Initiative to establish a DFG Priority Programme “Geometry at Infinity” (excerpt from original proposal from 15 October 2015)

Christian Bär Bernhard Hanke* Anna Wienhard Burkhard Wilking

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1. Summary

In almost all fields of mathematics, a consideration of asymptotics takes place. We may distinguish two different manifestations of this fundamental concept:

- Convergent sequences are used to approach mathematical objects whose precise properties are either not known or difficult to describe.
- Non-convergent or divergent sequences enable us to define new objects as “ideal” (artificial) limit points.

An illustrative example for the second aspect is projective geometry, originating in the development of perspective art in the Renaissance period. It is based on the addition of infinitely distant points to Euclidean space, which serve as virtual limit points of sequences that move to infinity on straight lines. In both cases many essential, sometimes drastic, features such as collapse, explosion of geometric quantities, or rigidity phenomena become apparent only if families and limits of geometric objects are taken into account or if these objects are considered in the large. These features are the subject of what we propose to call *Geometry at infinity*.

Recent developments of the last decades have shown that methods dealing with “finite” geometric objects like compact Riemannian or topological manifolds, finite groups and cell complexes, simplicial homology or index theory on compact manifolds are challenged and lead to completely new problems, concepts and theories when “infinite” structures come into play. Examples are asymptotic properties of Riemannian manifolds and their ideal boundaries, convergence of Riemannian manifolds, rigidity phenomena, homology based on infinite chains, K-theory of operator algebras and Brownian motion on non-compact manifolds, just to mention a few. Problems, concepts and results in these different fields bear many similarities that have not yet been explored to a full extent, partly due to the historic formation of different research communities, their regional separation and differences in scientific language. Furthermore, the study of limits and asymptotic properties often involve methods from geometry, analysis and topology at the same time. In order to improve our understanding of geometry at infinity, it is necessary to investigate problems jointly

*Coordinator of Priority Programme

and to combine different perspectives, mathematical descriptions and solution strategies from all of these areas.

The key feature of the proposed structured research initiative is to take the perspective "Geometry at infinity" as the guiding principle to address several key research problems. Taking this perspective we cross and transcend the frontiers of classical disciplines such as Riemannian geometry, global analysis, and algebraic topology, and bring together researchers from these disciplines to create an interdisciplinary research community. We expect that this more comprehensive approach will *lead to important and significant scientific progress* in this emerging field, *create a new research community in Germany*, with international radiance, and provide the structural framework to *train the next generation of researchers in an interdisciplinary environment*.

Our research will combine several classical areas of research in geometry: *Differential geometry*, *Geometric topology*, and *Global analysis*. The interrelation of these areas is emphasised by the following three cross sectional topics, that form the thematic backbone of our proposal:

Convergence **Compactifications** **Rigidity**

2. State of the art and preliminary work

2.1. Background

In this section we give brief overviews of the mathematical disciplines represented in our proposal, and indicate their relevance for geometry at infinity.

Differential geometry is the study of shapes of geometric objects. It has had an impact on lots of other fields in and outside mathematics. Differential geometry is also the mathematical language of general relativity and the two fields continue to stimulate one another. A central theme is the interaction of local properties of smooth Riemannian manifolds, expressed in terms of curvature, with global, topological properties of these objects.

Often one has to investigate families of geometric objects and limits of such families. Sequences of pointed manifolds may show collapsing phenomena, which are reflected by singularities in the limiting objects. The investigation of geometric flow equations, which, for example, model the development of distorted soap bubbles, lead to families of manifolds whose shape changes continuously in time. This can again induce singularities, for example when two soap bubbles touch each other before they merge. A general theory for singular spaces is based on the theory of metric measure spaces and of Alexandrov spaces.

Another interesting topic are geometric objects of infinite size. Here limits are used in order to study properties that are involved in passing "to infinity" inside such objects, and this leads to the notion of ideal limit points and compactifications. Both aspects, *limits and asymptotic properties of geometric objects*, play a prominent role in our proposal. A general reference is [40].

Geometric topology applies algebraic-combinatorial invariants, like the Euler characteristic or homology groups, to study geometric objects. One central goal is the classification of differentiable manifolds in terms of computable invariants of this kind.

Most relevant for us are *large scale geometry and topology*, see for example [59], which study macroscopic properties of non-compact spaces at arbitrary large scales while ignoring microscopic data. These areas stand opposite to infinitesimal methods, with classical topology in the middle. Major parts of large scale geometry are in fact concerned with the large scale geometry of groups.This subject of *geometric group theory* was initiated in the seminal work [39].

Large scale geometry has implications also for small ("compact") spaces, either by passing to infinite covers or to subspaces which are themselves non-compact, for example to leaves of foliations. Beyond the intrinsic interest in studying large geometric objects, the subject has numerous ramifications in representation theory, geometric and low dimensional topology, group theory, and random walks.

