# ACTIONS OF FINITE p-GROUPS ON HOMOLOGY MANIFOLDS

BERNHARD HANKE

Abstract. Let a cyclic group of odd prime order $p$ act on a $\mathbb{Z}_{(p)}$-Poincaré duality space $X$. We prove a relation between the Witt classes associated to the $\mathbb{F}_p$-cohomology rings of the fixed point set of this action and of $X$. This is applied to show a similar result for actions of finite $p$-groups on $\mathbb{Z}_{(p)}$-homology manifolds.

## 1. Introduction

Finding relations between invariants of the space on which a continuous group action is defined and the fixed point set of this action is of fundamental interest in the theory of transformation groups ([4, 7, 13, 15]). Let $p$ be an odd prime number. By a basic result of Smith theory proven first by G.E. Bredon, T. Chang and T. Skjelbred ([8, 12]), each component of the fixed point set of a $\mathbb{Z}/p$-action on an $\mathbb{F}_p$-Poincaré duality space fulfills Poincaré duality over $\mathbb{F}_p$. This immediately generalizes to finite $p$-group actions on $\mathbb{F}_p$-Poincaré duality spaces by a well known induction approach, using the fact that every nontrivial finite $p$-group contains a normal subgroup of order $p$. On the other hand, the above result by Bredon, Chang and Skjelbred leads one to explore relations between several symmetric inner product spaces associated to the cohomology structures that arise in this context - analogous to the study of intersection forms in the theory of smooth group actions using the Atiyah-Singer $G$-signature theorem. J.P. Alexander and G.C. Hamrick in [1] prove such a relation in the Witt ring $W(\mathbb{F}_p)$ for $\mathbb{Z}/p$-actions on integral Poincaré duality spaces. But contrary to the usual approach in Smith theory, this result does not immediately generalize to finite $p$-group actions, because the fixed point set of a $\mathbb{Z}/p$-action on an integral Poincaré duality space need not fulfill Poincaré duality over $\mathbb{Z}$ (cf. Remark 15). In this paper, however, we show that a treatment of simplicial finite $p$-group actions on orientable $\mathbb{Z}_{(p)}$-homology manifolds is possible. This includes the case of simplicial actions on closed orientable topological manifolds. Our argument proceeds in two steps. In Section 2 we weaken the assumption of integral Poincaré duality in the theorem of Alexander and Hamrick to Poincaré duality over $\mathbb{Z}_{(p)}$.

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In the following section, we provide a refined version of the classical Smith theory result in [6] concerning continuous $\mathbb{Z}/p$-actions on orientable $\mathbb{F}_p$-homology manifolds. Indeed, the fixed point set of such an action on an orientable $\mathbb{Z}_{(p)}$-homology manifold is not only an orientable $\mathbb{F}_p$-homology manifold as the theorem of Borel predicts, but an orientable $\mathbb{Z}_{(p)}$-homology manifold, as long as the action is simplicial. In combination with the results in the second section we can compare the Witt classes of total space and fixed set for simplicial actions of finite $p$-groups on such spaces, if $p$ is large compared to the total Betti number of the space that is acted upon. As an application, we provide examples of orientable closed topological manifolds with vanishing Euler characteristic where our results imply the existence of a fixed point for every simplicial action of a finite $p$-group, if $p$ is large. Furthermore, as a corollary of our results, we show that the union of the $4k$-dimensional fixed components of a simplicial finite $p$-group action ($p$ large) on an orientable $\mathbb{Z}_{(p)}$-homology manifold whose dimension is even and not divisible by four cannot consist of a single point. In the last section of this paper we will discuss the question to what extent our results can be generalized to continuous $p$-group actions that are not necessarily simplicial.

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## 2. Actions on $\mathbb{Z}_{(p)}$-Poincaré duality spaces

The symbol $p$ always stands for an odd prime number and $\mathbb{F}_p$ for the field with $p$ elements.

**Definition 1.** Let $k$ be a natural number and $R$ a commutative ring with unit. A triple $(A, B, \nu)$ consisting of a pair $(A, B)$ of finite dimensional CW complexes, $B \subset A$, and a (singular) homology class $\nu \in H_k(A, B; R)$ is called an *R-Poincaré duality pair (R-PD pair* for short*) of formal dimension k with orientation* $\nu$, if $A$ is connected, $H_*(A; R)$ is finitely generated and if the cap product map

$$\nu \cap - : H^*(A, B; R) \to H_{k-*}(A; R)$$

is an isomorphism.

(Compare the treatment of integral Poincaré complexes in [5]). Now suppose that $X$ is a $\mathbb{Z}/p$-CW complex and a $\mathbb{Z}_{(p)}$-PD space (i.e. $(X, \emptyset)$ is a $\mathbb{Z}_{(p)}$-PD pair) of even formal dimension $2n$, with orientation $\nu \in H_{2n}(X; \mathbb{Z}_{(p)})$. In particular, $X$ is an $\mathbb{F}_p$-PD space with the $\mathbb{F}_p$-reduction $\overline{\nu}$ of $\nu$ as orientation (cf. [5], Proposition I.2.1). Then the number of components $F \subset X^{\mathbb{Z}/p}$ of the fixed point set of this action is finite and each such $F$ is an $\mathbb{F}_p$-PD space of even formal dimension $n_F \leq 2n$ with orientation $\nu_F$ depending canonically on $\overline{\nu}$ ([4, 8, 12]). We hence obtain an element

$$w(X^{\mathbb{Z}/p}) = \sum_{F \subset X^{\mathbb{Z}/p}} w(H^{\mathrm{ev}}(F; \mathbb{F}_p)) \in W(\mathbb{F}_p)$$

in the Witt ring of nondegenerate symmetric bilinear forms over $\mathbb{F}_p$ (for the definition of the Witt ring and for a treatment of inner product spaces in general, see e.g. [16]). The summation in this formula is over the components of $X^{\mathbb{Z}/p}$ and $w(H^{\mathrm{ev}}(F; \mathbb{Z}/p))$ denotes the Witt class of the bilinear form induced on $H^{\mathrm{ev}}(F; \mathbb{F}_p)$ by the cup product operation and the orientation $\nu_F$. We shall prove the following $p$-local analogue of [1], Theorem 1.

**Theorem 2.** *The following equation holds in $W(\mathbb{F}_p)$:*

$$w(X^{\mathbb{Z}/p}) = w(\hat{H}^n(\mathbb{Z}/p; H^n(X; \mathbb{Z}_{(p)})/\operatorname{Tor})).$$

Here, the right hand side of the stated equation denotes the Witt class of the symmetric inner product on the Tate cohomology of $\mathbb{Z}/p$ with coefficients in the $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$-module $H^n(X, \mathbb{Z}_{(p)})/\operatorname{Tor}$ induced by the usual cup product in group cohomology and the coefficient pairing

$$(H^n(X; \mathbb{Z}_{(p)})/\operatorname{Tor}) \otimes (H^n(X; \mathbb{Z}_{(p)})/\operatorname{Tor}) \to H^{2n}(X; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$$

which is induced by the cup product on $H^*(X; \mathbb{Z}_{(p)})$ and the orientation $\nu$.

