INNER PRODUCTS AND $\mathbb{Z}/p$-ACTIONS ON POINCARÉ DUALITY SPACES

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ABSTRACT. Let $\mathbb{Z}/p$ act on an $\mathbb{F}_p$-Poincaré duality space X, where p is an odd prime number. We derive a formula that expresses the $\mathbb{F}_p$-Witt class of the fixed point set $X^{\mathbb{Z}/p}$ in terms of the $\mathbb{F}_p[\mathbb{Z}/p]$-algebra $H^*(X;\mathbb{F}_p)$, if $H^*(X;\mathbb{Z}_{(p)})$ does not contain $\mathbb{Z}/p$ as a direct summand. This extends previous work of Alexander and Hamrick, where the orientation class of X is supposed to be liftable to an integral class.

Given a prime p and a finite dimensional $\mathbb{Z}/p$-CW complex X which fulfills Poincaré duality over $\mathbb{F}_p$, a theorem of Bredon ([4]) and Chang and Skjelbred ([7]) predicts the fixed point set components of this $\mathbb{Z}/p$-action to be $\mathbb{F}_p$-Poincaré duality complexes, as well. Furthermore, the formal dimension (with $\mathbb{F}_p$-coefficients) of each fixed point component has the same parity as the formal dimension of X. It is the purpose of this paper to derive an analogue of the classical Atiyah-Singer-Segal G-signature formula in this context, if p is odd (the case $p = 2$ is easy, see Section 3). Our main result is Theorem 7 below and can be stated as follows.

Theorem. *Let the formal dimension of X be an even number $2m$ and let an $\mathbb{F}_p$-orientation ν of X be fixed. Additionally, assume that $H^*(X;\mathbb{Z}_{(p)})$ does not contain $\mathbb{Z}/p$ as a direct summand. For each component $F \subset X^{\mathbb{Z}/p}$, let $w(F,\rho_F) \in W(\mathbb{F}_p)$ denote the Witt class of the inner product on $H^{ev}(F;\mathbb{F}_p)$ induced by the cup product and the orientation ρ_F of F (which depends canonically on the orientation ν of X, cf. Theorem 3 below). Then the element*

$$\sum_{F \subset X^{\mathbb{Z}/p}} w(F,\rho_F)$$

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in the Witt ring of $\mathbb{F}_p$ is equal to the Witt class of the form $w/\operatorname{Ker}(w)$, where w is the symmetric bilinear form

$$\begin{aligned} H^m(\mathbb{Z}/p; H^m(X;\mathbb{F}_p)) \times H^m(\mathbb{Z}/p; H^m(X;\mathbb{F}_p)) &\to \\ H^{2m}(\mathbb{Z}/p; H^{2m}(X;\mathbb{F}_p)) &\to \mathbb{F}_p\,. \end{aligned}$$

This form is defined using the multiplicative structures of the group cohomology of $\mathbb{Z}/p$ and of $H^*(X;\mathbb{F}_p)$ and the orientation ν.

In particular, we do not assume that the orientation class of X can be lifted to an integral class or to a class with $\mathbb{Z}_{(p)}$-coefficients, as it is the case in analogous discussions in [1] and [12]. In this respect, the content of this paper should be viewed as complementary to these previous results.

The new feature of our approach is a careful analysis of the $\Lambda(s)$-module structure on $H^*(X_{\mathbb{Z}/p};\mathbb{F}_p)$, where $\Lambda(s)$ denotes the exterior algebra over a generator $s \in H^1(\mathbb{Z}/p;\mathbb{F}_p)$ and $X_{\mathbb{Z}/p}$ denotes the Borel construction (cf. Prop. 9). In the first part of this paper we develop a computational tool that proves to be very useful in order to carry out this analysis within the Leray-Serre spectral sequence for the Borel construction $X_{\mathbb{Z}/p}$, but might be of independent interest.

1. Connecting homomorphisms on spectral sequences

We take coefficients in a fixed commutative ring k with unit. Let $(A^*,\delta_A), (B^*,\delta_B)$ and (C^*,δ_C) be $\mathbb{Z}$-graded k-modules equipped with differentials of degree $+1$. Furthermore, we assume that each of these differential modules comes with a decreasing filtration

$$\ldots \supset \mathcal{F}_{\gamma-1}X^* \supset \mathcal{F}_\gamma X^* \supset \mathcal{F}_{\gamma+1}X^* \supset \ldots\,,$$

which is indexed over the integers and respected by the differentials, i.e. $\delta_X(\mathcal{F}_\gamma X^*) \subset \mathcal{F}_\gamma X^*$. Here X stands for either A, B or C. As usual, one can naturally associate spectral sequences $(E_*^{*,*}(X),\delta_*)$ to the filtered cochain complexes A^*, B^* and C^*. These have E_0-terms

$$E_0^{\gamma,\mu}(X) = \mathcal{F}_\gamma X^{\gamma+\mu}/\mathcal{F}_{\gamma+1}X^{\gamma+\mu}$$

and a limit term $E_\infty^{\gamma,\mu}(X)$. By definition, the last expression can be naturally identified with $\mathcal{F}_\gamma H^{\gamma+\mu}(X)/\mathcal{F}_{\gamma+1}H^{\gamma+\mu}(X)$ (using the induced filtration on $H^*(X)$), if $E_*^{*,*}(X)$ is convergent. For example, this is the case, if the filtration on X^* is exhaustive and bounded above, i.e. if $\bigcup_\gamma \mathcal{F}_\gamma X^* = X^*$ and for all n, we have $\mathcal{F}_\gamma(X^n) = 0$ for large γ (cf. [17], 5.5, where a detailed discussion of convergence properties of spectral sequences can be found). Now we additionally assume that we are given a short exact sequence

$$0 \to A^* \xrightarrow{\phi} B^* \xrightarrow{\psi} C^* \to 0 \tag{1}$$

of filtered cochain complexes with the property that for each $\gamma \in \mathbb{Z}$, the induced sequence

$$0 \to \mathcal{F}_\gamma A^* \to \mathcal{F}_\gamma B^* \to \mathcal{F}_\gamma C^* \to 0$$