Global analysis studies properties of differentiable manifolds by analytic methods. In experimental sciences it is very common that one measures spectra of emitted radiation of a physical system with the aim of extracting as much information about this system as possible. This inverse problem of *spectral geometry* ("Can one hear the shape of a drum"?, [46]) has been investigated intensively on compact manifolds and, to a lesser extent, on complete non-compact manifolds. One main question is which geometric quantities are determined by the spectrum of a natural differential operator such as the Laplacian.

Besides spectral geometry, *Brownian motion*, a standard concept of stochastic analysis, is becoming increasingly important in geometry. A good survey is provided by [38]. Often the spaces of interest are not smooth, but have singularities, such as algebraic varieties or spaces arising as limits of smooth objects. Spectral geometry and Brownian motion, both of which are well established on complete Riemannian manifolds, have good chances to work also in many singular situations.

2.2. Recent developments

Our proposal is based on the observation that many problems dealing with geometry at infinity are not associated with just one of the disciplines mentioned in the last section, but require a more comprehensive point of view. We therefore prefer to structure the recent developments according to the main areas of research listed at the end of the introduction (see Section 1), which are transversal to the classical scheme of mathematical disciplines.

Convergence of geometric and topological structures are central topics for geometry at infinity. An obvious example from differential geometry is Gromov-Hausdorff convergence and the determination of geometric structures arising in the limit, see [40]. This includes the discussion of singularities and the introduction of generalised, non-smooth geometric objects, which motivate more general points of view such as metric geometry or geometric topology. A major theme is the behaviour of geometric, topological and spectral invariants while passing to limits of geometric objects, possibly under additional restrictions on geometric quantities such as curvature, diameter or volume.

A natural source for families of geometric objects and the formation of limits are *geometric evolution equations*. The Ricci or (inverse)-mean curvature flows are parabolic equations that can be used to change the shape of a manifold. There have been spectacular breakthroughs during the last decade, such as the resolution of Thurston's geometrisation conjecture by Perelman and the resulting classification of three-dimensional manifolds, see [54, 55] for a reader friendly exposition. One key problem is the finite time singularity analysis, which can be viewed as a limiting process along the time parameter. For this aim one considers all possible blow up limits (if compactness can be established). Understanding the geometry of all blow up limits at infinity then roughly amounts to understanding the geometry of the original manifold near the singular set. This is an essential step if one wants to resolve the singularity by generalised surgery, which was successfully carried out by Perelman for the Ricci flow on general closed three-dimensional manifolds to prove the geometrisation conjecture in his famous series [60–62].

While in geometric group theory one generally defines invariants that capture a certain geometric behaviour at infinity, often the key ingredient in geometric evolution equations is a monotonic quantity. A prime example is Huisken's monotonicity formula [45] or the two Perelman entropies [60]. Entropy – a notion which originates from statistical physics and also plays a key role in large scale geometry and the synthetic definition of Ricci curvature – can be used in the context of Ricci flow to establish a local non-collapsing theorem.

In special situations one can show that the evolving manifold converges (after rescaling) to a spherical space form (Ricci flow) or to a "round point" (mean curvature flow). By a result of Böhm-Wilking [7] this is the case if one starts with a manifold with positive curvature operator, a result which has later been used by Brendle-Schoen [32] to solve a long-standing problem on classifying manifolds with positive curvature.

Convergence and limits also occur in the definition and discussion of *asymptotic invariants in geometry and topology* such as simplicial volume, L^2-invariants and volume entropy. Atiyah's L^2-index theorem relates index theoretic properties of closed manifolds to spectral properties of the Laplace operators on their universal covers [21]. Besides this analytic aspect, which is based on the solution operator for the heat equation, there are purely algebraic approaches to L^2-invariants, establishing them as topological invariants, see the monograph [52]. A major theme is the approximation of L^2-Betti numbers by classical homological invariants. Lück's approximation theorem relates L^2-Betti numbers of universal covers of aspherical finite CW-complexes with residually finite fundamental groups to the sequence of (normalised) rational Betti numbers of finite intermediate covering spaces.

In the study of locally homogeneous geometric structures, convergence of these structures is closely related to collapsing phenomena. For example, a sequence of hyperbolic manifolds, collapsing to a point, converges to a Euclidean manifold, if suitably scaled. This phenomenon can be formalised by considering hyperbolic, Euclidean and spherical geometry as sub-geometries of projective geometries, as proposed by Felix Klein in the nineteenth century. Projectively one can then deform hyperbolic geometry continuously such that it approaches Euclidean geometry, and similarly for spherical geometry. Such transitions of geometries play a role in the geometrisation of three-dimensional orbifolds [50], and are important in various other instances, for examples when transitioning between three-dimensional hyperbolic geometry and anti-de Sitter geometry, which also plays a role in theoretical physics.

The *spectrum* of elliptic operators on compact manifolds has many beautiful qualitative properties which are well understood: The spectrum is discrete, it determines geometric data such as the dimension and the volume and its growth rate is known [34]. None of this remains true in general when one passes to non-compact manifolds: The discrete part of the spectrum is complemented by the essential spectrum. The Laplacian on the Euclidean space shows that not even the dimension is determined by the spectrum.

