POSITIVE SCALAR CURVATURE WITH SYMMETRY

BERNHARD HANKE

ABSTRACT. We show an equivariant bordism principle for constructing metrics of positive scalar curvature that are invariant under a given group action. Furthermore, we develop a new codimension-2 surgery technique which removes singular strata from fixed point free S^1-manifolds while preserving equivariant positive scalar curvature. These results are applied to derive the following theorem: Each closed fixed point free S^1-manifold of dimension at least 6 whose isotropy groups have odd order and whose union of maximal orbits is simply connected and not spin, carries an S^1-invariant metric of positive scalar curvature.

CONTENTS

1. INTRODUCTION

A classical theme in differential geometry is the investigation of topological conditions that are necessary or sufficient for the existence of a particular kind of geometric structure on a given smooth manifold. In the context of Riemannian metrics of positive scalar curvature, this question has revealed a strong link between subtle differential topological invariants of smooth manifolds and their geometry. A prominent role in this context is played by an effective method for constructing metrics of positive scalar curvature described in the seminal work by Gromov-Lawson [7] and Schoen-Yau [26]: The class of smooth manifolds admitting metrics of positive scalar curvature is closed under surgery in codimension at least three. Based on this principle, the existence question for positive scalar curvature metrics can be translated into a bordism problem that is then discussed with the help of powerful algebraic-topological means. The effectiveness of this approach is illustrated by the following result, which provides a complete classification of simply connected closed manifolds of dimension at least 5 that admit metrics of positive scalar curvature.

Theorem. *Let M be a closed simply connected manifold of dimension at least 5.*

i.) (Gromov-Lawson [7]) *If M does not admit a spin structure, then M carries a metric of positive scalar curvature.*

ii.) (Lichnerowicz [14], Hitchin [8], Stolz [27]) *If M admits a spin structure, then M carries a metric of positive scalar curvature if and only if $\alpha(M) = 0$.*

Date: October 1, 2009; MSC 2000: primary 53C23, 53C25, 57R85; secondary 57R65.

Here $\alpha(M) \in \mathrm{KO}_n$ is an invariant (defined in [8]) closely related to $\hat{A}(M)$. For non-simply connected manifolds, there are refined versions of this obstruction with values in the K-theory of certain C^*-algebras associated to the fundamental group of M. A complete classification of manifolds admitting metrics of positive scalar curvature is not known, even in the case of finite fundamental groups.

We will investigate the positive scalar curvature question in an equivariant context: Given a closed smooth manifold M equipped with a smooth action of a compact Lie group G, does M admit a positive scalar curvature metric which is invariant under the G-action? Even if M admits a (nonequivariant) metric of positive scalar curvature, this may be a nontrivial problem: Bérard Bergery shows in [2], Example 9.1., that averaging a positive scalar curvature metric over a group of symmetries may destroy the positivity of the scalar curvature.

If G is finite and the action on M is free, this problem is equivalent to asking whether M/G, a closed manifold whose fundamental group is a certain extension of G (assuming that M is connected), admits a metric of positive scalar curvature - and this is a nonequivariant question. Another extreme case is that of (not necessarily free) actions of compact Lie groups whose identity components are non-abelian. This problem is completely settled by the following result.

Theorem. (Lawson-Yau [13]) *If a compact connected manifold M is equipped with an effective action of a compact Lie group G whose identity component is non-abelian, then it admits a Riemannian metric of positive scalar curvature which is invariant under the given G-action.*

In its original form, this theorem only states that M carries a Riemannian metric of positive scalar curvature, but one easily checks that the construction in *loc. cit.* yields a metric which is in fact also G-invariant.

The proofs of the Lawson-Yau theorem and of the Gromov-Lawson and Stolz theorems differ in an essential way: The first one yields an explicit metric of positive scalar curvature. In contrast, the proofs of the latter are based on structure results for the oriented and spin bordism rings which rely on homotopy theoretic considerations. In particular, they do not provide a direct description of the positive scalar curvature metrics in question.

Important aspects of the equivariant positive scalar curvature problem were discussed by Bérard Bergery in [2]. In this paper the Kazdan-Warner trichotomy, the Yamabe problem, the index obstruction to positive scalar curvature and the surgery principle for constructing metrics of positive scalar curvature are formulated in an equivariant context.

Further elaborations of the index theoretic obstruction in the case of S^1-manifolds are carried out by Lott in [15]. Rosenberg-Weinberger [25] provide an ad-hoc discussion of an equivariant bordism principle for constructing metrics of positive scalar curvature on simply connected manifolds which are invariant under spin preserving $\mathbb{Z}/p$-actions, see Theorem 2.3. in *loc. cit*. However, the proof contained in this paper requires more assumptions than stated in the theorem (see the discussion following Corollary 16 below). Relying on this (potentially problematic) result, Farsi [6] has discussed some instances of the equivariant positive scalar curvature problem on spin manifolds of low dimension equipped with $\mathbb{Z}/p$-actions.

Our work is devoted to a systematic exploration of the surgery and bordism techniques for constructing equivariant positive scalar curvature metrics. Part of the nonequivariant discussion can be translated more or less directly to the equivariant context. This applies in particular to the surgery principle of Gromov-Lawson and Schoen-Yau. The paper [2] formulates an equivariant analogue of this fundamental result (see [2], Theorem 11.1) without proof. In Section 2 of our work, we will recapitulate the essential steps of the argument in [7] and will explain how they translate to an equivariant setting.

The following Section 3 is devoted to a proof of the first main result, a general bordism principle for constructing equivariant positive scalar curvature metrics.

Theorem A. *Let Z be a compact connected oriented G-bordism between the closed G-manifolds X and Y. Assume the following:*

- i.) *The cohomogeneity of Z is at least 6,*
- ii.) *the inclusion of maximal orbits $Y_{max} \hookrightarrow Z_{max}$ is a nonequivariant 2-equivalence (i.e. a bijection on π_0, an isomorphism on π_1 and a surjection on π_2),*
- iii.) *each singular stratum of codimension 2 in Z meets Y.*

Then, if X admits a G-invariant metric of positive scalar curvature, the same is true for Y.

This statement makes clear that the bordism principle for constructing equivariant metrics of positive scalar curvature metrics does definitely not require a strong gap hypothesis as envisaged in [25], Remark 2.4, where the authors assume that for any two closed subgroups $H, K \subset G$ with $K \subset H$, the codimension of each component of $Z_{(H)}$ contained in the closure of $Z_{(K)}$ is either 0 or at least 3 in $Z_{(K)}$ (here $Z_{(H)}$ denotes the set of points in Z whose isotropy groups are conjugate to H).

Our Theorem A is almost a direct analogue of the corresponding nonequivariant result (see [28], Theorem 3.3). In particular the dimension restriction i.) and the connectivity restriction for the inclusion $Y_{max} \hookrightarrow Z_{max}$ stated in point ii.) translate to analogous requirements in the nonequivariant setting if $G = \{1\}$. However, if G is not trivial, we need an additional assumption on codimension-2 singular strata. The plausibility of such an assumption is illustrated in Proposition 17: Any equivariant handle decomposition of $\mathbb{Z}/p$-bordisms Z with codimension-2 fixed point components disjoint from Y contain handles of codimension 0 or 2 (independent of the connectivity of the map $Y_{max} \hookrightarrow Z_{max}$). This points towards a fundamental limitation of the method of equivariant handle decompositions to construct equivariant metrics of positive scalar curvature.

Theorem A is useful for constructing equivariant metrics of positive scalar curvature only if it can be combined with powerful structure results for geometric equivariant bordism groups, which imply that the manifold X in Theorem A can be assumed to admit an equivariant positive scalar curvature metric under some general assumptions on the manifold Y. Two main difficulties occur at this point. Firstly, explicit geometric generators of equivariant bordism groups are known only in a very limited number of cases. Secondly, whereas conditions i.) and ii.) in Theorem A can be achieved under fairly general assumptions on the manifold Y (by performing appropriate surgeries on Z_{max} - cf. the second proof of Proposition 33), it is a priori not clear under what circumstances condition iii.) holds.

In sections 4 and 5, we shall present a way to avoid the difficulties inherent in condition iii.) if $G = S^1$ and the G-action on Z is fixed point free. The idea we use is to alter a given bordism Z by cutting out equivariant tubes connecting Y with each of the codimension-2 singular strata in Z that are disjoint from Y. This replaces the bordism Z and the manifold Y by other manifolds Z' and Y' so that each codimension-2 singular stratum in Z' meets Y'. In particular, Theorem A can be applied to Z' (after some more manipulations of Z', but we omit these details here). We must now understand how Y can be recovered from Y'. A closer inspection of the situation shows that Y' is obtained from Y by adding certain codimension-2 singular strata with finite isotropies. Conversely, Y can be reconstructed from Y' by a Dehn-like codimension-2 surgery process that removes these additional singular strata and puts back free ones instead. The examination of this surgery step is the content of Section 4. In Theorem 25 of this section we show by a somewhat involved geometric argument that this surgery step preserves the existence of S^1-invariant positive scalar curvature metrics under fairly general assumptions. Roughly speaking, we replace the "bending outwards" process in the surgery step due to Gromov-Lawson and Schoen-Yau by a "bending inwards" process. We remark that this kind of positive scalar curvature preserving codimension-2 surgery only works under the additional S^1-symmetry.

Arguing in this rather roundabout manner, assumption iii.) of Theorem A is no longer a true obstacle against the construction of equivariant positive scalar curvature metrics on fixed point free S^1-manifolds. In combination with a classical theorem of Ossa [22], which states that fixed point free oriented S^1-manifolds satisfying condition $\mathbb{C}$ (cf. Definition 18) are oriented S^1-boundaries, we get the following equivariant version of the Gromov-Lawson theorem stated above.

Theorem B. *Let M be a closed fixed point free S^1-manifold satisfying condition $\mathbb{C}$ and of cohomogeneity at least 5. If the union of maximal orbits of M is simply connected and does not admit a spin structure, then M admits an S^1-invariant metric of positive scalar curvature.*

By Lemma 19, the manifold M satisfies condition $\mathbb{C}$, if all isotropy groups have odd order. We remark that no additional assumption on codimension-2 singular strata in M is needed in Theorem B. It is not clear at present to what extent Ossa's theorem can be generalized to the spin case, so that we will not discuss the corresponding S^1-equivariant analogue of Stolz' theorem in this paper.

This paper forms part of my habilitation thesis at the Ludwig-Maximilians-Universität München. I am grateful to D. Kotschick for constant help and encouragement, to L. Bérard Bergery for sending me a copy of [2] and to J. Rosenberg, T. Schick and S. Stolz for useful comments. The work on this paper was supported by a research grant within the DFG emphasis program "Global Differential Geometry".

2. Equivariant surgery theorem

Definition 1. Let G be a compact Lie group. An *equivariant G-handle* is a G-space of the form

$$G \times_H (D(V) \times D(W))$$

where $H \subset G$ is a closed subgroup and $D(V)$ and $D(W)$ are unit discs in some orthogonal H-representations V and W. We call $\dim V$ the *dimension* and $\dim W$ the *codimension* of the G-handle in question.

If Z is a smooth G-manifold with boundary and

$$\phi : G \times_H (S(V) \times D(W)) \hookrightarrow \partial Z$$

is a G-equivariant embedding, we can glue the given G-handle to Z along ϕ and obtain a G-space

$$Z' = Z \cup_\phi \big(G \times_H (D(V) \times D(W)) \big)$$

which can canonically be equipped with the structure of a smooth G-manifold by straightening corners. We say that Z' is obtained from Z by *attaching* the given G-handle. Correspondingly we call the subset

$$\phi(G \times_H (S(V) \times 0)) \subset \partial Z$$

the *attaching sphere* of the G-handle. If M is a smooth G-manifold, $Z := M \times (0, 1]$ and Z' is obtained from Z by attaching a G-handle of dimension $d \geq 0$ and codimension $c \geq 0$ to ∂Z

(which can be identified with M), we say that $\partial Z'$ is obtained from M by *equivariant surgery* of *dimension* $d-1$ and *codimension* c (in the case when G is a finite group, these numbers refer to the dimension and codimension of the attaching sphere in M). In this terminology, equivariant surgery of dimension -1 (i.e. $d=0$) amounts to adding a disjoint component of the form

$$G \times_H (D^0 \times S(W))$$

to M.

The equivariant generalization of the surgery result due to Gromov-Lawson and Schoen-Yau is fairly straightforward as was already remarked in [2].

Theorem 2 (cf. [2], Theorem 11.1). *Let M be a (not necessarily compact) G-manifold equipped with a G-invariant metric of positive scalar curvature and let N be obtained from M by equivariant surgery of codimension at least 3. Then N carries a G-invariant metric of positive scalar curvature which can be assumed to coincide with the given metric on M outside a prescribed neighbourhood (which may be arbitrarily small) around the attaching sphere in M.*

Proof. We start with an equivariant embedding

$$\phi : G \times_H (S(V) \times D(W)) \hookrightarrow M\,.$$

After blowing up the metric on M if we wish to increase the injectivity radius of M, we can (up to G-isotopy) assume that the radial lines in $D(W)$ are mapped to unit speed geodesics in M that are orthogonal to the attaching sphere

$$\phi(G \times_H (S(V) \times 0)) \subset M\,.$$

The generalization of the construction in [7] to the equivariant situation is possible because the crucial part of the argument uses the distance from the attaching sphere in M and in the equivariant context, this is automatically a G-equivariant function.

For completeness and because [2] contains only a sketch of the proof, we summarize the essential steps of the construction.

If the G-handle $G \times_H (D(V) \times D(W))$ is of dimension 0 (this case - which may very well occur - is usually skipped in the literature), then the corresponding surgery step amounts to adding a new component of the form

$$G \times_H (D^0 \times S(W))$$

to M. This new component can be written as the total space of a G-equivariant fibre bundle

$$S(W) \hookrightarrow G \times_H S(W) \to G/H$$

and an application of the O'Neill formula together with the assumption that $\dim S(W) \geq 2$ shows that this new component admits a G-invariant metric of positive scalar curvature.

