Non-negatively curved torus manifolds

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Outline

1. Non-negative curvature and torus manifolds
   - Definitions
   - Previous Work

2. Main results
   - Main results
   - Structure Results for torus manifolds
   - Proof of the main result

3. Applications
A torus manifold is a $2n$-dimensional orientable connected manifold $M$ together with a action of an $n$-dimensional torus such that $M^T \neq \emptyset$.

A Riemannian manifold $M$ is non-negatively curved if all triangles in $M$ are not “thinner” than a triangle in the Euclidean plane.
Goal

*Classify torus manifolds which admit an invariant metric of non-negative curvature.*
Previous Results

Theorem (Grove and Searle (1994))

A simply connected torus manifold with an invariant metric of positive sectional curvature is diffeomorphic to $S^{2n}$ or $\mathbb{C}P^{n}$.

Theorem (Hsiang and Kleiner (1989))

A 4-dimensional simply connected Riemannian manifold with positive sectional curvature and an isometric $S^1$-action is homeomorphic to $S^4$ or $\mathbb{C}P^2$. 
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A simply connected torus manifold with an invariant metric of positive sectional curvature is diffeomorphic to $S^{2n}$ or $\mathbb{C}P^n$.

Theorem (Hsiang and Kleiner (1989))

A 4-dimensional simply connected Riemannian manifold with positive sectional curvature and an isometric $S^1$-action is homeomorphic to $S^4$ or $\mathbb{C}P^2$.
Theorem (Kleiner (1990) and Searle and Yang (1994))

A 4-dimensional simply connected Riemannian manifold with non-negative sectional curvature and an isometric $S^1$-action is homeomorphic to $S^4$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ or $S^2 \times S^2$.

- Grove and Wilking (2013) classified 4-dimensional simply connected Riemannian manifolds with non-negative curvature and isometric $S^1$-action up to equivariant diffeomorphism.
- In particular, a 4-dimensional simply connected non-negatively curved torus manifold has at most four fixed points.
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Main Theorem

Theorem (W.)

Let $M$ be a simply connected torus manifold with $H^{\text{odd}}(M; \mathbb{Q}) = 0$ such that one of the following two conditions holds:

- $M$ admits an invariant metric of non-negative sectional curvature.
- $M$ is rationally elliptic.

Then $M$ has the same rational cohomology as a quotient of a free linear torus action on a product of spheres. If, moreover, $H^*(M; \mathbb{Z})$ is torsion-free or $H^{\text{odd}}(M; \mathbb{Z}) = 0$, then $M$ is homeomorphic to such a quotient.
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A simply connected topological space $X$ is called rationally elliptic, if

$$\sum_{i=0}^{\infty} \dim H^i(X; \mathbb{Q}) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \dim \pi_i(X) \otimes \mathbb{Q} < \infty.$$  

Conjecture (Bott)

A non-negatively curved manifold is rationally elliptic.

Theorem (Spindeler (2013))

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Discussion of assumptions

A rationally elliptic torus manifold $M$ has $\chi(M) = \chi(M^T) > 0$ and therefore $H^{\text{odd}}(M; \mathbb{Q}) = 0$. Hence, the assumption on the cohomology is not necessary in the main theorem.

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Towards a proof of the conjecture

Theorem

The conjecture holds for locally standard torus manifolds $M$ which satisfy

- The intersection of any collection of facets of $M/T$ is connected or empty, or
- $\dim M = 6$.

Proof.

- We first use the geometry of $M/T$ to show that all faces are contractible.
- Results of Masuda and Panov imply that $H^{\text{odd}}(M; \mathbb{Z}) = 0$.
- Hence, the statement follows from the main theorem.
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Masuda and Panov (2006) proved the following structure results for torus manifolds $M$ with $H^\text{odd}(M; \mathbb{Z}) = 0$:

- The torus action is locally standard, i.e. each $p \in M$ has an invariant neighborhood which is equivariantly diffeomorphic to an open subset of $\mathbb{C}^n$.
- $M/T$ is a manifold with corners.
- All faces $F$ of $M/T$ are acyclic, i.e. $\tilde{H}^*(F) = 0$. Therefore all $F$ are homology discs.
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Canonical models

- Denote by $\lambda(F)$ the isotropy group of a generic orbit in $F$.
- There is an equivariant homeomorphism

$$ (M/T \times T)/\sim \rightarrow M, $$

where $(x_1, t_1) \sim (x_2, t_2) \iff x_1 = x_2 \land t_1^{-1}t_2 \in \lambda(F(x_1))$

Therefore there is a principal torus bundle $Z_{M/T} \rightarrow M$, where $Z_{M/T}$ is the moment angle complex associated to $M/T$:

$$ Z_{M/T} = (M/T \times T^{\delta})/\sim, $$

where $(x_1, t_1) \sim (x_2, t_2) \iff x_1 = x_2 \land t_1^{-1}t_2 \in T^{\delta}(F(x_1))$ with $\delta(F) = \text{set of facets containing } F$. 

