THE STRONG NOVIKOV CONJECTURE FOR LOW DEGREE COHOMOLOGY

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ABSTRACT. We show that for each discrete group Γ, the rational assembly map

$$K_*(B\Gamma) \otimes \mathbb{Q} \to K_*(C^*_{max}\Gamma) \otimes \mathbb{Q}$$

is injective on classes dual to $\Lambda^* \subset H^*(B\Gamma; \mathbb{Q})$, where Λ^* is the subring generated by cohomology classes of degree at most 2. Our result implies homotopy invariance of higher signatures associated to classes in Λ^*. This consequence was first established by Connes-Gromov-Moscovici [4] and Mathai [9].

Our approach is based on the construction of flat twisting bundles out of sequences of almost flat bundles as first described in our work [5]. In contrast to the argument in [9], our approach is independent of (and indeed gives a new proof of) the result of Hilsum-Skandalis [6] on the homotopy invariance of the index of the signature operator twisted with bundles of small curvature.

1. INTRODUCTION

Throughout this paper, we use complex K-theory. Let Γ be a discrete group and denote by $C^*_{max}\Gamma$ the maximal group C^*-algebra of Γ. Recall the following form of the strong Novikov conjecture.

Conjecture 1.1. *The Baum-Connes assembly map*

$$A\colon K_*(B\Gamma) \to K_*(C^*_{max}\Gamma)$$

is injective after tensoring with the rationals.

The Chern character

$$\mathrm{ch}\colon K(-) \to H(-; \mathbb{Q})$$

is a natural transformation (of $\mathbb{Z}/2$-graded multiplicative (co-)homology theories) from K-homology to rational singular homology (both theories being defined in the homotopy theoretic sense). Let

$$\Lambda^*(\Gamma) \subset H^*(B\Gamma; \mathbb{Q})$$

be the subring generated by classes of degree at most 2.

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In this paper we verify the strong Novikov conjecture for classes dual to elements in $\Lambda^*(\Gamma)$.

Theorem 1.2. *Let $h \in K_*(B\Gamma)$ be a K-homology class such that the map*

$$\Lambda^*(\Gamma) \to \mathbb{Q}\,,\ \gamma \mapsto \langle \gamma, \mathrm{ch}(h) \rangle\,,$$

given by the Kronecker pairing (i.e. only elements of equal degree are paired) is nonzero. Then

$$A(h) \neq 0\,.$$

As a corollary we obtain the following result on homotopy invariance of higher signatures ($\mathcal{L}(M)$ denotes the L-polynomial of M).

Corollary 1.3 ([4], [9])**.** *Let M be a closed connected oriented smooth manifold, let Γ be a discrete group and let $f\colon M \to B\Gamma$ be a continuous map. Then for all $c \in \Lambda^*(\Gamma)$, the higher signature*

$$\langle \mathcal{L}(M) \cup f^*(c), [M] \rangle$$

is an oriented homotopy invariant.

The discussion of this result in [9] is based on a theorem of Hilsum-Skandalis [6] saying that the index of the signature operator twisted with Hilbert A-module bundles (A being a C^*-algebra) of small curvature is an oriented homotopy invariant. For flat twisting bundles this result was known before (see e.g. [7, 8, 10]). Our proof of Theorem 1.2 is independent from [6]. Indeed, we will illustrate in the last section of this paper how our methods allow the reduction of the Hilsum-Skandalis theorem to the case of flat twisting bundles.

2. Proof of the main theorem

By a standard suspension argument we may assume that $h \in K_0(B\Gamma)$. Because each discrete group is the direct limit of finitely presented groups and because the classifying space construction, the formation of C^*_{max} and the K-theory functors commute with direct limits, it is enough to treat the case of finitely presented Γ.

Due to the geometric description of K-homology by Baum-Douglas [2], elaborated in [3], there is triple (M, E, ϕ), where M is an even dimensional closed connected spin-manifold, $E \to M$ is a virtual (i.e. $\mathbb{Z}/2$-graded) hermitian complex vector bundle of finite dimension and $\phi : M \to B\Gamma$ is a continuous map, such that

$$\phi_*(E \cap [M]_K) = h\,.$$

(Strictly speaking, [3] only provides a spin^c-manifold; it is an exercise to obtain a spin manifold such that the associated spin^c-structure represents the right K-homology class). In the formula, we consider E as a class in $K^0(M)$, and $[M]_K$ is the K-theoretic orientation class of the spin manifold M.