Convergence of geometric objects naturally leads to the investigation of spaces with singularities. The programme, initiated by Cheeger, to extend spectral geometry to spaces with singularities has caused an enormous and fruitful research activity in the last thirty years. One milestone of global analysis however, the Cheeger-Müller theorem about equality of analytic and combinatorial torsion [36, 57], still resists such a generalisation.

For specific problems one will have to make precise the assumptions on singularities and consider particular types of stratified spaces. In the latter case spectral geometry is often carried out on the main stratum which is an incomplete manifold even if the whole singular space is compact. The geometry at infinity of the main stratum reflects the structure of the singularities.

Compactifications are a very efficient tool to investigate geometric and topological objects "in the large", which amounts to a systematic study of their asymptotic behaviour. Abstractly, compactifications arise when ideal limit points are added to a locally compact topological space. Local compactness implies that these points are placed "at infinity". In categorical-topological terms the one point and Stone-Čech compactifications are the extremal possibilities, but lack concrete geometric meaning and applicability. In hyperbolic geometry so called *ideal boundary points* yield geometrically meaningful structures. *Large scale and coarse geometry* extend this approach to more general spaces, including algebraic structures equipped with appropriate metrics and satisfying weak forms of hyperbolicity. The monograph [33] gives a good survey.

Singularities appearing in *geometric flows* as described in the previous section can be considered as naturally occurring compactifications of the complementary, regular part. From this point of view the neighbourhood of the singularity corresponds to a neighbourhood of infinity in the regular part.

The construction and investigation of compactifications or, more generally, bordifications builds on ideas and notions from very different areas ranging from abstract topology over differential geometry to operator algebras and probability.

Complete simply connected manifolds of nonpositive curvature have natural compactifications by their *geodesic boundaries*, and these compactifications play a crucial role for establishing structural results and rigidity theorems, with the Mostow-Prasad rigidity theorem as a celebrated example [56]. More generally, Gromov hyperbolic spaces admit natural bordifications which are particularly useful in the presence of large isometry groups. For CAT(0)-spaces, which generalise properties of non-positively curved manifolds, or even more generally for finitely generated groups whose large-scale geometry resembles CAT(0)-spaces, much less is known. Here one needs to identify appropriate bordifications whose dynamical properties are similar to the properties of the Gromov boundaries of hyperbolic spaces.

A particularly interesting class of non-simply connected manifolds of nonpositive curvature for which interesting compactifications have been constructed are locally symmetric spaces arising from Anosov representations. They deal with discrete groups which generalise the concept of convex cocompact subgroups in the context of higher rank symmetric spaces [15, 48]. The quotients of the symmetric spaces by these special groups are topologically tame, i.e. homeomorphic to the interiors of manifolds with boundary [43, 47].

Boundary value problems for Laplace type operators are a classical topic. For Dirac operators this is subtler because one has to consider nonlocal boundary conditions. One approach to identify the "contribution of infinity" to an index formula on a non-compact manifold consists of considering boundary value problems for an exhausting sequence of compacta. If the manifold is complete and has compact boundary, the theory has been developed to a satisfactory level by Ballmann-

Bär [1], after earlier work by many authors starting with the seminal work by Atiyah-Patodi-Singer [23]. If one allows the boundary to be non-compact as well, then not much is known, except in the case of bounded geometry [42]. A Lorentzian version of the Atiyah-Patodi-Singer index theorem has been proved recently [25] together with applications to quantum field theory [26].

In the theory of *Brownian motion* one constructs a probability measure, the Wiener measure, on certain spaces of paths in the manifold and one can then write the solutions to heat equations as path integrals (the Feynman-Kac formula), see e.g. [4]. Brownian motion is particularly interesting on non-compact manifolds. Properties of Brownian motion such as recurrence or stochastic completeness are related to curvature conditions, volume growth, the Liouville property or positivity of the minimum of the Laplace spectrum (see Grigor'yan's survey [38]), and also to L^2-invariants. A conjectured criterion for stochastic completeness in terms of volume growth was surprisingly disproved by Bär and Bessa in [3].

Since Brownian motion is based on continuous paths, it is suitable as a tool to study singular spaces. The approach to Wiener measures on Riemannian manifolds and the Feynman-Kac formula in [24] can probably be generalized to this setting.

If one passes from the heat equation to the Schrödinger equation from quantum mechanics, then unfortunately Wiener measure fails to yield a path integral formula for the solution. This is a major unsolved problem. Here another approach to path integrals seems more promising, that of finite dimensional approximation as developed in [4, 5].

Rigidity is a more abstract concept and one of the deepest and most fascinating aspects of geometry at infinity. Is is prototypically expressed in Mostow's rigidity theorem [56] saying that homotopy equivalent closed hyperbolic manifolds of dimension at least three are isometric. Roughly speaking, when these manifolds are deformed, hyperbolicity cannot be preserved: This property is "rigid". Rigidity is in some sense opposite to continuous deformations of geometric structures, which occur in geometric flows, for example. The interrelation of these two extreme perspectives, *flexibility and rigidity*, is essential for many classification problems in geometry.