From now on, we assume that $\dim V \geq 1$. Let g be the given metric on M. We consider the pull back metric $\phi^*(g)$ on

$$G \times_H (S(V) \times D(W))\,.$$

For $0 < \epsilon < 1$ let $\phi^*(g)_\epsilon$ denote this pull back metric restricted to

$$G \times_H (S(V) \times S_\epsilon(W))$$

where $S_\epsilon(W)$ is the ϵ-sphere in W (with respect to the given euclidean metric on W). Note that we have a G-fibre bundle

$$S_\epsilon(W) \hookrightarrow G \times_H (S(V) \times S_\epsilon(W)) \to G \times_H (S(V) \times 0)$$

which is nothing but the ϵ-sphere bundle (with respect to the metric $\phi^*(g)$) of the fibration

$$D(W) \hookrightarrow G \times_H (S(V) \times D(W)) \to G \times_H S(V) .$$

However, the metric $\phi^*(g)_\epsilon$ on $G \times_H (S(V) \times S_\epsilon(W))$ will not, in general, be a Riemannian submersion metric.

We now construct a Riemannian submersion metric h_ϵ on this bundle which is uniquely characterized by the following properties:

i.) On the base $G \times_H S(V)$ it is the restriction of $\phi^*(g)$,
ii.) on the fibres it is the usual round metric on $S_\epsilon(W)$,
iii.) the horizontal subspaces are determined by the normal connection of the embedding $\phi\big(G \times_H (S(V) \times 0)\big) \subset M$.

The fibres $S_\epsilon(W)$ are totally geodesic with respect to h_ϵ. We pick a G-invariant metric on $G \times_H D(V)$ which near the boundary is the product metric $\phi^*(g)|_{G \times_H S(V)} \oplus dr^2$ (where r denotes the radial coordinate in $D(V)$). Using this, we construct a G-invariant submersion metric $\overline{h}_\epsilon$ on the bundle

$$S_\epsilon(W) \hookrightarrow G \times_H (D(V) \times S_\epsilon(W)) \to G \times_H (D(V) \times 0)$$

which is a G-equivariant product $h_\epsilon \oplus dt^2$ near the boundary and has totally geodesic fibres $S_\epsilon(W)$. Because $\dim S(W) \geq 2$, an application of the O'Neill formula shows that there is a small $\epsilon_0 > 0$ such that $\overline{h}_\epsilon$ has positive scalar curvature for all $0 < \epsilon < \epsilon_0$.

The following fact is the equivariant version of the crucial "bending outwards" process described in the work of Gromov-Lawson [7] (see also the elaborations of this argument in [24] and [30]). Again, the proof in the equivariant case does not lead to any further complications so that we confine ourselves to a clear statement of the relevant fact.

There is a $T \in \mathbb{R}^+$ and a G-invariant metric γ of positive scalar curvature on

$$G \times_H (S(V) \times S(W)) \times [0, T]$$

which on the left hand part $G \times_H (S(V) \times S(W)) \times [0, \delta)$ (where $\delta > 0$ is some small number), is isometric to $\phi^*(g)$ restricted to a collar δ-neighbourhood of $G \times_H (S(V) \times S(W))$ in $G \times_H (S(V) \times D(W))$ and on the right hand part $G \times_H (S(V) \times S(W)) \times (T - \delta, T]$, is a Riemannian product metric $h_\epsilon \oplus dt^2$. Here ϵ can be chosen arbitrarily in some interval $(0, \epsilon_1)$ where $\epsilon_1 > 0$ is an appropriately chosen small constant.

We now pick some ϵ in the interval $(0, \min(\epsilon_0, \epsilon_1))$ and glue the piece

$$\big(G \times_H (S(V) \times S(W)) \times [0, T], \gamma\big)$$

to

$$\big(M \setminus \phi\big(G \times_H (S(V) \times D(W))\big), g\big)$$

along $G \times_H (S(V) \times S(W)) \times \{0\}$ and then glue the Riemannian manifold

$$\big(G \times_H (D(V) \times S_\epsilon(W)), \overline{h}_\epsilon\big)$$

along $G \times_H (S(V) \times S_\epsilon(W))$ to the boundary of the resulting space. The manifold thus obtained is G-diffeomorphic to N and carries a G-invariant metric of positive scalar curvature by construction.

$\square$

3. Equivariant bordism theorem

The reduction of the construction of (nonequivariant) positive scalar curvature metrics to a bordism problem relies on the fact that a bordism Z (which is always assumed to be compact) between two closed manifolds X and Y can be decomposed into a series of handle attachments and that the (co-)dimensions of these handles can be controlled in terms of the connectivity of the map $Y \hookrightarrow Z$. This uses the existence of Morse functions on Z and the technique of handle cancelations. For an exposition of these methods, see e.g. [11, 16, 19, 20]. If the inclusion $Y \hookrightarrow Z$ is a 2-equivalance, this implies that Y can be obtained from X by a series of surgeries of codimension at least 3 so that Y carries a metric of positive scalar curvature if X carries such a metric.

Not all of these steps carry over directly to the equivariant case. Indeed, it was observed by Wassermann [29] that Morse theory can be formulated in an equivariant setting and can be used to construct decompositions of closed smooth G-manifolds into G-handles. However, it is well known that handle cancelation does not work in the equivariant context in full generality. This leads to counterexamples to equivariant analogues of the h- and s-cobordism theorems (see e.g. [9, 10]) and is also a main obstacle against translating Wall's surgery theory to an equivariant setting. In order to circumvent these difficulties, one usually works with gap hypotheses of the form that each singular stratum which is properly contained in the closure of another singular stratum F must have a large enough codimension in F (cf. [1]) or one formulates the s-cobordism theorem in an isovariant context, see e.g. [17], Theorem 4.42. In some sense, Theorem A follows this isovariant viewpoint.

Our proof of this result is based on two observations. The first one is that we need to cancel only G-handles in Z of codimension less than 3 so that the full power of a handle cancelation machinery is not necessary. The second - Lemma 13 below - is that the codimensions of G-handles occurring in Z are related to the codimensions of the singular strata to which they are attached if the handle decomposition of Z is induced by a G-Morse function which is special in the sense of [18].

We start by recalling some important notions from equivariant differential topology. Let G be a compact Lie group and let M be a smooth G-manifold. For a closed subgroup $H \subset G$, we denote by (H) the conjugacy class of H in G. The set of conjugacy classes of subgroups of G is partially ordered by writing $(H) \leq (K)$ if and only if H is conjugate to a subgroup of K. For $x \in M$ let $G_x \subset G$ be the isotropy group of x. Furthermore, we use the following notation:

i.) $M_{(H)} := \{x \in M \mid (G_x) = (H)\}$,
ii.) $M^H := \{x \in M \mid H \subset G_x\} = \{x \in M \mid hx = x \text{ for all } h \in H\}$.

The space M^H is a closed submanifold of M, but usually consists of different orbit types and is in general not G-invariant unless G is abelian.

In contrast, $M_{(H)}$ is an (in general not closed) G-submanifold of M. The space $M_{(H)}$ is called the *H-orbit bundle* of M. It consists of all points in M with isotropy groups conjugate to H and these form exactly those G-orbits which are G-diffeomorphic to the left G-space G/H. There is a G-fibre bundle

$$G/H \hookrightarrow M_{(H)} \to M_{(H)}/G\,.$$

We cite

Proposition 3 ([5], Theorem (5.14)). *Suppose M is a G-manifold and M/G is connected. Then there exists a unique isotropy type (H) such that $M_{(H)}$ is open and dense in M. The space $M_{(H)}/G$ is connected. Each isotropy type (K) satisfies $(H) \leq (K)$. The set M^H intersects each orbit.*

We call the space G/H with H as in the last proposition the *principal (or maximal) orbit type* of M. Accordingly we call (H) the *minimal isotropy type*. From now on we denote the minimal isotropy type by (H_{min}) and use the shorthand notation

$$M_{max} := M_{(H_{min})}$$

for the union of maximal orbits in M. This subset is open and dense in M. If $(H) \neq (H_{min})$, we call each component of $M_{(H)}$ a *singular stratum* of the G-action. The *cohomogeneity*

$$\operatorname{coh}(M, G)$$

of a connected G-manifold M is the codimension of a principal orbit in M (this does not depend on the principal orbit chosen). Note the equality

$$\operatorname{coh}(M, G) = \dim M_{max}/G\,.$$

The following example illustrates how different singular strata in a given G-manifold can be related to each other.

Example 4. *Let V_2 and V_3 be the irreducible one dimensional complex $\mathbb{Z}/6$-representations of weights 2 and 3 respectively. Then the S^1-manifold*

$$M := S^1 \times_{\mathbb{Z}/6} (V_2 \times V_3)$$

has three singular strata with isotropy groups $\mathbb{Z}/2$, $\mathbb{Z}/3$ and $\mathbb{Z}/6$. The singular strata with isotropy $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are not closed in M. More precisely, the intersection of the closures of these two singular strata is the third singular stratum with isotropy $\mathbb{Z}/6$. The maximal orbit type of M is equal to S^1 (in other words, the given S^1-action is effective) and the cohomogeneity of M is equal to 4.

Now we give those notions and results from equivariant Morse theory that are important for our discussion. We will mainly refer to the papers of Mayer [18] and Wassermann [29].

Definition 5. Let M be a closed G-manifold and let

$$f : M \to \mathbb{R}$$

be a smooth G-equivariant map where $\mathbb{R}$ is equipped with the trivial G-action. An orbit

$$G/H \approx \mathcal{O} \subset M$$

is called *critical* if for one (and hence any) point $x \in \mathcal{O}$ the differential $D_x f$ is zero. The critical orbit $\mathcal{O}$ is called *nondegenerate* if for each $x \in \mathcal{O}$ the following holds: Let $N_x \subset M$ be a normal slice of $\mathcal{O}$ at x. Then the H-invariant function

$$f|_{N_x} : N_x \to \mathbb{R}$$

has a nondegenerate critical point at 0, i.e. the Hessian of $f|_{N_x}$ is nondegenerate. The index (resp. coindex) of the Hessian of $f|_{N_x}$ at 0 is called the *index* (resp. *coindex*) of f at the nondegenerate critical orbit $\mathcal{O}$. Note that the property of f being nondegenerate at $\mathcal{O}$ and the index do not depend on the choice of $x \in \mathcal{O}$ or the choice of normal slices. The function f is called a *G-Morse function* if it has only nondegenerate critical orbits.

If M contains just one orbit type, then M/G is a smooth manifold and it is clear that G-Morse functions $M \to \mathbb{R}$ and ordinary Morse functions $M/G \to \mathbb{R}$ are in one-to-one correspondence.

Similar to the nonequivariant case, we have

Lemma 6 ([29], Density Lemma 4.8.). *Let M be a closed G-manifold. Then the set of G-Morse functions is dense (and clearly open) in the set of smooth G-equivariant maps*

$$M \to \mathbb{R}$$

equipped with the C^∞-topology.

Passage through critical orbits is described by attaching equivariant handles:

Lemma 7 ([29], Theorem 4.6.). *Let M be a closed G-manifold, let*

$$f : M \to \mathbb{R}$$

be a G-Morse function and let $[a, b] \subset \mathbb{R}$ contain exactly one critical value of f, lying in (a, b). Then $f|_{f^{-1}[a,b]}$ has finitely many critical orbits

$$\mathcal{O}_1, \ldots, \mathcal{O}_r\,.$$

Let $d_1, \ldots, d_r$ be the indices and $c_1, \ldots, c_r$ be the coindices of f at these critical orbits. Then $f^{-1}((-\infty, b])$ is equivariantly diffeomorphic to $f^{-1}((-\infty, a])$ with finitely many disjoint G-handles

$$G \times_{H_1} (D(V_1) \times D(W_1)), \ldots, G \times_{H_r} (D(V_r) \times D(W_r))$$

attached. Here (H_i) is the isotropy type of the orbit $\mathcal{O}_i$ and $\dim V_i = d_i$. Furthermore, after a choice of appropriate coordinates, the Morse function f restricted to a slice $D(V_i) \times D(W_i)$ has the standard form

$$f(x_1, \ldots, x_{d_i}, y_1, \ldots, y_{c_i}) = f(0) - x_1^2 - \ldots - x_{d_i}^2 + y_1^2 + \ldots + y_{c_i}^2\,.$$

As an immediate corollary of these results, each closed G-manifold admits a G-handle decomposition.

Now let X and Y be closed G-manifolds and let Z be a G-bordism from X to Y, i.e. Z is a compact G-manifold whose boundary splits as the disjoint union of X and Y. In this special case we call a smooth G-equivariant map

$$f : Z \to \mathbb{R}$$

a *G-Morse function* if in addition to the previous requirements it satisfies

i.) $f(Z) \subset [0, 1]$, $f|_X = 0$, $f|_Y = 1$,

ii.) the critical values of f are different from 0 and 1.

For smooth G-equivariant maps $Z \to [0, 1]$ with these additional properties, analogues of the density Lemma 6 as well as of the passage-through-critical-orbits Lemma 7 hold.

Corollary 8. *Let Z be a compact G-bordism from X to Y where X and Y are closed G-manifolds. Then the manifold Z can be obtained from $X \times [0, 1]$ by successively attaching finitely many G-handles.*

If we wish to use this result for constructing G-invariant positive scalar curvature metrics, we need to control the codimensions of the handles occuring in such a G-handle decomposition. This is possible by use of *special* G-Morse functions that were introduced by Mayer in [18]. Note that if $f : M \to \mathbb{R}$ is a G-Morse function, then the restriction of f to every orbit bundle $M_{(H)}$ has the same critical orbits and is also a G-Morse function (see Satz 3.1 in [18]). However, it can be shown by easy examples that in general the indices of f and of the restricted Morse function do not coincide. This motivates the following definition.

Definition 9 ([18], Definition 2.1)**.** Let M be a closed G-manifold. A G-Morse function

$$f : M \to \mathbb{R}$$

is called *special* if for each critical orbit $\mathcal{O}$ the index of f at $\mathcal{O}$ is equal to the index of the restricted G-Morse function

$$f|_{M_{(H)}} : M_{(H)} \to \mathbb{R}$$

at $\mathcal{O}$. Here (H) is the isotropy type of $\mathcal{O}$.

For special G-Morse functions we have the following genericity statement.

Lemma 10 ([18], Satz 2.2 and the following Bemerkung)**.** *Let M be a closed G-manifold. Then the set of special G-Morse functions is dense in the set of all smooth G-equivariant functions $M \to \mathbb{R}$ in the C^1-topology (but in general not in the C^2-topology).*

A similar statement holds for special G-Morse functions defined on bordisms between two closed G-manifolds X and Y.