As Γ is finitely presented, we can and will assume that the map $\phi : M \to B\pi_1(M)$ induces an isomorphism of fundamental groups.

Now chose $h \in K_0(B\Gamma)$ as in the main theorem and let (M, E, ϕ) be a triple representing h. Let

$$\nu = E\Gamma \times_\Gamma C^*_{max}\Gamma$$

be the canonical flat $C^*_{max}\Gamma$-module bundle over $B\Gamma$.

Denoting by $S^\pm \to M$ the bundles of positive or negative complex spinors on M, respectively, the assembly map is described as follows.

Proposition 2.1. *The element*

$$A(h) \in K_0(C^*_{max}\Gamma)$$

is equal to the Mishchenko-Fomenko index of

$$D_{E\otimes\phi^*(\nu)} \colon \Gamma((S \otimes E)^+ \otimes \phi^*(\nu)) \to \Gamma((S \otimes E)^- \otimes \phi^*(\nu)) \,,$$

the Dirac operator on M twisted with the virtual bundle $E \otimes \phi^*(\nu)$. Here, E is equipped with an arbitrary hermitian connection.

This description of the assembly map will be used in order to show that $A(h) \neq 0$.

For that purpose, let $c \in H^*(B\Gamma, \mathbb{Q})$ be such that

$$\langle c, \mathrm{ch}(h)\rangle \neq 0 \in \mathbb{Q} \,.$$

We can assume that $c \in H^*(B\Gamma; \mathbb{Z})$. In order to keep the exposition transparent, let us first assume that $c \in H^2(B\Gamma; \mathbb{Z})$.

Let $L \to B\Gamma$ be a complex hermitean line bundle classified by c. We pick a unitary connection on the pull back bundle $L' = \phi^*(L) \to M$ and denote by $\omega \in \Omega^2(M; i\mathbb{R})$ its curvature form. Let $\pi : \widetilde{M} \to M$ be the universal cover. Because the universal cover of $B\Gamma$ is contractible, the bundle $\pi^*(L') \to \widetilde{M}$ is trivial. Fix a unitary trivialization $\pi^*(L') \cong \widetilde{M} \times \mathbb{C}$. With respect to this trivialization, the induced connection on $\pi^*(L')$ is given by a 1-form $\eta \in \Omega^1(\widetilde{M}; i\mathbb{R})$.

Because $U(1)$ is abelian, the curvature form of this connection is equal to $d\eta$ which in turn coincides with $\pi^*(\omega)$ by naturality. However, contrary to the form $\pi^*(\omega)$, the connection form η is in general not invariant under the deck transformation group (this would imply that ω represents the zero class in $H^2(M)$).

We will use the bundle $\pi^*(L')$ in order to construct a flat A-module bundle $W \to M$ with an appropriate C^*-algebra A along the lines of [5]. The flat bundle W will induce a holonomy representation

$$C^*_{max}\pi_1(M) \to A$$

whose induced map in K-theory will be used to detect nontriviality of the element $A(h)$ appearing in the main theorem.

The details are as follows. For $t \in [0,1]$ we consider the connection on $\widetilde{M} \times \mathbb{C}$ associated to the 1-form $t \cdot \eta$. The corresponding curvature form is equal to $t \cdot \pi^*(\omega)$ and this is invariant under deck transformations (note that this is in general not true for the forms $t \cdot \eta$ if $t \neq 0$).

We would like to use the bundle L' to construct a family of almost flat bundles $(P_t)_{t\in[0,1]}$ (cf. [5], Section 2) so that [5, Theorem 2.1] can be applied in order to obtain an "infinite product bundle"

$$V = \prod_{n\in\mathbb{N}} P_{1/n} \to M$$

whose quotient by the corresponding infinite sum bundle will be the desired flat bundle $W \to M$, cf. [5, Proposition 3.4]. However, by Chern-Weil theory it is in general impossible to produce a finite dimensional bundle on M whose curvature form is equal to $t \cdot \omega$, $0 < t < 1$ (the associated Chern class would not be integral).