The following proposition shows that the class of spaces covered by the last theorem is only slightly smaller than the class of all $\mathbb{F}_p$-PD spaces.

**Proposition 3.** *Let $(Y, \overline{\mu})$ be a $\mathbb{F}_p$-PD space of formal dimension $k$. If there is an element $\mu \in H_k(Y; \mathbb{Z}_{(p)})$ whose mod $p$ reduction is equal to $\overline{\mu}$, then $(Y, \mu)$ is a $\mathbb{Z}_{(p)}$-PD space.*

*Proof.* Let $\mu$ be represented by a chain $c \in S_*(Y; \mathbb{Z}_{(p)})$ in the singular chain complex of $Y$ with coefficients in $\mathbb{Z}_{(p)}$. By assumption, the mapping cone $C_*$ of

$$\begin{array}{rcl} S^*(Y; \mathbb{Z}_{(p)}) & \to & S_*(Y; \mathbb{Z}_{(p)}) \\ x & \mapsto & c \cap x \end{array}$$

has vanishing homology after reducing the coefficients mod $p$. Applying the universal coefficient theorem, this implies $H_*(C) = 0$ and hence the assertion. □

Notice that the liftability assumption of $\overline{\mu}$ in the last proposition holds for example if $H^1(Y; \mathbb{F}_p) = 0$ (by applying the universal coefficient theorem and the $\mathbb{F}_p$-Poincaré duality of $Y$). On the other hand, if $\phi : S^k \to S^k$ is a map of degree $j \cdot p+1$, where $j \in \mathbb{Z} \setminus \{0\}$, then the mapping torus $M = S^k \times [0,1]/(x, 0) \sim (\phi(x), 1)$ of this map is an $\mathbb{F}_p$-PD space of formal dimension $k+1$, but $H_{k+1}(M; \mathbb{Z}_{(p)}) = 0$.

Before we give the proof of Theorem 2, we need to discuss some preliminary facts on the Witt ring for $p$-torsion forms. Let $\mathbb{P}$ be the Prüfer group of type $p^\infty$, i.e. the $p$-primary part of $\mathbb{Q}/\mathbb{Z}$. Note that $\mathbb{P} \cong \mathbb{Q}/\mathbb{Z}_{(p)}$. We denote the Witt ring associated to nonsingular symmetric bilinear $p$-torsion forms by $W(\mathbb{P})$. Thus, an element $[F,\gamma]$ of $W(\mathbb{P})$ is represented by a pair $(F,\gamma)$, where $F$ is a finite abelian $p$-group and $\gamma : F \times F \to \mathbb{P}$ is a symmetric biadditive map that is nonsingular, i.e. the adjoint map $F \to \mathrm{Hom}(F,\mathbb{P})$ is injective (and therefore an isomorphism, because source and target of this map have the same finite cardinality). Two pairs of this kind are considered to be equivalent, if after adding appropriate split forms to both of them, they become isomorphic. Here, $(F,\gamma)$ is called *split*, if there is a subgroup $K \subset F$ such that $K^\perp = \{f \in F | \gamma(f,K) = 0\}$ coincides with $K$. As usual (cf. [2], Lemma 1.3) one proves the following important fact.

**Lemma 4.** *With the notation introduced so far, suppose that we have a subgroup $K \subset F$ that contains its orthogonal complement $K^\perp$. Then there is an induced form on $K/K^\perp$ that is Witt equivalent to $(F,\gamma)$.*

The inclusion $\mathbb{Z}/p \to \mathbb{P}$, $1 \mapsto 1/p$ induces a canonical isomorphism $\psi_p : W(\mathbb{F}_p) \to W(\mathbb{P})$ (cf. [2], Theorem 1.7.). We also have a canonical map $\partial_p : W(\mathbb{Q}) \to W(\mathbb{P})$ that is defined as follows. If $(V,\gamma)$ is an inner product space over $\mathbb{Q}$ (whose Witt class in $W(\mathbb{Q})$ we denote by $[V,\gamma]$), let $d = \dim V$ and choose a free $\mathbb{Z}_{(p)}$-module $L \subset V$ with $L \otimes \mathbb{Q} = V$ and $\gamma(L,L) \subset \mathbb{Z}_{(p)}$ (we call $L$ a *$p$-local lattice* in $V$). By the theory of free modules over principal ideal domains, $L$ has rank $d$. The dual lattice $L^+ = \{l \in V | \gamma(l,L) \subset \mathbb{Z}_{(p)}\} \supset L$ is again a free $\mathbb{Z}_{(p)}$-module of rank $d$. The quotient $L^+/L$ is a finite $p$-group with a nonsingular form $\gamma'$ induced by $\gamma$. $\partial_p([V,\gamma])$ is represented by $[L^+/L,\gamma']$. For the details of this construction we refer the reader to the analogous discussion of $W(\mathbb{Q}/\mathbb{Z})$ in [2]. We remark that the composition $\psi_p^{-1} \circ \partial_p : W(\mathbb{Q}) \to W(\mathbb{F}_p)$ coincides with the map $\psi^1$ in [16], Lemma (IV.1.2), associated to the $p$-adic valuation on $\mathbb{Q}$.

Now let $(A,B)$ be a $\mathbb{Z}_{(p)}$-PD pair of formal dimension $4k$, $k \in \mathbb{N}$, with orientation $[A,B] \in H_{4k}(A,B;\mathbb{Z}_{(p)})$. The canonical orientation of the $4k-1$ dimensional $\mathbb{Z}_{(p)}$-PD space $B$ is denoted by $[B]$. To the pair $(A,B)$ we associate two Witt classes in $W(\mathbb{P})$. In the following, let $i : B \hookrightarrow A$ and $j : A = (A,\emptyset) \hookrightarrow (A,B)$ denote the canonical inclusions. The *peripheral invariant* $\mathrm{per}(B) \in W(\mathbb{P})$ of $B$ is defined as $\partial_p([V,f])$, where

$$V = \mathrm{Im}(j^{2k} : H^{2k}(A,B;\mathbb{Q}) \to H^{2k}(A;\mathbb{Q}))$$

and for $x,y \in H^{2k}(A,B;\mathbb{Q})$, we set

$$f(j^*(x),j^*(y)) = < x \cup y, [A,B]_{\mathbb{Q}} > \in \mathbb{Q}\,.$$

Here and in the following, a lower subscript at a fundamental class indicates the coefficients of the corresponding homology module. One can prove that $f$ is well defined and yields an inner product on $V$. The peripheral invariant of $B$ seems to depend not only on $B$, but also on $A$. Proposition 5 below shows that this is not the case. The *linking form* $\mathrm{lk}(B) \in W(\mathbb{P})$ of $B$ is defined as $[W,g]$, where

$$W = \mathrm{Tor}\, H^{2k}(B;\mathbb{Z}_{(p)}) = \mathrm{Im}\,\beta_{(p)}$$

and for $x, y \in H^{2k-1}(B;\mathbb{P})$, we set

$$g(\beta_{(p)}(x), \beta_{(p)}(y)) = < x \cup \beta_{(p)}(y), [B]_{\mathbb{Z}_{(p)}} > ,$$

using the coefficient pairing $\mathbb{P} \times \mathbb{Z}_{(p)} \to \mathbb{P}$ for the cup product. Here, $\beta_{(p)}$ denotes the connecting homomorphism in cohomology associated to the exact sequence of coefficients

