4.1. RATIONAL DE RHAM THEORY

if \( \beta \mid_\tau = 0 \) for some face \( \tau \) of \( \sigma \), then \( \tilde{\beta} \) restricts to zero on the face spanned by \( \tau \) and \( v \).

Using this construction inductively, we now extend \( \alpha \in A^k(\partial \Delta^n) \) to \( \Delta^n \). For \( i = 0, \ldots, n \) set \( \sigma_i := \{(t_0, \ldots, t_n \in \Delta^n \mid t_i = 0\} \). Extend the restriction \( \alpha_0 \) of \( \alpha \) to \( \sigma_0 \) to \( \tilde{\alpha}_0 \in A^k(\Delta^n) \). Then \( \alpha_1 := \alpha - \tilde{\alpha}_0 \mid_{\partial \Delta} \) vanishes on \( \sigma_0 \). Extend \( \alpha_1 \mid_{\sigma_1} \) to \( \tilde{\alpha}_1 \in A^k(\Delta^n) \). Since \( \alpha_1 \) vanishes on \( \sigma_0 \cap \sigma_1 \), its extension \( \tilde{\alpha}_1 \) vanishes on \( \sigma_0 \). Thus \( \alpha_2 := \alpha_1 - \tilde{\alpha}_1 \mid_{\partial \Delta} = \alpha - (\tilde{\alpha}_0 + \tilde{\alpha}_1) \mid_{\partial \Delta} \) vanishes on \( \sigma_0 \cap \sigma_1 \). Continuing inductively, we find in the end a form \( \tilde{\alpha}_0 + \cdots + \tilde{\alpha}_n \in A^k(\Delta^n) \) which agrees with \( \alpha \) on \( \partial \Delta^n \).

Lemma 4.4 (rational homology of the sphere).

\((a_n)\) Let \( \alpha \in A^k(\Delta^n) \) be a closed form which vanishes on \( \partial \Delta^n \). If \( k = n \) assume in addition that \( \int_{\Delta^n} \alpha = 0 \). Then \( \alpha = d\beta \) for some \( \beta \in A^{k-1}(\Delta^n) \) which vanishes on \( \partial \Delta^n \).

\((b_n)\) Let \( \alpha \in A^k(\partial \Delta^n) \) be closed, \( k > 0 \). If \( k = n - 1 \) assume in addition that \( \int_{\partial \Delta^n} \alpha = 0 \). Then \( \alpha = d\beta \) for some \( \beta \in A^{k-1}(\partial \Delta^n) \).

Proof. \((a_0)\) is obvious and \((b_0), (b_1)\) are vacuous.

\((a_1)\) is obvious for \( k = 0 \), and for \( k = 1 \) it follows by integration: If \( p(t)dt \) is a (closed) 1-form on \([0, 1]\), then integration yields a unique polynomial \( q(t) \) with \( q' = p \) and \( q(0) = 0 \), and thus \( q(1) = \int_0^1 p(t)dt = 0 \).

\((a_{n-1}) \Rightarrow (b_n)\): Let \( \alpha \in A^k(\partial \Delta^n) \) be closed, \( k > 0 \). Pick a codimension 1 face \( \sigma \) of \( \Delta^n \). Since \( \partial \Delta^n \setminus \text{int} \sigma \) is a cone over \( \partial \sigma \), by the rational Poincaré lemma, \( \alpha \mid_{\partial \Delta^n \setminus \text{int} \sigma} = d\beta \) for some \( \beta \in A^{k-1}(\partial \Delta^n \setminus \text{int} \sigma) \). By the extension lemma we can extend \( \beta \) to \( \tilde{\beta} \in A^{k-1}(\partial \Delta^n) \). Then \( \alpha - d\tilde{\beta} \) is a closed form on \( \partial \Delta^n \) which
vanishes outside \( \sigma \) and on \( \partial \sigma \). Moreover, if \( k = n - 1 \), then

\[
\int_\sigma (\alpha - d\tilde{\beta}) = \int_{\partial \Delta^n} (\alpha - d\tilde{\beta}) = \int_{\partial \Delta^n} \alpha = 0
\]

by hypothesis. Thus by \((a_{n-1})\) we have \((\alpha - d\tilde{\beta})|_\sigma = d\gamma\) for some \( \gamma \in A^{k-1}(\sigma) \) which vanishes on \( \partial \sigma \). Extend \( \gamma \) by 0 to \( \tilde{\gamma} \in A^{k-1}(\partial \Delta) \). Then \( \alpha = d(\tilde{\beta} + \tilde{\gamma}) \).

\((b_n) \Rightarrow (a_n)\): Let \( \alpha \in A^k(\Delta^n) \) be a closed form which vanishes on \( \partial \Delta^n \). By the rational Poincaré lemma, \( \alpha = d\beta \) for some \( \beta \in A^{k-1}(\Delta^n) \) which may not vanish on \( \partial \Delta^n \). Since \( \alpha|_{\partial \Delta^n} = 0 \), the form \( \beta|_{\partial \Delta^n} = 0 \) is closed. If \( k = 1 \) this means that the function \( \beta \) is constant on \( \partial \Delta^n \), so by subtracting this constant we can achieve \( \beta|_{\partial \Delta^n} = 0 \). So suppose that \( k > 1 \). If \( k = n \), then \( \int_{\partial \Delta^n} \beta = \int_{\Delta^n} \alpha = 0 \) by hypothesis. Thus by \((b_n)\) we have \( \beta|_{\partial \Delta^n} = d\gamma \) for some \( \gamma \in A^{k-2}(\partial \Delta^n) \). By the extension lemma, we can extend \( \gamma \) to a form \( \tilde{\gamma} \in A^{k-2}(\Delta^n) \). Then \( d(\beta - d\tilde{\gamma}) = d\beta = \alpha \) and \( (\beta - d\tilde{\gamma})|_{\partial \Delta^n} = \beta|_{\partial \Delta^n} - d\gamma = 0 \).

**Problem 4.1.** Prove Lemmas 4.2, 4.3 and 4.4 for smooth differential forms instead of rational forms.

**The additive isomorphism.** Using the previous lemmas, we can now prove

**Proposition 4.5.**

(i) If \( \alpha \in A^n(K) \) is closed and \( \rho(\alpha) = 0 \) (i.e., \( \int_\sigma \alpha = 0 \) for all \( n \)-simplices \( \sigma \in K \)), then \( \alpha = d\beta \) for some \( \beta \in A^{n-1}(K) \) with \( \rho(\beta) = 0 \).

(ii) The map \( \rho : A^*(K) \to C^*(K; \mathbb{Q}) \) is surjective.

This proposition implies that \( \rho \) induces on homology an isomorphism of vector spaces: By (ii) we have a short exact sequence of
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cochain complexes

\[ 0 \longrightarrow \ker \rho \longrightarrow A^*(K) \overset{\rho}{\longrightarrow} C^*(K, \mathbb{Q}) \longrightarrow 0. \]

By (i) the cohomology of \( \ker \rho \) vanishes, so the long exact sequence in cohomology shows that \( \rho^* \) is an isomorphism.