We bypass this difficulty by allowing infinite dimensional bundles. Consider the Hilbert space bundle

$$\mu = \widetilde{M} \times_\Gamma l^2(\Gamma) \to M\,.$$

Here, Γ acts on the left of $l^2(\Gamma)$ by the formula

$$(\gamma\psi)(x) := \psi(x \cdot \gamma^{-1})\,,\ x, \gamma \in \Gamma\,.$$

The forms $t\eta \in \Omega^1(\widetilde{M})$ induce a family of connections ∇^t on μ, the connection ∇^t being induced by the Γ-invariant connection on $\widetilde{M} \times l^2(\Gamma)$ which on the subbundle

$$\widetilde{M} \times \mathbb{C} \cdot 1_\gamma \subset \widetilde{M} \times l^2(\Gamma)$$

(identified canonically with $\widetilde{M} \times \mathbb{C}$) coincides with $(\gamma^{-1})^*(t\eta)$.

We wish to regard the bundles (μ, ∇^t) as twisting bundles for the Dirac operator on M. The index of this Dirac operator will live in the K-theory of an appropriate C^*-algebra A_t that we think of as the holonomy algebra of μ with respect to the connection ∇^t.

To define these algebras, let us choose a base point $p \in M$ and a point $q \in \widetilde{M}$ above p. We identify the fibre over p with the Hilbert space $l^2(\Gamma)$.

Now let

$$A_t \subset B(l^2(\Gamma))$$

be the norm-linear closure of all maps $l^2(\Gamma) \to l^2(\Gamma)$ that arise from parallel transport with respect to ∇^t along closed curves in M based at p. Further, we define a bundle $P_t \to M$ whose fibre over $x \in M$ is given by the norm-linear closure (in $\mathrm{Hom}(\mu_p, \mu_x)$) of all isomorphisms $\mu_p \to \mu_x$ arising from parallel transport with respect to ∇^t along smooth curves connecting p with x. In this way we obtain, for each $t \in [0,1]$, a locally trivial bundle P_t consisting of free A_t-modules of rank 1 (the A_t-module structure given by precomposition) and equipped with A_t-linear connections. For the notions relevant in this context, we refer the reader e.g. to [11]. Parallel transport on P_t along a curve connecting x with x' is induced by parallel transport $\mu_x \to \mu_{x'}$ with respect to ∇^t. The bundle μ may be recovered from the "principal bundle" P_t as an associated bundle, i.e. $\mu = P_t \times_{A_t} l^2(\Gamma)$.

The next lemma is crucial for the calcuations which will follow.

Lemma 2.2. *Each of the algebras A_t carries a canonical trace*

$$\tau_t : A_t \to \mathbb{C}$$

given by

$$\tau_t(\psi) := \langle \psi(1_e), 1_e \rangle$$

where $1_e \in l^2(\Gamma)$ is the characteristic function of the neutral element and $\langle -, - \rangle$ is the inner product on $l^2(\Gamma)$.

Proof. Let γ and γ' be two closed smooth curves based at $p \in M$ and let ϕ_γ and $\phi_{\gamma'}$ be parallel transport along γ and γ'. We show that

$$\tau_t(\phi_\gamma \cdot \phi_{\gamma'}) = \tau_t(\phi_{\gamma'} \cdot \phi_\gamma) \, .$$

We will assume from now on that γ and γ' represent elements in $\pi_1(M,p)$ that are inverse to each other (otherwise, both sides of the above equation are zero). We lift the composed curves $\gamma \cdot \gamma'$ and $\gamma' \cdot \gamma$ to curves χ and χ' starting at q in the universal cover $\widetilde{M}$. By assumption, both χ and χ' are closed curves. We need to compare parallel transport of the element

$$(q, 1) \in \widetilde{M} \times \mathbb{C}$$

along χ and χ'. Denoting the result of parallel transport along these curves by (q, ξ_χ) and $(q, \xi_{\chi'})$, respectively, we have

$$\xi_\chi = \exp \Big(\int_{[0,1]} -t\eta_{\chi(\tau)}(\dot{\chi}(\tau)) d\tau \Big) \, ,$$

with the exponential map on the Lie group S^1. By use of Stoke's formula, the last expression is equal to

$$\exp \Big(\int_D -d(t\eta) dx \Big) = \exp \Big(\int_D -t\pi^*(\omega) dx \Big) \, ,$$

where $D : D^2 \to \widetilde{M}$ is a disk with boundary χ (recall that $\widetilde{M}$ is simply connected). However, the curve χ' is obtained (up to reparametrization) from