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{P} \to 0 .$$

One checks that $g$ is a well defined nonsingular symmetric bilinear form on $W$. Note that $\mathrm{lk}(B)$ can be defined for every $\mathbb{Z}_{(p)}$-PD space $B$ of formal dimension congruent to 3 mod 4. The following fundamental relation, which is a $p$-local analogue of [2], Theorem 2.1, will be important.

**Proposition 5.** *In $W(\mathbb{P})$ the following equation holds.*

$$\mathrm{per}(B) = -\mathrm{lk}(B) .$$

*In particular, the left hand side does not depend on the choice of $A$.*

*Proof.* We use the free $\mathbb{Z}_{(p)}$-module

$$L = j^*(H^{2k}(A,B;\mathbb{Z}_{(p)}))/\operatorname{Tor} \subset H^{2k}(A;\mathbb{Z}_{(p)})/\operatorname{Tor} \subset H^{2k}(A;\mathbb{Q})$$

for evaluating the map $\partial_p$ used in the definition of the invariant per. By Poincaré duality,

$$L^+ = (H^{2k}(A;\mathbb{Z}_{(p)})/\operatorname{Tor}) \cap (L \otimes \mathbb{Q}) .$$

We now set

$$\begin{aligned} H &= \operatorname{Tor} H^{2k}(B;\mathbb{Z}_{(p)}) \cap i^*(H^{2k}(A;\mathbb{Z}_{(p)}), \\ K &= i^*(\operatorname{Tor} H^{2k}(A;\mathbb{Z}_{(p)})) \subset H . \end{aligned}$$

Using exactness of the sequence

$$H^{2k}(A,B;\mathbb{Z}_{(p)}) \xrightarrow{j^*} H^{2k}(A;\mathbb{Z}_{(p)}) \xrightarrow{i^*} H^{2k}(B;\mathbb{Z}_{(p)}) ,$$

the map $i^*$ induces an isomorphism $L^+/L \to H/K$ .

**Lemma 6.** *With respect to the linking form on* $\operatorname{Tor} H^{2k}(B;\mathbb{Z}_{(p)})$*, the equation $H^\perp = K$ holds. In particular, there is an induced form on $H/K$ that represents the same element as* $\mathrm{lk}(B)$ *in $W(\mathbb{P})$.*

*Proof.* Let $h \in H$ and $k \in K$. By definition of $K$, there is a $k' \in H^{2k-1}(A;\mathbb{P})$ with $\beta_{(p)} \circ i^*(k') = i^* \circ \beta_{(p)}(k') = k$. For the bilinear form $g$ used in the definition of $\mathrm{lk}(B)$ we get

$$\begin{aligned} g(k,h) &= < i^*(k') \cup h, [B]_{\mathbb{Z}_{(p)}} > \\ &= < i^*(k') \cup h, \partial([A,B]_{\mathbb{Z}_{(p)}}) > \\ &= < \delta(i^*(k') \cup h), [A,B]_{\mathbb{Z}_{(p)}} > \\ &= < k' \cup \delta(h), [A,B]_{\mathbb{Z}_{(p)}} >, \end{aligned}$$

where $\partial$ and $\delta$ denote connecting homomorphisms of (co-)homology groups. But, as $h \in \operatorname{Im} i^*$ by definition, we get $\delta(h) = 0$ and hence $g(k,h) = 0$, i.e. $H \subset K^\perp$. For proving equality in the last statement, by the nonsingularity of $g$, it remains to show that $|G| = |H| \cdot |K|$, where $G = \operatorname{Tor} H^{2k}(B; \mathbb{Z}_{(p)})$. This will follow from $K \cong G/H$, an assertion that is shown as follows. By using the exact cohomology sequence

$$\ldots \to H^{2k}(A; \mathbb{Z}_{(p)}) \stackrel{i^*}{\to} H^{2k}(B; \mathbb{Z}_{(p)}) \stackrel{\delta}{\to} H^{2k+1}(A, B; \mathbb{Z}_{(p)}) \to \ldots$$

we get $G/H \cong \delta(G)$, which, by Poincaré duality, is isomorphic to

$$i_*(\operatorname{Tor}(H_{2k-1}(B; \mathbb{Z}_{(p)})) \subset H_{2k-1}(A; \mathbb{Z}_{(p)}) .$$

Now we use naturality of the part

$$0 \to \operatorname{Ext}(H_{2k-1}(-; \mathbb{Z}_{(p)}); \mathbb{Z}_{(p)}) \to H^{2k}(-; \mathbb{Z}_{(p)}) \to \ldots$$

of the universal coefficient sequence and the fact

$$\operatorname{Tor}(H^{2k}(A; \mathbb{Z}_{(p)})) \cong \operatorname{Ext}(H_{2k-1}(A; \mathbb{Z}_{(p)}); \mathbb{Z}_{(p)}) ,$$

(naturally in $A$) and get

$$K = i^*(\operatorname{Tor} H^{2k}(A; \mathbb{Z}_{(p)})) \cong \operatorname{Ext}(i_*; \mathbb{Z}_{(p)})(H_{2k-1}(A; \mathbb{Z}_{(p)})) .$$

In order to see that the last expression is indeed isomorphic to $i_*(\operatorname{Tor}(H_{2k-1}(B; \mathbb{Z}_{(p)})) \cong G/H$, note that in general for a linear map $f : M \to N$ between finitely generated $\mathbb{Z}_{(p)}$-modules,

$$\operatorname{Ext}(f; \mathbb{Z}_{(p)})(N) \cong f(\operatorname{Tor} M) .$$

The last part of the lemma is a simple consequence of Lemma 4. □

With the help of Lemma 6 we can use the isomorphism $L^+/L \cong H/K$, which was induced by $i^*$, to compare $\operatorname{per}(B)$ and $\operatorname{lk}(B)$. This will yield a proof of Proposition 5. Thus, let $h_1, h_2 \in H$ represent two elements in $H/K$. The form that is induced on $H/K$ by $i^*$ and the representative of $\operatorname{per}(B)$ on $L^+/L$ has the following description. We choose elements $x, y \in L^+ \subset H^{2k}(A; \mathbb{Z}_{(p)})/\operatorname{Tor} \subset H^{2k}(A; \mathbb{Q})$ with $i^*(x) = h_1$ and $i^*(y) = h_2$. Using the equality

$$L^+ = (H^{2k}(A; \mathbb{Z}_{(p)})/\operatorname{Tor}) \cap (L \otimes \mathbb{Q}),$$

there is a $z \in H^{2k}(A, B; \mathbb{Q})$ with $j^*(z) = x$. The class $\operatorname{per}(B)$ is represented by the bilinear form that sends the pair $(h_1 + K, h_2 + K)$ to

$$< \pi(z) \cup y, [A, B]_{\mathbb{Q}} > \in \mathbb{P},$$

where $\pi$ denotes reduction of coefficients $\mathbb{Q} \to \mathbb{P}$ and where we use the coefficient pairing $\mathbb{P} \times \mathbb{Z}_{(p)} \to \mathbb{P}$ for the cup product as before.