We get the following variant of Lemma 7.

Lemma 11. *Let M be a closed G-manifold, let*

$$f : M \to \mathbb{R}$$

be a special G-Morse function and let $[a, b] \subset \mathbb{R}$ contain exactly one critical value as before. Then $f|_{f^{-1}[a,b]}$ has finitely many critical orbits

$$\mathcal{O}_1, \ldots, \mathcal{O}_r\,.$$

Let $d_1, \ldots, d_r$ be the respective indices of f at these critical orbits. Then $f^{-1}((-\infty, b])$ is equivariantly diffeomorphic to $f^{-1}((-\infty, a])$ with a finite number of disjoint G-handles

$$G \times_{H_1} (D^{d_1} \times D(W_1))\,,\ \ldots\ ,\ G \times_{H_r} (D^{d_r} \times D(W_r))$$

attached. Here (H_i) is the isotropy type of the orbit $\mathcal{O}_i$ and H_i acts trivially on the unit discs D^{d_i} for $1 \leq i \leq r$.

A similar statement holds for special G-Morse functions defined on compact G-bordisms Z. From now on we will speak of G-handles of the form $G \times_H (D^d \times D(W))$ as *special* G-handles.

One pleasant feature of special G-Morse functions is that they lead directly to G-CW structures on the given manifolds, see [18], Satz 3.3.

Before we state and prove the main result of this section, we need three more preparatory lemmas. The first one says that we have some control on the order in which G-handles occur if we work with special G-Morse functions.

Lemma 12. *Let Z be a compact G-bordism as before, let*

$$Z \to [0, 1]$$

be a special G-Morse function and let $H \subset G$ be a closed subgroup. Then there is a special G-Morse function

$$f : Z \to [0, 1]$$

with the same critical orbits and the same indices on these critical orbits as the given G-Morse function, but with the following additional property: There are two noncritical values

$$0 < c < d < 1$$

of f so that for each critical orbit $\mathcal{O}$ of f the following equivalences hold:

$$\begin{aligned} f(\mathcal{O}) > d &\iff (K) \lneqq (H)\,, \\ c < f(\mathcal{O}) < d &\iff (K) = (H)\,. \end{aligned}$$

In these equivalences, (K) is the isotropy type of $\mathcal{O}$.

Proof. Assume that $\mathcal{U}$ is a critical orbit of isotropy type $(L) \leq (H)$. Passage through $\mathcal{U}$ corresponds to attaching a G-handle of the form

$$G \times_L (D(V) \times D(W))\,.$$

The isotropy types occuring in this G-handle are smaller than or equal to (L) and hence smaller than or equal to (H). Now let $\mathcal{O}$ be a critical orbit and let (K) be its isotropy type. Because we are dealing with special G-Morse functions, the attaching sphere of the handle associated to $\mathcal{O}$ is of the form

$$G \times_K (S^d \times 0)$$

with K acting trivially on S^d. If $(K) \nleqq (H)$, then this attaching sphere must be disjoint from the first G-handle associated to $\mathcal{U}$. The order in which the two G-handles are attached can therefore be interchanged by adapting the given special G-Morse function appropriately. Hence we can assume that there is a noncritical value c of f such that for each critical orbit $\mathcal{O}$ of f, we have

$$f(\mathcal{O}) > c \iff (K) \leq (H)$$

where (K) is the isotropy type of $\mathcal{O}$. As before we can now argue that of the remaining G-handles, those of isotropy exactly (H) are attached before those of isotropy type strictly smaller than (H). This proves the existence of the second noncritical value d with the stated property. □

The next lemma gives an important relation between the codimensions of G-handles and the codimensions of the associated G-strata. Again it is crucial to work with decompositions into special G-handles.

Lemma 13. *Let Z be a G-manifold with boundary and let*

$$Z' = Z \cup_\phi \left(G \times_H (D^d \times D(W))\right)$$

be obtained from Z by attaching a special G-handle. Let $F \subset Z_{(H)}$ be the component containing $\phi(G \times_H (D^d \times 0))$. Then the codimension of F in Z and the dimension of W are related by the inequality

$$\operatorname{codim} F \leq \dim W\,.$$

Proof. Because the action of H on $G \times (D^d \times D(W))$ is free, we have

$$\dim Z = \dim G + \dim W + d - \dim H\,.$$

But

$$\dim G + d - \dim H \leq \dim F$$

becauce H acts trivially on D^d and hence $G \times_H (D^d \times 0)$ is completely contained in F. Assuming $\dim F \leq \dim Z - \dim W - 1$, we therefore obtain

$$\dim Z \leq \dim F + \dim W \leq \dim Z - 1\,,$$

a contradiction. □

Finally we need an invariance statement for the G-homotopy type of certain singular strata under G-handle attachments.

Lemma 14. *Let Z be a G-manifold with boundary and let*

$$Z' = Z \cup_\phi \big(G \times_H (D(V) \times D(W))\big)$$

be obtained from Z by attaching a G-handle such that H acts trivially on W (this condition is in some sense dual to that of being a special G-handle). If $(K) \neq (H)$, then there exists a G-homotopy equivalence

$$Z_{(K)} \simeq Z'_{(K)}\,.$$

Proof. Because H acts trivially on W and $(K) \neq (H)$, we have

$$\big(G \times_H (0 \times D(W))\big)_{(K)} = \emptyset$$

and this (together with the fact that V and W are linear H-spaces) implies that the inclusion

$$\big(G \times_H (S(V) \times D(W))\big)_{(K)} \hookrightarrow \big(G \times_H (D(V) \times D(W))\big)_{(K)}$$

is a G-deformation retract. □

Now we can formulate a bordism principle - Theorem A from the introduction - for constructing G-invariant metrics of positive scalar curvature. In view of the Lawson-Yau theorem stated in the introduction, its main purpose is for Lie groups whose identity components are abelian. Recall that a singular stratum in a connected G-manifold M is a connected component of some $M_{(H)}$ where $(H) \neq (H_{min})$. Singular strata are G-invariant submanifolds, but need not be compact even if M is compact (cf. Example 4).

Theorem 15. *Let Z be a compact connected oriented G-bordism (with G acting by orientation preserving maps) between the closed G-manifolds X and Y. Assume the following:*

i.) $\mathrm{coh}(Z, G) \geq 6$,
ii.) *the inclusion $Y_{max} \hookrightarrow Z_{max}$ of maximal orbits is a (nonequivariant) 2-equivalence,*
iii.) *each singular stratum of codimension 2 in Z meets Y.*

Then, if X admits a G-invariant metric of positive scalar curvature, the same is true for Y.

Proof. Let

$$f : Z \to [0,1]$$

be a special G-Morse function. We will replace f by a special G-Morse function without critical orbits of coindex 0, 1 or 2. By Lemma 13, all critical orbits of f which are of this form are contained in singular strata of codimension less than 3 in Z. Because Z is oriented and G acts in an orientation preserving fashion, there are no codimension-1 singular strata in Z.

In a first step, we will take care of the singular strata of codimension 2 that contain critical orbits of coindex less than 3. Because f is special, the coindex of these critical orbits is exactly 2.

Let $F \subset Z$ be a singular stratum of codimension 2 (which need not be compact) of isotropy type (H) and containing coindex-2 critical orbits of f. We will remove the critical orbits of coindex 2 from F without changing f around the other singular strata in Z.

By applying Lemma 12 to the subgroup H, we can assume without loss of generality that there are noncritical values $0 < c < d < 1$ of f such that for each critical orbit $\mathcal{O}$ of f the equivalences

$$\begin{aligned} f(\mathcal{O}) > d &\iff (K) \lneqq (H)\,, \\ c < f(\mathcal{O}) < d &\iff (K) = (H) \end{aligned}$$

hold where (K) is the isotropy type of $\mathcal{O}$. Because F is connected and $F \cap Y \neq \emptyset$, each component of $F \cap f^{-1}[0,d]$ has nonempty intersection with $f^{-1}(d)$. Furthermore, the map

$$(f^{-1}[c,d])_{(H)} \hookrightarrow (f^{-1}[0,d])_{(H)}$$

is a G-homotopy equivalence by Lemma 14 (recall that f is special, so $-f$ induces a decomposition into G-handles of the form described in Lemma 14) and hence induces a bijection of connected components. This implies that each component $F_1, \ldots, F_k$ of $F \cap f^{-1}[c,d]$ has nonempty intersection with $f^{-1}(d)$. We will now concentrate on the partial bordism

$$P := f^{-1}[c,d] \subset Z\,.$$

By construction, the critical orbits of the special G-Morse function

$$f|_P : P \to [c,d]$$

are exactly those critical orbits of f which are of isotropy type (H). Hence the restriction $f|_{P_{(H)}}$ induces a finite G-handle decomposition of $P_{(H)}$ and all critical orbits of $f|_F$ are contained in $P_{(H)}$. It is therefore enough to remove all coindex-2 critical orbits of $f|_P$ which are contained in $F \cap P$, but without changing $f|_P$ near the boundary

$$\partial P := f^{-1}(c) \cup f^{-1}(d) \subset P$$

and around singular strata in P that are different from $F_1, \ldots, F_k$. We do this seperately for each component $C \subset F \cap P$.

Using the fact that C has nonempty intersection with $f^{-1}(d)$ (see above), we can use a (relative form of a) nonequivariant handle cancelation on the induced handle decomposition of C/G (see e.g. [20], Theorem 8.1) in order to obtain a G-Morse function

$$h : C \to [c,d]$$

without coindex 0 critical orbits, which coincides with $f|_C$ near $C \cap \partial P$ and outside a compact subset $K \subset C$. Note that C contains just one orbit type so that h is automatically a special G-Morse function. By (a relative form of) Lemma 10 we find a special G-Morse function

$$P \to [c,d]$$

whose restriction to C coincides with h and which is equal to $f|_P$ near ∂P and near the singular strata in P that are different from C. Because the codimension of C in P is 2, the critical orbits in C of this special G-Morse function are of coindex at least $1+2=3$.

This new special G-Morse function might have more critical orbits than f, but these do not lie on singular strata and are hence of minimal isotropy type. But critical orbits of minimal isotropy type (i.e. lying in Z_{max}) will be taken care of later.

We carry out the same process for all other components of $F \cap P$ and obtain a special G-Morse function $P \to [c,d]$ which coincides with $f|_P$ near ∂P and near those singular strata in P that are different from a component of $F \cap P$. We combine this new special G-Morse function on P with the old G-Morse function $f|_{Z \setminus P}$ in order to produce a special G-Morse function

$$Z \to [0,1]$$

with no critical orbits of coindex 2 in F, which coincides with f near the singular strata different from F.

After applying this procedure several times, we get a special G-Morse function $f : Z \to [0,1]$ such that no singular stratum of codimension 2 contains a critical orbit of coindex less than 3.

We now remove the critical orbits of coindex 0, 1 or 2 in Z_{max}. Arguing similarly as before we can assume that there is a noncritical value c of f so that

$$P := f^{-1}[c, 1]$$

contains exactly those critical orbits of f that are maximal. In particular, we get an induced handle decomposition of P_{max}. Because the inclusion

$$Y_{max} \hookrightarrow P_{max}$$

is a 2-equivalence (this uses assumption ii.) in Theorem 15 as well as Lemma 14), the same is true for the inclusion of orbit spaces

$$Y_{max}/G \hookrightarrow P_{max}/G$$

by comparing the long exact homotopy sequences for the respective $G/(H_{min})$-fibrations. We can now use nonequivariant handle cancelation on P_{max}/G to find a handle decomposition of this space without codimension 0, 1 or 2 handles. For the non-simply connected case, this is explained carefully in [16]. Note that by assumption, P_{max}/G is oriented and of dimension at least 6 so that the requirements for performing handle cancelation are fulfilled.

We end up with a special Morse function

$$f : Z \to [0, 1]$$

without critical orbits of coindex less than 3. The equivariant surgery principle (Theorem 2) finishes the proof of Theorem 15. □

The connectivity assumption ii.) of Theorem 15 reduces to a corresponding assumption in the nonequivariant bordism principle if G is trivial. It is clear that the map $Y_{max} \hookrightarrow Z_{max}$ is a 2-equivalence if the inclusion $Y \hookrightarrow Z$ is a 2-equivalence and Z does not contain singular strata of codimension 2 or 3. However, we have not been able to replace condition ii.) by a connectivity assumption on $Y \hookrightarrow Z$ if singular strata of codimension 2 or 3 occur in Z. Assumption iii.) can be viewed as the condition that the inclusion $Y \hookrightarrow Z$ restricted to singular strata of codimension 2 be a 0-equivalence (i.e. it induces a surjective map on π_0). We will make a few comments on the cohomogeneity restriction i.) of Theorem 15 at the end of this section.

The following corollary of Theorem 15 is immediate.

Corollary 16 (cf. [25], Theorem 2.3)**.** *Let $\mathbb{Z}/p$ act smoothly on a closed simply connected spin manifold M^n where $n \geq 5$ and p is an odd prime, preserving a spin structure. Assume furthermore that M is equivariantly cobordant to another (not necessarily connected) spin $\mathbb{Z}/p$-manifold M' by a bordism W whose fixed set $W^{\mathbb{Z}/p}$ does not contain components of codimension 2. If M' has an invariant metric of positive scalar curvature, then so does M.*

Proof. Because p is odd, W only contains fixed components of even codimension and so the singular strata in W have codimension at least 4. Because $\mathbb{Z}/p$ is a 0-dimensional Lie group, the cohomogeneity assumption in Theorem 15 is satisfied. We can kill $\pi_1(W)$ and $\pi_2(W)$ by performing surgeries on the free part of the interior of W and therefore assume that $M \hookrightarrow W$ is a 2-equivalence. This is equivalent to $M_{max} \hookrightarrow W_{max}$ being a 2-equivalence because the singular strata in W are of codimension at least 4. □

The original formulation in [25] requires that only the codimensions of singular strata contained in M be at least 4. The proof presented in *loc. cit.* argues that the fixed set $W^{\mathbb{Z}/p}$ can be built from $(M')^{\mathbb{Z}/p} \times [0, 1]$ by successive handle attachments. These handles are then thickened inside W and thus replaced by handles of codimension at least 4 in W if the codimension of $W^{\mathbb{Z}/p}$ in W (!) is assumed to be at least 4. The remaining (free) part of W is then constructed by successive handle attachments to the union of M' and the thickening of $W^{\mathbb{Z}/p}$ obtained before. The codimensions of these handles can be controlled by the topological assumptions on M. However, we do not understand how this proof works if there are codimension-2 components in $W^{\mathbb{Z}/p}$ that are disjoint from M (see the following Proposition 17). If we assume that each codimension-2 component in W touches M and

$$M \setminus M^{\mathbb{Z}/p} \hookrightarrow W \setminus W^{\mathbb{Z}/p}$$

is a 2-equivalence, then our Theorem 15 shows that an invariant metric of positive scalar curvature on M' can still be pushed through the bordism in order to produce one on M. So, in fact, our codimension-2 restriction is rather the opposite of the one proposed in [25], Theorem 2.3.