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Therefore there is a principal torus bundle $Z_{M/T} \rightarrow M$, where $Z_{M/T}$ is the moment angle complex associated to $M/T$:

$$Z_{M/T} = (M/T \times T^{\mathfrak{g}})/ \sim,$$

where $(x_1, t_1) \sim (x_2, t_2) \iff x_1 = x_2 \land t_1^{-1}t_2 \in T^{\mathfrak{g}}(F(x_1))$ with $\mathfrak{g}(F) =$ set of facets containing $F$. 
The face poset $\mathcal{P}(M/T)$ is defined to be the set of all faces of $M/T$ together with the ordering given by inclusion.

**Theorem (W.)**

Let $M_1$ and $M_2$ be two simply connected torus manifolds with $H^{\text{odd}}(M_i, \mathbb{Z}) = 0$. Then $M_1$ and $M_2$ are homeomorphic if $(\mathcal{P}(M_1/T), \lambda_1)$ and $(\mathcal{P}(M_2/T), \lambda_2)$ are isomorphic.
Proof.

If all faces of $M_i/T$, $i = 1, 2$ are contractible, then the statement follows, because every homeomorphism of the boundary of a contractible manifold extends to a homeomorphism of the contractible manifold.

If not all faces are contractible, then one can change the torus action on $M_i$ in such a way that all faces become contractible without effecting $(\mathcal{P}(M_i/T), \lambda_i)$.

Corollary

Let $M$ be a torus manifold homotopy equivalent to $\mathbb{C}P^n$. Then $M$ is homeomorphic to $\mathbb{C}P^n$. 

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Let $M$ be a torus manifold homotopy equivalent to $\mathbb{C}P^n$. Then $M$ is homeomorphic to $\mathbb{C}P^n$. 
By the structure results for torus manifolds, for the proof of the main theorem it is sufficient to determine the combinatorial type of $M/T$ and then to realize these combinatorial types by a simply connected torus manifold.
Lemma

Let $M$ be a torus manifold with $H^{\text{odd}}(M; \mathbb{Q}) = 0$ such that

- $M$ admits an invariant metric of non-negative sectional curvature, or
- $M$ is rationally elliptic.

Then all two-dimensional faces of $M/T$ have at most four vertices.
Lemma

Let \( M \) be a torus manifold with \( H^{odd}(M; \mathbb{Q}) = 0 \) such that all two-dimensional faces of \( M/T \) have at most four vertices. Then \( M/T \) is combinatorially equivalent to a product \( \prod_i \Sigma n_i \times \prod_i \Delta n_i \), where

\[ \Delta n_i \text{ is an } n_i\text{-dimensional simplex and } \Sigma n_i = S^{2n_i}/T. \]

Note that
\[ Z_{\Sigma n} = S^{2n} \text{ and } Z_{\Delta n} = S^{2n+1} \text{ and } Z_{Q_1 \times Q_2} = Z_{Q_1} \times Z_{Q_2}. \]

Therefore the theorem follows.
Lemma

Let $M$ be a torus manifold with $H^{odd}(M; \mathbb{Q}) = 0$ such that all two-dimensional faces of $M/T$ have at most four vertices. Then $M/T$ is combinatorially equivalent to a product $\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i}$, where

- $\Delta^{n_i}$ is an $n_i$-dimensional simplex and $\Sigma^{n_i}$ is $S^{2n_i}/T$.

Note that $Z_{\Sigma^n} = S^{2n}$ and $Z_{\Delta^n} = S^{2n+1}$ and $Z_{Q_1 \times Q_2} = Z_{Q_1} \times Z_{Q_2}$.

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- Therefore the theorem follows.
Orbit spaces in dimension 6.
Rigidity problem

Definition

A polytope $P$ is called rigid if the following holds:

- There is a quasitoric manifold $M_1$ with $M_1 / T = P$.
- If $M_2$ is another quasitoric manifold with $H^*(M_2) \cong H^*(M_1)$ and $M_2 / T = Q$, then $P$ and $Q$ are combinatorially equivalent.

Theorem (Choi, Panov and Suh (2010))

$P = \prod_i \Delta^{n_i}$ is rigid.
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**Theorem (Choi, Panov and Suh (2010))**

\[ P = \prod_i \Delta^{n_i} \text{ is rigid.} \]
The product $\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i}$ is rigid in the following sense:

**Theorem**

Let $M_1$ and $M_2$ be two simply connected torus manifolds with $H^{odd}(M_i, \mathbb{Z}) = 0$. If $M_1$ is rationally elliptic and $M_2$ is rationally homotopy equivalent to $M_1$, then $\mathcal{P}(M_1/T)$ and $\mathcal{P}(M_2/T)$ are isomorphic.

**Corollary**

Let $M$ be a torus manifold homotopy equivalent to $\prod_i \mathbb{C}P^{n_i}$, $n_i > 1$. Then $M$ is homeomorphic to $\prod_i \mathbb{C}P^{n_i}$. 
\[ \prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i} \] is rigid in the following sense:

**Theorem**

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**Corollary**

Let \( M \) be a torus manifold homotopy equivalent to \( \prod_i \mathbb{C}P^{n_i} \), \( n_i > 1 \). Then \( M \) is homeomorphic to \( \prod_i \mathbb{C}P^{n_i} \).
Thank you!