Now let $\xi \in H^{2k-1}(B; \mathbb{P})$ such that $\beta_{(p)}(\xi) = h_1$. Then $\operatorname{lk}(B)$ is represented by the bilinear form that sends the pair $(h_1 + K, h_2 + K)$ to

$$\begin{aligned} < \xi \cup h_2, [B]_{\mathbb{Z}_{(p)}} > &= < \delta(\xi \cup i^*(y)), [A,B]_{\mathbb{Z}_{(p)}} > \\ &= < \delta(\xi) \cup y, [A,B]_{\mathbb{Z}_{(p)}} > \\ &= < \delta(\xi) \cup y, [A,B]_{\mathbb{Q}} > \in \mathbb{P}. \end{aligned}$$

It remains to prove that $\pi(z) = -\delta(\xi)$. Let $c \in S^{2k}(A;\mathbb{Z}_{(p)})$ be a cochain representing $x$. Because $i^*(x)$ is a torsion element in $H^{2k}(B;\mathbb{Z}_{(p)})$, the restriction $c|B$ of $c$ to $B$ is a coboundary in $S^{2k}(B;\mathbb{Q})$. Using the fact that the restriction map $S^*(A;\mathbb{Q}) \to S^*(B;\mathbb{Q})$ is surjective, we hence get a $\lambda \in S^{2k-1}(A;\mathbb{Q})$ with $c|B = \delta(\lambda)|B$, where $\delta : S^{2k-1}(A;\mathbb{Q}) \to S^{2k}(A;\mathbb{Q})$ is the usual differential. By the definition of $\beta_{(p)}$, $\xi$ is represented by the $\mathbb{P}$-reduction of $\lambda|B$ (for a suitable choice of $\xi$). Hence, $\delta(\xi)$ is represented by the $\mathbb{P}$-reduction of $\delta(\lambda)$, giving a cocycle in $S^{2k}(A,B;\mathbb{P})$. On the other hand, we know that $c - \delta(\lambda)$ restricts to zero in $S^{2k}(B;\mathbb{Q})$ and therefore can be considered as a cocycle in $S^{2k}(A,B;\mathbb{Q})$ representing $z$. After reducing the coefficients to $\mathbb{P}$, $c$ is sent to zero, as $c \in S^{2k}(A;\mathbb{Z}_{(p)})$. Thus, $\pi(z)$ is represented by the $\mathbb{P}$-reduction of $-\delta(\lambda)$.

This completes the proof of Proposition 5. □

**Lemma 7.** *Let $(A,B,\mu)$ be a $\mathbb{Z}_{(p)}$-PD pair of formal dimension $k$. Furthermore, assume that $(A,B)$ is a free $G$-CW pair, where $G = \mathbb{Z}/p$. Then there is a $\mu' \in H_k(A/G,B/G;\mathbb{Z}_{(p)})$ such that $(A/G,B/G,\mu')$ is a $\mathbb{Z}_{(p)}$-PD pair and the orbit map $\pi : A \to A/G$ has degree $p$, i.e. $\pi_*(\mu) = p \cdot \mu'$.*

*Proof.* First, let $B = \emptyset$. By the argument given in [1], Lemma 13, $A/G$ is an $\mathbb{F}_p$-PD space of formal dimension $k$. By considering the spectral sequence for the Borel fibration $A \hookrightarrow EG \times_G A \to BG$ with coefficients in $\mathbb{Q}$, we get $H_k(A/G;\mathbb{Q}) \cong \mathbb{Q}$ and by Proposition 3, $A/G \simeq EG \times_G A$ is a $\mathbb{Z}_{(p)}$-PD space of formal dimension $k$. By applying the same spectral sequence with coefficients in $\mathbb{Z}_{(p)}$, the orbit map $A \to A/G$ has degree $p$ (this in turn defines the orientation $\mu'$). The case of $B \neq \emptyset$ follows by applying this argument to the double $A \cup_B A$ and to $B$ seperately and using a $p$-local version of [5], Theorem I.3.2. □

We now proceed with the proof of Theorem 2. We consider the truncated Borel construction

$$X^m_{\mathbb{Z}/p} = S^{2m-1} \times_{\mathbb{Z}/p} X ,$$

$m \in \mathbb{N} \setminus \{0\}$, where $S^{2m-1} \subset \mathbb{C}^m$ carries the standard free $\mathbb{Z}/p$-action induced by multiplication with $e^{2\pi i/p}$ in each of the $m$ complex coordinates. Notice that $(S^{2m-1} \times X, \emptyset)$ fulfills the requirements of Lemma 7. We deduce that $X^m_{\mathbb{Z}/p}$ is a $\mathbb{Z}_{(p)}$-PD space of formal dimension $2m+2n-1$ with an orientation that depends canonically on $\nu$, the orientation of $X$. From now on we suppose that $m+n$ is even and that $m \gg n$. Applying a similar argument as in [1], Proposition 6, we get the following equation in $W(\mathbb{F}_p) \cong W(\mathbb{P})$.

**Proposition 8.** $w(X^{\mathbb{Z}/p}) = \mathrm{lk}(X^m_{\mathbb{Z}/p})$.

Similar to the strategy in [1], the right hand side can be calculated by regarding the truncated Borel construction as part of a suitable $\mathbb{Z}_{(p)}$-PD pair and using Proposition 5. This construction is based on the Riemannian surface $R = \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_0^p + z_1^p + z_2^p = 0\}$ on which the map $(z_0 : z_1 : z_2) \mapsto (e^{2\pi i/p} z_0 : z_1 : z_2)$ induces a smooth $\mathbb{Z}/p$-action with exactly $p$ fixed points. We remove a $\mathbb{Z}/p$-invariant open subset of $R^m$ that is homeomorphic to $p^m$ copies of $B^{2m}$ with the standard $\mathbb{Z}/p$-action and contains the $p^m$ fixed points of the induced action on $R^m$. Thus, we obtain a $\mathbb{Z}_{(p)}$-PD pair $(M, \partial M, \mu)$ and hence a $\mathbb{Z}_{(p)}$-PD pair $(M \times X, \partial M \times X, \mu \times \nu)$, where $\mu \in H_{2m}(M, \partial M; \mathbb{Z}_{(p)})$ is induced by the standard complex orientation of $R^m$ and $\mu \times \nu$ denotes the cross product in homology. By Proposition 5 and Lemma 7, the following equation holds in $W(\mathbb{P})$, where the first minus sign comes from our orientation conventions.