is exact. In particular, we get an induced short exact sequence

$$0 \to E_0^{*,*}(A) \to E_0^{*,*}(B) \to E_0^{*,*}(C) \to 0$$

and, after applying the differential δ_0, a long exact sequence

$$\ldots \to E_1^{*,*}(A) \to E_1^{*,*}(B) \to E_1^{*,*}(C) \xrightarrow{\Gamma_1} E_1^{*,*+1}(A) \to \ldots$$

with a connecting homomorphism Γ_1 of bidegree $(0,1)$. Now, if $\Gamma_1 = 0$, we get a short exact sequence of E_1-terms and - after applying the differential δ_1 - an induced connecting homomorphism

$$\Gamma_2 : E_2^{*,*}(C) \to E_2^{*+1,*}(A)$$

of bidegree $(+1,0)$. Inductively, we can define connecting homomorphisms

$$\Gamma_r : E_r^{*,*}(C) \to E_r^{*+r-1,*-r+2}(A)$$

of bidegree $(r-1,-r+2)$ as long as

$$\Gamma_1 = \ldots = \Gamma_{r-1} = 0\,.$$

Note that the next theorem applies in particular to the first nonzero Γ_r.

Theorem 1. *The connecting homomorphism Γ_r is a map of spectral sequences, i.e. for all $\epsilon \in \{0,1,2,\ldots,\infty\}$, there are homomorphisms*

$$\Gamma_{r,\epsilon} : E_{r+\epsilon}^{*,*}(C) \to E_{r+\epsilon}^{*+r-1,*-r+2}(A)$$

with the following properties.

i. $\Gamma_{r,0} = \Gamma_r$.
ii. $\Gamma_{r,\epsilon} \circ \delta_{r+\epsilon} = -\delta_{r+\epsilon} \circ \Gamma_{r,\epsilon}$ *and - using this property -* $\Gamma_{r,\epsilon+1}$ *is induced by* $\Gamma_{r,\epsilon}$.
iii. *If $E_*(A)$ and $E_*(C)$ are convergent, then*

$$\Gamma_{r,\infty} : \mathcal{F}_\gamma H^*(C)/\mathcal{F}_{\gamma+1} H^*(C) \to \mathcal{F}_{\gamma+r-1} H^{*+1}(A)/\mathcal{F}_{\gamma+r} H^{*+1}(A)$$

is induced by the connecting homomorphism $H^*(C) \to H^{*+1}(A)$ associated to the short exact sequence (1).

Proof. We use a construction due to Eilenberg and Moore ([10], equ. (7.16)) and define a decreasing filtration on the mapping cone of the map ϕ by setting

$$\mathcal{F}_\gamma \operatorname{cone}(\phi)^n = \mathcal{F}_{\gamma+r-1}A^{n+1} \oplus \mathcal{F}_\gamma B^n \,.$$

Recall that the mapping cone is equipped with the differential

$$(a,b) \mapsto (-\delta_A(a), \delta_B(b) - \phi(a)) \,.$$

As in [10], we get for $\gamma \in \mathbb{Z}$

$$\operatorname{cone} E_{r-1}^{\gamma,*}(\phi) \cong E_{r-1}^{\gamma,*}(\operatorname{cone}(\phi))$$

and from this a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & E_r^{*,*}(B) & \longrightarrow & E_r^{*,*}(\operatorname{cone}(\phi)) & \xrightarrow{E_r(\pi)} & E_r^{*+r-1,*-r+2}(A) & \longrightarrow & \dots \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle E_r^{*,*}(\alpha)}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
\dots & \longrightarrow & E_r^{*,*}(B) & \longrightarrow & E_r^{*,*}(C) & \xrightarrow{\Gamma_r} & E_r^{*+r-1,*-r+2}(A) & \longrightarrow & \dots .
\end{array}$$

The map $\alpha : \operatorname{cone}(\phi)^* \to C^*$ is defined using the universal property of $\operatorname{cone}(\phi)^*$ and provides a factorisation

$$B^* \hookrightarrow \operatorname{cone}(\phi)^* \xrightarrow{\alpha} C^*$$

of ψ, where α is filtration preserving. The map $\pi : \operatorname{cone}(\phi)^* \to A^{*+1}$ is the projection map. This map anticommutes with the respective differentials and increases the filtration degree by $r-1$. Applying the five lemma, we see that

$$E_r^{*,*}(\alpha) : E_r^{*,*}(\operatorname{cone}(\phi)) \cong E_r^{*,*}(C) \,.$$

Hence, $E_{r+\epsilon}(\alpha)$ is an isomorphism for all $\epsilon \geq 0$. Assertions i., ii. and iii. in the theorem follow.

□

By applying the standard machinery of homological algebra, an analogue of Theorem 1 holds, if we replace A^*, B^* and C^* by chain complexes in an abelian category and work with the Grothendieck spectral sequence associated to the composition of two functors. This generalisation is mainly technical and we leave it to the interested reader.

Example 2. Let $F \hookrightarrow E \to B$ be a Serre fibration. The Leray-Serre spectral sequence (with its multiplicative structure) for this fibration can be constructed by a filtration on the cubical cochain complex $SC^*(E)$ associated to E (cf. [16]). In particular, the exact sequence of coefficients

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

induces a short exact sequence of filtered cochain complexes

$$0 \to SC^*(E;\mathbb{Z}/p) \to SC^*(E;\mathbb{Z}/p^2) \to SC^*(E;\mathbb{Z}/p) \to 0$$

in the sense explained before Theorem 1. Our considerations now imply the following fact: Let p be a prime. If the Bockstein operator

$$\beta : H^*(F;\mathbb{F}_p) \to H^{*+1}(F;\mathbb{F}_p)$$

is the zero map, we have operators

$$\Gamma_{2,\epsilon} : E_{2+\epsilon}^{\gamma,\mu}(E;\mathbb{F}_p) \to E_{2+\epsilon}^{\gamma+1,\mu}(E;\mathbb{F}_p)$$

for $\epsilon \in \{0,1,2,\ldots,\infty\}$. Note that these operators act as bigraded derivations (this follows from the definition of Γ_2). Furthermore,

$$\Gamma_{2,0} : H^\gamma(B;H^\mu(F;\mathbb{F}_p)) \to H^{\gamma+1}(B;H^\mu(F;\mathbb{F}_p))$$

is induced by the sequence of coefficients

$$0 \to H^*(F;\mathbb{Z}/p) \to H^*(F;\mathbb{Z}/p^2) \to H^*(F;\mathbb{Z}/p) \to 0\,,$$

which is exact by the assumption that $\beta = 0$. The operator $\Gamma_{2,\epsilon}$ restricted to $E_{2+\epsilon}^{*,0}(E;\mathbb{F}_p)$ is induced by the usual Bockstein operator on $H^*(B;\mathbb{F}_p)$. In this sense, $\Gamma_{2,\epsilon}$ reflects the 'Bockstein on the base' rather than the 'Bockstein on the fibre'.

