DISSOLVING FOUR-MANIFOLDS AND POSITIVE SCALAR CURVATURE

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ABSTRACT. We prove that many simply connected symplectic four-manifolds dissolve after connected sum with only one copy of $S^2 \times S^2$.

For any finite group G that acts freely on the three-sphere we construct closed smooth four-manifolds with fundamental group G which do not admit metrics of positive scalar curvature, but whose universal covers do admit such metrics.

1. INTRODUCTION

It is a classical result of Wall [23] that any two simply connected four-manifolds with isomorphic intersection forms become diffeomorphic after taking the connected sum with sufficiently many copies of $S^2 \times S^2$. It follows that any simply connected four-manifold is stably diffeomorphic to a connected sum of complex projective planes (with both orientations allowed) or to a connected sum of copies of $S^2 \times S^2$ and of the $K3$ surface. In general, it is a very hard problem to determine the minimal number of copies of $S^2 \times S^2$ required. Gauge theory shows that this number is usually positive, but, in spite of various attempts, it has not led to any further lower bounds, compare [2, 3]. This can be taken as evidence for the conjecture that one copy of $S^2 \times S^2$ always suffices, which is only known in very few cases, for example for elliptic complex surfaces, see [13].

One purpose of this paper is to prove that many simply connected symplectic four-manifolds constructed from algebraic surfaces by symplectic sums along submanifolds [6] become diffeomorphic to standard manifolds after taking the connected sum with only one copy of $S^2 \times S^2$. We focus on the spin manifolds of nonnegative signature constructed by J. Park [16], but our argument applies to many other cases. There are several reasons for looking at these particular manifolds. Firstly, in the spin case there are

gauge theoretic invariants [2, 3] which could, in theory, obstruct the kind of result we seek. Secondly, simply connected algebraic surfaces or symplectic manifolds with nonnegative signature are quite difficult to construct, and are considered to be more exotic than the ones of negative signature. In the spin case, a standard four-manifold of positive signature is a connected sum of copies of $S^2 \times S^2$ and of the $K3$ surface with the non-complex orientation. While Wall's result implies that any simply connected four-manifold with positive signature becomes diffeomorphic to such a connected sum after sufficiently many stabilizations, it is perhaps mildly surprising that a symplectic manifold built from algebraic surfaces splits off $K3$ surfaces with the "wrong" orientation already after a single stabilization. Finally, we are interested in spin manifolds of zero signature because of an application to questions about the existence of positive scalar curvature metrics.

In Section 3 we give examples of both spin and non-spin four-manifolds with finite fundamental groups which do not admit metrics of positive scalar curvature, although their universal covers do admit such metrics. The possible fundamental groups are all finite groups which act freely on S^3, including all the finite cyclic groups. Previously, such an example with fundamental group of order two was given by LeBrun [11], whose result is analogous to one obtained by Bérard Bergery [1] in high dimensions. The case of odd order fundamental groups is more interesting, because it disproves the following:

Conjecture 1 (Rosenberg [19], 1.2). *Assume that M^n is a connected closed manifold and $\pi_1(M)$ is finite of odd order. Then M admits a metric of positive scalar curvature, if and only if its universal cover does.*

In fact, in dimensions ≥ 5, Rosenberg [19] proved this conjecture for cyclic groups, and thus our examples with finite cyclic fundamental groups of odd orders do not have higher-dimensional analogs. Kwasik–Schultz [10] have confirmed Rosenberg's conjecture for some classes of non-cyclic groups in dimensions ≥ 5.

Since the beginning of Seiberg–Witten theory [24], it has been known that there are many simply connected four-manifolds which do not admit metrics of positive scalar curvature, although the Gromov–Lawson conjecture [8], which is true in dimensions ≥ 5, predicts that they should have such metrics. Our results in Section 3 show that the situation is similar for Conjecture 1.

2. DISSOLVING SPIN FOUR-MANIFOLDS

In this section we prove that there are many spin symplectic four-manifolds which upon taking the connected sum with just one copy of $S^2 \times S^2$ dissolve into connected sums of copies of $S^2 \times S^2$ (in the zero signature case), or of

copies of $S^2 \times S^2$ and of the $K3$ surface (in the case of non-zero signature). In fact, it will turn out that the requirement to dissolve after a single stabilization does not substantially influence the geography of Chern numbers. See Remark 8 for the non-spin case.

Our proof is based on the following result of Gompf [5] (Lemma 4 and Corollary 5) elaborating on earlier work of Mandelbaum [12]. We only state the special case that we need.