**Proof of Proposition 4.5.** (i) For each \( n \)-simplex \( \sigma \in K \) the \( n \)-form \( \alpha \) vanishes on \( \partial \sigma \) by degree reasons. Hence, by Lemma 4.4 (a), \( \alpha|_{\sigma} = d\beta_{\sigma} \) for some \( \beta_{\sigma} \in A^{n-1}(\sigma) \) which vanishes on \( \partial \sigma \). The collection \( (\beta_{\sigma}) \) defines an element in \( A^{n-1}(K^{(n)}) \) which vanishes on the \((n-1)\)-skeleton \( K^{(n-1)} \). Using the extension lemma repeatedly, we extend this form to a form \( \beta_n \in A^{n-1}(K) \). Note that \( \alpha - d\beta_n \) vanishes on \( K^{(n)} \). So for each \((n+1)\)-simplex \( \sigma \in K \) the \( n \)-form \( (\alpha - d\beta_n)|_{\sigma} \) is closed and vanishes on \( \partial \sigma \). Again by Lemma 4.4 (a), \( (\alpha - d\beta)|_{\sigma} = d\gamma_{\sigma} \) for some \( \gamma_{\sigma} \in A^{n-1}(\sigma) \) which vanishes on \( \partial \sigma \). The collection \( (\gamma_{\sigma}) \) defines an element in \( A^{n-1}(K^{(n+1)}) \) which vanishes on \( K^{(n)} \) and can be extended to a form \( \beta_{n+1} \in A^{n-1}(K) \). Now \( \alpha - d\beta_n - d\beta_{n+1} \) vanishes on \( K^{(n+1)} \). Continuing inductively, we construct forms \( \beta_i \in A^{n-1}(K), i = n, n+1, \ldots \), such that \( \beta_i \) vanishes on \( K^{(i-1)} \) and \( \alpha - d(\beta_n + \cdots + \beta_i) \) vanishes on \( K^{(i)} \). Hence the sum \( \sum_{i=n}^{\infty} \beta_i \) stabilizes on each \( K^{(j)} \) and defines an element \( \beta \in A^{n-1}(K) \) which vanishes on \( K^{(n-1)} \) and satisfies \( d\beta = \alpha \).

(ii) The rational \( n \)-form \( (\text{vol} \Delta^n)^{-1} dt_1 \wedge \cdots \wedge dt_n \) on the standard simplex \( \Delta^n \) has integral 1. Thus on every \( n \)-simplex \( \sigma \in K \) there exists a rational \( n \)-form with integral 1. This form can be extended by 0 to the rest of \( K^{(n)} \) and then by the extension lemma to all of \( K \). The resulting form \( \alpha_{\sigma} \in A^n(K) \) satisfies \( \int_{\sigma} \alpha_{\sigma} = 1 \) and \( \int_{\tau} \alpha_{\sigma} = 0 \) for all \( n \)-simplices \( \tau \neq \sigma \). Since such a form exists for each \( n \)-simplex \( \sigma \in K \), this proves surjectivity of the map \( \rho \).

**Naturality under subdivision.** Consider more generally a linear cell complex \( K \), i.e., a collection of convex polyhedra with
linear identifications of some faces. Let $C^*(K, \mathbb{Q})$ be its cellular cohomology, defined analogously to simplicial cohomology. For a subdivision $K'$ of $K$ which is a simplicial complex (e.g., the barycentric subdivision), the proof of Theorem 4.1 shows that the chain map

$$\rho : A^*(K') \rightarrow C^*(K, \mathbb{Q})$$

given by integration over cells induces an isomorphism of vector spaces on cohomology.

Next consider a simplicial complex $K$ and a subdivision $K'$ in which every new vertex has rational coordinates in $K$. Then restriction of forms induces a DGA map $A^*(K) \rightarrow A^*(K')$ which fits into the commutative diagram

$$
\begin{array}{ccc}
A^*(K) & \rightarrow & A^*(K') \\
\downarrow \rho & & \downarrow \rho' \\
C^*(K) & \leftarrow & C^*(K')
\end{array}
$$

where the lower horizontal map is dual to the subdivision map $C_*(K) \rightarrow C_*(K')$. Since this map induces an isomorphism on cohomology, and so do the vertical maps by Theorem 4.1, this shows that the map $A^*(K) \rightarrow A^*(K')$ induces an algebra isomorphism on cohomology.

For a simplicial complex $K$, denote by $|K|$ its geometric realization, i.e., the associated topological space. The canonical map $C^*(|K|) \rightarrow C^*(K)$ from singular to simplicial cochains induces an algebra isomorphism on cohomology. It follows from Theorem 4.1 that the map $\rho : A^*(K) \rightarrow C^*(|K|)$ induced by integration over singular simplices induces an isomorphism of vector spaces on cohomology.

**Remark** 4.6. Any continuous map $f : |K| \rightarrow |L|$ between the geometric realizations of two simplicial complexes is homotopic to
a simplicial map \( \phi : K' \to L \) for a sufficiently fine subdivision \( K' \)
of \( K \), which we can choose to be rational. We will see later that
the associated map \( \phi : A^*(L) \to A^*(K') \) is independent of choices
up to DGA homotopies.

**Multiplicativity of the isomorphism \( \rho^* \).** The map \( \rho : A^*(K) \to C^*(|K|) \) is not an algebra map. To show that it becomes
an algebra map on cohomology, recall that the cup product on \( H^*(|K|) \) is given by the composition

\[
H^*(|K|) \otimes H^*(|K|) \overset{\times}{\longrightarrow} H^*(|K| \times |K|) \overset{\Delta^*}{\longrightarrow} H^*(|K|),
\]

where \( \times \) is the cross product and \( \Delta : |K| \hookrightarrow |K| \times |K| \) is the
inclusion of the diagonal. Now \( |K| \times |K| \) is the geometric realization of the linear cell complex \( K \times K \) defined by the product
cells, and to \( \alpha, \beta \in C^*(K) \) we can canonically associate the chain
\( \alpha \times \beta \in C^*(K \times K) \) whose value on a product of simplices is

\[
\langle \alpha \times \beta, \sigma \times \tau \rangle = \langle \alpha, \sigma \rangle \langle \beta, \tau \rangle.
\]

Under the isomorphism between cellular and singular cohomology
this induces the cross product on \( H^*(|K|) \).

Now consider two rational cocycles \( \alpha, \beta \in A^*(K) \). Let \( (K \times K)' \)
be a rational subdivision of \( K \times K \) which is a simplicial complex.
Then each simplex \( \nu \) of \( (K \times K)' \) lies in a product cell \( \sigma \times \tau \) such
that its vertices are rational. Thus the form \( \alpha_\sigma \otimes \beta_\tau \in A^*(\sigma \times \tau) \cong
A^*(\sigma) \otimes A^*(\tau) \) restricts to a rational form on \( \nu \), and these forms
fit together to a form \( \alpha \otimes \beta \in A^*((K \times K)') \).

The map \( \rho : A^*((K \times K)') \to C^*(K \times K) \) sends \( \alpha \otimes \beta \) to the
cochain whose value on \( \sigma \times \tau \) equals \( (\int_\sigma \alpha)(\int_\tau \beta) \), so we have

\[
\rho(\alpha \otimes \beta) = \rho(\alpha) \times \rho(\beta) \in C^*(K \times K).
\]

Restricting both sides to the diagonal, we see that the cocycle
\( \rho(\alpha \wedge \beta) \) represents the cup product \([\rho(\alpha) \cup [\rho(\beta)]\). This proves
that $\rho^*$ is an algebra isomorphism and hence concludes the proof of Theorem 4.1.