2. Actions of $\mathbb{Z}/p$ on Poincaré duality spaces

In this section we shall apply the discussion of the last section to the cohomology theory of transformation groups. Let p be an odd prime number and let X be a finite dimensional $\mathbb{Z}/p$-CW complex that is an (oriented) Poincaré duality complex over $\mathbb{F}_p$. By definition, this means that $H^*(X;\mathbb{F}_p)$ is finitely generated over $\mathbb{F}_p$, there is given a natural number n, the *formal dimension* of X, and an element $\nu \in H_n(X;\mathbb{F}_p)$, the *orientation* of X, such that

$$H^i(X;\mathbb{F}_p) \times H^{n-i}(X;\mathbb{F}_p) \xrightarrow{\cup} H^n(X;\mathbb{F}_p) \xrightarrow{-\cap\nu} \mathbb{F}_p$$

is a nonsingular bilinear form for all $i \in \mathbb{Z}$. In particular, $H^{>n}(X;\mathbb{F}_p) = 0$. Let $X^{\mathbb{Z}/p}$ denote the fixed point set of the $\mathbb{Z}/p$-space X. We recall the following fundamental result.

Theorem 3 ([4, 7, 11])**.** *Each component of $X^{\mathbb{Z}/p}$ fulfills Poincaré duality over $\mathbb{F}_p$ and has formal dimension equal to $n \mod 2$. The orientation of each component of $X^{\mathbb{Z}/p}$ can be chosen to depend canonically on the orientation of X.*

We remark that the components of $X^{\mathbb{Z}/p}$ do not fulfill Poincaré duality over $\mathbb{Z}$, in general, even if X is a sphere, unless the $\mathbb{Z}/p$-action is assumed to be locally linear. Examples are provided by the converses of the P.A. Smith theorems ([13], Corollary 3.1).

Similar to the signature for topological manifolds, we can associate an invariant to an $\mathbb{F}_p$-Poincaré duality complex using a nonsingular bilinear form on its $\mathbb{F}_p$-cohomology.

Definition 4. Let Y be an $\mathbb{F}_p$-Poincaré duality complex of even formal dimension and let

$$\rho \in H_*(Y;\mathbb{F}_p) = \operatorname{Hom}(H^*(Y;\mathbb{F}_p),\mathbb{F}_p)$$

be an orientation of Y. We denote by $w(Y,\rho) \in W(\mathbb{F}_p)$ the Witt class of the nondegenerate symmetric bilinear form

$$H^{ev}(Y;\mathbb{F}_p) \times H^{ev}(Y;\mathbb{F}_p) \to \mathbb{F}_p\,, \quad (x,y) \mapsto \rho(x \cup y)\,,$$

where $H^{ev}(Y;\mathbb{F}_p) = \bigoplus_i H^{2i}(Y;\mathbb{F}_p)$.

For the definition and the properties of the Witt ring $W(k)$ associated to a commutative ring with unit k, see for example [2, 15]. For our purposes we recall

$$W(\mathbb{F}_p) = \begin{cases} \mathbb{Z}/4, \text{ if } p \equiv 3 \mod 4\,, \\ \mathbb{Z}/2[\mathbb{Z}/2], \text{ if } p \equiv 1 \mod 4\,, \end{cases}$$

as rings. Note that $w(Y,\rho) = 0$, if the formal dimension of Y is not divisible by four. If the formal dimension of Y is equal to $4m$ with $m \in \{0,1,2,\ldots\}$, then $w(Y,\rho)$ is equal to the Witt class of the induced form on $H^{2m}(Y;\mathbb{F}_p)$, because the form restricted to $\oplus_{i\neq m} H^{2i}(Y;\mathbb{F}_p)$ is split.

Theorem 3 motivates the search for a formula that relates the Witt classes of the components of $X^{\mathbb{Z}/p}$ to invariants associated to the cohomology of X. Results of this type can be regarded as analogues (for actions on Poincaré duality spaces) of the classical G-signature theorem. In [1], the authors derive such a relation, if X fulfills Poincaré duality over $\mathbb{Z}$. This assumption is weakened in [12] to X satisfying Poincaré duality over $\mathbb{Z}_{(p)}$, a property that is in fact equivalent to the orientation class of the $\mathbb{F}_p$-Poincaré duality space X being liftable to homology with coefficients in $\mathbb{Z}_{(p)}$ (see [12], Proposition 3).

Example 5. Let M be an oriented closed differentiable manifold of dimension m and let

$$f : M \to M$$

be a map of degree $\zeta \cdot p + 1$, where $\zeta \in \mathbb{Z}$ is a nonzero number. Let

$$N = M \times [0,1]/(m,0) \sim (f(m),1)$$

be the mapping torus of f. One checks that N is (homotopy equivalent to) an $\mathbb{F}_p$-Poincaré duality complex of formal dimension $m+1$. But the orientation class in $H_{m+1}(N;\mathbb{F}_p)$ cannot be lifted to a class with coefficients in $\mathbb{Z}_{(p)}$, because $H_{m+1}(N;\mathbb{Z}_{(p)}) = 0$. Hence the study of

$\mathbb{Z}/p$-actions on N (concerning the comparison of Witt classes) cannot be carried out using any of the previously known results.