Proposition 2 (Gompf [5])**.** *Let M and N be simply connected oriented 4-manifolds containing the same embedded surface F of genus $g \geq 1$ with zero selfintersection. Assume that F has simply connected complement in M, and that M is spin. Denote by P the sum of M and N along F. Then $P \# (S^2 \times S^2)$ is diffeomorphic to $M \# N \# 2g(S^2 \times S^2)$.*

This implies in particular that the connected sum of any simply connected spin elliptic surface with $S^2 \times S^2$ is diffeomorphic to a connected sum of copies of the $K3$ surface and of $S^2 \times S^2$, see [12], or [5] Corollary 8.

Here is our main result about spin manifolds of zero signature, which we will use for our application to questions about positive scalar curvature.

Theorem 3. *There are infinitely many integers l for which the following statements hold:*

    (1) *There are infinitely many symplectic manifolds X_i homeomorphic to the connected sum of l copies of $S^2 \times S^2$, which are pairwise non-diffeomorphic.*

    (2) *The connected sum of each X_i with $S^2 \times S^2$ is diffeomorphic to the connected sum of $l + 1$ copies of $S^2 \times S^2$.*

Proof. If one does not insist on the second property, the required examples have been constructed by J. Park [16]. For certain l, Park first constructs a symplectic manifold X homeomorphic to the connected sum of l copies of $S^2 \times S^2$. Then he shows using the knot surgery of Fintushel and Stern that one can change the smooth structure of X to infinitely many different ones, all of which support symplectic structures.

We will show that for infinitely many values of l the first part of Park's construction yields a manifold X which dissolves after connected sum with just one copy of $S^2 \times S^2$. Then we construct infinitely many homeomorphic non-diffeomorphic examples X_i from X by varying one of the building blocks. Our construction allows us to check that all the X_i do indeed dissolve upon connected sum with only one copy of $S^2 \times S^2$.

We now recall the construction of Park [16]. One begins with a simply connected spin algebraic surface Y of positive signature containing a smooth holomorphic curve F of genus g of zero selfintersection, together with an embedded 2-sphere S intersecting F transversely in one point. Such

examples are provided by Persson–Peters–Xiao [17]. The existence of S ensures that the complement of F in Y is simply connected.

The second building block is a simply connected spin symplectic manifold Z containing a symplectically embedded copy of F also with zero selfintersection, and a symplectically embedded torus T of zero selfintersection which is disjoint from F. Park exhibits concrete examples for each g.

Let $X(k,n)$ be the manifold obtained by symplectically summing k copies of Y and one copy of Z along F and then symplectically summing the result with the simply connected spin elliptic surface $E(2n)$ without multiple fibers along T and a fiber in $E(2n)$. The manifold $X(k,n)$ is spin and symplectic by construction. It is also simply connected because F has simply connected complement in Y, as does the fiber in $E(2n)$. We shall think of Z as the central component, with k copies of Y and one copy of $E(2n)$ attached to Z, rather than being attached to each other.

By Novikov additivity the signature of $X(k,n)$ is the sum of the signatures of all the building blocks. Thus, if we choose

$$(1) \qquad\qquad n = \frac{1}{16}(k\sigma(Y) + \sigma(Z)) \,,$$

which is positive for large enough k because $\sigma(Y)$ is positive, then $X(k,n)$ has zero signature. (Note that by Rochlin's theorem $\sigma(Y)$ and $\sigma(Z)$ are divisible by 16.) By Freedman's classification [4], $X(k,n)$ is then homeomorphic to a connected sum of l copies of $S^2 \times S^2$, where $l = \frac{1}{2}b_2(X(k,n))$.

Now consider the connected sum $X(k,n)\#(S^2 \times S^2)$. By using Gompf's result, Proposition 2 above, repeatedly we see that

$$X(k,n)\#(S^2 \times S^2) \cong kY \# Z \# E(2n) \#(2gk - k + 2)(S^2 \times S^2) \,.$$

As $E(2n)$ dissolves upon connected sum with $S^2 \times S^2$, we obtain

$$X(k,n)\#(S^2 \times S^2) \cong kY \# Z \# nK3 \#(2gk - k + 1 + n)(S^2 \times S^2) \,.$$

Recall that n grows linearly with k according to (1).