χ by applying the deck transformation corresponding to γ'. This implies - grace to the invariance of the form $\pi^*(\omega)$ under deck transformation - that the last expression is equal to $\xi_{\chi'}$. Using continuity and linearity of τ_t, this shows our assertion. □

We now equip E with a unitary connection and consider the twisted Dirac operator

$$D_{E\otimes P_t} : \Gamma((S\otimes E)^+ \otimes P_t) \to \Gamma((S\otimes E)^- \otimes P_t)\,.$$

The index $\operatorname{ind}(D_{E\otimes P_t}) \in K_0(A_t)$ satisfies (up to sign) the equation

$$\tau_t(\operatorname{ind}(D_{E\otimes P_t})) = \langle \operatorname{ch}(\mathcal{A}(M)) \cup \operatorname{ch}(E) \cup \operatorname{ch}_{\tau_t}(P_t), [M]\rangle$$

by the Mishchenko-Fomenko index theorem (see [11, Theorem 6.9]). Because the Poincaré dual of $\operatorname{ch}(\mathcal{A}(M))$ is equal to $\operatorname{ch}([M]_K)$, the last expression equals $\langle \operatorname{ch}_{\tau_t}(P_t), \operatorname{ch}([E]\cap[M]_K)\rangle$. Recall that the choice of (M, E, ϕ) implies $\phi_* \operatorname{ch}([E]\cap[M]_K) = h$. The construction of the connection ∇^t and the definition of $\operatorname{ch}_{\tau_t}$ (see [11, Definition 5.1.]) show

$$\operatorname{ch}_{\tau_t}(P_t) = \exp(t\phi^* c) \in H^*(M;\mathbb{R})$$

so that finally

$$\tau_t(\operatorname{ind}(D_{E\otimes P_t})) = \langle \exp(t\phi^* c), \operatorname{ch}([E]\cap[M]_K)\rangle = \langle \exp(tc), \operatorname{ch}(h)\rangle \in \mathbb{R}[t]\,,$$

a polynomial which is different from zero by the assumption $\langle c, \operatorname{ch}(h)\rangle \neq 0$.

We wish to use this calculation in order to detect nontriviality of $A(h) \in K_0(C^*_{max}\Gamma)$. This is done by constructing a flat bundle on M out of the sequence of bundles $(P_{1/n})_{n\in\mathbb{N}}$ with connections. This sequence is almost flat in the sense of [5, Section 2]. Therefore, applying [5, Theorem 2.1.] we obtain a smooth "infinite product bundle"

$$V = \prod_{n\in\mathbb{N}} P_{1/n} \to M$$

equipped with a connection that we may think of as the product of the connections on $P_{1/n}$. After passing to the quotient by the infinite sum of the bundles $P_{1/n}$, we obtain a smooth bundle

$$W = \big(\prod P_{1/n}\big)/\big(\bigoplus P_{1/n}\big) \to M$$

of free modules of rank 1 over the C^*-algebra

$$A = \big(\prod A_{1/n}\big)/\big(\overline{\bigoplus A_{1/n}}\big)$$

which carries an induced flat connection. For more details of this discussion, we refer the reader to [5], especially to Section 2 and the statements before Proposition 3.4.

The flat bundle W induces a unitary holonomy representation of $\pi_1(M)$. Because ϕ induces an isomorphism $\pi_1(M) \cong \Gamma$ by the choice of the triple (M, ϕ, E), we hence obtain a C^*-algebra map

$$\psi\colon C^*_{max}\Gamma \to A$$

using the universal property of C^*_{max}. The induced map in K-theory can be analyzed in terms of the KK-theoretic description of the index map (cf. [5, Lemma 3.1]). One concludes that $\psi_*(A(h)) \in K_0(A)$ is equal to the index of the twisted Dirac operator

$$D_{E\otimes W}\colon \Gamma((S\otimes E)^+ \otimes W) \to \Gamma((S\otimes E)^- \otimes W)$$

and this in turn equals the image of the index of the twisted Dirac operator

$$D_{E\otimes V}\colon \Gamma((S\otimes E)^+ \otimes V) \to \Gamma((S\otimes E)^- \otimes V)$$

under the canonical map $p_*\colon K_0(\prod A_{1/n}) \to K_0(A)$ induced by the canonical projection $p\colon \prod A_{1/n} \to A$.