In their seminal paper [30] Besson-Courtois-Gallot extended Mostow's rigidity theorem to a rigidity result for negatively curved locally symmetric spaces. Their approach is based on the notion of *volume entropy*, which measures the growth rate of balls in universal covering spaces as a function of the radius. A purely topological rigidity statement is the Borel conjecture (1953) claiming topological rigidity of aspherical manifolds. Together with the Novikov conjecture (1965) on the homotopy invariance of higher signatures it still represents a fundamental challenge for the subject. In a slightly different setting, rigidity appears in the *spectral geometry of non-compact manifolds*, where essential spectra of geometric operators are stable under compact perturbations of the underlying manifolds. It is a major goal to find systematic, axiomatic settings for rigidity phenomena and to prove general theorems.

Here an especially efficient tool are *isomorphism conjectures* relating analytic, geometric and homological invariants of groups. This corresponds to the observation that rigidity of geometric objects often is governed by large scale properties of their fundamental groups, which are studied in geometric group theory. These properties can in turn be related to ideal boundaries and hence to various forms of compactifications, another main theme of our proposal. This perspective has not only led to tremendous progress for the classical isomorphism conjectures, but establishes links to at first unexpected applications in algebra and geometry. In recent years Lück and his collaborators [27–29] made great progress on the class of groups for which the Farrell-Jones conjecture is true. This conjecture plays a fundamental role for the classification of non-simply connected manifolds. The proofs are based on methods from homotopy theory, controlled topology, geometric group theory and flows on metric spaces, thus combining methods from completely different areas.

Generalisations of the *Atiyah-Singer index theorem* from compact to non-compact manifolds motivated and shaped the field from the very beginning. This topic, which runs under the name *large scale* or *coarse index theory*, uses advanced functional analytic methods, and is eventually based on the heat kernel proof of the Atiyah-Singer index theorem of Atiyah-Bott-Patodi [22]. It was systematically developed by Roe [63]. An important application of large scale index theory is the classification of *Riemannian metrics of positive scalar curvature* on non-simply connected or non-compact manifolds.

A pervasive theme for rigidity questions are *aspherical manifolds*, whose topology is controlled solely by the fundamental group. Closed aspherical manifolds necessarily have infinite fundamental groups, whose large scale properties may have non-trivial geometric implications for the given manifold. One conjecture in this direction predicts that closed aspherical manifolds do not carry metrics of positive scalar curvature. For spin manifolds work of Rosenberg [64] implies that this conjecture follows from the Novikov conjecture.

One possible approach to this question, which avoids the Novikov conjecture, is by interpolating between hyperbolic and aspherical manifolds. This can be done using different notions of largeness as defined by Gromov and Lawson [41]. Brunnbauer-Hanke [9] showed that this and many other largeness properties just depend on the image of the fundamental class in the homology of the fundamental group. Hanke-Kotschick-Roe-Schick and Hanke-Schick [16, 17] link this *homological invariance* of geometric-topological largeness properties to large scale index theory.

2.3. Perspectives

Here we present a sample of some possible future developments in the main thematic areas of our proposal. We restrict ourselves to some rough outlines, and reserve detailed elaborations to the research proposals that will be prepared in the course of the priority programme

Convergence. The goal to understand limits of Riemannian manifolds under certain curvature restrictions remains central. A specific problem concerns *collapse with lower sectional curvature bound*. There are several special cases where we hope that progress can be made in the near future. One particular case might be the situation where one has a lower bound on the lowest eigenvalue of the curvature operator rather than on sectional curvature. Work of Tuschmann's group [44] suggests that this case should be surprisingly rigid. This might be related to the fact that open manifolds with nonnegative curvature operator are far more rigid than open manifolds with nonnegative sectional curvature. Work of Simon [65] indicates that the Ricci flow could help to understand the analysis in this case.

In another direction it is an interesting project to investigate spin manifolds with non-vanishing $\hat{A}$-genus. It is an open question of Lott whether such a manifold can have a collapse with lower sectional curvature and upper diameter bound. Preliminary work of Cabezas-Rivas and Wilking suggests that the generic fibre of any such collapse has to be an infranil manifold. If in this particular case one can show that collapse is happening along a singular Riemannian foliation, one might use topological methods to give an affirmative answer to Lott's question.

Singular spaces are an important theme in the context of convergence of smooth manifolds. One goal in this direction is to prove a *Cheeger-Müller theorem for singular manifolds*. So far the technique in [51] works only if the Laplace operators on the manifold pieces making up the singular manifold have compact resolvents. However there are spaces, for example those with hyperbolic ends, where this condition is violated. Using *scattering methods* it should be possible to extend those methods accordingly. A rougher concept of singular space is that of an *Alexandrov space*. These spaces have become important as limits of Riemannian manifolds with lower sectional curvature bounds. Even more generally, *metric measure spaces* can be studied from a geometric and an analytic point of view. It is possible to give sense to lower Ricci curvature bounds, one can discuss tangent spaces, the Hausdorff dimension and even prove an Obata theorem. The link to optimal transportation as explored by Sturm and coauthors is particularly appealing, see e.g. [31].

