Invariant metrics of positive scalar curvature on $S^1$-manifolds

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Outline

1. Introduction
2. The case where $M^{S^1}$ has codimension two
3. The case where $\text{codim } M^{S^1} \geq 4$
Introduction

The case where $M^{S^1}$ has codimension two

The case where $\text{codim } M^{S^1} \geq 4$

Summary

Geometric meaning of scalar curvature

A basic question

Known results
Let $(M, g)$ be a Riemannian manifold.

- The scalar curvature of $M$ is a function $\text{scal} : M \to \mathbb{R}$
- For small $r > 0$ and $x \in M$ we have:

$$\text{vol}(B_r(x)) = \text{vol}_{\text{euclid}}(B_r(0))(1 - \frac{\text{scal}(x)}{6(n + 2)} r^2 + O(r^4))$$
Scalar curvature

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The case where $\text{codim } M^{S^1} \geq 4$

Summary

Geometric meaning of scalar curvature

A basic question

Known results

A basic question

Question

Assume that a compact connected Lie group $G$ acts effectively on a closed connected manifold $M$.

Does there exist an $G$-invariant metric of positive scalar curvature on $M$?
Theorem (Gromov-Lawson 1980)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and $M$ does not admit a spin-structure.
Then $M$ admits a metric of positive scalar curvature.
If $M$ is spin and admits a metric of positive scalar curvature, then

- the Dirac-operator $D$ on $M$ is invertible (Lichnerowicz 1963).
- Hence its index vanishes.
- $\text{ind } D = \hat{A}(M)$ is an invariant of the spin-bordism type of $M$ (Atiyah-Singer 1968).
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- $\text{ind } D = \hat{A}(M)$ is an invariant of the spin-bordism type of $M$ (Atiyah-Singer 1968).
Theorem (Stolz 1992)

Assume that $\pi_1(M) = 0$, $\dim M \geq 5$ and $M$ admits a spin structure.
Then $M$ admits a metric of positive scalar curvature if and only if $\alpha(M) = 0$. 
Proof.

1. If $M$ is constructed from $N$ by a surgery of codimension at least three and $N$ admits a metric of positive scalar curvature, then the same holds for $M$. (Gromov-Lawson, Schoen-Yau)

2. Hence, $M$ admits a metric of positive scalar curvature, if and only if its class in a certain bordism group can be represented by a manifold with such a metric.

3. Find all bordism classes which can be represented by such manifolds.
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3. Find all bordism classes which can be represented by such manifolds.
Non-abelian groups

Theorem (Lawson-Yau 1974)

If $G$ is non-abelian, then there is always a $G$-invariant metric of positive scalar curvature on $M$.

Therefore in the following we assume that $G = T$ is a torus or $G = S^1$. 
Non-abelian groups

Theorem (Lawson-Yau 1974)

*If G is non-abelian, then there is always a G-invariant metric of positive scalar curvature on M.*

Therefore in the following we assume that $G = T$ is a torus or $G = S^1$.
Theorem (Bérard Bergery 1983)

Assume that a torus $T$ acts freely on $M$. Then $M$ admits an invariant metric of positive scalar curvature if and only if $M/T$ admits a metric of positive scalar curvature.
Examples

- Exist manifolds which admit a non-trivial $S^1$-action but no metric of positive scalar curvature:
  - Exotic spheres with $\alpha(\Sigma) \neq 0$ (Bredon, Schultz, Joseph 1967-1981)

- Exist $S^1$-manifolds which admit metrics of positive scalar curvature but no invariant such metric:
  - Simply connected $S^1$-bundles over $K3$-surfaces (Bérard Bergery).
Examples

- ∃ manifolds which admit a non-trivial $S^1$-action but no metric of positive scalar curvature:
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First main theorem

Theorem (2013)

Let $M$ be a connected $(G \times S^1)$-manifold such that $\text{codim } M^{S^1} = 2$. Then $M$ admits a $(G \times S^1)$-invariant metric of positive scalar curvature.

Corollary

Every torus manifold admits an invariant metric of positive scalar curvature.
Theorem (2013)

Let $M$ be a connected $(G \times S^1)$-manifold such that $\text{codim } M^{S^1} = 2$. Then $M$ admits a $(G \times S^1)$-invariant metric of positive scalar curvature.

Corollary

Every torus manifold admits an invariant metric of positive scalar curvature.
Let $Z = M - N(F, M)$, where $F \subset M^{S^1}$ component with codim $F = 2$.

$\exists$ a $(G \times S^1)$-handle decomposition of $Z$ without handles of codimension zero.
The proof of the Theorem

Let \( Z = M - N(F, M) \), where \( F \subset M^{S^1} \) component with codim \( F = 2 \).

\( \exists \) a \((G \times S^1)\)-handle decomposition of \( Z \) without handles of codimension zero.
Z × D^2 is (G × S^1 × S^1)-manifold.

∃ a (G × S^1 × S^1)-handle decomposition of Z × D^2 without handles of codimension < 3.