The following proposition shows that in the case of codimension-2 fixed components in Z that do not touch Y, any special $\mathbb{Z}/p$-handle decomposition of Z contains handles of codimension 2. This points towards a clear limitation of using (conventional) equivariant handle decompositions for constructing equivariant positive scalar curvature metrics and shows that a new idea is needed in order to deal with assumption iii.) in Theorem 15.

Proposition 17. *Let Z be a compact $\mathbb{Z}/p$-bordism (p an odd prime) between the closed $\mathbb{Z}/p$-manifolds X and Y. Assume that Z contains a fixed component of codimension 2 that is disjoint from Y. Then any $\mathbb{Z}/p$-handle decomposition of Z (starting from $X \times [0,1]$) contains handles of codimension 0 or codimension 2.*

Proof. Each such handle decomposition is associated to a $\mathbb{Z}/p$-Morse function

$$f : Z \to \mathbb{R}\,.$$

By assumption, the restriction of f to the fixed point set $Z^{\mathbb{Z}/p}$ must have a local maximum $x \in F$ where $F \subset Z^{\mathbb{Z}/p}$ is a fixed component of codimension 2 in Z which is disjoint from Y. Then x is a critical orbit of f of coindex 0 or 2 (it is of coindex exactly 2 if f happens to be special). ☐

One might speculate that Theorem 15 could be used to give an alternative proof of the Lawson-Yau theorem stated in the introduction. However, we have not been able to carry this out. We remark again that in the proof of the Lawson-Yau theorem, an explicit metric with the required properties is constructed, whereas the metric on Y prescribed by Theorem 15 depends on the given bordism Z and is therefore difficult to make explicit.

We conclude this section with some remarks on the cohomogeneity assumption i.) in Theorem 15. The necessity of this assumption is obvious if G is finite because a similar dimension restriction appears in the nonequivariant case. We will therefore concentrate on S^1-actions.

Let M be a closed symplectic non-spin simply connected 4-manifold with $b_2^+(M) \geq 2$. Then M does not admit a positive scalar curvature metric by Seiberg-Witten theory (see [21], Corollary 2.3.8 in combination with Theorem 3.3.29). By [2], Theorem C (cf. Lemma 32 below), the free S^1-manifold $Y := M \times S^1$ (with S^1 acting trivially on M) does not admit an S^1-invariant metric of positive scalar curvature, because the quotient manifold

$$(M \times S^1)/S^1 = M$$

does not admit such a metric. Moreover, the oriented bordism group Ω_4^{SO} is generated by the bordism class of $\mathbb{C}P^2$ and hence the manifold M is oriented bordant to a manifold X which carries a metric of positive scalar curvature. The inclusion $M \hookrightarrow W$ into such a bordism can be assumed to be a 2-equivalence by performing appropriate surgeries on W, since M is simply connected, is not spin and $\dim W = 5$. The manifold $W \times S^1$ is then an S^1-bordism between $X \times S^1$ and $M \times S^1$, and moreover the S^1-manifold $X \times S^1$ admits an invariant metric of positive scalar curvature and the inclusion

$$M \times S^1 \hookrightarrow W \times S^1$$

is a 2-equivalence. However, the cohomogeneity of $W \times S^1$ is equal to 5 (and in particular smaller than 6).

4. Resolution of singularities

We now specialize our discussion to fixed point free S^1-manifolds. It turns out that in this case, the assumption on codimension-2 singular strata in Theorem 15 can be dropped if Y_{max} is simply connected and not spin. This improvement is based on a new surgery technique which enables us to remove codimension-2 singular strata from fixed point free S^1-manifolds while preserving a given invariant positive scalar curvature metric. In this section, we will work out the details of this procedure before we discuss the improvement of Theorem 15 in the next Section 5.

We will first describe the surgery step and discuss some technical conditions needed in the geometric analysis of this situation before we finally state and prove the preservation of invariant positive scalar curvature metrics under the surgery in Theorem 25.

Let M be a closed fixed point free S^1-manifold of dimension $n \geq 6$ and let

$$\phi : S^1 \times_H (S^{n-3} \times D(W)) \hookrightarrow M$$

be an S^1-equivariant embedding where $H \subset S^1$ is a finite subgroup and W is a one dimensional unitary effective H-representation. The group H acts trivially on S^{n-3}.

Because $S(W)/H$ can be identified with S^1 and because $n \geq 6$, the S^1-principal bundle

$$S^1 \hookrightarrow S^1 \times_H (S^{n-3} \times S(W)) \to S^{n-3} \times S(W)/H$$

is trivial. After a choice of trivialization

$$\chi : S^1 \times_H (S^{n-3} \times S(W)) \cong S^1 \times S^{n-3} \times S(W)/H$$

and considering $S(W)/H$ as the boundary of D^2, we can glue the free S^1-manifold $S^1 \times S^{n-3} \times D^2$ back to $M \setminus \operatorname{im}(\phi)$ to get a new S^1-manifold M'. The manifold M' is constructed from M by removing the singular stratum $F := \phi(S^1 \times_H (S^{n-3} \times 0))$ via a kind of codimension-2 surgery.This surgery step is different from the equivariant surgery described at the beginning of Section 2. We will refer to M' as being obtained from M by a *resolution* of the codimension-2 singular stratum F. We remark that in general this surgery step depends on the choice of the trivialization χ.

Before proceeding, we need to impose some restrictions on the S^1-actions under consideration.

Definition 18 (cf. [22], p. 46)**.** Let M be a compact S^1-manifold. We say that M satisfies *condition* $\mathbb{C}$ if for each closed subgroup $H \subset S^1$, the S^1-equivariant normal bundle of the closed submanifold $M^H \subset M$ (which may contain different isotropy types) is equipped with the structure of a complex S^1-bundle such that the following compatibility condition holds: If $K, H \subset S^1$ are two closed subgroups and $K \subset H$, then the restriction of the normal bundle of $M^K \subset M$ to M^H is a direct summand of the normal bundle of $M^H \subset M$ as a complex S^1-bundle.

Note that the singular strata of an S^1-manifold satisfying condition $\mathbb{C}$ always have even codimension. The following lemma shows that S^1-actions satisfying condition $\mathbb{C}$ occur naturally in many situations.

Lemma 19. *Let M be a compact S^1-manifold for which all finite isotropy groups have odd order. Then M satisfies condition $\mathbb{C}$.*

Proof. Let $H \subset S^1$ be a closed subgroup and let $\nu \to M^H$ be the normal bundle of M^H in M. For each $x \in M^H$, the fibre ν_x is a real H-representation with $\nu_x^H = 0$. This H-representation has a decomposition

$$\nu_x = (E_1 \otimes V_1) \oplus \ldots \oplus (E_k \otimes V_k)$$

where E_i, $1 \leq i \leq k$ are real vector spaces with trivial H-action and V_i are pairwise different nontrivial irreducible real H-representations. Because $H = S^1$ or H is of odd order, each V_i has real dimension 2 with H acting as a rotation action. Hence, each V_i carries the structure of a one dimensional complex H-representation with $g \in H \subset S^1$ acting by

$$(g, v) \mapsto g^{\xi_i} \cdot v\,.$$

Here $\xi_i \in \mathbb{Z}$ is different from 0 and not a multiple of $|H|$ if H is a finite group. The complex structure on V_i is uniquely determined if we require that $\xi_i > 0$ if $H = S^1$, resp. $\xi_i = 1, \ldots, \frac{|H|-1}{2} \mod |H|$ if H is finite. We conclude that ν_x carries a canonical induced structure of a complex H-representation. □

Lemma 20. *Let Z be a compact S^1-bordism satisfying condition $\mathbb{C}$ and assume that Z is decomposed into special S^1-handles of the form*

$$S^1 \times_H (D^d \times D(W))\,.$$

Then each of these S^1-handles is equipped with the following additional structure: If $K \subset S^1$ is a closed subgroup with $K \subset H$ and we decompose the orthogonal H-representation W as

$$W = W^K \oplus (W^K)^\perp\,,$$

then the H-representation $(W^K)^\perp$, which is H-invariant because S^1 is abelian, carries the structure of a unitary H-representation that is compatible with the given orthogonal H-structure.

Proof. This holds because each such S^1-handle is equivariantly embedded in Z and the normal bundle of the K-fixed set

$$S^1 \times_H (D^d \times D(W)^K) = (S^1 \times_H (D^d \times D(W))^K \subset S^1 \times_H (D^d \times D(W))$$

has fibre $(W^K)^\perp$. □

The next definition is of a rather technical nature. These properties of S^1-actions play a crucial role in our proofs, but do not appear in the final theorems.

Definition 21. Let M be a manifold equipped with an S^1-action τ and with an S^1-invariant Riemannian metric g.

i.) The metric is called *scaled* if the S^1-action is fixed point free and the vector field on M generated by the action has constant length, called the *scale* of the action.

ii.) We call g *normally symmetric in codimension* 2 if the following holds: Let $H \subset S^1$ be a closed subgroup and let $F \subset M^H$ be a component of the H-fixed subset in M which is of codimension 2 in M (recall that F is a closed S^1-invariant submanifold of M). Then there exists an S^1-invariant tubular neighbourhood $N_F \subset M$ of F together with a second isometric S^1-action σ_F on N_F that commutes with τ and has fixed point set F.

If an S^1-invariant metric on M has scale s, then the length of an orbit with isotropy H is exactly $(2\pi s)/|H|$. We remind the reader of the following special case of the O'Neill formula (see [3], 9.37): Let

$$S^1 \hookrightarrow E \xrightarrow{\pi} B$$

be a Riemannian submersion with totally geodesic fibres S^1 (necessarily of constant length, if B is connected). Then the scalar curvatures of E and B are related by the formula

$$s_E(x) = s_B(\pi(x)) - \|A(x)\|^2$$

where $x \in E$ and A is the tensor field on E defined in [3], 9.20. This makes clear why we prefer to work with invariant metrics of positive scalar curvature which are scaled: If a free S^1-manifold E of dimension at least 3 is equipped with such a metric, then the orbits are totally geodesic submanifolds of E (see [3], Theorem 9.59) and the induced metric on the orbit space E/S^1 has positive scalar curvature. This observation will be used at various places in the following discussion. In general, if the vertical part (tangent to the fibres) of the metric on E is multiplied by a constant factor ϵ^2 (we call the resulting Riemannian manifold E_ϵ), then the norm $\|A\|^2$ is multiplied by ϵ^2. In particular, shrinking the fibres S^1 increases the scalar curvature on E and if $s_B > 0$ and B is compact, then there is some constant $\epsilon_0 > 0$ so that $s_{E_\epsilon} > 0$, if $0 < \epsilon < \epsilon_0$. Also, if an S^1-invariant metric of positive scalar curvature is scaled, then the scale can be decreased arbitrarily while preserving the positive scalar curvature property.

Note that the additional normal symmetries described in part ii.) of Definition 21 need only be defined locally around the respective fixed point sets. These additional symmetries will considerably faciliate our later analytic arguments. We consider normally symmetric S^1-invariant metrics as the "naturally occuring" ones. This will become clear in the proof of Lemma 24. As a first illustration we provide the following example.

Example 22. *Let V be a unitary S^1-representation. Then the induced S^1-invariant metric on the representation sphere $S(V)$ (equipped with the restricted S^1-action) is normally symmetric in codimension 2.*

Neither of the restrictions formulated in Definition 21 are serious. This is the content of the following two lemmas.

Lemma 23. *Let M be a closed fixed point free S^1-manifold of dimension at least 3 which is equipped with an invariant metric of positive scalar curvture. Then M also carries an invariant metric of positive scalar curvature which is scaled. If the original metric is normally symmetric in codimension 2, the same can be assumed for the new metric.*

Proof. Let

$$X : M \to TM$$

be the vector field generated by the S^1-action. Because the action is fixed point free, X has no zeros and defines a 1-dimensional subbundle $\mathcal{V} \subset TM$. Let the smooth function $f : M \to \mathbb{R}$ be defined by

$$f(p) := \|X(p)\|_g \, .$$

We split TM into $\mathcal{V}$ and its orthogonal complement $\mathcal{H}$. For a moment we restrict attention to a tube

$$S^1 \times_H D(V) \subset M$$

of the action. After pulling back the metric g along the orbit map

$$S^1 \times D(V) \to S^1 \times_H D(V)\,,$$

we get a metric on $S^1 \times D(V)$ which is invariant under the free S^1-action on the first factor. Let h be the induced quotient metric on $D(V)$. We set $n := \dim M$. The argument from [2], Section 9, shows that the metric

$$f^{\frac{2}{n-2}} \cdot h$$

on $D(V)$ has positive scalar curvature.

After this preparation, let dt^2 be the metric on $\mathcal{V}$ with respect to which X has constant length 1. An application of the O'Neill formula together with the previous local argument shows that there is a constant $\epsilon_0 > 0$ so that the metric

$$(\epsilon^2 \cdot dt^2) \oplus (f^{\frac{2}{n-2}} \cdot g|_{\mathcal{H}})$$

on M has positive scalar curvature if $0 < \epsilon < \epsilon_0$. By construction, this new metric has all of the required properties. If g is normally symmetric, this follows because the additional S^1-actions σ_F around the different codimension-2 subsets $F \subset M$ respect the decomposition $TM = \mathcal{V} \oplus \mathcal{H}$. $\square$

Concerning the construction of normally symmetric metrics of positive scalar curvature, we have the following variant of the surgery principle from Section 2. We formulate this result directly in the form needed in Section 5.