**Relation to the smooth de Rham complex.** To a smooth manifold $M$ we can associate two other commutative DGAs. The first one is the usual de Rham complex $\Omega^*(M)$. For the second one, pick a smooth triangulation of $M$, i.e., a homeomorphism from a simplicial complex $M_\Delta \subset \mathbb{R}^N$ to $M$ which is a smooth embedding on each simplex. We define the complex of piecewise smooth differential forms on $M_\Delta$ as

$$A^*_C(M_\Delta) := \left\{ (\omega_\sigma) \in \prod_{\sigma \in M_\Delta} A^*_C(\sigma) \mid \omega_\sigma|_\tau = \omega_\tau \text{ whenever } \tau \text{ is a face of } \sigma \right\},$$

where $A^*_C(\sigma) = \Omega^*(\sigma)$ is the space of smooth differential forms on the simplex $\sigma$, and the wedge product and exterior derivative are defined on each simplex. Thus we obtain three commutative DGAs over $\mathbb{R}$ which fit into the following commutative diagram:

$$\Omega^*(M) \xrightarrow{i} A^*_C(M_\Delta) \xleftarrow{j} A^*(M_\Delta) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\rho} C^*(M_\Delta; \mathbb{R})$$

(4.1)

Here the horizontal maps are the obvious inclusions (where $i$ is given by restriction to simplices), and the downward maps to simplicial cohomology are given by integration over simplices.

**Problem 4.2.** Prove that the canonical inclusion $i : \Omega^*(M) \hookrightarrow A^*_C(M_\Delta)$ induces an isomorphism on cohomology.

**Hint:** Consider two adjacent simplices $\sigma_1, \sigma_2$ with forms $\alpha_1, \alpha_2$. Use the relative Poincaré lemma to modify $\alpha_2$ to $\tilde{\alpha}_2$ such that $\alpha_1$ and $\tilde{\alpha}_2$ fit together to a smooth form on $\sigma_1 \cup \sigma_2$.

By Theorem 4.1 the map $\rho : A^*(M_\Delta) \otimes_{\mathbb{Q}} \mathbb{R} \to C^*(M_\Delta; \mathbb{R})$ induces an isomorphism on cohomology. An analogous proof, based on
Problem 4.1 shows that the map \( \rho : A_{C^\infty}(M_\Delta) \to C^*(M_\Delta; \mathbb{R}) \) induces an isomorphism on cohomology. Combining this with Problem 4.2, we have thus proved

**Theorem 4.7.** All the maps in the commutative diagram (4.1) induce algebra isomorphisms on cohomology.

In particular, we recover de Rham’s theorem that integration over simplices \( \rho : \Omega^*(M) \to C^*(M_\Delta; \mathbb{R}) \) induces an algebra isomorphism on cohomology \( H^*_\text{dR}(M) \xrightarrow{\sim} H_\text{sing}(M; \mathbb{R}) \).

### 4.2 Minimal models for differential graded algebras

In this section we construct a minimal model for a differential graded algebra (DGA). Recall that \( R \) is now always a field of characteristic zero (mostly \( \mathbb{Q} \) or \( \mathbb{R} \)) and all our DGAs are assumed **commutative**:

**Definition.** A **differential graded algebra (DGA)** is a graded \( R \)-vector space \( A = \bigoplus_{k \geq 0} A^k \) with bilinear product (sometimes written as \( \wedge \) and sometimes omitted) and a linear map \( d : A^k \to A^{k+1} \) of degree +1 such that

(i) the product is associative and graded commutative, i.e. \( ab = (-1)^{|a||b|} ba \), and has a unit \( 1 \in A^0 \);

(ii) \( d \) is a derivation, i.e. \( d(ab) = (da)b + (-1)^{|a|} a \, db \);

(iii) \( d \) is a differential, i.e. \( d^2 = 0 \).

We denote the homology of \( A \) with respect to \( d \) by \( H^*(A) \). Throughout this chapter, we assume that our DGAs are **connected**, i.e. \( H^0(A) \cong R \), and **simply connected**, i.e. \( H^1(A) = 0 \).
**Definition.** A DGA $M$ is called *minimal* if

(i) $M$ is free as a graded commutative algebra;

(ii) $M^1 = 0$;

(iii) $d(M) \subset M^+ \wedge M^+$, where $M^+ := \bigoplus_{k > 0} M^k$.

Conditions (i) and (ii) mean that, as an algebra, $M$ is an exterior algebra on generators of (even or odd) degrees $\geq 2$. Condition (iii) means that each $da$ is *decomposable*, i.e., a sum of products of elements of positive degrees.

**Definition.** A *minimal model* for a DGA $A$ is a minimal DGA $M$ together with a DGA map $\rho : M \to A$ which induces an isomorphism on cohomology.

In this section we will prove that every (connected and simply connected) DGA has a minimal model, and in the next section we will prove that the minimal model is unique up to DGA isomorphism.

**Hirsch extensions.** We shall see in Section 4.6 that, in a precise sense, minimal DGAs are the algebraic analogues of Postnikov towers. Hirsch extensions correspond under this analogy to principal fibrations.

**Definition.** A *Hirsch extension* of degree $k$ of a DGA $A$ is a DGA

$$A \otimes_d \Lambda(V)_k,$$

where

(i) $\Lambda(V)_k$ is the exterior algebra over a graded vector space homogeneous in degree $k$;

(ii) $d|_A$ is the differential of $A$ and $d : V \to A^{k+1}$. 

By condition (ii) we have a DGA embedding

\[ A \hookrightarrow A \otimes_d \Lambda(V)_k, \quad A \mapsto a \otimes 1. \]

We will write \( a \) for \( a \otimes 1 \) and \( v \) for \( 1 \otimes v \). Note that \( dv \in A^{k+1} \) is closed for each \( v \in V \), so \( d \) induces a map \( d : V \to H^{k+1}(A) \).

Two Hirsch extensions of the same degree \( k \) are called *equivalent* if there exists a DGA isomorphism \( \phi \) such that the following diagram commutes:

\[
\begin{array}{ccc}
A \otimes_d \Lambda(V)_k & \xrightarrow{\phi} & A \otimes_{d'} \Lambda(V')_k.
\end{array}
\]

**Lemma 4.8.** Two Hirsch extensions \( A \otimes_d \Lambda(V)_k \) and \( A \otimes_{d'} \Lambda(V')_k \) of a DGA \( A \) with \( A^0 = R \) are equivalent if and only if there exists a vector space isomorphism \( \psi : V \to V' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\psi} & H^{k+1}(A).
\end{array}
\]

**Proof.** Suppose first that \( \phi : A \otimes_d \Lambda(V)_k \to A \otimes_{d'} \Lambda(V')_k \) is a DGA isomorphism extending the identity on \( A \). Since \( A^0 = R \), by degree reasons we must have \( \phi(v) = a(v) + \psi(v) \) for linear maps \( a : V \to A^k \) and \( \psi : V \to V' \). Thus \( \phi \) is an isomorphism if and only if \( \psi \) is an isomorphism. Since \( dv = \phi(dv) = d'(\phi v) = d'a(v) + d'\psi(v) \), we have \([dv] = [d'\psi(v)] \in H^{k+1}(A)\).
Conversely, suppose that $\psi : V \to V'$ is an isomorphism such that $[dv] = [d'\psi(v)] \in H^{k+1}(A)$. Then $dv - d'\psi(v) = da(v)$ for a map $a : V \to A^k$ which we can choose to be linear by defining it on a basis. Then $\phi(v) := a(v) + \psi(v)$ extends via the identity on $A$ to a DGA isomorphism $\phi : A \otimes_d \Lambda(V)_k \to A \otimes_{d'} \Lambda(V')_k$.