∂(Z × D^2) = SN(F, M) × D^2 ∪ Z × S^1 admits invariant metric of positive scalar curvature

diag(S^1 × S^1) acts freely on ∂(Z × D^2) with orbit space M.
Z \times D^2 \text{ is } (G \times S^1 \times S^1)-\text{manifold.}

\exists \text{ a } (G \times S^1 \times S^1)-\text{handle decomposition of } Z \times D^2 \text{ without handles of codimension } < 3.

\partial(Z \times D^2) = SN(F, M) \times D^2 \cup Z \times S^1 \text{ admits invariant metric of positive scalar curvature}

\text{diag}(S^1 \times S^1) \text{ acts freely on } \partial(Z \times D^2) \text{ with orbit space } M.
\begin{itemize}
\item $Z \times D^2$ is $(G \times S^1 \times S^1)$-manifold.
\item $\exists$ a $(G \times S^1 \times S^1)$-handle decomposition of $Z \times D^2$ without handles of codimension $< 3$.
\item $\partial(Z \times D^2) = SN(F, M) \times D^2 \cup Z \times S^1$ admits invariant metric of positive scalar curvature
\item $\text{diag}(S^1 \times S^1)$ acts freely on $\partial(Z \times D^2)$ with orbit space $M$.
\end{itemize}
Corollary

Let $M$ be an effective $S^1$-manifold $\dim M \geq 5$.

- Assume that the principal orbits in $M$ are null-homotopic.
- If $\tilde{M}$ is spin, assume that the lifted $S^1$-action on $\tilde{M}$ is of odd type.

Then $M$ admits a non-invariant metric of positive scalar curvature.

Corollary (Ono 1991)

Let $M$ be a spin manifold with an effective $S^1$-action of odd type, then $\alpha(M) = 0$. 
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Some more corollaries

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Then $M$ admits a non-invariant metric of positive scalar curvature.

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Let $M$ be a spin manifold with an effective $S^1$-action of odd type, then $\alpha(M) = 0$. 

A related result of M. Bendersky

Theorem (Bendersky, Ochanine, Ono 1990-1992)

Let $M$ be a spin manifold with effective $S^1$-action of odd type, then the Ochanine-genus of $M$ vanishes.

- Bendersky’s paper was in final form almost exactly 25 years ago on April 2nd, 1990.
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Proof of Corollary 2

Corollary (Ono 1991)

Let $M$ be a spin manifold with an effective $S^1$-action of odd type, then $\alpha(M) = 0$.

- A neighborhood of a principal orbit in $M$ is equivariantly diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$.
- Equivariant surgery on such an orbit produces $S^1$-manifold $N$ with codim $N^{S^1} = 2$. 

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Invariant psc-metrics on $S^1$-manifolds
Proof of Corollary 2

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Introduction

The case where $M^{S^1}$ has codimension two

The case where $\text{codim } M^{S^1} \geq 4$

Summary

The first theorem

Some corollaries

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Invariant psc-metrics on $S^1$-manifolds
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- Equivariant surgery on such an orbit produces $S^1$-manifold $N$ with codim $N_{S^1} = 2$.
- If $M$ is spin and $S^1$-action on $M$ of odd type, then $N$ is spin.
Proof of Corollary 1

Corollary

Let $M$ be an effective $S^1$-manifold $\dim M \geq 5$.

- Assume that the principal orbits in $M$ are null-homotopic.
- If $\tilde{M}$ is spin, assume that the lifted $S^1$-action on $\tilde{M}$ is of odd type.

Then $M$ admits a non-invariant metric of positive scalar curvature.

- First construct $N$ as in the proof of the previous corollary.

- If principal orbits are null-homotopic, then $N \cong M \# S^2 \times S^{n-2}$.

- So by surgery on $S^2$ we can recover $M$. 
Proof of Corollary 1

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Invariant psc-metrics on $S^1$-manifolds
Obstructions to positive scalar curvature and to $S^1$-actions

**Corollary**

Let $M$ be a manifold with $\dim M \geq 5$, $\chi(M) \neq 0$ and non-spin universal covering. If $M$ does not admit a metric of positive scalar curvature then there is no non-trivial $S^1$-action on $M$.