$$-p^m \cdot lk(X^m_{\mathbb{Z}/p}) = lk(\partial M \times_{\mathbb{Z}/p} X) = -\mathrm{per}(\partial M \times_{\mathbb{Z}/p} X). \tag{1}$$

If we let $(V, f)$ with $V \subset H^{m+n}(M \times X; \mathbb{Q})$ as in the definition of $\mathrm{per}(\partial M \times X)$, we get by Lemma 7

$$\mathrm{per}(\partial M \times_{\mathbb{Z}/p} X) = \partial_p([1/p] \cdot [V^{\mathbb{Z}/p}, f]). \tag{2}$$

Here, $[1/p]$ denotes the Witt class of the form $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, $(x, y) \mapsto 1/p \cdot xy$. Because $H^{>1}(M, \partial M; \mathbb{Q}) \cong H^{>1}(R^m; \mathbb{Q})$ by excision, the Künneth theorem implies that

$$H^{>2n+1}(M \times X, \partial M \times X; \mathbb{Q}) \cong H^{>2n+1}(R^m \times X; \mathbb{Q}).$$

Furthermore, as $m \gg n$ and $H^*(\partial M; \mathbb{Z}_{(p)})$ is concentrated in degrees 0 and $2m-1$ and by using the long exact cohomology sequence, we have $H^{m+n}(M \times X, \partial M \times X; \mathbb{Q}) \cong H^{m+n}(M \times X; \mathbb{Q})$. Theorem 2 therefore follows from Proposition 8, from equations (1) and (2) above, from the fact that $W(\mathbb{F}_p)$ is a 2-group (cf. [16]) and from the following lemma. □

**Lemma 9.** *In $W(\mathbb{P})$ we have*

$$\partial_p([1/p] \cdot [H^{m+n}(R^m \times X, \mathbb{Q})^{\mathbb{Z}/p}, (\mu \times \nu) \circ \cup]) = p^m \cdot w(\hat{H}^n(\mathbb{Z}/p; H^n(X; \mathbb{Z}_{(p)})/\operatorname{Tor})).$$

*Proof.* Let $D^*$ denote an $\mathbb{N}$-graded $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$-module consisting of finitely generated free $\mathbb{Z}_{(p)}$-modules in every degree together with a graded $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$-linear map $\mathrm{mult} : D^* \otimes_{\mathbb{Z}_{(p)}} D^* \to D^*$ (with the diagonal $\mathbb{Z}/p$-action). We assume that there is a natural number $l$ and an isomorphism of $\mathbb{Z}_{(p)}$-modules $\alpha : D^{2l} \cong \mathbb{Z}_{(p)}$, such that the form

$$\gamma : D^* \otimes D^* \xrightarrow{mult.} D^* \xrightarrow{\alpha} \mathbb{Z}_{(p)}$$

is nonsingular ($\alpha$ extended trivially onto all of $D^*$). In particular $D^{>2l} = 0$. Because the cohomology $\hat{H}^*(\mathbb{Z}/p; D^*)$ is an algebra over $\hat{H}^*(\mathbb{Z}/p; \mathbb{Z}_{(p)}) \cong \mathbb{F}_p[t, t^{-1}]$, $\deg(t) = 2$, we get a $\mathbb{Z}/2$-graded $\mathbb{F}_p$-algebra

$$H^\star = \hat{H}^*(\mathbb{Z}/p; D^*)_{t=1}$$

by evaluating $t$ at 1 (i.e. by factoring out the ideal in $\hat{H}^*(\mathbb{Z}/p; D^*)$ generated by $(t-1)$). Here, $\star$ stands for the induced $\mathbb{Z}/2$-graduation (recall that $\deg(t) = 2$). One checks that the bilinear form on $H^\star$ induced by the group cohomology cup product on $\hat{H}^*(\mathbb{Z}/p; D^*)$ and the coefficient pairing $\gamma$ from above is nonsingular. After restricting this bilinear form to the elements of even degree in $H^\star$, we get a nonsingular symmetric bilinear form over $\mathbb{F}_p$ whose Witt class we denote by $\mathrm{lk}(D^*, \alpha) \in W(\mathbb{F}_p)$. This is the *linking form* of $(D^*, \alpha)$. One proves the following.

- $\mathrm{lk}(D^*, \alpha) = w(\hat{H}^l(\mathbb{Z}/p; D^l))$ in $W(\mathbb{F}_p)$, where the form on the right is induced by the group cohomology cup product and the coefficient pairing $\gamma$. This follows from the fact that $\hat{H}^*(\mathbb{Z}/p; D^{\neq l}) \cong \hat{H}^*(\mathbb{Z}/p; D^{<l}) \oplus \hat{H}^*(\mathbb{Z}/p; D^{>l})$ which are paired to each other.
- If $l$ is even, then in $W(\mathbb{F}_p) \cong W(\mathbb{P})$ we have
$$w(\hat{H}^l(\mathbb{Z}/p; D^l)) = \partial_p([1/p] \cdot [(D^l \otimes \mathbb{Q})^{\mathbb{Z}/p}]) \,,$$
where on the right hand side we use the form induced by $\gamma$ (it is easy to show that this is a nonsingular form over $\mathbb{Q}$). This equation is proven by evaluating the map $\partial_p : W(\mathbb{Q}) \to W(\mathbb{P})$ using the free $\mathbb{Z}_{(p)}$-module $L = (1 + g + \ldots + g^{p-1}) \cdot D^l \subset (D^l \otimes \mathbb{Q})^{\mathbb{Z}/p}$ (here $g$ denotes a generator of $\mathbb{Z}/p$). One checks that $L^+ = (D^l)^{\mathbb{Z}/p}$ (with respect to the form $1/p \cdot (\gamma|...)$). Now the claim follows from $\hat{H}^l(\mathbb{Z}/p; D^l) \cong L^+/L$ as $l$ is even.
- Assume that $E^*$ is another $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$-module with the same properties as $D^*$. In particular there is a natural number $m$ and an isomorphism $\rho : E^{2m} \cong \mathbb{Z}_{(p)}$ that induces a nonsingular form similar to $\gamma$. Then
$$\mathrm{lk}(D^* \otimes E^*, \alpha \otimes \rho) = \mathrm{lk}(D^*, \alpha) \cdot \mathrm{lk}(E^*, \rho) \,.$$
For showing this fact we first note that
$$H^\star \cong \hat{H}^*(\mathbb{Z}/p; D^* \otimes \mathbb{F}_p)_{t=1;s=0} \,,$$
where $s$ is a generator of $\hat{H}^1(\mathbb{Z}/p; \mathbb{F}_p)$. This is proven by the universal coefficient theorem with help of the Bockstein operator $\beta$ on $\hat{H}^*(\mathbb{Z}/p; D^* \otimes \mathbb{F}_p)$ induced by the exact sequence of coefficients $0 \to D^* \otimes \mathbb{F}_p \to D^* \otimes \mathbb{Z}/p^2 \to D^* \otimes \mathbb{F}_p \to 0$. Note that $\beta(s) = t$ in $\hat{H}^*(\mathbb{Z}/p; \mathbb{F}_p)$ (with a suitable choice of $s$). Now we use the Künneth theorem for group cohomology with coefficients $(D^* \otimes \mathbb{F}_p) \otimes (E^* \otimes \mathbb{F}_p)$ and observe that the induced bilinear form on
$$(H^*(\mathbb{Z}/p; D^* \otimes \mathbb{F}_p)_{t=1,s=0})^{odd} \otimes (H^*(\mathbb{Z}/p; E^* \otimes \mathbb{F}_p)_{t=1,s=0})^{odd}$$
is split - as is every tensor product of two alternate forms.