In the following, we need some explicit calculations of Tate cohomology groups. Let g be a generator of $\mathbb{Z}/p$. Recall (cf. [3], Chapter 1) that for a (graded) $\mathbb{F}_p[\mathbb{Z}/p]$-module V considered to be concentrated in degree d, the Tate cohomology $\hat{H}^*(\mathbb{Z}/p;V)$ can be calculated using an explicit cochain model

$$(V \otimes_{\mathbb{F}_p} \Lambda(\sigma) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\tau,\tau^{-1}],\delta)\,,$$

where $\deg(\sigma)=1$, $\deg(\tau)=2$ and $\Lambda(\sigma)$ is the exterior algebra over $\mathbb{F}_p$ generated by σ. The differential δ is given by the formulas

$$\begin{aligned}\delta(v\otimes 1\otimes\tau^i) &= (-1)^{d+1}(1-g)v\otimes\sigma\otimes\tau^i\,,\\ \delta(v\otimes\sigma\otimes\tau^i) &= (-1)^{d}(1-g)^{p-1}v\otimes 1\otimes\tau^{i+1}\end{aligned}$$

for $i\in\mathbb{Z}$. Further, if U, W are other $\mathbb{F}_p[\mathbb{Z}/p]$-modules, considered to be concentrated in degrees c and $c+d$, respectively and if we are given an $\mathbb{F}_p[\mathbb{Z}/p]$-linear map $\eta: U\otimes V\to W$ (using the diagonal $\mathbb{Z}/p$-operation on the left hand side), then the induced pairing

$$\hat{H}^*(\mathbb{Z}/p;U)\otimes\hat{H}^*(\mathbb{Z}/p;V)\to\hat{H}^*(\mathbb{Z}/p;W)$$

has an explicit description on the cochain level, given by

$$\begin{aligned}(u\otimes 1\otimes\tau^i, v\otimes 1\otimes\tau^j) &\mapsto \eta(u\otimes v)\otimes 1\otimes\tau^{i+j}\,,\\ (u\otimes 1\otimes\tau^i, v\otimes \sigma\otimes\tau^j) &\mapsto \eta(u\otimes v)\otimes \sigma\otimes\tau^{i+j}\,,\\ (u\otimes \sigma\otimes\tau^i, v\otimes 1\otimes\tau^j) &\mapsto (-1)^d\eta(u\otimes gv)\otimes \sigma\otimes\tau^{i+j}\,,\\ (u\otimes \sigma\otimes\tau^i, v\otimes \sigma\otimes\tau^j) &\mapsto (-1)^d\sum_{0\le\lambda<\mu\le p-1}\eta(g^\lambda u\otimes g^\mu v)\otimes 1\otimes\tau^{i+j+1}.\end{aligned}$$

If the $\mathbb{Z}/p$-action on W is trivial, the last formula simplifies to

$$(u\otimes\sigma\otimes\tau^i, v\otimes\sigma\otimes\tau^j)\mapsto(-1)^{d+1}\eta(u\otimes g(1-g)^{p-2}v)\otimes 1\otimes\tau^{i+j+1}\,.$$

The next proposition is well known (cf. [8], pp. 638, ff.). But our method of proof is different from previous ones and is tightly connected with the forthcoming discussion.

Proposition 6. *Let V be a finitely generated $\mathbb{Z}/p^2[\mathbb{Z}/p]$-module that is free over $\mathbb{Z}/p^2$. Let g be a generator of $\mathbb{Z}/p$. Then there is an $\mathbb{F}_p[\mathbb{Z}/p]$-linear splitting*

$$V\otimes\mathbb{F}_p = V_1\oplus V_{p-1}\oplus V_p\,.$$

where V_i is a free $\mathbb{F}_p[\xi]/(1-\xi)^i$-module and ξ acts as multiplication by g.

Proof. By applying the structure theorem for finitely generated modules over the principal ideal domain $\mathbb{F}_p[\xi]$, we obtain an $\mathbb{F}_p[\mathbb{Z}/p]$-linear splitting

$$V \otimes \mathbb{F}_p \cong V_1 \oplus V_2 \oplus \ldots \oplus V_p \,,$$

where each V_i is free over $\mathbb{F}_p[\xi]/(1-\xi)^i$. In our situation, however, the summands V_i for $i \neq 1, p-1, p$ cannot occur: Because V is free over $\mathbb{Z}/p^2$, we have a short exact sequence of $\mathbb{Z}/p^2[\mathbb{Z}/p]$-modules

$$0 \to V \otimes \mathbb{F}_p \to V \to V \otimes \mathbb{F}_p \to 0$$

which induces a connecting homomorphism

$$\delta : \hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p) \to \hat{H}^{*+1}(\mathbb{Z}/p; V \otimes \mathbb{F}_p)$$

of Tate cohomology groups. Using this fact, $\hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p)$ can be shown to be a free $\Lambda(s)$-module, where s is a generator of $\hat{H}^1(\mathbb{Z}/p; \mathbb{F}_p)$ as follows. Let π denote the canonical projection

$$\hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p) \to \hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p)/s\hat{H}^{*-1}(\mathbb{Z}/p; V \otimes \mathbb{F}_p)$$

and let $t \in \hat{H}^2(\mathbb{Z}/p; \mathbb{F}_p)$ be the image of s under the Bockstein operator. Note that t is invertible in $\hat{H}^*(\mathbb{Z}/p; \mathbb{F}_p)$. The map

$$\begin{aligned} \phi : \hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p)/(s) \otimes \Lambda(s) &\to \hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p) \\ \pi(x) \otimes f(s) &\mapsto f(s) \cdot \frac{\delta(sx)}{t} \end{aligned}$$

is well defined, grading preserving and $\Lambda(s) \otimes \mathbb{F}_p[t, t^{-1}]$-linear. We claim that this map is an isomorphism. Suppose that

$$f(s) \cdot \frac{\delta(sx)}{t} = 0 \,.$$

It follows $x \in s \cdot \hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p)$ (using $\delta(sx) = tx - s\delta(x)$), which proves injectivity of ϕ. Now, let $x \in \hat{H}^*(\mathbb{Z}/p; V \otimes \mathbb{F}_p)$. Then

$$\pi(x) \otimes 1 + \pi(\frac{\delta(x)}{t}) \otimes s$$

is a preimage of x under ϕ which shows that ϕ is surjective.