- The only known obstruction to a metric of positive scalar curvature on a manifold as above comes from the minimal hypersurface method of Schoen and Yau (1979).
- This gives obstructions for manifolds of dimensions $n \leq 8$.
- Without using scalar curvature we can prove that there is a similar obstruction to non-trivial $S^1$-actions.
- This works in all dimensions.
Obstructions to positive scalar curvature and to $S^1$-actions

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3. The case where codim $M^{S^1} \geq 4$
The case where codim $M^{S^1} \geq 4$

In this part assume that $\pi_1(M_{\text{max}}) = 0$, codim $M^{S^1} \geq 4$ and that the action satisfies the following condition:

**Condition C**

- For all subgroups $H \subset S^1$, $N(M^H, M)$ is a $S^1$-equivariant complex vector bundle.
- For $H \subset K \subset S^1$, there is an isomorphism of $S^1$-equivariant complex vector bundles

$$N(M^K, M) \cong N(M^K, M^H) \oplus N(M^H, M)|_{M^K}.$$ 

This condition is always satisfied if no isotropy group of a point in $M$ has even order.
The case where codim $M^{S^1} \geq 4$

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Some notations

- Let $\Omega_{\geq 4,n}^{C,SO,S^1}$ the bordism group of oriented $n$-manifolds as above
- Let $\Omega_{\geq 4,n}^{C,Spin,S^1}$ the bordism group of $n$-Spin-manifolds as above
We want to prove a bordism principle for these actions. Here singular strata of codimension two in the bordisms cause some problems.
This has been dealt with essentially by Hanke (2008).
A invariant metric $g$ is called *normally symmetric in codimension two* if

- For each component $F \subset M^H$ with $\text{codim} F = 2$,
  - $\exists$ an invariant neighborhood $U$ of $F$ in $M$
  - and an $S^1$-action on $U$ which
    - has $U^{S^1} = F$
    - commutes with the original $S^1$-action and
    - leaves $g$ invariant.

- If $\text{codim} M(\mathbb{Z}_2) > 2$, then any metric $g$ can be deformed to a normally symmetric metric.
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- If codim $M(Z_2) > 2$, then any metric $g$ can be deformed to a normally symmetric metric.
The bordism principle

Theorem

If \( \dim M \geq 6 \) and \( M_{\text{max}} \) is not spin, then \( M \) admits a normally symmetric metric of positive scalar curvature if and only if its class in \( \Omega_{\geq 4,n}^{C,SO,S^1} \) can be represented by a manifold which admits such a metric.

Theorem

If \( \dim M \geq 6 \) and \( M \) is spin, then \( M \) admits a normally symmetric metric of positive scalar curvature if and only if its class in \( \Omega_{\geq 4,n}^{C,\text{Spin},S^1} \) can be represented by a manifold which admits such a metric.
The case where $M^{S^1}$ has codimension two

The case where $\text{codim } M^{S^1} \geq 4$

Summary

Some definitions
The bordism principle
The existence result

The bordism principle

Theorem

If $\dim M \geq 6$ and $M_{\max}$ is not spin,
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Theorem

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Existence results

Theorem (2015)

If \( \dim M \geq 6 \) and

- \( M_{\text{max}} \) is not spin, or
- \( M \) is spin and the \( S^1 \)-action of odd type,

then there is an \( \ell \in \mathbb{N} \) such that the equivariant connected sum of \( 2^\ell \) copies of \( M \) admits an invariant normally symmetric metric of positive scalar curvature.

- In the first case \( \ell \) can be taken to be 1.
- If the action is semi-free, \( \ell \) can be taken to be 1.
Existence results

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Existence results II

**Theorem (2015)**

If $\dim M \geq 6$, $M$ is spin and the $S^1$-action of even type, then $\hat{A}_{S^1}(M/S^1) = 0$ if and only if there is an $\ell \in \mathbb{N}$ such that the equivariant connected sum of $2^\ell$ copies of $M$ admits an invariant normally symmetric metric of positive scalar curvature.

- $\hat{A}_{S^1}(M/S^1)$ is a $\mathbb{Z}[\frac{1}{2}]$-valued equivariant bordism invariant of $M$.
- For free actions it is the $\hat{A}$-genus of the orbit space.
- For semi-free actions it was defined by Lott (2000).
Existence results II

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If $\dim M \geq 6$, $M$ is spin and the $S^1$-action of even type, then $\hat{A}_{S^1}(M/S^1) = 0$ if and only if there is an $\ell \in \mathbb{N}$ such that the equivariant connected sum of $2^\ell$ copies of $M$ admits an invariant normally symmetric metric of positive scalar curvature.

- $\hat{A}_{S^1}(M/S^1)$ is a $\mathbb{Z}[\frac{1}{2}]$-valued equivariant bordism invariant of $M$.
- For free actions it is the $\hat{A}$-genus of the orbit space.
- For semi-free actions it was defined by Lott (2000).
Corollary (Atiyah-Hirzebruch 1970)

Let $M$ be a spin-manifold with $\dim M \geq 6$ which admits a non-trivial $S^1$-action which satisfies Condition C. Then $\hat{A}(M) = 0$.

- We may assume that $\dim M = 4k$.
- Since $\hat{A}_{S^1}(M/S^1) \neq 0$ implies $\dim M = 4k + 1$, $2^\ell M$ is equivariantly spin-bordant to an $S^1$-manifold $N$ with an invariant metric of positive scalar curvature.
- Hence, $2^\ell \hat{A}(M) = \hat{A}(N) = 0 \Rightarrow \hat{A}(M) = 0$
A corollary

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Michael Wiemeler
Invariant psc-metrics on $S^1$-manifolds
Summary

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