Lemma 24. *Let Z be a compact S^1-bordism satisfying condition $\mathbb{C}$ between the closed S^1-manifolds X and Y. Assume that X carries an invariant metric of positive scalar curvature which is normally symmetric in codimension 2. If Z admits a decomposition into special S^1-handles (starting from $X \times [0,1]$) of codimension at least 3, then Y also carries an invariant metric of positive scalar curvature which is normally symmetric in codimension 2.*

Proof. The decomposition of Z into special S^1-handles of codimension at least 3 implies that Y is obtained from X by performing equivariant surgeries of codimension at least 3 along embedded submanifolds of the form

$$S^1 \times_H (S^{d-1} \times D(W))\,.$$

If $d = 0$, then this attaching locus is empty and the surgery step produces a new component $S := S^1 \times_H (D^0 \times S(W))$. We show at first that these S^1-manifolds S admit S^1-invariant metrics of positive scalar curvature which are normally symmetric in codimension 2.

Let $K \subset S^1$ be closed and let $F = S^K \subset S$ be a component of codimension 2. Then $K \subset H$ (otherwise S^K would be empty) and W admits an orthogonal splitting

$$W = W^K \oplus (W^K)^\perp\,.$$

Because Z satisfies condition $\mathbb{C}$, Lemma 20 implies that $(W^K)^\perp$ has the induced structure of a unitary H-representation.

We now consider the orthogonal S^1-action on

$$W = W^K \oplus (W^K)^\perp$$

which is given by complex multiplication on the second summand. The restriction of this S^1-action to $S(W)$ commutes with the H-action. Furthermore, this S^1-action on $S(W)$ has fixed point set $S(W)^K$ and leaves the usual round metric on $S(W)$ invariant. In this way we obtain an S^1-action σ on S by letting S^1 act on the $S(W)$-factor of $S^1 \times_H (D^0 \times S(W))$. This action commutes with the original S^1 action on S (acting on the S^1-factor), leaves the positive scalar curvature metric on S constructed in the proof of Theorem 2 invariant (recall that this is a Riemannian submersion metric on the fibre bundle $S(W) \hookrightarrow S^1 \times_H (D^0 \times S(W)) \to S^1/H$) and has fixed point set

$$S^1 \times_H (D^0 \times S(W^K)) = S^K .$$

This finishes the discussion of S^1-handles of dimension 0 in Z.

From now on, we concentrate on S^1-handles in Z that are of dimension at least 1.

We need to prove the following fact: Let M be a closed S^1-manifold equipped with an invariant metric of positive scalar curvature which is normally symmetric in codimension 2. If M' is obtained from M by equivariant surgery of codimension at least 3 along an S^1-equivariant embedding

$$\phi : S^1 \times_H (S^{d-1} \times D(W)) \hookrightarrow M$$

with $d-1 \geq 0$ (i.e. $S^{d-1} \neq \emptyset$), then M' also carries an invariant metric of positive scalar curvature which is normally symmetric in codimension 2. We argue as follows.

At first notice that it suffices to treat the case when there is a closed subgroup $K \subset S^1$ and a codimension-2 component $F \subset M^K$ (for brevity we will call such submanifolds *fixed codimension-*2 *components*) such that

$$\phi(S^1 \times_H (S^{d-1} \times 0)) \cap F \neq \emptyset .$$

Otherwise we could assume that $\mathrm{im}(\phi)$ would be disjoint from any fixed codimension-2 component of M and this would imply that the S^1-handle $S^1 \times_H (D^d \times D(W))$ would have no fixed codimension-2 components, either. Hence there would be nothing to prove.

We may assume (up to G-isotopy) that ϕ maps the radial lines in the fibres of the S^1-equivariant fibre bundle

$$D(W) \hookrightarrow S^1 \times_H (S^{d-1} \times D(W)) \to S^1 \times_H (S^{d-1} \times 0)$$

to unit speed geodesics in M that are orthogonal to $\phi(S^1 \times_H (S^{d-1} \times 0))$. Now let

$$F_i \subset M^{K_i} , i = 1, \ldots , r ,$$

be those fixed codimension-2 components (with certain - necessarily distinct - subgroups $K_i \subset S^1$) which intersect $\phi(S^1 \times_H (S^{d-1} \times 0))$ nontrivially. This implies that $K_i \subset H$ for all i and

$$\phi(S^1 \times_H (S^{d-1} \times 0)) \subset \bigcap_i F_i .$$

Without loss of generality we may assume that

$$\mathrm{im}(\phi) \subset \mathrm{N_{F_i}}$$

for all $i = 1, \ldots , r$, where N_{F_i} are tubular neighbourhoods of F_i which are equipped with additional actions σ_i as described in Definition 21.

The map ϕ and the actions $\sigma_1, \ldots , \sigma_r$ induce S^1-actions Σ_i on $S^1 \times_H (S^{d-1} \times D(W))$ which are induced by orthogonal S^1-actions on W (with respect to the standard Euclidean metric on W) commuting with the H-action on W and with fixed points sets W^{K_i}. Let g be the given metric on M and $\phi^*(g)$ be the induced metric on $S^1 \times_H (S^{d-1} \times D(W))$. This metric enjoys the following S^1-symmetries:

i.) It is invariant under the S^1-action on the first factor of $S^1 \times_H (S^{d-1} \times D(W))$,

ii.) for each $i = 1, \ldots , r$ it is invariant under the action Σ_i.

The constructions in the proof of Theorem 2 preserve all these S^1-symmetries of $\phi^*(g)$: This is clear for the submersion metrics on the sphere bundles

$$S_\epsilon(W) \hookrightarrow S^1 \times_H (S^{d-1} \times S_\epsilon(W)) \to S^1 \times_H (S^{d-1} \times 0)$$

which are constructed in the proof of Theorem 2 and hence also for the metric on

$$S^1 \times_H (S^{d-1} \times S(W)) \times [0, T]$$

which interpolates (via the bending outward process) between the metric $\phi^*(g)$ and a submersion metric of the above form (with small ϵ).

This finishes the proof of Lemma 24 $\square$

The main result of this section reads as follows.

Theorem 25. *Let M^n be a closed fixed point free S^1-manifold of dimension $n \geq 6$ and let*

$$\phi : S^1 \times_H (S^{n-3} \times D(W)) \hookrightarrow M$$

be an S^1-equivariant embedding where W is a unitary effective H-representation of dimension 1. Let the manifold M' be obtained from M by resolving the singular stratum $\phi(S^1 \times_H (S^{n-3} \times 0)) \subset M$. If the manifold M admits an invariant metric of positive scalar curvature which is scaled and normally symmetric in codimension 2, then also M' admits such a metric.

The remainder of this section is devoted to a proof of Theorem 25.

Let g be an invariant metric of positive scalar curvature on M which is scaled and normally symmetric in codimension 2. As before we assume that ϕ maps the radial lines in $D(W)$ to unit speed geodesics in M orthogonal to $\phi(S^1 \times_H (S^{n-3} \times 0)) \subset M$ and that $\mathrm{im}(\phi)$ is contained in an S^1-invariant tubular neighbourhood of $\phi(S^1 \times_H (S^{n-3} \times 0))$ that is equipped with an additional S^1-action σ as described in Definition 21. We still denote the original S^1-action on M by τ. We thus obtain corresponding actions τ and σ on

$$N := S^1 \times_H (S^{n-3} \times D(W))$$

that are induced by rotation actions on the factors S^1 and $D(W)$ respectively. Because W is a unitary H-representation, the induced H-action on $D(W)$ commutes with σ (this also follows from the requirement that the actions τ and σ commute).

Recall that the induced metric $\phi^*(g)$ (that we will denote by g from now on) on the total space of the S^1-equivariant fibre bundle

$$D(W) \hookrightarrow N \to S^1 \times_H (S^{n-3} \times 0)$$

need not be a Riemannian submersion metric and - contrary to the surgery principle explained in Section 2 - the consideration of such a metric does not seem to be of much use for our purposes because the representation sphere $S(W)$ is of dimension one and hence does not carry a Riemannian metric of positive scalar curvature.

From now on, we will write S instead of S^{n-3} for simplicity. On the orbit space

$$N/(S^1, \tau) = S \times D(W)/H$$

we obtain an induced metric of positive scalar curvature away from the singular locus $S \times 0$ because g is scaled. In a first step we will deform this metric near the singular locus so that it can be extended to a smooth metric of positive scalar curvature on $S \times D^2$. Here we identify the cone factor $D(W)/H$ with D^2. Roughly speaking, this deformation is possible because the tip of the cone $D(W)/H$ can be viewed as a source of a large amount of positive scalar curvature which can be distributed over a neighbourhood of the singularity in $D(W)/H$.

One of the main technical problems is the extension of the induced metric on $S \times (D(W) \setminus 0)/H$ to a smooth metric on $S \times D^2$ in a well controlled way. At this point we make essential use of the additional rotation symmetry (induced by σ) of this metric.

The metric g can be pulled back along the quotient map

$$S^1 \times (S \times D(W)) \to S^1 \times_H (S \times D(W))$$

and yields a metric on $S^1 \times (S \times D(W))$ which is invariant under the usual S^1-rotation actions on S^1 and on $D(W)$.

Ater dividing out the (free) S^1-action on the S^1-factor (this action is lifted from τ), we get an induced metric h on $S \times D(W)$ which is of positive scalar curvature (because the action τ is scaled) and invariant under the rotation action on the $D(W)$-factor.

The space $N/(S^1, \tau)$ can now be identified with the orbit space of $S \times D(W)$ under the H-action on $D(W)$. We consider the (in general not orthogonal) canonical splitting of bundles over $S \times D(W)$

$$T(S \times D(W)) \cong TS \oplus T(D(W))\,.$$

Using local coordinates $u_1, \ldots, u_m$ (where $m = n - 3$) on S and polar coordinates (r, θ) on $D(W) \setminus 0$, the restriction of h to $S \times (D(W) \setminus 0)$ can be written as

$$\sum_{1 \le i \le j \le m} \alpha_{ij}(u, r) du_i du_j + \sum_{1 \le i \le m} \beta_i(u, r) du_i d\theta + dr^2 + \gamma(u, r)^2 d\theta^2\,.$$

Summands of the form $du_i dr$ are not needed here because the radial lines in $(S \times D(W), h)$ are orthogonal to $S \times S^1_r$ for each $0 < r \le 1$. Here and in what follows, S^1_r will be the r-sphere in $D(W) = D^2 \subset \mathbb{C}$. Furthermore, because h is invariant under rotation of $D(W)$, the coefficient γ does not depend on θ. For later use we note

Lemma 26. *Each of the summands*

$$\sum_{1 \le i \le m} \beta_i(u, r) du_i d\theta$$

and

$$dr^2 + \gamma(u, r)^2 d\theta^2$$

extends to a smooth $(0, 2)$*-tensor on* $S \times D(W)$.

Proof. Under the conversion of polar into cartesian coordinates

$$\begin{aligned} x &= r \cos\theta\,, \\ y &= r \sin\theta \end{aligned}$$

on $D(W)$, the first of the displayed terms is transformed into the sum of those components of the metric 2-tensor of $(S \times (D(W) \setminus 0), h)$ containing $du_i dx$ or $du_i dy$ and the second of the displayed terms is transformed into the sum of those components containing dx^2, $dxdy$ and dy^2. Because $h|_{S \times (D(W) \setminus 0)}$ obviously extends to $S \times D(W)$, each of these seperate summands extends to a smooth $(0, 2)$-tensor on $S \times D(W)$. $\square$

Using the standard identification $D(W)/H = D^2$ which is induced by the map

$$S^1 \to S^1\,,\; x \mapsto x^{|H|}\,,$$

(here we use effectiveness of the H-action on W) we get a homeomorphism

$$S \times D(W)/H \approx S \times D^2\,,$$

which is a diffeomorphism on $S \times (D(W) \setminus 0)/H$. Hence (and because W is a unitary H-representation), the metric h induces a metric q on $S \times (D^2 \setminus 0)$ which is given by

$$q = \sum \alpha_{ij} du_i du_j + \frac{1}{|H|} \sum \beta_i du_i d\theta + dr^2 + \frac{\gamma^2}{|H|^2} d\theta^2$$

using again polar coordinates (r, θ) on $D^2 \setminus 0$ and the fact that γ is independent of θ. This metric cannot be extended to a smooth metric on $S \times D^2$. One reason is that the partial derivative

$$\frac{1}{|H|} \cdot \frac{\partial \gamma(u,r)}{\partial r}|_{r=0} = \frac{1}{|H|}$$

is different from 1 (this fact corresponds to the conelike form of this metric). For a general comparison of metrics given in polar and cartesian coordinates, see [23], 1.3.4. However, this failure can be remedied by using a “bending inwards”-process of the following form.

We fix the slope

$$c := \frac{\sqrt{|H|^2 - 1}}{|H|}$$

and consider the affine function

$$\kappa : [0,1] \to \mathbb{R}\,,\; r \mapsto -cr + c\,.$$

Taking into account that

$$\frac{1}{|H|^2} + c^2 = 1\,,$$

the metric q away from the singular locus $S \times 0$ is the induced metric on the hypersurface

$$\{((u, r, \theta), t) \in (S \times (D^2 \setminus 0), \widetilde{q}) \times (\mathbb{R}, dt^2) \mid (r, t) = (r, \kappa(r))\}\,.$$

Here we use the metric

$$\widetilde{q} := \sum \alpha_{ij} du_i du_j + \frac{1}{|H|} \sum_i \beta_i du_i d\theta + \frac{1}{|H|^2}(dr^2 + \gamma^2 d\theta^2)$$

on $S \times (D^2 \setminus 0)$ and the canonical metric dt^2 on $\mathbb{R}$.

The first statement of the following lemma is the main reason why g was assumed to be normally symmetric.

Lemma 27. *The metric $\widetilde{q}$ extends to a smooth metric on $S \times D^2$. With respect to this metric, the subset $S \times 0 \subset S \times D^2$ is a totally geodesic submanifold and the radial lines in $S \times D^2$ are geodesics of constant speed $\frac{1}{|H|}$*

Proof. The first claim follows from Lemma 26. Moreover, $S \times 0$ is a totally geoedesic submanifold because $\tilde{q}$ is invariant under the rotation action on the D^2-factor induced by σ.