In particular, using the above splitting of $V \otimes \mathbb{F}_p$, each $\hat{H}^*(\mathbb{Z}/p; V_i)$ has to be a projective $\Lambda(s)$-module, hence a free $\Lambda(s)$-module, because $\Lambda(s)$ is a local ring with unique maximal ideal (s).

But now one checks by a direct calculation, using the formulas written down previously, that for $i \neq 1, p-1, p$, multiplication by s is the zero map on $\hat{H}^*(\mathbb{Z}/p; V_i)$ and $\hat{H}^*(\mathbb{Z}/p; V_i) \neq 0$. Note that on the cochain level, (left) multiplication by $s \in \hat{H}^1(\mathbb{Z}/p;\mathbb{F}_p)$ on $\hat{H}^*(\mathbb{F}_p;V_i)$ is (up to sign) given by the maps

$$v \otimes 1 \otimes \tau^i \mapsto \xi v \otimes \sigma \otimes \tau^i\,, \qquad v \otimes \sigma \otimes \tau^i \mapsto \xi(1-\xi)^{p-2} v \otimes 1 \otimes \tau^{i+1}\,.$$

This concludes the proof of the proposition. □

Using the notation of the last proposition, suppose that on $V \otimes \mathbb{F}_p$, we are given a nonsingular $(-1)^\epsilon$-symmetric bilinear form γ, where $\epsilon = \pm 1$. It follows from [14], Theorem 3.2., that the direct sum decomposition

$$V \otimes \mathbb{F}_p = V_1 \oplus V_2 \oplus \ldots \oplus V_p$$

can be assumed to be orthogonal with respect to γ. We denote the nonsingular bilinear form induced on V_i by γ_i.

We will now suppose additionally to the standing assumptions that the formal dimension of X is an even number $2m$ and that the Bockstein operator $\beta : H^*(X;\mathbb{F}_p) \to H^{*+1}(X;\mathbb{F}_p)$ is the zero map. This last requirement is equivalent to saying that $H^*(X;\mathbb{Z}_{(p)})$ does not contain $\mathbb{Z}/p$ as a direct summand. For simplicity, we also assume that X is connected. In particular, $H^{2m}(X;\mathbb{F}_p) \cong \mathbb{F}_p$ with the trivial $\mathbb{Z}/p$-operation. Furthermore, we fix an orientation ν of X. Let g be a generator of $\mathbb{Z}/p$ as before. Set $V^* = H^*(X;\mathbb{Z}/p^2)$. By our assumption on β, this is a finitely generated $\mathbb{Z}/p^2[\mathbb{Z}/p]$-module that is free over $\mathbb{Z}/p^2$. Further, $V^m \otimes \mathbb{F}_p$ (that is concentrated in degree m), comes equipped with a nondegenerate bilinear form γ that is symmetric if m is even and antisymmetric, if m is odd. This form is invariant under the induced $\mathbb{Z}/p$-action on V. We define $\overline{w}(X,\nu) \in W(\mathbb{F}_p)$ as the Witt class of the nonsingular symmetric bilinear form

$$\hat{H}^m(\mathbb{Z}/p;V_i^m) \times \hat{H}^m(\mathbb{Z}/p;V_i^m) \to \hat{H}^{2m}(\mathbb{Z}/p;\mathbb{F}_p) \cong \mathbb{F}_p\,,$$

where we choose $i = 1$, if m is even, and $i = p-1$, if m is odd, and use the induced bilinear form γ_i on V_i. That these forms are nonsingular can be checked using the above explicit formulas for the product structure of the Tate cohomology groups.

Now we can state our main result.

Theorem 7. *In $W(\mathbb{F}_p)$ the following equation is valid with the canonical orientation ρ_F of each component F in $X^{\mathbb{Z}/p}$:*

$$\overline{w}(X,\nu) = \sum_{F \subset X^{\mathbb{Z}/p}} w(F,\rho_F)\,.$$

Remark 8. The left hand side of the last equation coincides with the Witt class of the form $w/\operatorname{Ker} w$ mentioned in the introduction for the following reason. The kernel of the symmetric bilinear form

$$\begin{aligned} \hat{H}^m(\mathbb{Z}/p; H^m(X;\mathbb{F}_p)) \times \hat{H}^m(\mathbb{Z}/p; H^m(X;\mathbb{F}_p)) &\to \\ \hat{H}^{2m}(\mathbb{Z}/p; H^{2m}(X;\mathbb{F}_p)) &\to \mathbb{F}_p \end{aligned}$$

coincides with $\hat{H}^m(\mathbb{Z}/p; V^m_{p-1})$, if m is even, and with $\hat{H}^m(\mathbb{Z}/p; V^m_1)$, if m is odd, because $\hat{H}^m(\mathbb{Z}/p; V^m_i) = s \cdot \hat{H}^{m-1}(\mathbb{Z}/p; V^m_i)$ in the respective cases. Furthermore, $\hat{H}^*(\mathbb{Z}/p; V^m_p) = 0$.

The proof of Theorem 7 proceeds in several steps. We write

$$H^*(\mathbb{Z}/p;\mathbb{F}_p) \cong \Lambda(s) \otimes_{\mathbb{F}_p} \mathbb{F}_p[t]$$

were s and t carry gradings 1 and 2, respectively, and t is the image of s under the Bockstein map. We will consider the Leray-Serre spectral sequence $E^{*,*}_*(X)$ with coefficients $\mathbb{F}_p$ associated with the Borel fibration

$$X \hookrightarrow X_{\mathbb{Z}/p} \to B\mathbb{Z}/p\,,$$

where $X_{\mathbb{Z}/p} = X \times_{\mathbb{Z}/p} E\mathbb{Z}/p$. Note that all $E^{*,*}_r(X)$ with $r \geq 2$ are (bigraded) modules over $H^*(\mathbb{Z}/p;\mathbb{F}_p)$, hence the statement in the following proposition makes sense. Recall that we assume throughout that the Bockstein operator acts trivially on $H^*(X;\mathbb{F}_p)$.