We have a short exact sequence

$$K_0(\bigoplus A_{1/n}) \to K_0(\prod A_{1/n}) \xrightarrow{p_*} K_0(A)$$

where the left hand group is canonically isomorphic to the algebraic direct sum $\bigoplus_{n\in\mathbb{N}} K_0(A_{1/n})$. Furthermore, the traces $\tau_{1/n}\colon A_{1/n} \to \mathbb{C}$ all have norm 1 and hence induce a trace

$$\prod A_{1/n} \to \prod \mathbb{C}\,.$$

Using the canonical isomorphism $K_0(\bigoplus A_{1/n}) \cong \bigoplus K_0(A_{1/n})$, we finally get a trace map

$$\tau\colon \operatorname{im} p_* \to \big(\prod \mathbb{C})/(\bigoplus \mathbb{C}\big)\,.$$

Note that the direct sum on the right is understood in the algebraic sense: Any element has only finitely many components different from 0. Assuming $\operatorname{ind}(D_{E\otimes W}) = 0$ we conclude that the element in $\prod \mathbb{C}/\bigoplus \mathbb{C}$ represented by

$$(\operatorname{ind}(D_{E\otimes P_{1/n}}))_{n\in\mathbb{N}}$$

is equal to zero. Combining this with our previous calculation, this means that the polynomial

$$\langle \exp(tc), \operatorname{ch}(h)\rangle \in \mathbb{R}[t]$$

is equal to zero for all but finitely many values $t = 1/n$, $n \in \mathbb{N}$. This implies that this polynomial is identically zero and hence in particular

$$\langle c, \operatorname{ch}(h)\rangle = 0$$

in contradiction to our assumption. The proof of the main theorem is therefore complete, if $c \in H^2(B\Gamma;\mathbb{Z})$.

We will now discuss the case of general $c \in \Lambda^*(\Gamma)$ (still assuming $h \in K_0(B\Gamma)$). If

$$c = c_1 \cup \ldots \cup c_k$$

is a product of classes in $H^2(B\Gamma;\mathbb{Z})$, we replace the bundle L in the above argument by the tensor product bundle

$$L := L_1 \otimes L_2 \otimes \ldots \otimes L_k \to B\Gamma$$

where the line bundle $L_i \to B\Gamma$ is classified by c_i. In an analogous fashion as before, we get bundles

$$P_{t_1,\ldots,t_k} \to M$$

of Hilbert-$A_{t_1,t_2,\ldots,t_k}$-modules working with the connection

$$t_1\eta(1) + t_2\eta(2) + \ldots + t_k\eta(k)$$

on the bundle $\widetilde{M} \times \mathbb{C}$ where $\eta(i)$ is a connection induced from L_i and where each $t_i \in [0,1]$.

Assuming that $A(h) = 0$ we can conclude in this case that the polynomial

$$\langle \exp(t_1c_1) \cdot \ldots \cdot \exp(t_kc_k), \mathrm{ch}(h)\rangle \in \mathbb{R}[t_1,\ldots,t_k]$$

is equal to zero for all but finitely many

$$(t_1,\ldots,t_k) = (1/n_1,\ldots,1/n_k)$$

where $n_i \in \mathbb{N}$. Hence this polynomial is identically 0 and in particular

$$\langle c_1 \cup \ldots \cup c_k, \mathrm{ch}(h)\rangle = 0$$

again contradicting our assumption. In the most general case, c being a sum

$$c = c(1) + \ldots + c(k)$$

where each $c(i)$ is a product of two dimensional classes in $H^2(B\Gamma;\mathbb{Z})$, the assumption

$$\langle c, \mathrm{ch}(h)\rangle \neq 0$$

implies that already for one summand $c(i)$, we have $\langle c(i), \mathrm{ch}(h)\rangle \neq 0$ so that we are reduced to the previous case.

Because we already used a suspension argument in order to restrict attention to classes in $K_0(B\Gamma)$, this finishes the proof of the main theorem.

3. Hilsum-Skandalis revisited

If M is an oriented Riemannian manifold of even dimension, recall that $d + d^*$, the sum of the exterior differential and its formal adjoint, defines the signature operator

$$D^{sign}\colon \Gamma(\Lambda^*_+(M)) \to \Gamma(\Lambda^*_-(M))\,,$$

where the $\pm$-signs indicate ± 1-eigenspaces of the Hodge star operator.