An central objective in the field of *geometric evolution equations* is to develop refined techniques for the resolution of singularities in mean curvature or Ricci flow. One can formulate curvature conditions where one can expect, for example, that only codimension one or codimenison two type singularities can occur, but it certainly requires a lot of additional tools to make the analysis work. In this context the work of Cabezas-Rivas and Wilking [12] might be of interest, where structure results for the Ricci flow on open manifolds with nonnegative complex curvature were proven. Furthermore even in the case of unbounded curvature, existence was established and a partial long time analysis was carried out.

Rather recent *interactions between group theory and L^2-invariants* come from open problems about approximation questions of L^2-Betti numbers in prime characteristic, generalising Lück's approximation theorem, which holds for characteristic zero. Possibly this opens the way to an "L^2-cohomology with finite field coefficients". Part of this programme is an improved understanding of relations between the first L^2-Betti number of a group and classical invariants based on ergodic theory, like its cost and rank gradient. Also relations between the L^2-torsion of universal coverings of aspherical manifolds and the growth of the homology with arbitrary field coefficients for towers of finite intermediate coverings will be treated. Recent results [53] give some hope that further steps in this direction are possible.

In the field of *spectral geometry* the problem that the spectrum of a geometric operator on a non-compact manifold may not contain any information about the geometry of the manifold (for example the Laplacian on $\mathbb{R}^n$) can be circumvented by considering an exhaustion by compact balls and the corresponding Dirichlet problem for the Laplacian. Then a volume renormalisation of the corresponding eigenvalue counting functions converges to a function as the radius of the balls goes to infinity. This function can be seen as a spectral density and from it one can at least read off the dimension of the Euclidean space. The procedure in this example should be generalised to suitable classes of non-compact manifolds. One would then try to find isospectral-type invariants and investigate the geometric content of this spectral density.

Another relevant field of study is to extend the approach to path integrals by finite dimensional approximation so as to cover manifolds with boundary as well as non-compact manifolds with bounded geometry. An ambitious goal would be to find a mathematically rigorous *path integral formula for solutions of the Schrödinger equation*.

Compactifications. There is no general construction to compactify locally symmetric spaces of higher rank. New constructions for compactifications of locally symmetric spaces of higher rank, which arise as quotients by Anosov representations, puts this now in reach. One can employ a combination of these new constructions with techniques from the classical compactification of arithmetic (finite volume) quotients to provide answers in a more general setting. One way to unify the various classical boundary constructions for symmetric spaces of higher rank is to consider the horoboundary construction for various Finsler (non-Riemmanian) metrics on symmetric spaces.

Another important theme is the study of the space of all group actions of a given group on a symmetric space, or in other words the investigation of *representation varieties*. This relates to compactifications in two ways. First, boundary maps play a crucial role in the characterisation of such group actions. Various properties of such boundary maps relate to precise geometric properties of the group action. This has been described for maximal representations [10, 11] and Hitchin representations, but the potential of these boundary maps has not yet employed to the full extent. Second, the representation variety itself is a non-compact space, which can be compactified by adding ideal points, as described in work of Paulin and Parreau. Such ideal points describe limiting actions on buildings, which are special kinds of singular spaces. For maximal representations or Hitchin representations, an interpretation of these boundary points through geometric objects on the surface is still missing.

The theory of *boundary value problems* of geometrically motivated differential operators for non-compact boundaries should be developed further, in particular under non-local boundary conditions for Dirac type operators. This is tightly connected with *asymptotic aspects of spectra on noncompact manifolds*.

In the probabilistic description of asymptotics the concept of stochastic completeness of *Brownian motion* can be refined by considering escape rates which measure the speed at which random paths approach infinity. These should be compared to geometric properties such as volume growth rates. In particular there should be close relations to growth conditions of fundamental groups and hence to large scale invariants of groups, a very interesting and largely unexplored question. Heat kernel estimates like the ones obtained by Grigor'yan should enable one to make progress in this direction. This will however require joint efforts by participants with different backgrounds.

Rigidity. There is a unifying approach to the Farrell-Jones and Baum-Connes *isomorphism conjectures* developed by Davis-Lück using generalised assembly maps [35]. Roughly speaking, given a group, one tries to minimise the input of non-homological ("large scale") information, "assembled" in a homological way, that is needed to describe non-homological structures derived from this group. Within this framework new types of isomorphism conjectures and their implications for geometric and algebraic problems can be considered. The investigation can be started with groups of hyperbolic nature according to Bridson's universe of groups - a useful organising principle in geometry group theory - where the already known methods seem most promising.