Proposition 9. *For all $r \geq 2$, the localized terms $E^{*,*}_r(X)[t^{-1}]$ are finitely generated free graded $\Lambda(s) \otimes \mathbb{F}_p[t,t^{-1}]$-modules.*

Proof. The proof is similar to the proof of Proposition 6, but uses the operators $\Gamma_{2,\epsilon}$ from Example 2 in the first section. These are $\mathbb{F}_p[t]$-linear, as $\beta(t) = 0$ in $H^*(\mathbb{Z}/p;\mathbb{F}_p)$. Therefore, we get induced operators on $E_{2+\epsilon}(X)[t^{-1}]$ that we denote by the same symbols $\Gamma_{2,\epsilon}$. Let π denote the canonical projection

$$E_r(X)[t^{-1}] \to E_r(X)[t^{-1}] \;/\; (s)\,.$$

It is clear that the map

$$\begin{aligned} \phi : (E_r(X)[t^{-1}]/(s))^{*,*} \otimes \Lambda(s) &\to E^{*,*}_r(X)[t^{-1}] \\ \pi(x) \otimes f(s) &\mapsto f(s)\frac{\Gamma_{2,r-2}(sx)}{t} \end{aligned}$$

is well defined, grading preserving and $\Lambda(s) \otimes \mathbb{F}_p[t,t^{-1}]$-linear. In an analogous fashion as in the proof of Proposition 6, one proves that ϕ is an isomorphism. Because $E_r(X)[t^{-1}]/(s)$ is free over the graded field $\mathbb{F}_p[t,t^{-1}]$, the assertion follows. □

In [3], Remark 5.2.4, it is claimed that the conclusion of Proposition 9 above holds without the additional assumption on the Bockstein operator on $H^*(X;\mathbb{F}_p)$. This is not correct, in general. Let $p \geq 5$. We can construct a smooth closed manifold Y with a smooth $\mathbb{Z}/p$-action such that $H^1(Y;\mathbb{F}_p)$ as an $\mathbb{F}_p[\mathbb{Z}/p]$-module is isomorphic to V_2 (compare the proof of Proposition 6). The E_2-term $E_2(Y_{\mathbb{Z}/p})[t^{-1}]$ of the localized spectral sequence (with $\mathbb{F}_p$-coefficients) for the Borel construction is not free over $\Lambda(s)$.

For abbreviation, we set

$$\overline{E}_r^{*,*}(X) = (E_r(X)[t^{-1}]/(s))^{*,*}$$

and denote the induced differentials by $\overline{\delta}_r$.

Note that the orientation $\nu : H^*(X;\mathbb{F}_p) \to \mathbb{F}_p$ induces $\mathbb{F}_p[t,t^{-1}]$-linear (not necessarily surjective) homomorphisms ('orientations')

$$\mathcal{O}_r : \overline{E}_r^{*,*}(X) \to \mathbb{F}_p[t,t^{-1}]$$

for $r \in \{1,2,3,\ldots,\infty\}$ that lower the bidegree by $(0,2m)$ and satisfy

$$\mathcal{O}_r(\overline{E}_r^{\gamma,\mu}(X)) = 0\,,$$

if γ is odd or if $\mu \neq 2m$. We will prove the following assertion by induction on r, where the superscript ev denotes restriction to even total degree as before.

Proposition 10. *The forms* $(-,-)_r : \overline{E}_r^{ev}(X) \times \overline{E}_r^{ev}(X) \to \mathbb{F}_p[t,t^{-1}]$ *defined by*

$$(x,y) \mapsto \mathcal{O}_r(x \cdot y)$$

are nonsingular and Witt equivalent (in $W(\mathbb{F}_p[t,t^{-1}])$*) for all* $r \geq 2$.

Proof. First, we prove that the form $(-,-)_2$ is nonsingular. Using the decomposition

$$H^*(X;\mathbb{F}_p) = V_1^* \oplus V_{p-1}^* \oplus V_p^*$$

(cf. Proposition 6), one gets isomorphisms

$$\overline{E}_2^{\gamma,\mu}(X) \cong \begin{cases} \hat{H}^\gamma(\mathbb{Z}/p; V_1^\mu), & \text{for } \gamma \text{ even} \;, \\ \hat{H}^\gamma(\mathbb{Z}/p; V_{p-1}^\mu), & \text{for } \gamma \text{ odd} \;. \end{cases}$$

(This is usual dimension shifting for the $\mathbb{F}_p[\mathbb{Z}/p]$-module V_{p-1}^*). Using this description, nonsingularity of $(-,-)_2$ is checked using the same calculation as carried out for the definition of $\overline{w}(X,\nu)$ above.

Now let $r \geq 2$. We will prove that

$$\overline{\delta}_r(\overline{E}_r(X)) = (\operatorname{Ker}\overline{\delta}_r)^\perp$$

with respect to $(-,-)_r$ and that there is a canonical isomorphism

$$\operatorname{Ker}(\overline{\delta}_r)/\operatorname{Im}(\overline{\delta}_r) \cong \overline{E}_{r+1}(X)\,.$$

These two statements complete the induction step (after restriction to elements of even total degree), cf. [2], Lemma 1.3. The left hand side in the first statement is contained in the right hand side, because the differential $\overline{\delta}_r$ is a derivation with respect to the multiplication on $\overline{E}_r(X)$ and because $\mathcal{O}_r(\overline{E}_r^{*,<2m}(X)) = 0$. The full equality now holds, because both sides in this equation have the same dimension over $\mathbb{F}_p[t, t^{-1}]$.

The second statement is shown as follows. The isomorphism ϕ in the proof of Proposition 9 is an isomorphism of differential algebras, if we use the differential $\overline{\delta}_r \otimes \mathrm{id}$ on the left and the differential δ_r which is induced by the differential on $E_r(X)$ on the right. Consequently, ϕ induces $\Lambda(s)$-linear isomorphisms $\mathrm{Ker}\,\overline{\delta}_r \otimes \Lambda(s) \cong \mathrm{Ker}\,\delta_r$, $\mathrm{Im}\,\overline{\delta}_r \otimes \Lambda(s) \cong \mathrm{Im}\,\delta_r$ and therefore a $\Lambda(s)$-linear isomorphism $(\mathrm{Ker}\,\overline{\delta}_r / \mathrm{Im}\,\overline{\delta}_r) \otimes \Lambda(s) \cong E_{r+1}(X)[t^{-1}]$. From this, the assertion is immediate.