The following lemma is the algebraic analogue of the decomposition of a Postnikov tower into principal fibrations.

**Lemma 4.9.** A DGA $M$ is minimal if and only if $M = \bigcup_{n \geq 0} M(n)$, where

$$R = M(0) = M(1) \subset M(2) \subset \ldots$$

and each $M(n) \subset M(n+1)$ is a Hirsch extension of degree $n+1$.

**Proof.** If $M$ is minimal, denote by $M(n) \subset M$ the subalgebra generated by elements of degrees $\neq n$. Then $M = \bigcup_{n \geq 0} M(n)$, each $M(n)$ is free as an algebra (since $M$ is free), and $M(n+1) \cong M(n) \otimes \Lambda(V)_{n+1}$ as an algebra, where $V = M^{n+1}/M^{n+1} \cap (M^+ \wedge M^+)$ is the vector space of indecomposable elements of degree $n+1$. Since each $dv$ is decomposable and $M$ has no elements of degree 1, it follows that

$$d\left(M(n+1)\right) \subset M(n)$$

for all $n$, so each $M(n) \subset M(n+1)$ is a Hirsch extension. Conversely, if $M = \bigcup_{n \geq 0} M(n)$, where $R = M(0) = M(1)$ and each $M(n) \subset M(n+1)$ is a Hirsch extension of degree $n+1$, then $M$ is free as an algebra, $M^0 = R$, and $M^1 = 0$. Moreover, $d\left(M(n+1)\right) \subset M(n)$ implies that each $dv$ is decomposable, so $M$ is minimal.

**Existence of a minimal model.**
4.2. MINIMAL MODELS FOR DIFFERENTIAL GRADED ALGEBRAS

**Proposition 4.10.** Every (connected and simply connected) DGA has a minimal model.

**Proof.** Given a (connected and simply connected) DGA \((A, d)\), we will construct a sequence of Hirsch extensions

\[ R = M(0) = M(1) \subset M(2) \subset \cdots \]

and DGA maps \(\rho_n : M(n) \to A\) which are \(n\)-minimal models in the following sense:

1. \(M(n)\) is minimal and generated by elements of degree \(\leq n\);
2. \(\rho_n^* : H^k(M(n)) \to H^k(A)\) is an isomorphism for all \(k \leq n\);
3. \(\rho_n^* : H^{n+1}(M(n)) \to H^{n+1}(A)\) is injective.

The minimal model is then \(M := \bigcup_{n \geq 0} M(n)\) with the induced maps \(d\) and \(\rho\). Note the similarity to the iterative construction of a Postnikov tower.

To begin the construction, we set \(M(0) = M(1) := R\) with trivial differential and \(\rho_0 = \rho_1 : R \to A\) mapping 1 to 1.

Now suppose inductively that for \(n \geq 1\) we have already constructed an \(n\)-minimal model \(\rho_n : M(n) \to A\). We pick

- cycles \(a_i \in A^{n+1}\) whose cohomology classes \([a_i]\) form the basis for a complement of \(\text{im } \rho_n^* \subset H^{n+1}(A)\);
- cycles \(z_j \in M(n)^{n+2}\) whose cohomology classes \([z_j]\) form a basis for the kernel of \(\rho_n^* : H^{n+2}(M(n)) \to H^{n+2}(A)\);
- elements \(b_j \in A^{n+1}\) with \(db_j = \rho_n(z_j)\) (which exists by the choice of \(z_j\)).

We define the Hirsch extension

\[ M(n+1) := M(n) \otimes_d \Lambda[x_i, y_j], \quad |x_i| = |y_j| = n+1, \quad dx_i = 0, \quad dy_j = z_j \]
and extend $\rho_n : M(n) \to A$ to an algebra map

$$\rho_{n+1} : M(n + 1) \to A, \quad \rho_{n+1}(x_i) := a_i, \ \rho_{n+1}(y_j) := b_j.$$ 

Since $dx_i = da_i = 0$ and $d\rho_{n+1}(y_j) = db_j = \rho_n(z_j) = \rho_{n+1}(dy_j)$, $\rho_{n+1}$ is a DGA map.

It remains to check properties (ii) and (iii) for $\rho_{n+1}$. Since $\rho_{n+1} = \rho_n$ on the sub-DGA $M(n) \subset M(n + 1)$, the map $\rho^*_n : H^k(M(n + 1)) \to H^k(A)$ is an isomorphism for all $k \leq n$. It is an isomorphism for $k = n + 1$ in view of the diagram

\[
\begin{array}{ccc}
H^{n+1}(M(n + 1)) & = & H^{n+1}(M(n)) \oplus \langle x_i \rangle \\
\downarrow \rho^*_n & & \downarrow \rho^*_n \\
H^{n+1}(A) & = & \text{im } \rho^*_n \oplus \langle [a_i] \rangle,
\end{array}
\]

where $\langle x_i \rangle$ denotes the linear span of the elements $x_i$. Finally, since

\[
d : M(n+1)^{n+1} = M(n)^{n+1} \oplus \langle x_i, y_j \rangle \to M(n+1)^{n+2} = M(n)^{n+2}
\]

maps $x_i \to 0$ and $y_j \to z_j$ (where the last equality holds because $M(n)$ has no elements of degree 1), we have

\[
H^{n+2}(M(n+1)) = H^{n+2}(M(n)) \big/ \langle [z_j] \rangle = H^{n+2}(M(n)) \big/ \ker \rho^*_n,
\]

and the map $\rho^*_{n+1} : H^{n+2}(M(n + 1)) \to H^{n+2}(A)$ induced by $\rho^*_n$ is injective. This proves the inductive step and hence Proposition 4.10. □

**Quasi-isomorphisms of minimal DGAs.** A DGA morphism that induces an isomorphism on cohomology is called a *quasi-isomorphism*. The following lemma will be an important ingredient in the proof of uniqueness of minimal models.

**Lemma 4.11.** Every quasi-isomorphism between two minimal DGAs is an isomorphism.
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Proof. Let \( \rho : M \to M' \) be a quasi-isomorphism between minimal DGAs. As before, let \( M(n) \subset M \) be the sub-DGA generated by elements of degree \( \leq n \). Then \( \rho \) restricts to quasi-isomorphisms \( \rho_n : M(n) \to M'(n) \) for all \( n \geq 0 \). We will prove by induction that \( \text{ech} \ \rho_n \) is an isomorphism. For \( n = 0 \) this is clear because \( M(0) = M(1) = R \) and \( \rho(1) = 1 \). Since \( M(n) = M(n-1) \otimes \Lambda(V)_n \) is a Hirsch extension of \( M(n-1) \), the inductive step follows from the more general Lemma 4.12 below.

Lemma 4.12. Let

\[ \phi : A = \bar{A} \otimes \Lambda(V)_n \to B = \bar{B} \otimes \Lambda(W)_n \]

be a quasi-isomorphism between degree \( n \) Hirsch extensions of two DGAs \( \bar{A}, \bar{B} \) that restricts to an isomorphism \( \bar{\phi} : \bar{A} \to \bar{B} \). Then \( \phi \) is an isomorphism.