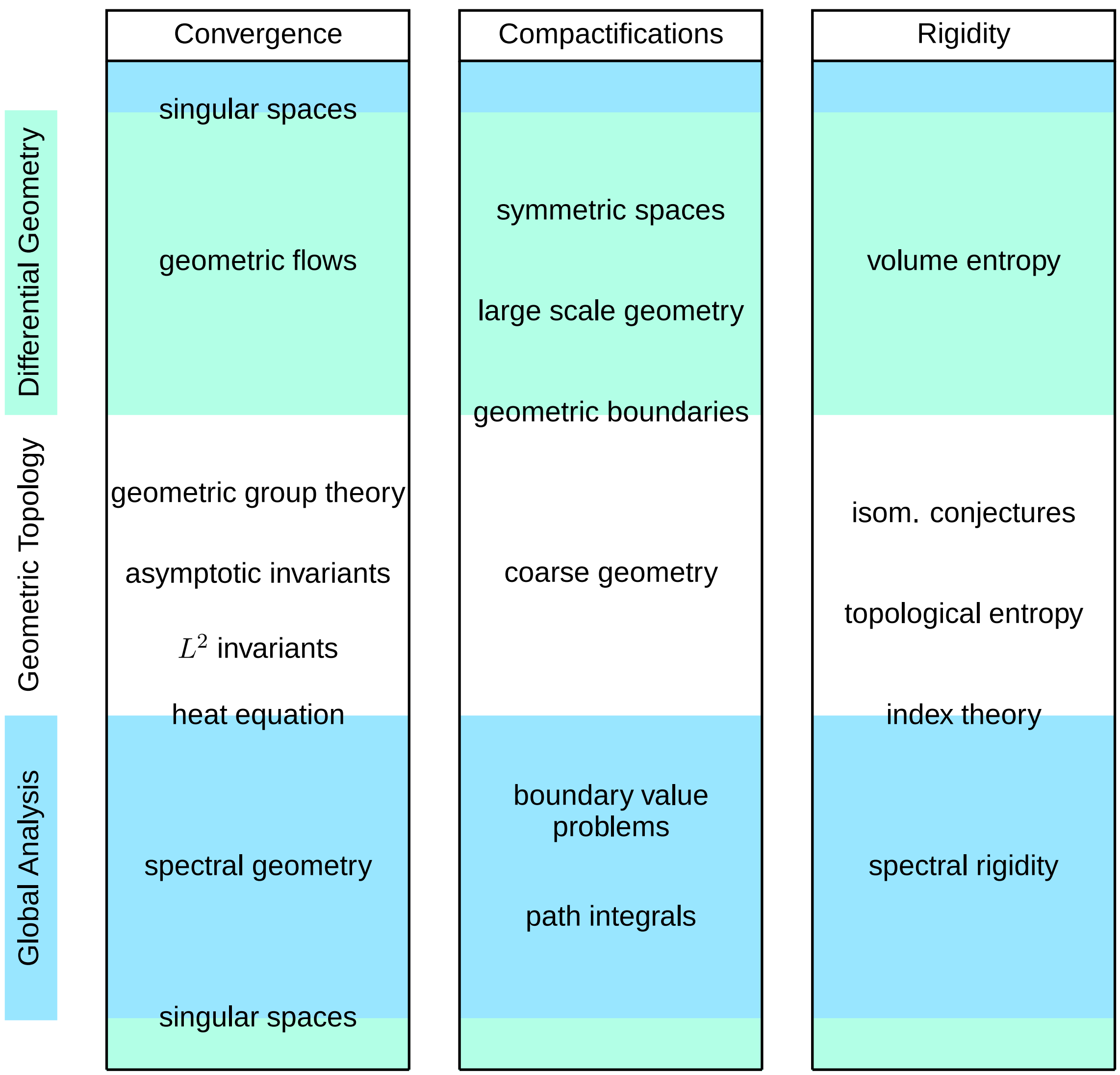
The (moduli-)space of *metrics of positive scalar curvature* on closed manifolds remains a challenging and attractive field of study. The large scale index methods from Xie-Yu [66] and ongoing work of Botvinnik-Ebert-Randal-Williams based on homological stability [37] can probably be combined to obtain refined information on higher homotopy groups of the (moduli) space of positive scalar curvature metrics for non-simply connected manifolds, compare [8, 18].

Intricate analytic problems occur in the study of *foliations*. These objects are interesting in their own right, but also because they can give new insight into the structure of three-manifolds or symplectic manifolds, among others. It is not known which open manifolds occur as leaves of foliated compact manifolds. Leaves of a fixed compact manifold come equipped with a canonical

Riemannian metric up to quasi-isometry so that this problem seems tailor-made for large scale geometric tools.

A different perspective on rigidity occurs in *spectral geometry*, where it is well known that the essential spectrum of a geometric operator does not change if the underlying manifold is modified in a compact subset. A concrete question concerns the behaviour of the essential spectrum under small perturbations on non-compact subsets (rigidity of the essential spectrum).

2.4. Schematic overview of research topics



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5. Merits of the proposal taking into account the objectives of the programme

5.1. Originality of research questions in terms of topic and methodology

Important developments in natural sciences were often initiated by empirical observations, experiments or problems in applications. Mathematics firstly provides a precise language to formulate these problems and secondly develops theories to solve them. This has resulted in a variety of specialised mathematical disciplines, many of which are nowadays independent fields of research. On the one hand this differentiation leads to efficient tools and deep theories. On the other hand this may enlarge the distance to the problems that stand at the beginning of these theories.