□

We now recall the localization theorem (cf. [3, 9]) that in our case states that the inclusion $X^{\mathbb{Z}/p} \to X$ induces an isomorphism of graded $\mathbb{F}_p[t, t^{-1}]$-algebras

$$H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]/(s) \cong H^*(X^{\mathbb{Z}/p}; \mathbb{F}_p) \otimes \mathbb{F}_p[t, t^{-1}] .$$

As before, the grading superscript $*$ on the left hand side takes into account that the ideal generated by s is homogenous. By construction of the Leray-Serre spectral sequence, we have an induced filtration $\mathcal{F}_*$ on $H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)$ that in turn induces a filtration $\mathcal{F}_*$ on $H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]$ by declaring

$$x \in \mathcal{F}_\gamma(H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]) \Leftrightarrow t^c \cdot x \in \mathcal{F}_{\gamma+2c}(H^{*+2c}(X_{\mathbb{Z}/p}; \mathbb{F}_p)) \text{ for } c \gg 0 .$$

This makes sense, as multiplication with t induces isomorphisms

$$\mathcal{F}_\gamma(H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)) \cong \mathcal{F}_{\gamma+2}(H^{*+2}(X_{\mathbb{Z}/p}; \mathbb{F}_p)),$$

if $\gamma \geq 2m + 1$.

Finally, we get an induced filtration $\mathcal{F}_*$ on $H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]/(s)$ by setting $\mathcal{F}_\gamma(H^i(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]/(s))$ equal to

$$\mathcal{F}_\gamma(H^i(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]) \;/\; s \cdot \mathcal{F}_{\gamma-1}(H^{i-1}(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]) .$$

The following fact follows immediately.

Lemma 11. *There are canonical graded isomorphisms*

$$\mathcal{F}_\gamma(H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]/(s))/\mathcal{F}_{\gamma+1}(H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]/(s)) \cong \overline{E}_\infty^{\gamma,*-\gamma}(X) .$$

The map $\mathcal{O}_\infty : \overline{E}_\infty^{*,*}(X) \to \mathbb{F}_p[t, t^{-1}]$ constructed earlier induces a linear map

$$\mathcal{O} : H^*(X_{\mathbb{Z}/p}; \mathbb{F}_p)[t^{-1}]/(s) \to \overline{E}_\infty^{*-2m,2m}(X) \stackrel{\mathcal{O}_\infty}{\to} \mathbb{F}_p[t, t^{-1}] ,$$

where the first map is the canonical projection. Using this map, we obtain a bilinear form

$$H^*(X_{\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]/(s) \times H^*(X_{\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]/(s) \to \mathbb{F}_p[t,t^{-1}]$$

that (restricted to elements of even degree) is nonsingular, using Proposition 10. We claim that this pairing (restricted to elements of even degree) is Witt equivalent to the previously defined pairing on $\overline{E}^{ev}_\infty(X)$ restricted to $\oplus_{i\in\mathbb{Z}}\overline{E}^{2i+m,m}_\infty(X)$. Observe that

$$\bigoplus_i \mathcal{F}_{2i+m+1}(H^{2i+2m}(H^*(X_{\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]/(s)) = \bigoplus_i (\mathcal{F}_{2i+m}(H^{2i+2m}(H^*(X_{\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]/(s)))^\perp\,,$$

cf. Lemma 11. Hence, the form in question is Witt equivalent to the form induced on the quotient of these two modules which is indeed isomorphic to $\oplus_i\overline{E}^{2i+m,m}_\infty(X)$. Now let $F\subset X^{\mathbb{Z}/p}$ be a component of formal dimension n_F (which is automatically even). Using the argument in [4], one can show that the map

$$\begin{aligned} H^{n_F}(F;\mathbb{F}_p)\otimes\mathbb{F}_p[t,t^{-1}] &\subset H^*(X^{\mathbb{Z}/p};\mathbb{F}_p)\otimes\mathbb{F}_p[t,t^{-1}] \\ &\cong H^*(X_{\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]/(s) \xrightarrow{\mathcal{O}} \mathbb{F}_p[t,t^{-1}] \end{aligned}$$

is not the zero map, hence an isomorphism. After evaluating at $t=1$, this is exactly the orientation that we referred to in the statement of Theorem 3. Using these maps, we obtain a nonsingular symmetric bilinear form on

$$\bigoplus_{F\subset X^{\mathbb{Z}/p}} H^{ev}(F;\mathbb{F}_p)\otimes\mathbb{F}_p[t,t^{-1}] \cong H^{ev}(X^{\mathbb{Z}/p};\mathbb{F}_p)\otimes\mathbb{F}_p[t,t^{-1}]\,.$$

Using Proposition 10, we finally get the following sequence of equations in $W(\mathbb{F}_p[t,t^{-1}])$, where we only write down the representing module of an element in $W(\mathbb{F}_p[t,t^{-1}])$.

$$\begin{aligned} \bigoplus_i \overline{E}^{2i+m,m}_2(X) &= \overline{E}^{ev}_2(X) = \overline{E}^{ev}_\infty(X) = \\ = \bigoplus_i \overline{E}^{2i+m,m}_\infty(X) &= H^{ev}(X_{\mathbb{Z}/p};\mathbb{F}_p)[t^{-1}]/(s) = \\ = H^{ev}(X^{\mathbb{Z}/p};\mathbb{F}_p)\otimes\mathbb{F}_p[t,t^{-1}] &= \bigoplus_{F\subset X^{\mathbb{Z}/p}} H^{ev}(F;\mathbb{F}_p)\otimes\mathbb{F}_p[t,t^{-1}]\,. \end{aligned}$$

Theorem 7 now follows from evaluating this equation at $t=1$.