Using these properties the proof of the lemma is complete (by setting $D^* = H^*(X;\mathbb{Z}_{(p)})/\operatorname{Tor}$ and $E^* = H^*(R^m;\mathbb{Z}_{(p)})/\operatorname{Tor})$), if we show that

$$\operatorname{lk}(H^*(R;\mathbb{Z}_{(p)})/\operatorname{Tor},\rho) = p\cdot[1],$$

where $\rho$ is induced by the standard orientation of $R$. But this follows by applying [3], Theorem II.4.1, to the $\mathbb{Z}/p$-space $R$. □

## 3. Actions on $\mathbb{Z}_{(p)}$-homology manifolds

The assumptions in Theorem 2 are weak enough to allow for an induction approach to actions of finite $p$-groups on oriented $\mathbb{Z}_{(p)}$-homology manifolds, where $p$ is an odd prime number as before. We will be concerned mainly with the case of simplicial complexes and simplicial actions.

**Definition 10.** Let $R$ be a commutative ring with unit. A pair $(Y,\nu)$ consisting of a finite connected simplicial complex $Y$ and an element $\nu\in H_k(Y;R)$, where $k\geq 0$, is called an *oriented $R$-homology manifold of dimension $k$*, if for all $y\in Y$ we have isomorphisms of (reduced) homology groups

$$H_*(Y,Y-y;R)\cong \widetilde{H}_*(S^k;R)$$

and if, under the map $H_k(Y;R)\to H_k(Y,Y-y;R)$ and this isomorphism, $\nu$ is mapped to a generator of $\widetilde{H}_k(S^k;R)$.

We recall the following basic fact.

**Proposition 11.** *If $(Y,\nu)$ is an oriented $R$-homology manifold of dimension $k$, then $(Y,\nu)$ is an $R$-PD space of formal dimension $k$.*

For a proof, see [9], Corollary V.10.2. Cf. also [17], Chapter 8, §§62-65.

**Definition 12.** We call an action of a finite group $G$ on a simplicial complex *simplicial*, if each element of $G$ acts as a simplicial map.

It is well known that the second barycentric subdivision of a simplicial complex $K$ with a simplicial $G$-action is regular in the sense of ([15], p. 231). This new $G$-space has the structure of a $G$-CW complex in a canonical way and the fixed point set of the corresponding action is a simplicial subcomplex of the second barycentric subdivision of $K$.

The following Proposition is a refinement of a classical result from Smith theory. It is one of the main ingredients for dealing with finite $p$-group actions by induction later on.

**Proposition 13.** *Let $Y$ be an oriented $\mathbb{Z}_{(p)}$-homology manifold of dimension $k$ and let $\mathbb{Z}/p$ act simplicially on $Y$. Then each component of $Y^{\mathbb{Z}/p}$ is an orientable $\mathbb{Z}_{(p)}$-homology manifold of dimension equal to $k$ mod 2.*

*Proof.* By the universal coefficient theorem it is immediate that $Y$ is an oriented $\mathbb{F}_p$-homology manifold of dimension $k$. Choose a component $F \subset Y^{\mathbb{Z}/p}$. By a well known fact from Smith theory, $F$ is an oriented $\mathbb{F}_p$-homology manifold of dimension $n_F \leq k$ and congruent to $k$ mod 2 ([6], Theorem V.2.2). Let $\nu_F \in H_{n_F}(F;\mathbb{F}_p)$ be an orientation of $F$. By the universal coefficient theorem again, it is clear that for each $x \in F$

$$H_*(F, F - x; \mathbb{Z}_{(p)}) \cong \widetilde{H}_*(S^{n_F}; \mathbb{Z}_{(p)}).$$

Thus we are reduced to showing that $\nu_F$ can be lifted to an element in $H_{n_F}(F;\mathbb{Z}_{(p)})$.

Let $\nu_F$ be represented by a simplicial chain

$$c = \sum_{i\in I} \alpha_i \sigma_i \in C_{n_F}(F;\mathbb{F}_p),$$

$\alpha_i \in \mathbb{F}_p$. By Definition 10, every $n_F$-simplex $\sigma$ of $F$ occurs in $c$ with a nontrivial coefficient. Now the following two facts hold:

- Every $(n_F - 1)$-simplex of $F$ is face of exactly two $n_F$-simplices of $F$ (if not, take an interior point of an $(n_F - 1)$-simplex that fails this property. Then it is easy to see that $H_{n_F}(F, F - x; \mathbb{F}_p) \neq \widetilde{H}_{n_F}(S^{n_F}; \mathbb{F}_p)$).
- Every two $(n_F - 1)$-simplices of $F$ can be joined by a chain of $n_F$-simplices in $F$. Indeed, this requirement defines at least an equivalence relation on the set $S$ of $(n_F - 1)$-simplices in $F$. Let $E \subset S$ be a (nonempty) equivalence class and let $M$ be the set of $n_F$-simplices of $F$ having elements in $E$ as faces. $M$ is not empty by the first point proven previously. Now

$$c' = \sum_{i\in I} \alpha'_i \sigma_i,$$

where $\alpha'_i = \alpha_i$, if $\sigma_i \in M$, and $\alpha'_i = 0$ otherwise, is a nontrivial cycle in $C_{n_F}(F;\mathbb{F}_p)$ by the definition of the equivalence relation above and by the fact that $c$ is a cycle. Because $F$ does not contain any $(n_F + 1)$-simplices - an interior point of such a simplex would not fulfil the requirement in Definition 10 - $c'$ represents a nonzero element in $H_{n_F}(F;\mathbb{F}_p)$. If $q$ is the number of equivalence classes in $S$, we therefore get

$$H_{n_F}(F;\mathbb{F}_p) \cong (\mathbb{F}_p)^q$$

and it follows $q = 1$.