The novelty of our proposal lies in regarding geometry at infinity as a *comprehensive, unifying aspect* of modern geometry, which is motivated by fundamental observations and problems. It combines several lines of research in which spectacular progress could be achieved during the last decades. Bringing these lines together will result not only in an *attractive and diverse research environment*, but also in a *new perspective on geometry* in general. This underlines the *timeliness* of our proposal and identifies geometry at infinity as an *emerging field of research* which will prepare the ground for endeavours by a *new generation of scientists*.

Our *methodology* emphasises ideas and solution strategies that usually occur in different, separate contexts, but in fact bear many similarities. We will illustrate this by two examples.

The first concerns *entropy*, a fundamental notion from statistical physics. In its original definition it measures the number of microscopically attainable states in a given macroscopic system, and hence it is closely linked to the theory of Brownian motion. The abstract, stochastic definition uses the logarithm of the volume of the space of configurations of a given macroscopic physical system. In geometry this point of view has been implemented in the definition of *topological entropy* of continuous self maps of compact Hausdorff spaces, as the average information needed to describe the long time behaviour of iterates of these maps. In the case of time one maps of geodesic flows, there are lower bounds of the topological entropy in terms of the exponential growth rate of the volumes of balls in the universal cover of the given space. This exponential growth rate is the so called *volume entropy*. All of these notions appear in several parts of our proposal, from the rigidity results of Besson-Courtois-Gallot based on volume entropy, and local non-collapsing theorems for the Ricci flow to the definition of lower Ricci curvature bounds on general metric spaces in terms of transport inequalities.

Another technique that links many parts of our proposal is the *heat kernel method*, originally developed to solve the heat equation describing the distribution of heat in physical media. It occurs prominently at various places of our proposal, because it governs the distribution of local data over larger geometric structures not only in physics, but as well in abstract mathematical settings. For example the heat kernel proof of the Atiyah-Singer index theorem makes transparent how local curvature information spreads out to finally induce one single invariant which reflects the global shape of a given object. In the theory of Brownian motion the heat kernel is interpreted as a probability density for the transition between random points on the given manifold.This adds a stochastic aspect to our considerations, which also play a key role in the theory of path integrals via the Feynman-Kac formula.

From this point of view we ask for a more systematic perspective on passing from microscopic to macroscopic information on large scale objects. For example, the volume growth of balls in Riemannian manifolds is determined on small scales by the scalar curvature and on large scales by the volume entropy. Can the transition between these extreme cases be made more precise?

5.2. Delimination of scope taking into account the duration of a Priority Programme

In order to ensure feasibility of the proposed priority programme it is necessary to restrict its overall scope and to define the thematic boundary of the research initiative. As described above the focus of our proposal lies in bringing researches from differential geometry, geometric topology, and global analysis together to attack research problems in the three proposed research themes *Convergence*, *Compactifications*, and *Rigidity*. Of course these themes also play a role in other areas of mathematics, which are only of minor importance for the present proposal. In particular symplectic geometry and dynamical systems are mathematical areas where the proposed themes are of importance. We will take into account methods and results and interact with researchers in these areas, but they do not lie in the main focus of the priority programme.

Other areas which we expect to have fruitful interactions with are mathematical physics and probability theory. Several members of the programme committee are actively involved in projects with physicists, and several members of the research communities we plan to address already had cooperations and interactions with researchers from these areas.

We apply for a total duration of six years, distributed over *two periods of three years*. The first three years will include the building up phase, the second three years will be necessary to consolidate the progress and successes of the first funding period.

5.3. Coherence of planned research activities

Research in this proposed priority programme will be carried out in two interacting frameworks, through *individual research projects* and in *coordinated activities*. All fields of research represented in our proposal are strongly represented in Germany so that we are able to involve a number of scientists of the *highest international reputation*.

In addition to the official call of the DFG, we will communicate to colleagues at all levels (senior and junior) the opportunity to submit individual research projects. We will encourage colleagues to submit research proposals with more than one principal investigator, pairing for example researchers from different areas or at different career stages.

For the coordinated research activities we plan to hold *annual workshops* and *learning seminars*, and - on a more spontaneous basis - to offer the opportunity to organise *networking seminars*, and to profit from a *visiting programme* within the priority programme. We will organise two large *international conferences* at the end of each funding period of three years. These conferences will play a central role for the international visibility of the priority programme.

The priority programme will maintain a *webpage*, which will also serve as a platform of communication between the individual members. The programme committee will hold regular meetings (virtual and in person) to discuss the governance of the priority programme (e.g. topics of future conferences, workshops and seminars, use of the coordinator and network funds).