For proving the last claim, we must show that

$$\nabla_{\partial_r} \partial_r = 0$$

where ∇ is the connection induced by $\tilde{q}$. But this follows directly from the Koszul formula, the definition of $\tilde{q}$ and the fact that radial lines are (unit speed) geodesics for the metric q. □

In this picture, the bending inwards process amounts to replacing the preceding hypersurface by a smooth Riemannian submanifod

$$\Sigma \subset (S \times D^2, \widetilde{q}) \times (\mathbb{R}, dt^2)$$

as follows. Let

$$\lambda : [0,1] \to \mathbb{R}_{\geq 0}\,, r \mapsto \lambda(r)\,,$$

be a smooth function which has the following properties: There is a positive constant C so that

i.) the function $[-1,1] \mapsto \mathbb{R}$, $s \mapsto \lambda(|s|)$, is smooth,

ii.) the second derivative of λ is nonpositive and bounded in absolute value by $2C$,

iii.) there are positive constants ϵ, μ so that $\lambda(r) = \kappa(r)$, if $\mu+\epsilon \leq r \leq 1$ and $\lambda(r) = \lambda(0) - Cr^2$ for $0 \leq r \leq \mu$.

The numbers μ, ϵ and C are to be specified later. For brevity we will refer to such a function λ as a profile of *width* μ, with *bending parameter* C and of *adjusting length* ϵ. The following fact is elementary.

Lemma 28. *Let $C, \mu > 0$ be chosen such that*

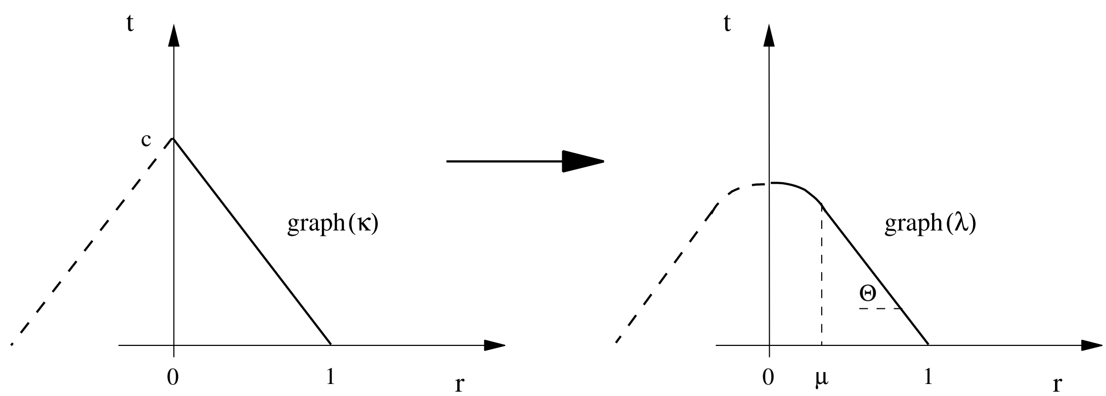
$$2C \cdot \mu > c\,.$$

Then for any $\epsilon > 0$, there exists a profile λ with bending parameter C, of width smaller than μ and of adjusting length smaller than ϵ.

For $r \in [0,1]$ we denote by $\Theta(r) \in [0, \frac{\pi}{2})$ the angle between the graph of λ and the r-axis at the point $(r, \lambda(r))$ and by

$$k(r) = \frac{d\Theta}{d\xi}$$

the curvature of $\mathrm{graph}(\lambda)$ at $(r, \lambda(r))$. Here ξ parametrizes the graph of λ as a unit speed curve in $\mathbb{R}^2$ with initial point $(1,0)$. Because $\lambda'' \leq 0$, the function $k(r)$ is nonpositive for any profile λ.



Depending on a given profile λ, we now consider the smooth Riemannian submanifold

$$\Sigma := \{((u,r,\theta),t) \in (S \times D^2, \widetilde{q}) \times (\mathbb{R}, dt^2) \mid (r,t) \in \mathrm{graph}(\lambda)\}\,.$$

Let $U \subset S$ be an open subset and

$$(e_1, \ldots, e_m)\,,\ e_i \in \Gamma(TU)\,,$$

be an orthonormal frame with respect to the metric $\widetilde{q}|_{S\times 0}(= h|_{S\times 0})$. We get a family $(e_1, \ldots, e_m)$ of orthonormal fields on $U \times D^2$ that are tangential to the submanifolds $S \times S^1_r$, $r \in (0,1]$, by parallel transport along radial lines. The fields

$$e'_{m+1}(u, r, \theta) := \frac{|H|}{\gamma(u,r)} \cdot \partial_\theta\,,\ e_{m+2} := |H| \cdot \partial_r$$

complete this to a family of vector fields

$$(e_1, \ldots, e_m, e'_{m+1}, e_{m+2})$$

defined on $U \times (D^2 \setminus 0)$ and of unit length. This need not be an orthonormal frame. Indeed, e'_{m+1} is orthogonal to e_{m+2}, but it need not be orthogonal to the fields $e_1, \ldots, e_m$, as the submanifolds $\{u\} \times D^2 \subset U \times D^2$ need not be totally geodesic. However, there is a vector field e_{m+1} on $U \times (D^2 \setminus 0)$ that is tangential to the submanifolds $S \times S^1_r$, that satisfies

$$\|e_{m+1}(u, r, \theta) - e'_{m+1}(u, r, \theta)\| = O(r)$$

for each fixed (u, θ) and that yields an orthonormal frame

$$(e_1, \ldots, e_m, e_{m+1}, e_{m+2})$$

on $U \times (D^2 \setminus 0)$: The vector $e_{m+1}(u, r, \theta)$ is given by parallel transport along the radial line

$$l_\theta := \{(u, r, \theta) \mid r \in [0,1]\} \subset U \times D^2$$

of the vector in $T_{(u,0)}(U \times D^2)$ which is tangential to $\{u\} \times D^2$ (hence orthogonal to $S \times 0$) and includes an angle of $\pi/2$ with l_θ. We denote by $s_{S\times D^2}$ the scalar curvature of $(S \times D^2, \widetilde{q})$ and by $s_{S\times S^1_r}$ the scalar curvature of $(S \times S^1_r, \widetilde{q}|_{S\times S^1_r})$. Furthermore, for $r \in (0,1]$, let

$$K_{S\times S^1_r}(u, \theta) = \sum_{1\le i\le m+1} \langle \nabla_{e_i} e_i, e_{m+2}\rangle$$

denote the mean curvature of the submanifold

$$S \times S^1_r \subset (S \times D^2, \widetilde{q})$$

at (u, θ). With these specifications, the scalar curvature s_Σ of Σ at (u, r, θ) is given by the following formula.

Lemma 29 ([30], formula (4.1))**.** *At the point $(u, r, \theta) \in U \times D^2$, we have*

$$s_\Sigma = \cos^2\Theta(r) \cdot s_{S\times D^2} + \sin^2\Theta(r) \cdot s_{S\times S^1_r} + 2k(r)\sin\Theta(r) \cdot K_{S\times S^1_r}\,.$$

We need to describe the behavior of the relevant geometric quantities near $r = 0$.

Lemma 30. i.) *For fixed (u, θ) and varying r, we have $K_{S\times S^1_r} = -\frac{|H|}{r} + O(1)$. Furthermore, the remainder term $O(1)$ depends continuously on (u, θ) and hence - using the normal symmetry of the metric in question - defines a continuous function $S \times D^2 \to \mathbb{R}$.*

ii.) *The scalar curvature $s_{S\times S^1_r}$ is uniformly bounded with respect to $r > 0$ and all (u, θ).*

Proof. For fixed (u, θ) and varying $r \in (0,1]$, we consider the previously described orthonormal base $(e_1, \ldots, e_m, e_{m+1}, e_{m+2})$ of $T_{(u,r,\theta)}(S \times D^2)$. Recall that $e_1, \ldots, e_m$ are actually smooth vector fields on the whole of $U \times D^2$. For $1 \le i, j \le m+1$, let

$$\alpha_{ij}(r) := \langle \nabla_{e_i} e_j, e_{m+2}\rangle$$

be the components of the second fundamental form of $S \times S^1_r \subset (S \times D^2, \widetilde{q})$ at the point (u, r, θ). In a first step we show that for $1 \le i, j \le m$ we have

i.) $\alpha_{ij}(r) = O(r)$,
ii.) $\alpha_{m+1,j}(r) = O(1)$,
iii.) $\alpha_{m+1,m+1}(r) = -\frac{|H|}{r} + O(1)$.

These expansions are proved as follows. For each $1 \leq i, j \leq m$ the expression $\nabla_{e_i} e_j$ defines a smooth vector field on $U \times D^2$ which is tangential to $S \times 0$, because this is a totally geodesic submanifold of $S \times D^2$ (see Lemma 27). This proves equation i.).

For equation ii.) notice that the restriction

$$e_{m+1}|_{U \times (0,1] \times \{\theta\}}$$

can be extended to a smooth vector field on $U \times [-1, 1] \times \{\theta\}$ by parallel transport along radial lines. Hence, the smooth function

$$\alpha_{m+1,j}(u, -, \theta) : (0, 1] \to \mathbb{R}$$

extends to $[-1, 1]$.

For proving the third equation, it is enough (using points i.) and ii.)) to show the asymptotic expansion

$$\langle \nabla_{e'_{m+1}} e'_{m+1}, e_{m+2} \rangle = -\frac{|H|}{r} + O(1)$$

we recall (see Lemma 27) that the radial lines

$$[0, 1] \to (S \times D^2, \widetilde{q}) \, , \; r \mapsto (u, r, \theta)$$

are geodesics of constant speed $\frac{1}{|H|}$. We can therefore use the proof of [7], Lemma 1, applied to the submanifold $\{u\} \times D^2 \subset S \times D^2$ with the restricted metric.

It is clear, that the remainder terms in the expansions i.), ii.) and iii.) depend continuously on (u, θ).

Using these asymptotic expansions, the first claim of Lemma 30 is immediate and the second claim follows from the Gauss equation

$$s_{S \times S^1_r} = 2 \cdot \Big(\sum_{1 \leq i < j \leq m+1} K(e_i, e_j) + \alpha_{ii}\alpha_{jj} - \alpha_{ij}^2 \Big) \, ,$$

where $K(e_i, e_j)$ denotes the sectional curvature of the plane spanned by e_i and e_j. Here it is crucial that

$$\alpha_{ii}(r)\alpha_{m+1,m+1}(r) = O(1)$$

for all $1 \leq i \leq m$ and that $\widetilde{q}$ is a smooth metric on the whole of $S \times D^2$ which is (of course) bounded in the C^2-norm so that these sectional curvatures are uniformly bounded on $S \times D^2$. □

We remark that in principle, this proof works for any smooth metric on $S \times D^2$ with respect to which $S \times 0$ is totally geodesic. The normal symmetry of g was mainly used in order to construct such a metric from the conelike metric on $S \times (D(W)/H)$ induced by g.

We are now in a position to show that the profile λ can be chosen such that the corresponding hypersurface Σ has positive scalar curvature. First we observe that for a profile λ with bending parameter C, Lemma 30 implies for all (u, θ) the important asymptotic expansion

$$2k(r) \sin\Theta(r) \cdot K_{S \times S^1_r}(u, \theta) = (-4C + O(r)) \cdot (2Cr + O(r^2)) \cdot \big(-\frac{|H|}{r} + O(1) \big) = 8|H|C^2 + O(r) \, .$$

Furthermore, the function on the left hand side extends to a continuous function on $S \times D^2$.

Lemma 31. *Let λ_1, λ_2 be two profiles with bending parameters C_1 and C_2 and of widths μ_1 and μ_2, respectively. If $C_1 \leq C_2$, then*

$$0 \geq 2k_1(r)\sin\Theta_1(r) \geq 2k_2(r)\sin\Theta_2(r)$$

for $0 \leq r \leq \min(\mu_1, \mu_2)$.

Proof. for $0 \leq r \leq \min(\mu_1, \mu_2)$, both profiles are in the standard form

$$\lambda_1(r) = \lambda_1(0) - C_1 r^2\,,\ \lambda_2(r) = \lambda_2(0) - C_2 r^2\,.$$

□

This lemma together with the preceding asymptotic expansion make clear that there are $\overline{C}, \overline{\mu} > 0$ so that for any profile λ with bending parameter $C \geq \overline{C}$ and width $\mu \leq \overline{\mu}$, we have

$$2k(r)\sin\Theta(r)\cdot K_{S\times S^1_r}(u,\theta) \geq |s_{S\times D^2}(u,r,\theta)| + |s_{S\times S^1_r}(u,\theta)| + 1$$

for all $0 < r \leq \mu$ and all (u,θ). Note that the right hand side of this equation as well as the value of $K_{S\times S^1_r}(u,r,\theta)$ are independent of the particular profile λ. Using Lemma 28 we can choose a profile with data C, μ satisfying these conditions. Furthermore, we get

$$K_{S\times S^1_r}(u,\theta) < 0$$

for $0 < r \leq \overline{\mu}$ and all (u,θ).

Again referring to Lemma 28, we can assume that the adjusting length of λ is arbitrarily small. This is helpful because there is a constant $s > 0$ so that for any profile λ we have

$$\cos^2\Theta(r)\cdot s_{S\times D^2}(u,r,\theta) + \sin^2\Theta(r)\cdot s_{S\times S^1_r}(u,\theta) \geq s$$

as long as $r \geq \mu + \epsilon$ (this is the region where $\lambda = \kappa$, so in particular the left hand side of this inequality is the scalar curvature of the metric q). Here we use the fact that $s_{(S\times(D^2\setminus 0),q)}$ is bounded below by the same constant s as the scalar curvature of $(S \times (D(W) \setminus 0), h)$ - but this metric on $S \times (D(W) \setminus 0)$ of course extends to a positive scalar curvature metric on $S \times D(W)$ and consequently $s > 0$. Using the inequality $|\lambda''| \leq 2C$, which holds for any profile with bending parameter C, we know that for small enough ϵ, the angle $\Theta(r)$ changes so little in the region $r \in [\mu, \mu + \epsilon]$ that the previous sum is larger than $s/2 > 0$ for all $r \geq \mu$. Because we can additionally assume that $\mu + \epsilon \leq \overline{\mu}$, we conclude (by Lemma 29) that the hypersurface Σ has positive scalar curvature for $r \geq \mu$ (this uses $k(r) \leq 0$ for any profile and any r). But once $r \leq \mu$, we are in the safe region where the last term

$$2k(r)\sin\Theta(r)\cdot K_{S\times S^1_r}(u,\theta)$$

dominates - with a margin of at least 1 - the sum of the absolute values of the first two terms in the expression of the scalar curvature of Σ in Lemma 29. Hence (and with a continuity argument for $r = 0$) with this profile λ, the induced metric on Σ is of positive scalar curvature.