Wall [23] proved that any two simply connected 4-manifolds with isomorphic intersection forms become diffeomorphic after some number of stabilizations with $S^2 \times S^2$. Thus, if k is large enough, the connected sum of Y and $(2gk - k + 1 + n)(S^2 \times S^2)$ will be diffeomorphic to a connected sum of copies of $S^2 \times S^2$ and of copies of the $K3$ surface with the non-complex orientation. We can then break up all the k copies of Y inductively. Moreover, pairing the resulting copies of the $K3$ surface with the non-complex orientation with copies of $K3$ with the complex orientation, we obtain further copies of $S^2 \times S^2$. The number of copies of $S^2 \times S^2$ that we split off grows with k, so by Wall's result we may assume that there are enough of them to break up Z as well. As $X(k,n)$ is spin of zero signature,

we finally see that for all large enough k and n given by (1), the manifold $X(k,n)\#(S^2 \times S^2)$ is diffeomorphic to a connected sum of copies of $S^2 \times S^2$.

It remains to show that there are infinitely many symplectic manifolds homeomorphic but non-diffeomorphic to $X(n,k)$, all of which dissolve upon connected sum with $S^2 \times S^2$. For this we would like to replace the elliptic surface $E(2n)$ without multiple fibers by one with multiple fibers obtained by logarithmic transformation. However, in this case the general fiber becomes divisible in homology, in particular its complement is no longer simply connected. This could introduce fundamental group in our 4-manifold, and could obstruct the application of Proposition 2.

To circumvent this problem we argue as follows. Think of the elliptic surface $E(2n)$ as the fiber sum of $E(2n-2)$ with $E(2) = K3$. Instead of attaching $E(2n)$ to Z in the above construction, we attach $K3$ to Z and then attach $E(2n-2)$ to the $K3$ surface. However, these attachments are not performed along parallel two-tori in $K3$. We exploit the existence of two disjoint Gompf nuclei in $K3$, each of which contains a torus of selfintersection zero and a transverse 2-sphere intersecting the torus once. Gompf proved in [6], Section 3, that the $K3$ surface has a symplectic structure which makes the tori in two disjoint nuclei into symplectic submanifolds. Therefore, the above construction, using summation along tori in different nuclei, can be performed symplectically. Now, the boundary circle of a normal disk in a tubular neighbourhood of such a torus is null-homotopic along the transverse 2-sphere inside the nucleus, so that summing $K3$ to Z (with copies of Y attached) does not introduce any fundamental group. Then, attaching the elliptic surfaces with multiple fibers obtained by logarithmic transformations on $E(2n-2)$ along a torus in a different nucleus of the $K3$ surface, this torus will again have simply connected complement in its own nucleus, making it irrelevant that the fiber of the elliptic surface does not have simply connected complement. Thus Proposition 2 is applicable to all these summations.

The logarithmic transformations on $E(2n-2)$ produce infinitely many distinct smooth structures on the topological manifold underlying $E(2n-2)$, which are detected by Seiberg–Witten invariants, cf. [24]. This difference in the Seiberg–Witten invariants survives the symplectic sum operation along a fiber, because of the gluing formulas due to Morgan–Mrówka–Szabó [14] and Morgan–Szabó–Taubes [15]. Thus, we can produce infinitely many symplectic spin manifolds homeomorphic but non-diffeomorphic to $X(k,n)$. All these dissolve upon taking the connected sum with only one copy of $S^2 \times S^2$ by the same argument as for $X(k,n)$. $\square$

We can easily adapt the above argument to spin manifolds of nonzero signature. Starting from the examples constructed in the proof of Theorem 3, we can symplectically sum in extra copies of Y along F to make the signature positive, or we can use larger elliptic surfaces, with n larger than the value given by (1), to make the signature negative. This will give manifolds which still dissolve after only one stabilization.

To cover a large area of the geography more systematically, we proceed as follows. We use the coordinates $c_1^2 = 2e + 3\sigma$ and $\chi = \frac{1}{4}(e + \sigma)$. The spin condition implies, via Rochlin's theorem, that

$$(2) \qquad\qquad c_1^2 \equiv 8\chi \pmod{16}.$$

As spin symplectic manifolds are automatically minimal, they satisfy $c_1^2 \geq 0$ by the work of Taubes [21]. It is clear that in the simply connected case one must have $\chi > 0$. Thus, we try to cover lattice points in the first quadrant of the (χ, c_1^2)–plane subject to the congruence (2) with spin symplectic manifolds which dissolve after only one stabilization.