Proof. We will show that \( \phi : A^n \to B^n \) is bijective. Since \( \phi = \bar{\phi} : \bar{A}^n \to \bar{B}^n \) is bijective, this implies that \( \phi \) induces an isomorphism \( V \cong A^n/\bar{A}^n \to B^n/\bar{B}^n \cong W \), which in turns implies that \( \phi \) is an algebra isomorphism (surjectivity is clear, and for injectivity consider the quotient \( B/\langle \bar{B}^+ \rangle \cong \Lambda(W) \)).

For injectivity, consider \( a \in A^n \) with \( \phi a = 0 \). Then \( da \in \bar{A}^{n+1} \) satisfies \( \bar{\phi}da = d\phi a = 0 \), so injectivity of \( \bar{\phi} \) yields \( da = 0 \). Since \( [\phi a] = 0 \in H^n(B) \), injectivity of \( \phi^* \) implies \( a = d\bar{a} \) for some \( \bar{a} \in A^{n-1} = \bar{A}^{n-1} \). Then \( a = d\bar{a} \in \bar{A}^n \) satisfies \( \bar{\phi}a = \phi a = 0 \), so injectivity of \( \bar{\phi} \) yields \( a = 0 \).

For surjectivity, consider \( w \in W \). Then \( dw \in \bar{B}^{n+1} \), so surjectivity of \( \bar{\phi} \) implies \( dw = \phi a \) for some \( a \in \bar{A}^n \). Now \( da \in \bar{A}^{n+2} \) satisfies \( \bar{\phi}da = d\phi a = d(dw) = 0 \), so injectivity of \( \bar{\phi} \) yields \( da = 0 \). Since \( [\phi a] = [dw] = 0 \in H^{n+1}(B) \), injectivity of \( \phi^* \) implies \( a = da_1 \) for some \( a_1 \in A^n \). Now \( d\phi a_1 = \phi da_1 = \phi a = dw \), thus \( d(w - \phi a_1) = \).
0, and surjectivity of $\phi^*$ implies that $[w - \phi a_1] = [\phi a_2] \in H^n(B)$ for some cycle $a_2 \in A^n$, i.e., $w - \phi a_1 - \phi a_2 = db$ for some $b \in B^{n-1} = \bar{B}^{n-1}$. Surjectivity of $\bar{\phi}$ yields $a_3 \in \tilde{A}^{n-1}$ with $\bar{\phi} a_3 = b$, and hence $w = \phi(a_1 + a_2 + da_3)$.

**Mapping cones.** In the proof of uniqueness we will use the following algebraic construction. The *cone* of a cochain map $\phi : (A, d_A) \to (B, d_B)$ between cochain complexes is the cochain complex

$$A \oplus B[-1], \quad D := \begin{pmatrix} d_A & 0 \\ \phi & -d_B \end{pmatrix},$$

where $B[-1]^k := B^{k-1}$. It has the subcomplex $(B[-1] \cong 0 \oplus B[-1], -d_B)$ with quotient complex $(A \oplus B[-1]/B[-1], D) \cong (A, d_A)$, so with the *relative cohomology*

$$H^k(A, B) := H^k(A \oplus B[-1], D)$$

the associated long exact sequence in cohomology (with the obvious differentials) becomes

$$\cdots \longrightarrow H^k(B[-1]) \longrightarrow H^k(A \oplus B[-1]) \longrightarrow H^k(A \oplus B[-1]/B[-1]) \longrightarrow \cdots \cong \downarrow \quad = \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow \quad \cong \downarrow$$

$$\cdots \longrightarrow H^{k-1}(B) \longrightarrow H^k(A, B) \longrightarrow H^k(A) \longrightarrow \phi^*$$

This long exact sequence is clearly functorial for commutative diagrams of cochain maps

$$\begin{array}{ccc}
A & \phi \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \phi' \longrightarrow & B'.
\end{array}$$

**Problem 4.3.** Show: If the cochain map $\phi : (A, d_A) \to (B, d_B)$ is surjective, then the inclusion $(\ker \phi \oplus 0, d_A) \hookrightarrow (A \oplus B[-1], D)$
induces on cohomology an isomorphism $H^*(\ker \phi) \cong H^*(A, B)$ and the long exact sequence (4.2) becomes the long exact sequence of the pair $\ker \phi \subset A$.

The algebraic cone construction has the following topological counterpart. Recall that the cone of a space $X$ is the space $CX := ([0, 1] \times X)/0 \times X$. More generally, the *mapping cone* of a continuous map $f : X \to Y$ is the space

$$C_f := (CX \amalg Y)/CX \ni (1, x) \sim f(x) \in Y,$$

i.e., the space obtained by attaching $CX$ to $Y$ via the map $f : 1 \times X \to Y$. The cone $C_f$ contains $Y$ as a subspace and the quotient $C_f/Y = CX/1 \times X = SX$ is the suspension of $X$.

**Problem 4.4.** Prove that the canonical map $S : C_k(X) \to C_{k+1}(SX)$ that associates to a singular simplex $\sigma : \Delta^k \to X$ the simplex (after triangulation of the domain) $S\sigma : [0, 1] \times \Delta^n \to SX$, $(s, t) \mapsto [s, \sigma(t)]$ induces isomorphisms on (co)homology

$$S_* : H_k(X) \xrightarrow{\cong} H_{k+1}(SX), \quad S^* : H^{k+1}(SX) \xrightarrow{\cong} H^k(X).$$

**Problem 4.5.** Show that with the *relative cohomology* $H^k(Y, X) := H^k(C_f)$

the long exact sequence in reduced cohomology of the pair $Y \subset C_f$ becomes

$$\cdots \tilde{H}^k(C_f/Y) \longrightarrow \tilde{H}^k(C_f) \longrightarrow \tilde{H}^k(Y) \longrightarrow \tilde{H}^{k+1}(C_f/Y) \cdots \cong \quad = \quad = \quad \cong \downarrow$$

$$\cdots \tilde{H}^{k-1}(X) \longrightarrow \tilde{H}^k(Y, X) \longrightarrow \tilde{H}^k(Y) \xrightarrow{f^*} \tilde{H}^k(X) \cdots (4.3)$$

Moreover, this sequence agrees with the sequence (4.2) for the chain map $\phi = f^* : A = \tilde{C}^*(Y) \to B = \tilde{C}^*(X)$. 
4.3 Homotopy theory of DGAs

Homotopies of DGA maps.

**Definition.** A *homotopy* between DGA maps \( f, g : A \to B \) is a DGA map

\[
H : A \to B[t, dt]
\]

satisfying \( H|_{t=0} = f \) and \( H|_{t=1} = g \), where \( B[t, dt] \) denotes the (non-minimal) DGA

\[
B[t, dt] := \Lambda[t, dt] \otimes B, \quad |t| = 0, |dt| = 1, d(t) = dt, d(dt) = 0,
\]

and the restriction to \( t = i \) is given by setting \( t = i \) and \( dt = 0 \).

Note that if \( B = A^*(K) \) is the space of rational forms on a simplicial complex \( K \), then \( B[t, dt] \) is the space of rational forms on the cell complex \( [0, 1] \times K \). Moreover, if \( h : [0, 1] \times K \to L \) is a simplicial map defining a homotopy between \( h|_{0 \times K} = f \) and \( h|_{1 \times K} = g \), then \( H = h^* : A^*(L) \to A^*([0, 1] \times K) = B[t, dt] \) is a homotopy between \( f^* \) and \( g^* \) in the sense of the definition.