The projection

$$\mathbb{R}^2 \to \mathbb{R}^2\,,\ (r,t) \mapsto (r,0)$$

induces a diffeomorphism

$$\Sigma \approx S \times D^2$$

and we finally get an induced metric on $S \times D^2$ which is of positive scalar curvature and coincides near the boundary $S \times \partial D^2$ with the metric on $S \times S(W)/H$ induced by g.

In order to complete the proof of Theorem 25, we choose a trivialization of the S^1-principal bundle

$$S^1 \hookrightarrow S^1 \times_H (S^{n-3} \times S(W)) \to S^{n-3} \times S(W)/H\,.$$

Now we pick an S^1-principal connection ω on the total space of the trivial S^1-principal bundle

$$S^1 \hookrightarrow S^1 \times (S^{n-3} \times D^2) \to S^{n-3} \times D^2$$

which - after applying the above trivialization - coincides near the boundary $S^1 \times (S^{n-3} \times S^1)$ with the S^1-connection on the total space of

$$S^1 \hookrightarrow S^1 \times_H (S^{n-3} \times (D(W) \setminus 0)) \to S^{n-3} \times (D(W) \setminus 0)/H$$

which is induced by viewing g as a Riemannian submersion metric on this fibre bundle.

For $\epsilon > 0$ we now consider the associated Riemannian submersion metric on

$$S^1 \hookrightarrow S^1 \times (S^{n-3} \times D^2) \to S^{n-3} \times D^2$$

with fibres S^1 of constant length ϵ, horizontal subspaces induced by ω and the smooth positive scalar curvature metric on $S^{n-3} \times D^2$ constructed before. It follows from O'Neill that for small enough ϵ, this metric has positive scalar curvature. Without loss of generality (possibly after shrinking the orbits in $M \setminus N$) we can assume that the metric on $M \setminus N$ is of the same scale (without violating the positive scalar curvature property). Hence the two metrics on $M \setminus N$ and on $S^1 \times (S^{n-3} \times D^2)$ can be combined such as to define an S^1-invariant metric of positive scalar curvature on M'. This completes the proof of Theorem 25.

5. Fixed point free S^1-manifolds

The main purpose of this section is to prove Theorem B from the introduction (see Theorem 35 below). This proof is summarized as follows. A theorem of Ossa [22] states that any oriented fixed point free S^1-manifold M satisfying condition $\mathbb{C}$ is the boundary of an oriented S^1-manifold W (possibly with fixed points). However, our Theorem A cannot be applied directly because W may contain singular strata of codimension 2 that are disjoint from M. Different ideas are needed to handle this problem. At first, we remove tubular neighbourhoods of the components of the fixed point set W^{S^1} (which are disjoint from M as $M^{S^1} = \emptyset$). This produces new boundary components of W carrying invariant metrics of positive scalar curvature. This last statement follows from the O'Neill formula if the codimension of the corresponding fixed component in W is larger than 2, and from the structure of the oriented bordism ring of free S^1-manifolds, which will be explained in the proof of Proposition 33 below, if this codimension is equal to 2 (we can assume that the action on W is effective). Let Z be the resulting fixed point free bordism. Unfortunately, the remaining codimension-2 singular strata can be embedded in Z in a complicated way. For example, their closures may have nonempty intersections with each other (cf. Example 4). The resulting problems can be circumvented by cutting out small equivariant tubes in Z_{max} connecting M with those codimension-2 singular strata that are disjoint from M. In this way, we add new singular strata to M, but in any case Theorem A can be applied. Consequently, the manifold M' obtained from M by adding these singular strata (and adding certain free 2-handles, but we ignore this step for the moment) admits an S^1-invariant metric of positive scalar curvature. It turns out that M can be recovered from M' by performing surgery steps as explained in the previous Section 4. In particular, thanks to Theorem 25, the original manifold M admits an invariant metric of positive scalar curvature. The details of this argument are explained in the proof of Theorem 34.

Let us start with the following observation concerning free G-manifolds.

Lemma 32. *Let M be a closed manifold equipped with a free G-action. If the identity component of G is abelian, then the following assertions are equivalent:*

i.) *M admits a G-invariant metric of positive scalar curvature.*
ii.) *M/G admits a metric of positive scalar curvature.*

Proof. The case of finite G is immediate. The general case is Theorem C in [2]. □

Note that this fact is not true for connected nonabelian G. An easy counterexample is given by $M = \mathrm{SU}(2) \times S^1$ with a bi-invariant Riemannian metric on $\mathrm{SU}(2)$ (which has positive scalar curvature) and $\mathrm{SU}(2)$ acting freely on the first factor in $\mathrm{SU}(2) \times S^1$.

Before we state the next proposition, we remind the reader of the following basic fact (cf. [4]). Let M and N be closed oriented manifolds equipped with free orientation preserving G-actions. Then the following assertions are equivalent:

i.) There is a compact oriented G-bordism W between M and N such that G acts freely and orientation preserving on W.
ii.) Consider the orbit manifolds M/G and N/G together with the maps $f_M : M/G \to BG$ and $f_N : N/G \to BG$ classifying the respective G-principal bundles. Then $f_M : M/G \to BG$ and $f_N : N/G \to BG$ define the same bordism class in $\Omega_*^{SO}(BG)$.

The following proposition contains our first general existence result of invariant metrics of positive scalar curvature on S^1-manifolds.

Proposition 33. *Let M be a closed oriented free S^1-manifold of dimension at least 6 which is simply connected and does not admit a spin structure. Then M carries an S^1-invariant metric of positive scalar curvature.*

Proof. We give two proofs of this fact.

The long exact homotopy sequence of the S^1-fibration

$$S^1 \hookrightarrow M \xrightarrow{\pi} M/S^1$$

shows that M/S^1 is also simply connected. Furthermore,

$$TM \cong \pi^*(T(M/S^1)) \oplus \underline{\mathbb{R}}$$

with a trivial line bundle $\underline{\mathbb{R}}$, and hence M/S^1 does not admit a spin structure by an easy characteristic class calculation. By the Gromov-Lawson theorem stated in the introduction, M/S^1 admits a metric of positive scalar curvature and by Lemma 32, the manifold M admits an S^1-invariant metric of positive scalar curvature.

The second proof is independent of [2] and a little longer, but can later be generalized to a wider class of actions.

By a standard use of the Atiyah-Hirzebruch spectral sequence,

$$\Omega_*^{SO}(BS^1) \cong \Omega_*^{SO}[x_1, x_2, \ldots]$$

is a graded polynomial ring in indeterminates x_i of degree $2i$ (recall that $BS^1 = \mathbb{C}P^\infty$). The variable x_i can be assumed to correspond to the free S^1-bordism class represented by the sphere

$$S^{2i+1} \subset \mathbb{C}^{i+1}$$

equipped with the standard free S^1-action. Obviously, these manifolds carry S^1-invariant metrics of positive scalar curvature. Together with the fact that each element in Ω_*^{SO} can be represented by a manifold admitting a positive scalar curvature metric (see [7]), this implies that each element in $\Omega_*^{SO}(BS^1)$ is represented by a free S^1-manifold carrying an invariant metric of positive scalar curvature.

We conclude that the given manifold M is bordant to an S^1-manifold admitting an invariant metric of positive scalar curvature and moreover the bordism W can be assumed to be an oriented free S^1-manifold. We need to show that W can be improved in such a way that the inclusion $M \hookrightarrow W$ is a 2-equivalence.

Because $\mathbb{C}P^\infty$ is simply connected, we can kill the fundamental group in W/S^1 by surgeries over (i.e. with reference maps to) $\mathbb{C}P^\infty$.

Comparing the long exact homotopy sequences induced by the commutative diagram of fibrations

$$\begin{array}{ccccc} S^1 & \longrightarrow & M & \longrightarrow & M/S^1 \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & W & \longrightarrow & W/S^1 \end{array}$$

we see that the new bordism W is simply connected.

Now let $c \in \pi_2(W/S^1)$ represent an element in the cokernel of the map

$$\pi_2(M/S^1) \to \pi_2(W/S^1)\,.$$

We can represent c by an embedded 2-sphere $S^2 \subset W/S^1$ (recall that $\dim W/S^1 \geq 5$). Let λ be the image of c under the map

$$\pi_2(W/S^1) \to \pi_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$$

which is induced by the reference map $W/S^1 \to \mathbb{C}P^\infty$. Before we can kill c by surgery over $\mathbb{C}P^\infty$ we must make sure that $\lambda = 0$ (this would be automatic if we replaced S^1 by a finite group G because then $\pi_2(BG) = 0$) and that the normal bundle of $S^2 \subset W/S^1$ is trivial.

In order to achieve these requirements we consider the commutative diagram

$$\begin{array}{ccccc} \pi_2(M/S^1) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(M) = \{1\} \\ \downarrow & & {\scriptstyle =}\downarrow & & \\ \pi_2(W/S^1) & \longrightarrow & \pi_1(S^1) & & \\ \downarrow & & {\scriptstyle =}\downarrow & & \\ \pi_2(\mathbb{C}P^\infty) & \xrightarrow{\cong} & \pi_1(S^1) & & \end{array}$$

which is induced by the composition

$$M/S^1 \hookrightarrow W/S^1 \to \mathbb{C}P^\infty$$

and whose horizontal maps are connecting homomorphisms in the long exact homotopy sequences of the respective S^1-fibrations. Because the first horizontal map is surjective, we find an element

$$y \in \pi_2(M/S^1)$$

which goes to λ under the map

$$\pi_2(M/S^1) \to \pi_1(S^1)\,.$$

After replacing c by $c - y$ we can therefore assume that $\lambda = 0$. If the second Stiefel-Whitney class of W/S^1 evaluated on (the new) c is nontrivial, we pick an element $x \in \pi_2(M)$ on which the second Stiefel Whitney class of M evaluates nontrivially (a spherical class in $H_2(M;\mathbb{Z})$ with this property exists, because M is not spin and simply connected). Now we replace c by $c + \eta(x)$ where

$$\eta : \pi_2(M) \to \pi_2(M/S^1) \to \pi_2(W/S^1)$$

is the obvious map. This will preserve the property that $\lambda = 0$ because the composition

$$\pi_2(M) \to \pi_2(M/S^1) \to \pi_1(S^1)$$

is zero.

This shows that we can indeed kill c by surgery over $\mathbb{C}P^\infty$. Because $\pi_2(W/S^1)$ is finitely generated, we can therefore (after finitely many surgery steps) assume that the inclusion

$$M/S^1 \hookrightarrow W/S^1$$

is a 2-equivalence and the same is then true for the inclusion $M \hookrightarrow W$. An application of Theorem 15 finishes the proof of Proposition 33. □

Before we generalize the last proposition to fixed point free S^1-manifolds, we show that assumption iii.) in Theorem 15 can be avoided in the case of fixed point free S^1-actions whose union of maximal orbits is simply connected and not spin. Here the surgery procedure explained in Section 4 will be used.

Theorem 34. *Let Z be a compact connected oriented fixed point free S^1-bordism between the closed S^1-manifolds X and Y. Assume that Z satisfies condition $\mathbb{C}$ and that the following hold:*

- i.) *The cohomogeneity of Z is at least 6,*
- ii.) *the union of maximal orbits Y_{max} is simply connected and does not admit a spin structure.*

Then, if X admits an S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension 2, the same is true for Y.

Proof. Let $n = \dim X$ (i.e. $\dim Z = n + 1$). By Lemma 23 we may assume that the given metric on X is scaled. Now let

$$F \subset Z$$

be a codimension-2 singular stratum in Z (i.e. $\dim F = n - 1$) which has empty intersection with Y and is therefore problematic in view of Theorem 15. By assumption, the isotropy group $H \subset S^1$ of F is finite. Let $\Omega \subset F$ be an orbit. It follows from the slice theorem that Ω has an S^1-invariant closed tubular neighbourhood N in Z which is S^1-diffeomorphic to

$$S^1 \times_H (D^{n-2} \times D(W))$$

where W is a one dimensional unitary H-representation (because the given action on Z satisfies condition $\mathbb{C}$) and S^1 acts only on the S^1-factor. We can assume that the S^1-action on Z is effective and hence H acts effectively on W. The idea is to alter Z by cutting out an equivariant tube in Z_{max} which connects N and Y.