For a constant c let R_c denote the set of lattice points (x, y) in the plane satisfying $y \equiv 8x \pmod{16}$, $x > 0$, $y \geq 0$, and

$$(3) \qquad\qquad y \leq 2x - 16,$$
$$(4) \qquad\qquad y \leq 8x - c.$$

Proposition 4. *There exists a constant c such that all lattice points in R_c are realized as the Chern invariants (χ, c_1^2) of infinitely many pairwise non-diffeomorphic simply connected symplectic spin manifolds, all of which dissolve upon taking the connected sum with $S^2 \times S^2$.*

Proof. The argument is modelled on the above proof of Theorem 3 and the proof of Proposition 1 in [16].

Let H be a spin Horikawa surface with $c_1^2(H) = 8$ and $\chi(H) = 7$. This is a symplectic genus two fibration over the two-sphere; we fix a fiber F of this fibration. The complement of F in H contains the Milnor fiber of the $(2, 3, 7)$-singularity, and one can find a torus T of zero selfintersection with a transverse 2-sphere inside the Milnor fiber. We can deform the symplectic structure of H so that T is a symplectic submanifold. Let $H(k, n)$ be the manifold obtained by taking the fiber sum of k copies of H along F and then summing the resulting manifold with the elliptic surface $E(2n)$ along T. This is a symplectic spin manifold, which is simply connected because of the existence of the transverse 2-spheres for F and T. We have $c_1^2(H(k, n)) = 16k - 8$ and $\chi(H(k, n)) = 8k + 2n - 1$. By varying $k \geq 1$ and $n \geq 1$, we can cover all the lattice points in R_c with odd x using these manifolds.

Now consider the connected sum $H(k,n)\#(S^2 \times S^2)$. By using Gompf's result, Proposition 2, repeatedly we see that

$$H(k,n)\#(S^2 \times S^2) \cong kH\#E(2n)\#(3k-1)(S^2 \times S^2) \,.$$

As $E(2n)$ dissolves upon connected sum with $S^2 \times S^2$, we obtain

$$H(k,n)\#(S^2 \times S^2) \cong kH\#nK3\#(3k+n-2)(S^2 \times S^2) \,.$$

By the result of Wall [23] there is a k_0 such that $H\#k_0(S^2 \times S^2)$ dissolves. Therefore, $H(k,n)\#(S^2 \times S^2)$ dissolves as soon as $3k+n-2 \geq k_0$, which is equivalent to (4) with $c = 16k_0 + 32$.

Finally, as $n \geq 1$, we can perform logarithmic transformations on the elliptic building blocks to achieve infinitely many distinct smooth structures on the topological manifold underlying $H(k,n)$. All these smooth structures admit symplectic structures and still dissolve after only one stabilization. This completes the proof for odd values of x.

To cover the lattice points with even x in R_c, we use $H'(k,n)$, obtained from the above $H(k,n)$ by summing in an additional copy of H, summed along T, not along F. The resulting manifolds are again simply connected because H contains a transverse 2-sphere for T in the complement of F. This construction covers all lattice points with $y > 0$. For $y = 0$ we can just use the spin elliptic surfaces $E(2n)$ themselves. The rest of the argument is as for the case of odd x. $\qquad\square$

This leads to the following geography result:

Theorem 5. *There is a line of slope > 8 in the (x,y)-plane such that every lattice point in the first quadrant which is below this line and satisfies $y \equiv 8x \pmod{16}$ is realized by the Chern invariants (χ, c_1^2) of infinitely many pairwise nondiffeomorphic simply connected symplectic spin manifolds, all of which dissolve upon taking the connected sum with $S^2 \times S^2$.*

Proof. Let Y and Z be the building blocks from the proof of Theorem 3. We sum k copies of Y to Z along the surface F of genus g. Then we sum the $H(l,n)$ and $H'(l,n)$ from the region R_c in Proposition 4 to the resulting manifolds by summation along the torus T in Z and in the elliptic piece of $H(l,n)$ or $H'(l,n)$. In all these summations the complement of the surface along which the summation is performed is simply connected in at least one of the summands, so that the resulting manifolds are simply connected.

By the proofs of Theorem 3 and of Proposition 4, these manifolds have all the desired properties as soon as $3l + n$ is large enough. It is easy to see that varying k and letting (l,n) range over the parameters of the $H(l,n)$ or $H'(l,n)$ in R_c, the Chern invariants cover all the lattice points in the claimed area, because Y has positive signature. (We may have to increase

the constant c from Proposition 4 in order to ensure that $3l + n$ is always large enough.) $\square$

Remark 6. Theorem 5 should be compared to the main theorem of Park [16], who proved a version of it without the requirement that the manifolds in question dissolve after only one stabilization. Park argues that by using many copies of Y, one can push the slope up to approximately the slope $\frac{c_1^2(Y)}{\chi(Y)}$ of Y, and [17] provides a construction for Y with slope > 8.76. However, the summation of Y to itself, or to Z, is performed along a surface of (unknown) genus $g > 1$, so that the Chern numbers are not additive. Instead, asymptotically for large k, the best slope one can obtain is approximately

$$\frac{c_1^2(Y) + 8(g - 1)}{\chi(Y) + g - 1} > 8 \, ,$$

both in Park's result and in ours. This is smaller than $\frac{c_1^2(Y)}{\chi(Y)}$.