We define an “integration” map

\[
\int_0^t : B[t, dt] \to B[t, dt], \quad \int_0^t t^i \otimes b := 0, \quad \int_0^t t^i dt \otimes b := \frac{t^{i+1}b}{i+1},
\]

and its restriction to \( t = 1 \),

\[
\int_0^1 : B[t, dt] \to B, \quad \int_0^1 t^i \otimes b := 0, \quad \int_0^1 t^i dt \otimes b := \frac{b}{i+1}.
\]

**Problem 4.6.** (a) For each \( \beta \in B[t, dt] \) we have

\[
d\left( \int_0^t \beta \right) + \int_0^t d\beta = \beta - \beta|_{t=0}. \tag{4.4}
\]
(b) For a homotopy $H : A \to B[t, dt]$ from $f$ to $g$ and each $a \in A$ we have the homotopy formula

$$d \left( \int_0^1 H(a) \right) + \int_0^1 dH(a) = g(a) - f(a). \quad (4.5)$$

Note that if $h : [0, 1] \times K \to L$ is a simplicial map defining a homotopy between $h|_{0 \times K} = f$ and $h|_{1 \times K} = g$, then $\int_0^1 h^* : A^*(L) \to A^*(K)$ is the chain homotopy between $f^*$ and $g^*$ that we used in the proof of the rational Poincaré lemma.

**Obstruction theory in topology.**

To do.

**Obstruction theory for DGAs.**

The following is the main technical result in the obstruction theory of DGAs. It is stated in enough generality to cover all the applications below. An important special case which suffices for most applications is $C = D$ and $\nu = 1$.

**Proposition 4.13.** (a) Given a Hirsch extension $A \hookrightarrow A \otimes_d \Lambda(V)_n$, DGA maps $f, g, \phi$ as in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow & & \downarrow \phi \\
A \otimes_d \Lambda(V)_n & \xrightarrow{f} & C \xrightarrow{\nu} D,
\end{array}
$$

and a homotopy $H : A \to C[t, dt]$ from $f|_A$ to $\phi \circ g$, there exists an obstruction class $\sigma \in H^{n+1}(B, C; V^*)$ which vanishes if and only if there is an extension of $g$ to a DGA map $\tilde{g} : A \otimes_d \Lambda(V)_n \to B$ and an extension $\tilde{H} : A \otimes_d \Lambda(V)_n \to C[t, dt]$ of $H$ to a homotopy from $f$ to $\phi \circ \tilde{g}$.

(b) Suppose that in addition we are given maps $\mu, \nu$ as in the diagram such that $\mu = \nu \circ \phi$ is surjective, and $\nu \circ H : A \to D[t, dt]$ is the constant homotopy at $\nu \circ f|_A = \mu \circ g$. Then
for \( \sigma = 0 \) we can choose the extensions \( \tilde{g} \) and \( \tilde{H} \) in (a) such that \( \nu \circ \tilde{H} : A \otimes \Lambda(V)_n \to D[t, dt] \) is the constant homotopy at \( \nu \circ f = \mu \circ \tilde{g} \).

Proof. (a) Recall that \( H^{n+1}(B, C) \) is the cohomology of the cone of \( \phi \),

\[
B \oplus C[-1], \quad D := \begin{pmatrix} d_B & 0 \\ \phi & -d_C \end{pmatrix}.
\]

For \( v \in V \) we define

\[
\tilde{\sigma}(v) := \left( g(dv), f(v) + \int_0^1 H(dv) \right) \in B^{n+1} \oplus C^n.
\]

By the homotopy formula (applied to \( a = dv \)) we obtain

\[
D\tilde{\sigma}(v) = \left( dg(dv), \phi g(dv) - df(v) - d \int_0^1 H(dv) \right) = (0, 0),
\]

so \( \tilde{\sigma}(v) \) is a cocycle. The cohomology classes \( \sigma(v) := [\tilde{\sigma}(v)] \) for \( v \) give rise to a linear map \( \sigma : V \to H^{n+1}(B, C) \), or equivalently, to an element \( \sigma \in H^{n+1}(B, C; V^*) \).

Suppose that \( \sigma = 0 \), i.e.,

\[
\tilde{\sigma}(v) = D\left( b(v), c(v) \right) = \left( db(v), \phi b(v) - dc(v) \right)
\]

for maps \( b : V \to B^n \) and \( c : V \to C^{n-1} \) which we can choose to be linear in \( v \). Comparing the first components yields \( db(v) = g(dv) \), so \( \tilde{g} := b \) gives rise to a DGA map \( A \otimes_d \Lambda(V)_n \to B \) extending \( g \).

The second components yield

\[
\phi b(v) = f(v) + \int_0^1 H(dv) + dc(v).
\]

(4.6)
The map \( \tilde{H} : V \to C[t, dt]^{n-1} \)

\[
\tilde{H}(v) := f(v) + \int_0^t H(dv) + d(tc(v))
\]
satisfies \( \tilde{H}(v)|_{t=0} = f(v) \), \( \tilde{H}(v)|_{t=1} = \phi b(v) = \phi \tilde{g}(v) \) by equation (4.6), and

\[
d\tilde{H}(v) = df(v) - \int_0^t dH(dv) + H(dv) - H(dv)|_{t=0} = H(dv)
\]

by the homotopy formula and \( H|_{t=0} = f \). So \( \tilde{H} \) extends \( H \) to the desired homotopy \( \tilde{H} : A \otimes_d \Lambda(V)_n \to C[t, dt] \) from \( f \) to \( \phi \circ \tilde{g} \).

Conversely, if \( \tilde{g} \) and \( \tilde{H} \) are extensions of \( g \) and \( H \) with the required properties, then

\[
\psi(v) := \left( \tilde{g}(v), \int_0^1 \tilde{H}(v) \right)
\]
is a primitive of the obstruction cycle \( \tilde{\sigma}(v) \) because

\[
D\psi(v) = \left( d\tilde{g}(v), \phi \tilde{g}(v) - d \int_0^1 \tilde{H}(v) \right)
\]

\[
= \left( g(dv), f(v) + \int_0^1 d\tilde{H}(v) \right)
\]

\[
= \tilde{\sigma}(v),
\]

where we have used the chain map property of \( \tilde{g} \) in the first component and the homotopy formula in the second component.

(b) Suppose now that in addition we are given maps \( \mu, \nu \) as in the proposition. If \( \sigma = 0 \), then as above there exist linear maps \( b : V \to B^n \) and \( c : V \to C^{n-1} \) satisfying \( db(v) = g(dv) \) and equation (4.6). Since \( \nu \circ H \) is a constant homotopy, we have \( \nu \int_0^1 H(dv) = \int_0^1 \nu H(dv) = 0 \), so applying \( \nu \) to the equation (4.6) we obtain

\[
\mu b(v) = \nu \phi b(v) = \nu f(v) + \nu dc(v).
\]
Since \( \mu \) is surjective, we find a linear map \( \tilde{b} : V \to B^{n-1} \) such that \( \mu \circ \tilde{b} = \nu \circ c \) and the previous equation becomes

\[
\nu f(v) = \mu b(v) - d\mu b(v) = \mu (b - d\tilde{b})(v),
\]

so \( \tilde{g}(v) := b(v) - d\tilde{b}(v) \) extends \( g \) to a DGA map \( \tilde{g} : A \otimes_d \Lambda(V)_n \to B \) satisfying \( \mu \circ \tilde{g} = \nu \circ f \). The map \( \tilde{H} : V \to C[t, dt]^{n-1} \) defined by