We write

$$\partial N = S^1 \times_H \big((D^{n-2} \times S(W)) \cup (S^{n-3} \times D(W)\big)$$

and use the H-invariant subset

$$T := \{e^{\frac{2\pi i \omega}{|H|}} \mid \omega \in \bigcup_{k=0}^{|H|-1} [k, k + \frac{1}{2|H|}]\} \subset S(W) = S^1$$

to define the S^1-invariant submanifold

$$B := S^1 \times_H \left(D^{n-2}_{[0,1/2]} \times T\right) \subset \partial N$$

(the subscript at D^{n-2} indicates restriction of the radial coordinate). The S^1-principal bundle

$$S^1 \hookrightarrow B \to D^{n-2}_{[0,1/2]} \times T/H \approx D^{n-1}$$

is trivial. Hence, by the connectivity of Z_{max}/S^1, there exists an orientation preserving S^1-equivariant embedding

$$\Psi : (S^1 \times D^{n-1}) \times [0,1] \to Z_{max}$$

(with S^1-acting freely on the S^1-factor) which restricts to an S^1-equivariant diffeomorphism

$$S^1 \times D^{n-1} \times \{0\} \approx B \subset \partial N$$

and to an embedding

$$S^1 \times D^{n-1} \times \{1\} \hookrightarrow Y$$

and satisfies

$$\Psi\big(S^1 \times D^{n-1} \times (0,1)\big) \subset Z \setminus (Y \cup N)\,.$$

We now consider the S^1-bordism

$$Z' := Z \setminus \big(N \cup \mathrm{im}(\Psi)\big)\,.$$

In this bordism, the manifold Y is replaced by another manifold Y' which contains a new codimension-2 singular stratum

$$\Sigma := S^1 \times_H (S^{n-3} \times 0) \subset S^1 \times_H (S^{n-3} \times D(W)) \subset \partial N\,.$$

We claim that Y can be recovered from Y' by resolving Σ. The argument goes as follows: The construction of Y' yields an embedding

$$\phi' : S^1 \times_H (S^{n-3} \times D(W)) \hookrightarrow Y'$$

of a tubular neighbourhood of $\Sigma \subset Y'$ and the manifold $Y' \setminus \mathrm{im}(\phi')$ can be written as

$$\big(Y \setminus \Psi(S^1 \times D^{n-1} \times \{1\})\big) \cup_{S^1 \times S^{n-2} \times \{1\}} \big(S^1 \times \partial D^{n-1} \times [0,1]\big) \cup_{\partial B} A$$

where

$$A := S^1 \times_H \big((D^{n-2} \times S(W)) \setminus (D^{n-2}_{[0,1/2]} \times T)\big) \subset \partial N\,.$$

The S^1-action on A is free and the quotient space A/S^1 is diffeomorphic to $(D^{n-2} \times S^1) \setminus D$ where $D = B/S^1$ is a submanifold of $D^{n-2} \times S^1$ diffeomorphic to D^{n-1}. Because $n \geq 6$ by assumption, all principal S^1-bundles over A/S^1 are isomorphic and hence there is an S^1-equivariant diffeomorphism.

$$A \approx S^1 \times \big((D^{n-2} \times S^1) \setminus D\big)\,.$$

We conclude that there is an S^1-equivariant diffeomorphism

$$Y' \setminus \mathrm{im}(\phi') \approx \mathrm{Y} \setminus \mathrm{im}(\phi)$$

where

$$\phi : S^1 \times (S^{n-3} \times D^2) \hookrightarrow Y_{max}$$

is an S^1-equivariant orientation preserving embedding whose image is contained in the S^1-equivariant coordinate chart

$$\Psi(S^1 \times D^{n-1} \times \{1\}) \subset Y\,.$$

(Note the standard decomposition $S^{n-1} = (D^{n-2} \times S^1) \cup (S^{n-3} \times D^2)$.) It follows that we can write Y as

$$\big(Y' \setminus \mathrm{im}(\phi')\big) \cup \big(S^1 \times (S^{n-3} \times D^2)\big)$$

and this proves that Y can be recovered from Y' by a resolution of Σ.

In particular (using Theorem 25), if we can show that Y' admits a scaled S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension 2, the same holds for Y.

Because the embedded S^1-manifold

$$\phi(S^1 \times (S^{n-3} \times 0)) \subset Y_{max}$$

is contained in an S^1-equivariant coordinate chart, it can be assumed to be disjoint from some embedded 2-sphere in Y_{max} with nontrivial normal bundle (such a 2-sphere exists because Y_{max} does not admit a spin structure). This implies that Y'_{max} does not admit a spin structure, either. We would like Y'_{max} to be simply connected, too. However, this need not be the case due to the existence of a non-nullhomotopic linking sphere

$$S^1 \subset Y'_{max}/S^1$$

of $\Sigma/S^1 = S^{n-3} \subset Y'_{max}$. But this problem can be solved as follows: Before we perfom the cutting-out procedure on Z, we attach a free S^1-equivariant 2-handle $S^1 \times (D^2 \times D^{n-2})$ to

$$S^1 \times (S^{n-3} \times D^2) \subset Y_{max} \subset \partial Z$$

(here we suppress the identification ϕ) along

$$S^1 \times (S^1 \times D^{n-2}) \subset S^1 \times (S^{n-3} \times D^2)$$

where $\{1\} \times (S^1 \times D^{n-2})$ is identified with a small tubular neighbourhood of $\{1\} \times \{p\} \times S^1_{1/2} \subset \{1\} \times S^{n-3} \times D^2$. Here, $p \in S^{n-3}$ is an arbitrary point and $S^1_{1/2} \subset D^2$ is the circle of radius $1/2$. The space which is obtained from Z by attaching this free 2-handle is denoted by $\widetilde{Z}$. By construction, we can attach a further free S^1-equivariant 3-handle to $\widetilde{Z}$ which may be canceled against the previously attached 2-handle. Furthermore (by a backward use of the Seifert-van Kampen theorem)

$$\pi_1(\widetilde{Y} \setminus \big(S^1 \times (S^{n-3} \times D^2_{[0,1/4]})\big) = \{1\}$$

where $\widetilde{Y}$ is the space obtained from Y by performing the surgery associated to the additional free 2-handle (this can be assumed not to affect the subset $S^1 \times (S^{n-3} \times D^2_{[0,1/4]}) \subset Y$). The old cutting out process on Z can also be performed on the new bordism $\widetilde{Z}$ because it can be assumed only to affect the part

$$S^1 \times (S^{n-3} \times D^2_{[0,1/4]}) \subset \widetilde{Y}\,.$$

The same procedure (i.e. attaching a free dummy 2-handle and cutting out a certain part of the bordism) is now applied to all other singular strata of codimension 2 in Z. In this way, we end up with an S^1-bordism Z' in which all singular strata of codimension 2 have nonempty intersection with Y' and Y'_{max} is simply connected and does not admit a spin structure.

We now attach equivariant handles to Z'_{max} as in the proof of Proposition 33 to make sure that the inclusion

$$Y'_{max} \hookrightarrow Z'_{max}$$

is 2-connected. Here we note that Z'_{max} has finitely generated fundamental group and homology groups so that in any case, only finitely many surgery steps on Z'_{max}/S^1 are needed.

Theorem 15 together with its refinement formulated in Lemma 24 implies that Y' admits a scaled S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension 2. By Theorem 25, this also holds for the manifold obtained from Y' by resolving the singularities created by the cutting out processes. But then, as explained before, the attachment of the dummy free 2-handles (which we can assume to be disjoint from each other) can be neutralized by attaching free 3-handles. These handles have codimension $\operatorname{coh}(Y, S^1) - 2$ which is at least 3 by assumption. Therefore, using the equivariant surgery principle, Theorem 2, this step also preserves the S^1-invariant scalar curvature metric and the resulting space - which can be identified with Y - indeed carries an S^1-invariant metric of positive scalar curvature. □

The following is Theorem B from the introduction.

Theorem 35. *Let M be a closed fixed point free S^1-manifold satisfying condition $\mathbb{C}$ and of cohomogeneity at least 5. If M_{max} is simply connected and does not admit a spin structure, then M admits an S^1-invariant metric of positive scalar curvature.*

Proof. Without loss of generality, the given S^1-action on M is effective. Because M satisfies condition $\mathbb{C}$, the singular strata in M are of codimension at least 2 and hence M is simply connected by a general position argument. In particular, it is an orientable S^1-manifold. By [22], Satz 1, the manifold M is the boundary of an oriented (connected) S^1-manifold W satisfying condition $\mathbb{C}$. Let

$$F_1, \ldots, F_k$$

be the components of W^{S^1}. By assumption, these are disjoint from M. We cut out pairwise disjoint S^1-invariant tubular neighbourhoods N_i of F_i in W. This yields an oriented fixed point free bordism satisfying condition $\mathbb{C}$ from M to another S^1-manifold with components ∂N_i, $1 \leq i \leq k$. Each N_i is the total space of a unitary S^1-equivariant fibre bundle

$$V_i \hookrightarrow N_i \to F_i$$

with a unitary S^1 representation V_i. If the codimension of F_i is larger than 2, then ∂N_i carries an S^1-invariant metric of positive scalar curvature which is normally symmetric in codimension 2 by the O'Neill formula in combination with $\dim S(V_i) \geq 2$ and Example 22. If $\operatorname{codim} F_i = 2$, then, because the action on W is effective, the S^1-action on V_i is effective and therefore the induced action on ∂N_i is free. As explained in the second proof of Proposition 33, ∂N_i is then freely and oriented bordant to a free S^1-manifold admitting an invariant metric of positive scalar curvature. Now Theorem 35 follows from Theorem 34. □

One might ask whether Theorem 35 can be proven without the somewhat involved discussion of codimension-2 singular strata in Section 4 and Theorem 34 under the assumption that M does not contain such strata. But a closer look at [22] reveals that in general (depending on the dimensions of the isotypical summands of the normal representations around the singular strata in M) the zero bordism W does contain codimension-2 singular strata with finite isotropies, even if M does not.

Unfortunately, we do not have such a general existence result for S^1-manifolds with fixed points. One can check that the oriented S^1-bordism ring (always restricting to actions satisfying condition $\mathbb{C}$) is generated by S^1-manifolds admitting S^1-invariant metrics of positive scalar curvature. This follows from an inspection of the generators constructed in [12]. However, if we express a given S^1-manifold without codimension-2 singular strata in terms of these generators, it might happen that generators with codimension-2 singular strata do appear and this leads to S^1-handle decompositions of the given S^1-bordism containing handles of codimension 0 or 2, cf. Proposition 17.

However, we do not know if a surgery principle as explained in Section 4 exists for S^1-manifolds with fixed points (note that invariant metrics on such manifolds can never be scaled).

In some special situations, one can construct the necessary bordisms by hand. For example, we have the following result for semifree S^1-manifolds with isolated fixed points.

Theorem 36. *Let M be a closed simply connected non-spin manifold of even dimension at least 6 and equipped with a semifree S^1-action (i.e. the action has either free or fixed orbits) with only isolated fixed points. Then M admits an S^1-invariant metric of positive scalar curvature.*

Proof. Let $2n$ be the dimension of M. After removing small invariant discs around the fixed points and dividing out the free S^1-action, we get a zero bordism over BS^1 of a disjoint union of copies of $\pm\mathbb{C}P^{n-1}$ where the reference maps to BS^1 classify the tautological line bundle over $\mathbb{C}P^{n-1}$. This classifying map $\mathbb{C}P^{n-1} \to BS^1$ generates a $\mathbb{Z}$-summand in $\Omega^{SO}_{2n-2}(BS^1)$ and therefore we get as many $-$-signs as we get $+$-signs. By pairwise connecting a positively oriented fixed point with a negatively oriented one by thin tubes we obtain an oriented S^1-bordism W from a free S^1-manifold N to the given manifold. Using the structure of $\Omega^{SO}_*(BS^1)$ (cf. the second proof of Proposition 33) we can assume (possibly after adding a free oriented S^1-bordism to N) that N has an invariant metric of positive scalar curvature. Furthermore, by an argument similar to the second proof of Proposition 33, the inclusion $M_{max} \hookrightarrow W_{max}$ can be assumed to be a 2-equivalence. Theorem 15 now implies that the manifold M admits an S^1-invariant metric of positive scalar curvature because assumption iii.) obviously holds. $\square$

REFERENCES

[1] S. Araki, K. Kawakubo, *Equivariant s-cobordism theorems*, J. Math. Soc. Japan **40** (1988), 349-367.

[2] L. Bérard Bergery, *Scalar curvature and isometry group*, in: *Spectra of Riemannian manifolds*, Kaigai Publications, Tokyo (1983), 9-28.

[3] A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete **10**, Springer-Verlag 1987.

[4] P. Conner, E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete **33**, Springer-Verlag 1964.

[5] T. tom Dieck, *Transformation groups*, de Gruyter studies in mathematics **8**, Walter de Gruyter 1987.

[6] C. Farsi, *Equivariant spin bordism in low dimensions*, Topology Appl. **43** (1992), 167-180.

[7] M. Gromov, H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. Math. **111** (1980), 209-230.

[8] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1-55.

[9] K. Kawakubo, *Stable equivalence of G-manifolds*, in *Homotopy theory and related topics* (Kyoto, 1984), Adv. Stud. Pure Math. **9**, 27-40, North-Holland, Amsterdam, 1987.

[10] K. Kawakubo, *G-s-Cobordism theorems do not hold in general for many compact Lie groups G*, in *Transformation groups* (Osaka 1987), Lecture Notes in Mathematics **1375**, Springer Verlag.

[11] A. Kosinski, *Differential manifolds*, Volume 138 in pure and applied mathematics, Academic Press 1993.

[12] C. Kosniowski, M. Yahia, *Unitary bordism of circle actions*, Proc. Edinb. Math. Soc. (2) **26** (1983), 97-105.

[13] H. B. Lawson, S. T. Yau, *Scalar curvature, non-abelian group actions and the degree of symmetry of exotic spheres*, Comment. Math. Helv. **49** (1974), 232-244.

[14] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris, Sér. A-B **257** (1963), 7-9.

[15] J. Lott, *Signatures and higher signatures of S^1-quotients*, Math. Ann. **316** (2000), 617-657.

[16] W. Lück, *A basic introduction to surgery theory*, in: *Topology of high dimensional manifolds* (Trieste, 2001), ICTP Lect. Notes **9**, 1-224.

[17] W. Lück, *Transformation groups and algebraic K-theory*, LNM **1408**, Springer-Verlag.

[18] K.-H. Mayer, *G-invariante Morsefunktionen*, Manuscripta Math. **63** (1989), 99-114.

[19] J. Milnor, *Morse theory*, Annals of Mathematical studies **51** (1963), Princeton University Press.

[20] J. Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press 1965.

[21] L. Nicolaescu, *Notes on Seiberg-Witten theory*, Graduate Studies in Mathematics **28** (2000), AMS.

[22] E. Ossa, *Fixpunktfreie S^1-Aktionen*, Math. Ann. **186** (1970), 45-52.

[23] P. Petersen, *Riemannian Geometry*, Springer GTM **171**, 1997.

[24] J. Rosenberg, S. Stolz, *Metrics of positive scalar curvature and connections with surgery*, in: *Surveys on Surgery Theory* (vol. 2), S. Cappell, A. Ranicki, J. Rosenberg (eds.), Annals of Mathematical Studies **149** (2001), Princeton University Press.

[25] J. Rosenberg, S. Weinberger, *Higher G-indices and applications*, Ann. Sci. École Norm. Sup. (4) **21** (1988), 479-495.

[26] R. Schoen, S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159-183.

[27] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. Math. **136** (1992), 511-540.

[28] S. Stolz, *Positive scalar curvature metrics - existence and classification questions*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 625–636.

[29] A. Wassermann, *Equivariant differential topology*, Topology **8** (1969), 127-150.

[30] J. Wehrheim, *Positive Skalarkrümmung auf Viermannigfaltigkeiten*, Diplomarbeit (2003), Universität München.