These two observations (which show that $F$ is a so called pseudomanifold) imply that either

$$H_{n_F}(F;\mathbb{Z}) \cong \mathbb{Z}$$

or

$$\begin{aligned} H_{n_F}(F;\mathbb{Z}) &= 0 \\ 2 \cdot \operatorname{Tor} H_{n_F - 1}(F;\mathbb{Z}) &= 0 \end{aligned}$$

(cf. [19], Exercise 4.E.2). But in the second case we would have $H_{n_F}(F;\mathbb{F}_p) = 0$ by the universal coefficient theorem. Hence, the first case applies and the canonical map $H_{n_F}(F;\mathbb{Z}_{(p)}) \to H_{n_F}(F;\mathbb{F}_p) \cong \mathbb{F}_p$ is surjective. This proves our assertion. □

Now we are able to prove a Witt class comparison result for actions of finite $p$-groups on homology manifolds

**Theorem 14.** *Let $X$ be an oriented $2n$-dimensional $\mathbb{Z}_{(p)}$-homology manifold, $n \in \mathbb{N}$. Assume that $p \geq \dim_{\mathbb{F}_p} H^*(X, \mathbb{F}_p)+1$ and let a finite $p$-group $G$ act simplicially on $X$. Then, with appropriate orientations of the fixed point set components of $X$, we have*

$$w(H^{\mathrm{ev}}(X;\mathbb{F}_p)) = \sum_{F \subset X^G} w(H^{\mathrm{ev}}(F;\mathbb{F}_p))$$

*in $W(\mathbb{F}_p)$.*

*Proof.* We apply induction on the order of $G$. If $|G| = p$, we use Theorem 2 twice, once with the given $G$-action and once with the trivial action. Note that by our assumption on $p$, $\mathbb{Z}/p$ acts trivially on $H^n(X;\mathbb{Z})/\operatorname{Tor}$ (cf. [11], §74) and therefore on $H^n(X;\mathbb{Z}_{(p)})/\operatorname{Tor}$. This means that the right hand side of the equation in Theorem 2 is the same for both of these $\mathbb{Z}/p$-actions. If $|G| > p$, we choose a normal subgroup $\mathbb{Z}/p \subset G$ that exists by group theory. By Proposition 13, every component $F$ of the fixed point set of the action restricted to this subgroup is an even dimensional orientable $\mathbb{Z}_{(p)}$-homology manifold again. By Smith theory, the dimension of $H^*(F;\mathbb{F}_p)$ is smaller than or equal to that of $H^*(X;\mathbb{F}_p)$. By normality of $\mathbb{Z}/p \subset G$, we have an induced $G$-action on $X^{\mathbb{Z}/p}$ and because $X^{\mathbb{Z}/p}$ consists of less than $p$ components (by our assumption on the Betti numbers of $X$), we have an induced $G$-action on $F$. We can now apply the induction hypothesis to the $G/(\mathbb{Z}/p)$-space $F$ and obtain

$$w(H^{\mathrm{ev}}(F;\mathbb{F}_p)) = \sum_{H \subset (X^G \cap F)} w(H^{\mathrm{ev}}(H;\mathbb{F}_p)),$$

where the summation is over connected components. After summing over all components $F \subset X^{\mathbb{Z}/p}$ we get the assertion of Theorem 14. □

*Remark* 15. In the context of locally linear actions on orientable topological manifolds, the last result can be derived with the results in [1] alone, because then the various fixed point set components are orientable topological manifolds again and hence integral Poincaré duality spaces. However, many naturally occuring examples of simplicial actions need not be locally linear such as joins or suspensions of - possibly smooth - actions on $\mathbb{F}_p$-homology spheres.

The invariant $w(H^{\mathrm{ev}}(X;\mathbb{F}_p))$ occuring in Theorem 14 is $\operatorname{sgn}(X) \cdot [1]$, where sgn is the signature. This is true, because, if $n$ is even,

$$w(H^{\mathrm{ev}}(X;\mathbb{F}_p)) = w((H^n(X;\mathbb{Z})/\operatorname{Tor}) \otimes \mathbb{F}_p)$$

by applying Theorem 2 with the trivial $\mathbb{Z}/p$-action on $X$ (or by using a direct argument based on the universal coefficient theorem). Now one uses the fact that the two ring homomorphisms

$$\mathbb{Z} \cong W(\mathbb{Z}) \overset{-\otimes\mathbb{R}}{\to} W(\mathbb{R}) \overset{\mathrm{sgn}}{\to} \mathbb{Z} \overset{1\mapsto[1]}{\to} W(\mathbb{F}_p)$$

and

$$W(\mathbb{Z}) \overset{-\otimes\mathbb{F}_p}{\to} W(\mathbb{F}_p)$$

are equal, which can be checked easily on a generator of $W(\mathbb{Z})$.

**Corollary 16.** *Let $X$ and $G$ be as in the Theorem 14. Then the following hold:*

i. *If $n$ is odd, then the union of all components of $X^G$ whose dimension is divisible by four cannot consist of one single point.*
ii. *If $\mathrm{sgn}(X) \cdot [1]$ does not vanish in $W(\mathbb{F}_p)$, then $X^G$ contains at least one component whose dimension is divisible by four.*

These are generalizations of well known facts for smooth (resp. locally linear) actions to the case of simplicial actions.

The assumption on the size of $p$ in Theorem 14 is essential - even in the case of smooth actions - as the following example shows.

*Example* 17. Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$ and let $X = \mathbb{C}P^{p-1}$. If $g_1$ and $g_2$ denote generators of the factors of $G$, the assignment

$$\begin{aligned} g_1 \cdot (z_0 : z_1 : \ldots : z_{p-1}) &= (z_0 : z_1^{2\pi i/p} : \ldots : z_{p-1}^{(p-1)2\pi i/p}) \\ g_2 \cdot (z_0 : z_1 : \ldots : z_{p-1}) &= (z_1 : \ldots : z_{p-1} : z_0) \end{aligned}$$

defines a smooth $G$-action on $X$. We have $\mathrm{sgn}(X) = \pm 1$ (depending on the orientation of $X$), in particular $w(H^{\mathrm{ev}}(X;\mathbb{F}_p))$ does not vanish in $W(\mathbb{F}_p)$. However, $X^G = \emptyset$. In fact, the fixed point set of the action restricted to $\mathbb{Z}/p \times \{0\} \subset G$ consists of exactly $p$ points (consistent with the prediction of Theorem 2), but these $p$ points are permuted by the induced action of $\{0\} \times \mathbb{Z}/p$. Note that $\dim_{\mathbb{F}_p} H^{\mathrm{ev}}(X;\mathbb{F}_p) = \dim_{\mathbb{F}_p} H^*(X;\mathbb{F}_p) = p$ in this example and that $G$ acts trivially on $H^*(X;\mathbb{Z}_{(p)})$. One can take this as indication that there is no obvious generalization of Theorem 14 to the case of smaller $p$.