\[
\tilde{H}(v) := f(v) + \int_0^t H(dv) + d(tc(v) - t\phi b(v))
\]

satisfies \( \tilde{H}(v)|_{t=0} = f(v) \),

\[
\tilde{H}(v)|_{t=1} = f(v) + \int_0^1 H(dv) + dc(v) - d\phi b(v) = \phi b(v) - \phi d\tilde{b}(v) = \phi \tilde{g}(v)
\]

by equation (4.6), and

\[
d\tilde{H}(v) = df(v) - \int_0^t dH(dv) + H(dv) - H(dv)|_{t=0} = H(dv)
\]

by the homotopy formula and \( H|_{t=0} = f \). So \( \tilde{H} \) extends \( H \) to a homotopy \( \tilde{H} : A \otimes_d \Lambda(V)_n \to C[t, dt] \) from \( f \) to \( \phi \circ \tilde{g} \). Moreover, \( \nu \phi b = \mu b = \nu c \) implies

\[
\nu \tilde{H}(v) = \nu f(v) + d(tc(v) - t\phi b(v)) = \nu f(v),
\]

so \( \nu \circ \tilde{H} \) is the constant homotopy at \( \nu \circ f \).

Applications of obstruction theory to minimal DGAs.

**Theorem 4.14.** (a) Given a minimal DGA \( M \) and DGA maps \( f, \phi \) as in the diagram

\[
\begin{array}{c}
M \xrightarrow{f} C \xrightarrow{\nu} D, \\
\end{array}
\]

\[
B \xrightarrow{\mu} C \xrightarrow{\nu} D,
\]

and a weak equivalence \( \phi : C \to D \) such that the triangle

\[
\begin{array}{ccc}
M & \to & C \\
\phi \downarrow & & \downarrow \nu \\
B & \to & D
\end{array}
\]

is commutative.

\[
\phi_0 = \mu \circ \phi_1 = \phi \circ f
\]

and

\[
\phi_2 = \nu \circ \phi_1 = \phi \circ f
\]

are weak equivalences. Furthermore, \( \nu \circ \phi b = \mu b = \nu c \) implies

\[
\nu \phi b = \mu b = \nu c
\]

so \( \nu \circ \phi b = \mu b = \nu c \).
with \( \phi \) inducing an isomorphism on cohomology, there exists a DGA map \( \tilde{f} : M \to B \) such that \( \phi \circ \tilde{f} \) is homotopic to \( f \).

(b) Suppose that in addition we are given maps \( \mu, \nu \) as in the diagram such that \( \mu = \nu \circ \phi \) is surjective. Then we can choose \( \tilde{f} \) in (a) such that \( \mu \circ \tilde{f} = \nu \circ f \).

**Proof.** If \( \phi \) induces an isomorphism on cohomology, then \( H^*(B, C') = 0 \). Hence all obstruction classes in Proposition 4.13 vanish and the theorem follows by induction over Hirsch extensions. \( \square \)

**Corollary 4.15.** For \( M \) minimal and \( A \) arbitrary, homotopy defines an equivalence relation on the set of DGA maps \( M \to A \).

**Proof.** Consider two homotopies

\[
H : M \to A[t, dt], \quad J : M \to A[s, ds], \quad H|_{t=1} = J|_{s=0}.
\]

Their sum defines a map

\[
H \oplus J : M \to A \otimes \tilde{C},
\]

where

\[
\tilde{C} := \left\{ (\alpha, \beta) \in \Lambda[t, dt] \oplus \Lambda[s, ds] \ \middle| \ \alpha|_{t=1} = \beta|_{s=0} \right\}
\]

is the DGA of rational forms on the union of two lines \( \{s(t - 1) = 0\} \subset \mathbb{R}^2 \); see Figure **fig:homotopy**. This DGA is isomorphic to the quotient DGA

\[
C := \Lambda[t, s, dt, ds] / \langle s(t - 1), (t - 1)ds, s dt \rangle
\]

via the DGA isomorphism

\[
\psi : \tilde{C} \xrightarrow{\cong} C, \quad (\alpha, \beta) \mapsto \alpha + \beta - \alpha|_{t=1}.
\]
Consider now the diagram
\[
\begin{array}{ccc}
A \otimes \Lambda[t, s, dt, ds] & \xrightarrow{\psi \circ (H \oplus J)} & A \otimes C. \\
\downarrow \pi & & \\
M & \xrightarrow{\psi \circ (H \oplus J)} & A \otimes C.
\end{array}
\]
By Problem 4.7 below, the (surjective) quotient map \( \pi : A \otimes \Lambda[t, s, dt, ds] \to A \otimes C \) induces an isomorphism on cohomology.

Hence Theorem 4.14 (with \( C = D \) and \( \nu = 1 \)) provides a DGA map \( \rho : M \to A \otimes \Lambda[t, s, dt, ds] \) such that \( \pi \circ \rho = \psi \circ (H \oplus J) \).

This map restricts at \( s = t \) to a DGA map
\[
\tilde{H} := \rho|_{s=t} : M \to A \otimes \Lambda[t, s, dt, ds]/\langle s - t, ds - dt \rangle \cong A[t, dt]
\]
whose restrictions at \( t = 0,1 \) satisfy
\[
\begin{align*}
\tilde{H}|_{t=0} &= \psi \circ (H \oplus J)|_{s=t=0} = H|_{t=0} + J|_{s=0} - H|_{t=1} = H|_{t=0}, \\
\tilde{H}|_{t=1} &= \psi \circ (H \oplus J)|_{s=t=1} = H|_{t=1} + J|_{s=1} - H|_{t=1} = J|_{t=1}.
\end{align*}
\]

\[\square\]

**Problem 4.7.** (a) Prove that the DGAs \( \Lambda[t, s, dt, ds] \) and its quotient \( C \) are both acyclic.

(b) Conclude that for every DGA \( A \), the quotient map \( A \otimes \Lambda[t, s, dt, ds] \to A \otimes C \) induces an isomorphism on cohomology.

For a DGAs \( M, A \) with \( M \) minimal, we denote by \([M, A]\) the set of homotopy classes of DGA maps \( M \to A \).

**Corollary 4.16.** If a DGA map \( \phi : B \to C \) induces an isomorphism on cohomology and \( M \) is a minimal DGA, then composition with \( \phi \) induces a bijection
\[
\phi \circ : [M, B] \xrightarrow{\cong} [M, C].
\]
Proof. For surjectivity, consider a DGA morphism \( f : M \to C \). Then Theorem 4.14 provides a DGA morphism \( \tilde{f} : M \to B \) such that \( \phi \circ \tilde{f} \) is homotopic to \( f \).