3. Further remarks

The invariant $\overline{w}(X,\nu)$ on the left hand side of the equation in Theorem 7 can be interpreted as a torsion linking form on the group cohomology of $\mathbb{Z}/p$ in the following way. We have a Bockstein operator

$$\delta : \hat{H}^{m-1}(\mathbb{Z}/p; H^m(X;\mathbb{F}_p)) \to \hat{H}^m(\mathbb{Z}/p; H^m(X;\mathbb{F}_p))$$

as before, because $H^m(X;\mathbb{Z}/p^2)$ is a free $\mathbb{Z}/p^2$-module. Let $T = \operatorname{Im}\delta$ and define a bilinear form $\gamma : T \times T \to \mathbb{F}_p$ as follows:

$$\gamma(\delta(x),\delta(y)) = x \cup \delta(y) \in \hat{H}^{2m-1}(\mathbb{Z}/p; H^{2m}(X;\mathbb{F}_p)) \cong \mathbb{F}_p \,.$$

As the Bockstein operator acts trivially on $\hat{H}^{2m-2}(\mathbb{Z}/p;\mathbb{F}_p)$, this bilinear form is well defined and symmetric (note the derivation property of δ). Furthermore, it can easily be checked that γ is nonsingular. Using the fact that $\operatorname{Im}\delta = \hat{H}^m(\mathbb{Z}/p; V_1^m)$, if m is even, and $\operatorname{Im}\delta = \hat{H}^m(\mathbb{Z}/p; V_{p-1}^m)$, if m is odd, (in the notation introduced before Theorem 7), it follows that the Witt class of γ coincides with $\overline{w}(X,\nu)$.

If the orientation class ν of X lifts to a class in $H^{2m}(X;\mathbb{Z}_{(p)})$, then X is a $\mathbb{Z}_{(p)}$-Poincaré duality space (see [12], Proposition 3) and the right hand side of Theorem 7 can be expressed as the Witt class of the evident nonsingular symmetric bilinear form

$$\hat{H}^m(\mathbb{Z}/p; H^m(X;\mathbb{Z}_{(p)})/\operatorname{Tor}) \times \hat{H}^m(\mathbb{Z}/p; H^m(X;\mathbb{Z}_{(p)})/\operatorname{Tor}) \to \mathbb{F}_p \,,$$

cf. [1] and [12] (for this, no further assumption on the Bockstein of $H^*(X;\mathbb{F}_p)$ is needed). If, in addition, the Bockstein operator on $H^*(X;\mathbb{F}_p)$ is trivial, it follows from our discussion that the Witt class of the above form is equal to $\overline{w}(X,\nu)$. We sketch a direct proof of this fact, if the induced action of $\mathbb{Z}/p$ on $H^m(X;\mathbb{F}_p)$ is trivial (which implies that the induced action on $H^m(X;\mathbb{Z}_{(p)})$ is also trivial). It is enough to restrict attention to even m. Then the form $\overline{w}(X,\nu)$ is represented by the usual inner product on $V = H^m(X;\mathbb{F}_p)$. Let

$$r : H^m(X;\mathbb{Z}_{(p)}) \to H^m(X;\mathbb{F}_p)$$

be the reduction of coefficients and let $W = \operatorname{Im} r \subset H^m(X;\mathbb{F}_p)$. Because X is a $\mathbb{Z}_{(p)}$-Poincaré duality space, $W^\perp = r(\operatorname{Tor} H^m(X;\mathbb{Z}_{(p)}))$ with respect to the inner product on V. Hence, the inner product spaces V and $W/W^\perp \cong (H^m(X;\mathbb{Z}_{(p)})/\operatorname{Tor}) \otimes \mathbb{F}_p$ are Witt equivalent. The latter is isomorphic to $\hat{H}^m(\mathbb{Z}/p; H^m(X;\mathbb{Z}_{(p)})/\operatorname{Tor})$.

For $p=2$ a comparison result for Witt classes can be proven without any assumption on the Bockstein operator. The proof is easier than the one discussed in this paper due to the simple structure of $W(\mathbb{Z}/2) \cong \mathbb{Z}/2$ and due to the fact, that the difference of the total Betti numbers (with $\mathbb{F}_p$-coefficients) of a $\mathbb{Z}/p$-space and its fixed point set is divisible by two.

A class of spaces that do not fulfill Poincaré duality over $\mathbb{Z}_{(p)}$, but for which Theorem 7 can be applied, are finite dimensional $\mathbb{Z}/p$-CW complexes X that are (nonequivariantly) homotopy equivalent to mapping tori N as described in example 5, if $p|\zeta$ and $\beta(H^*(M;\mathbb{F}_p)) = 0$. For example, the union of the $4k$-dimensional components of the fixed set of such an action cannot consist of exactly one point, if $p \geq \dim H^*(X;\mathbb{F}_p)$ and $\dim X \equiv 2 \mod 4$, as $\overline{w}(X,\nu) = 0$ in this case (cf. Prop. 6). Here, ν denotes the orientation of the complex X. More generally, let N be an oriented closed smooth manifold of dimension $m+1$ and let $h \in H_m(N;\mathbb{Z})$ be a nonzero homology class. It is well known that this class can be represented by an embedded oriented submanifold $M^m \hookrightarrow N$. The space $N \setminus \tilde{M}$, where $\tilde{M}$ is a tubular neighbourhood of M in N, is an oriented manifold whose boundary is oriented diffeomorphic to the disjoint union $M \cup \overline{M}$, where $\overline{M}$ is M with its orientation reversed. We identify these boundary components by a map $\overline{M} \to M$ of degree $-(\zeta \cdot p + 1)$, where $p|\zeta$, to get an $\mathbb{F}_p$-Poincaré duality complex X. If the Bockstein operator on $H^*(N;\mathbb{F}_p)$ is the zero map, any finite dimensional $\mathbb{Z}/p$-CW complex that is (nonequivariantly) homotopy equivalent to X can be studied using Theorem 7.

It is still an open problem, if there is a G-signature formula for $\mathbb{Z}/p$-actions on $\mathbb{F}_p$-Poincaré duality complexes X without any further restrictions on X. More specifically one might ask, if the theorem in the introduction holds in general, if $H^*(X;\mathbb{F}_p)$ does contain a direct summand $\mathbb{Z}/p$. This generalization seems particularly plausible, if the induced $\mathbb{Z}/p$-operation on $H^*(X;\mathbb{F}_p)$ is trivial.

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