Workshops. We plan to hold two *workshops* per year, which will focus on specific aspects of our research programme, but always emphasising the interplay of different mathematical fields. These workshops will be organised by various members of the priority programme. One of the two annual workshops should be organised by younger scientists in the programme, who are not yet on tenured positions, and some of these workshops will especially be directed to female researchers (“Women in geometry”), see Section 5.5.

A model for what we envision is the 2014 Oberwolfach workshop “Analysis, geometry and topology of positive scalar curvature metrics” organised by Bernd Ammann, Bernhard Hanke and André Neves. It brought together mathematicians working in global analysis, algebraic and geometric topology, and general relativity that usually do not meet at conferences. Instead of presenting talks at a highly, specialised level, the speakers succeeded in addressing a diverse audience while communicating guiding problems, solution strategies, recent results and problems for further study.

Learning Seminars. An especially important goal of our priority programme is to give all the participating researchers the opportunity to get to know concepts outside their research area in form of *learning seminars*. Christian Bär, Bernhard Hanke, and Anna Wienhard have jointly been organising block seminars for advanced undergraduate and graduate students for many years. Topics in the last years included optimal transport, Cartan-Kähler geometry, random walks in Riemannian manifolds, and L^2-invariants. These block seminars have been very successful and may serve as a model for our learning seminars. They will take place around two times each year at different levels, with the general feature being the accessibility for participants with different backgrounds. In this way they will help to bridge the gap between the different mathematical areas and help younger researchers to get trained immediately in a more interdisciplinary environment.

International Conferences. The priority programme as a whole will be represented by two large *international conferences* at the end of each funding period of three years. Besides the participating researchers from Germany, we will also invite leading international experts. The conferences will increase the international visibility and will also serve as an opportunity to review the progress and successes made during the funding periods.

The visiting programme will be addressed in Section 5.4.
The networking seminars will be addressed in Section 5.5.

5.4. Strategies for collaborating across disciplines and locations

A thorough exploration of the topics covered by our proposal requires the combination of efforts from research groups with different scientific backgrounds and located at several places.

The programme committee will manage funds for a *visiting programme* to encourage visits between research groups in the priority programme, which will enhance the connectivity and collaborations across the members of the priority programme. These visits can vary in duration from a few days to a few months, and should involve research at all levels (graduate students through full professors). In addition to funds within individual projects, members of the priority programme can apply for travel support by submitting a short application to the programme committee, including a description of the goal of the visit, a budget, and an invitation letter of the host member.

Another crucial role for the interaction across the disciplines and locations will be the learning seminars described in Section 5.3. Both of these activities will combine researchers in *different fields*, *at different levels* and based *at different locations*.

5.5 Early career support, promotion of female researchers, family-friendly policies

Early career support will be implemented by two schemes. Firstly, on the level of individual research projects we will encourage the formation of *junior-senior-teams* consisting, for example, of one experienced principal investigator and one postdoc. Secondly, we will encourage young researchers to engage in the *independent organisation* of scientific events, *learning seminars*, *mutual visits*, and in particular in the organisation of *networking seminars*. Senior members in our programme may serve as contact persons in order to clarify logistic or thematic issues coming along with the planning of such events.

The *networking seminars* should provide the young researchers in the priority programme with the opportunity to meet regularly, present their work and start collaborations among themselves. The goal is that the young members of the programme will grow into a community of researchers which persists and sustains the collaborative and interdisciplinary scope of the proposed programme beyond the funding period. Good networks among young scientists are often key to a successful scientific career and sometimes initiate completely new directions of research.

For the promotion of female researchers we will encourage participating scientists to include female graduate students and postdocs in their research proposals. Moreover, we will make an enhanced effort to increase the number of women among organisers, speakers and participants in workshops and conferences within the priority programme.

There are several female scientists at different career stages who already expressed their interest in becoming part of the priority programme, and we hope that many of them will do so. They will serve as role models and mentors for younger female scientists and graduate students who will participate in activities of the priority programme.

It is especially important to retain female scientists at transition points of their careers, in particular to stay in science at the doctoral and early postdoctoral level. In addition to the networking seminars, we plan to run some of the workshops in our priority programme under the header “Women in geometry”. These workshops, which will provide additional networking activities for young female scientists, will be organised by female scientists at an advanced doctoral or at a postdoctoral level. Senior female scientists in the network will provide organisational and scientific assistance. Besides the scientific component, they should also showcase successful female scientists in geometry, topology, and analysis, and give female scientists at all levels the possibility to socialise and build lasting mentorship relations. These workshops should take place at intervals of one and a half to two years so that the participants can meet on a regular basis and make lasting contacts.

We will reserve part of our budget to install family-friendly policies to ensure that nobody, independent of gender, is prevented from attending scientific events, because child care cannot be organised. Part of the funds is reserved to support *arrangements for child care* for those mathematicians who otherwise cannot attend a conference or workshop. Travel costs for small children of less than two years and, if possible, an accompanying person (partner, grandparent or nanny) might be paid from this budget, if a person otherwise cannot take part at a conference, for example breast-feeding mothers. Child care on-site at larger conferences and workshops will be organised, if necessary, so that our scientific activities and family life can be combined in the best way possible.