For injectivity, consider two DGA morphisms \( f_0, f_1 : M \to B \) and a homotopy \( H : M \to C[t, dt] \) from \( \phi \circ f_0 \) to \( \phi \circ f_1 \). Consider the diagram

\[
\begin{array}{cccccc}
B[t, dt] & \xrightarrow{\phi \oplus ev_0 \oplus ev_1} & \ker \mu & \xrightarrow{\phi^* \oplus 1 \oplus 1} & H^*(B) & \\
0 & \xrightarrow{\phi^* \oplus 1 \oplus 1} & C[t, dt] \oplus B \oplus B & \xrightarrow{\mu^*} & C \oplus C & \to 0,
\end{array}
\]

where the map \( \mu(\gamma, b_0, b_1) := (\gamma|_{t=0} - \phi(b_0), \gamma|_{t=1} - \phi(b_1)) \) is surjective and thus the lower row exact. On cohomology we obtain the diagram with exact lower row

\[
\begin{array}{cccccc}
0 & \xrightarrow{\phi^* \oplus 1 \oplus 1} & H^*(\ker \mu) & \xrightarrow{\mu^*} & H^*(C) \oplus H^*(B) \oplus H^*(B) & \xrightarrow{\mu^*} & H^*(C) \oplus H^*(C) & \to 0,
\end{array}
\]

where exactness of the lower row follows from exactness of the long exact sequence and surjectivity of the map \( \mu^*(c, b_0, b_1) = (c - \phi^*(b_0), c - \phi^*(b_1)) \), which in turn follows from surjectivity of \( \phi^* \). We see that

\[
H^*(\ker \mu) = \ker \mu^* = \{ (\phi^*(b), b, b) \mid b \in H^*(B) \}
\]

and the map \( \phi^* \oplus 1 \oplus 1 : H^*(B) \to H^*(\ker \mu) \) is an isomorphism. Thus in the diagram

\[
\begin{array}{cccccc}
B[t, dt] & \xrightarrow{\phi \oplus ev_0 \oplus ev_1} & \ker \mu & \xrightarrow{\phi^* \oplus 1 \oplus 1} & H^*(B) & \\
M \xrightarrow{H \oplus f_0 \oplus f_1} \ker \mu & \xrightarrow{\pi_B \oplus B} & B \oplus B
\end{array}
\]
the map \( \phi \oplus \text{ev}_0 \oplus \text{ev}_1 \) induces an isomorphism on cohomology. Since the map \( \text{ev}_0 \oplus \text{ev}_1 \) is surjective, Theorem 4.14 provides a DGA map \( \widetilde{H} : M \to B[t, dt] \) such that

\[
(\text{ev}_0 \oplus \text{ev}_1) \circ \widetilde{H} = \pi_{B \oplus B} \circ (H \oplus f_0 \oplus f_1) = f_0 \oplus f_1,
\]
i.e., \( \widetilde{H}\big|_{t=0} = f_0 \) and \( \widetilde{H}\big|_{t=1} = f_1 \).

\[\square\]

**Uniqueness of the minimal model.**

**Theorem 4.17.** Given two minimal models

\[
\begin{array}{ccc}
M' & \downarrow \rho' \\
\rho & \downarrow \\
M & \rightarrow A
\end{array}
\]

for a DGA \( A \), there exists a DGA isomorphism \( I : M \to M' \), unique up to homotopy, such that \( \rho' \circ I \) is homotopic to \( \rho \).

**Proof.** Since \( \rho' \) induces an isomorphism on cohomology, Corollary 4.16 provides a DGA map \( I : M \to M' \), unique up to homotopy, such that \( \rho' \circ I \) is homotopic to \( \rho \). Since \( \rho \) and \( \rho' \) induce isomorphisms on cohomology, so does \( I \), so by Lemma 4.11 it is a DGA isomorphism. \[\square\]

More generally, we have

**Corollary 4.18.** Given two minimal models \( \rho_A : M_A \to A \) and \( \rho_B : M_B \to B \) and a DGA map \( f : A \to B \), there exists a DGA map \( M_f : M_A \to M_B \), unique up to homotopy, such that \( \rho_B \circ M_f \) is homotopic to \( f \circ \rho_A \).

**Proof.** Since \( \rho_B \) induces an isomorphism on cohomology, this fol-
lows directly from Corollary 4.16 applied to the diagram

\[
\begin{array}{ccc}
M_B & \xrightarrow{\rho_B} & B \\
\downarrow & & \\
M_A & \xrightarrow{f \circ \rho_A} & B.
\end{array}
\]

\[\square\]

**Remark 4.19.** While the space \( M \) of a minimal model \( \rho : M \to A \) is unique up to isomorphism, the map \( \rho \) is in general only unique up to homotopy. This is related to the fact that there exist nontrivial maps between minimal DGAs that are homotopic to zero, see the following problem.

**Problem 4.8.** Show that the DGA map

\[
\phi : M \to N, \quad \phi(a) = \phi(b) := 0, \quad \phi(c) := u^2
\]

between the minimal DGAs

\[
M := \Lambda[a, b, c], \quad |a| = 2, \ |b| = 3, \ |c| = 4, \ da = db = 0, \ dc = ab
\]

and

\[
N := \Lambda[u, v], \quad |u| = 2, \ |v| = 3, \ du = 0, \ dv = u^2
\]

is homotopic to zero.

*Explain topological analogue*
4.4

4.5 Obstruction theory

4.6 Principal fibrations and Postnikov towers

4.7

4.8

4.9

4.10
Chapter 5

Equivariant Homology and the String bracket
Chapter 6

Applications in Symplectic Geometry
Appendix A

Some Algebraic Topology

This appendix collects some standard results from algebraic topology that are used in the lecture. All the proofs can be found in Hatcher’s book [10].

Universal coefficient theorems

Theorem A.1 (Universal coefficient theorem for homology). For each pair of spaces \((X, A)\) and any abelian group \(G\) there are natural split exact sequences

\[
0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to \text{Tor}(H_{n-1}(X, A), G) \to 0,
\]

where \(H_n(X, A)\) denotes homology with integer coefficients and \(\text{Tor}(H_{n-1}(X, A), G) = 0\) whenever \(H_{n-1}(X, A)\) or \(G\) is free.

Theorem A.2 (Universal coefficient theorem for cohomology). For each pair of spaces \((X, A)\) and any abelian group \(G\) there are natural split exact sequences

\[
0 \to \text{Ext}(H_{n-1}(X, A), G) \to H^n(X, A; G) \to \text{Hom}(H_n(X, A), G) \to 0,
\]

where \(H_n(X, A)\) denotes homology with integer coefficients and \(\text{Tor}(H_{n-1}(X, A), G) = 0\) whenever \(H_{n-1}(X, A)\) is free.
Hurewicz theorems

**Theorem A.3** (absolute Hurewicz theorem). If $X$ is an $(n-1)$-connected space with $n \geq 2$, then $\tilde{H}_i(X) = 0$ for all $i < n$ and the Hurewicz map gives an isomorphism

$$h : \pi_n(X, x_0) \cong H_n(X).$$

**Theorem A.4** (relative Hurewicz theorem). If $(X, A)$ is an $(n-1)$-connected pair of path connected spaces with $n \geq 2$ and $A \neq 0$, then the Hurewicz map gives an isomorphism

$$h' : \pi'_n(X, A, x_0) \cong H_n(X, A),$$

where $\pi'_n(X, A, x_0)$ denotes the quotient of $\pi'_n(X, A, x_0)$ by the normal subgroup generated by elements of the form $\gamma f - f$ for $\gamma \in \pi_1(A, x_0)$ and $f \in \pi_n(X, A, x_0)$.

Whitehead theorems

**Theorem A.5** (Whitehead’s theorem for homotopy groups). A map $f : X \to Y$ between connected CW complexes that induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ on all homotopy groups is a homotopy equivalence.

Together with the relative Hurewicz theorem (applied to the mapping cylinder of $f$), this implies

**Theorem A.6** (Whitehead’s theorem for homology groups). A map $f : X \to Y$ between connected abelian CW complexes that induces isomorphisms $f_* : H_n(X) \to H_n(Y)$ on all homology groups is a homotopy equivalence.
Bibliography


[7] Fukaya, Oh, Ohta, Ono, ...