If $E \to M$ is a Hilbert A-module bundle over M, where A is some C^*-algebra, we obtain the twisted signature operator

$$D_E^{sign} \colon \Gamma(\Lambda^*_+(M) \otimes E) \to \Gamma(\Lambda^*_-(M) \otimes E)$$

which has an index $\operatorname{ind} D_E^{sign} \in K_0(A)$. The following theorem says that this class is an oriented homotopy invariant, if E has small curvature.

Theorem 3.1 (Hilsum-Skandalis [6]). *Let M and M' be closed oriented Riemannian manifolds of the same dimension and let $h \colon M' \to M$ be an orientation preserving homotopy equivalence. Then there exists a constant $c > 0$ with the following property: If $E \to M$ is a Hilbert A-module bundle with connection ∇ so that the associated curvature form $\Omega_\nabla \in \Omega^2(M; \operatorname{End} E)$ satisfies the bound*

$$\|\Omega_\nabla\| < c\,,$$

(the norm being defined by the maximum norm on the unit sphere bundle in $\Lambda^2 M$ and the operator norm on each fibre $\operatorname{End}(E_x, E_x)$), then we have

$$\operatorname{ind}(D^{sign}_{f^*(E)}) = \operatorname{ind}(D^{sign}_E)\,.$$

If E is flat, i.e. $\Omega_\nabla = 0$, this result was proved in [7, 8, 10]. In this section, we will explain briefly how this special case implies the general statement of Theorem 3.1. Our argument is again based on the construction of a flat bundle out of a sequence of almost flat bundles [5].

Assuming that no c with the stated property exists, we find a sequence of C^*-algebras A_n and a sequence of Hilbert A_n-module bundles $E_n \to M$ with connections ∇_n so that $\|\Omega_{\nabla_n}\| < \frac{1}{n}$, but

$$\operatorname{ind}(D^{sign}_{f^*(E_n)}) \neq \operatorname{ind}(D^{sign}_{E_n})$$

for all n. We obtain an almost flat sequence

$$f^*(E_n) \cup E_n \to M' \cup M$$

of Hilbert A_n-module bundles in the sense of [5, Section 2] over the disjoint union $M' \cup M$. Applying again the methods in [5], we obtain a flat Hilbert A-module bundle

$$W = \big(\prod f^*(E_n) \cup \prod E_n\big) / \big(\bigoplus f^*(E_n) \cup \bigoplus E_n\big) \to M' \cup M$$

where

$$A = \big(\prod A_n\big) / \big(\overline{\bigoplus A_n}\big)\,.$$

The index $\operatorname{ind}(D^{sign}_W)$ of the signature operator on $(-M') \cup M$ (the minus-sign indicates reversal of orientation) twisted by W, vanishes by the results in [7, 8, 10].

This conclusion leads to the following contradiction. We have a canonical isomorphism

$$K_0(\bigoplus A_n) \cong \bigoplus K_0(A_n)$$

and - assuming that each A_n is unital and stable, the last property easily being achieved by replacing each A_n by $A_n \otimes \mathbb{K}(l^2(\mathbb{N}))$ - a canonical isomorphism

$$K_0(\prod A_n) \cong \prod K_0(A_n)\,,$$

compare the proof of [5, Proposition 3.6]. The signature operator on $-M' \cup M$ twisted with the non-flat Hilbert $\prod A_n$-module bundle

$$V := \prod f^*(E_n) \cup \prod E_n$$

has an index

$$\operatorname{ind}(D_V^{sign}) \in K_0(\prod A_n) \cong \prod K_0(A_n)$$

which is different from 0 for infinitely many factors by our assumption on the bundles E_n. On the other hand, under the canonical map

$$p_*\colon K_0(\prod A_n) \to K_0(A)$$

this index maps to the index of the signature operator twisted with the bundle W and this index was identified as 0. Now the contradiction arises from the fact that by the long exact K-theory sequence, the kernel of p_* is equal to $K_0(\bigoplus A_n) = \bigoplus K_0(A_n)$ (algebraic direct sum) and therefore $\operatorname{ind}(D_V^{sign})$ is not contained in it.

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