If one is only interested in the existence of fixed points, the conclusion of the second part of Corollary 16 is not interesting, if $\chi(X)$, the Euler characteristic of $X$, does not vanish, because then the Lefschetz fixed point theorem gives the same result (by induction on $|G|$, similar as in the proof of Theorem 14). On the other hand, if $\chi(X) = 0$, then $\mathrm{sgn}(X)$ is even (by Poincaré duality) and thus $\mathrm{sgn}(X) \cdot [1]$ does not vanish, if and only if $p \equiv 3 \bmod 4$ (otherwise, $[1]$ has order 2 in $W(\mathbb{F}_p)$) and $\mathrm{sgn}(X)$ is not divisible by 4. The following fairly general example shows that this situation can in fact occur in a lot of cases.

*Example* 18. Let $X$ be an arbitrary orientable closed $4n$-dimensional (topological) manifold with nonnegative signature. Recall that the complex projective space (with a suitable orientation) $\mathbb{C}P^{2n}$ has real dimension $4n$ and signature $-1$. Now set

$$Y = X\sharp(\mathbb{C}P^{2n}\sharp\ldots\sharp\mathbb{C}P^{2n}),$$

the connected sum of $X$ with $sgn(X)+2$ copies of $\mathbb{C}P^{2n}$. $Y$ is an orientable $4n$-dimensional manifold with signature $-2$. By Poincaré duality, $\chi(Y)$ is even, say $2m$. We set

$$Z = Y\sharp((S^1\times S^{4n-1})\sharp\ldots\sharp(S^1\times S^{4n-1})),$$

the connected sum of $Y$ with $m$ copies of $S^1\times S^{4n-1}$, if $m\geq 0$ and

$$Z = Y\sharp((S^2\times S^{4n-2})\sharp\ldots\sharp(S^2\times S^{4n-2})),$$

if $m<0$. One checks easily that $Z$ still has signature $-2$ and Euler characteristic 0. So, every simplicial action of a $p$-group with $p\equiv 3 \bmod 4$ and $p\geq \dim_{\mathbb{F}_p} H^*(Z,\mathbb{F}_p)+1$ on $Z$ (if it can be suitably triangulated) has a fixed point, although the Lefschetz fixed point theorem does not apply. A similar construction works for the case of negative signature of $X$.

## 4. Concluding remarks

For $p=2$, each element in $W(\mathbb{F}_p)$ is determined by the parity of the dimension of the underlying vector space. Therefore a Witt class comparison result in this case for actions on arbitrary $\mathbb{F}_p$-PD spaces is almost immediate.

With regard to the results derived in the last two sections it seems reasonable to ask about similar results for more general types of actions than those respecting a cellular or simplicial structure. Theorem 2 should hold for a large class of actions and spaces, if one uses a suitable cohomology theory in order to apply the localization theorem (that is used implicitely in the proof of Proposition 8). The first choice here is sheaf theoretic cohomology and Borel-Moore homology and a notion of Poincaré duality pairs based on these theories. More delicate is a generalization of Proposition 13 to arbitrary oriented $\mathbb{Z}_{(p)}$-homology manifolds (as defined in [9], Definition V.9.1) in such a way that an induction approach to actions of finite $p$-groups is possible. The question one has to deal with can be formulated as follows: *Is an oriented $\mathbb{F}_p$-homology manifold $M$ automatically an orientable $\mathbb{Z}_{(p)}$-homology manifold, if $p$ is an odd prime number?* Of course, without the $\mathbb{Z}_{(p)}$-orientability requirement of $M$, this is indeed the case (by the universal coefficient theorem) and if $p=2$, then the answer to the original question is negative (by the existence of nonorientable closed manifolds). G.E. Bredon in [10], p. 43, pointed out the question, if the orientable cover of a homology manifold (for arbitrary coefficients) is always single or double sheeted. If this is true, then Proposition 13 should hold for $\mathbb{Z}/p$-actions on arbitrary compact oriented $\mathbb{Z}_{(p)}$-homology manifolds as well. Using the methods in [9] we can show:

**Proposition 19.** *Let $M$ be a second countable connected compact oriented $\mathbb{F}_p$-homology manifold, where $p$ is an odd prime. Assume that the stalks of the homology sheaf $\mathcal{H}_*(M;\mathbb{Z})$ (cf. [9], p. 293) are finitely generated and that the integral cohomological dimension $\dim_{\mathbb{Z}} M < \infty$. Then $M$ is an orientable $\mathbb{Z}_{(p)}$-homology manifold.*

*Proof.* Let $n$ be the dimension of $M$. As $M$ is a compact $\mathbb{F}_p$-homology manifold and $\mathcal{H}_*(M;\mathbb{Z})$ has finitely generated stalks, there is a multiplicative subset $S \subset \mathbb{Z}$ generated by finitely many primes $\neq p$ such that $\mathcal{H}_*(M;S^{-1}\mathbb{Z})$ has stalks isomorphic to $S^{-1}\mathbb{Z}$ in degree $n$ and zero otherwise. By [9], Theorem V.16.15, $M$ is an $S^{-1}\mathbb{Z}$-homology manifold of dimension $n$. Because each automorphism of $S^{-1}\mathbb{Z}$ has order two (only finitely many primes have been inverted) and so is still visible after factoring out the ideal generated by $p$, the action of $\pi_1(M)$ on $\mathcal{H}_*(M;S^{-1}\mathbb{Z})$ is trivial (as $M$ is orientable over $\mathbb{F}_p$). Therefore $\mathcal{H}_*(M;S^{-1}\mathbb{Z})$ is a constant sheaf, i.e. $M$ is orientable over $S^{-1}\mathbb{Z}$ and hence over $\mathbb{Z}_{(p)}$. □

Consequently, a generalization of Theorem 14 to arbitrary topological actions of a finite $p$-group $G$ on a second countable connected compact oriented $\mathbb{Z}_{(p)}$-homology manifold $X$ with $\dim_{\mathbb{Z}} X < \infty$ should hold under the following additional assumption: There is a chain $G_0 < G_1 < \ldots < G_l = G$ of subgroups that are normal in $G$ such that $G_0 \cong \mathbb{Z}/p$ and for all $i$ we have $G_i/G_{i-1} \cong \mathbb{Z}/p$ and the stalks of $\mathcal{H}_*(X^{G_i};\mathbb{Z})$ are finitely generated. Indeed, each component $F$ of $X^{G_0}$ is a second countable $\mathbb{F}_p$-homology manifold of even codimension in $X$ with $\dim_{\mathbb{Z}} F < \infty$ ([7], Proposition II.16.9, Theorem V.20.1 and Proposition V.20.2) and we can apply Proposition 19 to $F$. By induction we can proceed as before.

It is not clear to the author if the results presented in this paper can be generalized to the case of topological actions in full generality.

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Ludwig-Maximilians-Universität
Theresienstrasse 39
80333 München, Germany
*E-mail address*: hanke@rz.mathematik.uni-muenchen.de