Thus, considering only manifolds which dissolve after a single stabilization does not alter the geography in any essential way.

Remark 7. For the spin manifolds in Theorems 3 and 5, the 2-torsion instanton invariants of [2, 3] are defined, but vanish because the manifolds dissolve after only one stabilization. Therefore [3], their Donaldson polynomials are all even. Under certain technical hypotheses, Fintushel–Stern [3] proved the evenness of Donaldson polynomials for manifolds which do not necessarily dissolve after the first stabilization.

Remark 8. Most of our examples also have the property that they dissolve upon connected sum with a single copy of $\mathbb{C}P^2$. This means that they are almost completely decomposable in the sense of Mandelbaum [13]. These are the first examples of irreducible four-manifolds of non-negative signature which are almost completely decomposable.

One can use the arguments in the proofs of Theorems 3 and 5 to exhibit non-spin almost completely decomposable minimal symplectic manifolds of positive and of zero signature by using non-spin elliptic surfaces instead of spin ones, or by constructing similar irreducible manifolds starting from non-spin building blocks, rather than the spin ones of Persson-Peters-Xiao [17]. The geography statements one obtains for irreducible almost completely decomposable four-manifolds in the non-spin case are rather stronger than Theorem 5.

3. POSITIVE SCALAR CURVATURE AND FINITE COVERINGS

Let M be a smooth Riemannian manifold of positive scalar curvature. Clearly, if $\bar{M} \to M$ is an unramified covering of M, then the pulled back

metric on $\bar{M}$ also has positive scalar curvature. On the other hand, if M is a smooth manifold and a finite cover $\bar{M}$ of M admits a Riemannian metric of positive scalar curvature, then it is not usually true that M also admits such a metric. One might try to average the metric on $\bar{M}$ and consider the induced metric on M, but this approach has turned out to be too naive. Using an index-theoretic obstruction with values in the K-theory of a certain C^*-algebra associated to the fundamental group of the manifold under consideration, Rosenberg [18] exhibited a triple cover $\bar{M} \rightarrow M$ of a closed five-dimensional manifold for which $\bar{M}$ admits a metric of positive scalar curvature, but M does not. In this example M has infinite fundamental group. Rosenberg pointed out in the same paper that the situation is very different for manifolds with finite fundamental groups. Examples constructed by Bérard Bergery [1] show that there are high-dimensional closed smooth manifolds with fundamental group $\mathbb{Z}/2$ which do not admit metrics of positive scalar curvature, although their universal covers do. This led Rosenberg [19] to formulate Conjecture 1 for odd order finite fundamental groups.

We now disprove this conjecture in dimension 4. More generally, we show the following:

Theorem 9. *For any nontrivial finite group G which acts freely on S^3, there are closed smooth four-manifolds M with fundamental group G which do not admit metrics of positive scalar curvature, but whose universal covers do admit such metrics. The manifolds M can be taken to be either spin or non-spin.*

Every cyclic group acts freely on S^3 and among odd order groups there are no others by Hopf's theorem [25]. Thus, Theorem 9 disproves Rosenberg's conjecture exactly for all finite cyclic groups of odd order. Moreover, there are both spin and non-spin counterexamples. The non-spin case of Theorem 9 with $G = \mathbb{Z}/2$, which is not relevant to Rosenberg's conjecture, was proved by LeBrun [11]. His argument works only for non-spin manifolds, whereas the one of Bérard Bergery [1] in high dimensions works only for spin manifolds.

Proof of Theorem 9. Let G be a finite group of order $d > 1$ acting freely on S^3, and let L be the quotient S^3/G. On the product $L \times S^1$ one can perform surgery to kill the fundamental group of the second factor in such a way that the resulting 4-manifold N is spin. It is obviously a rational homology sphere.

The following is well-known, compare [7, 22].

Lemma 10. *The universal cover $\tilde{N}$ of N is diffeomorphic to the connected sum of $d - 1$ copies of $S^2 \times S^2$.*

We first exhibit the claimed non-spin examples. Let $X = E(2n + 1)$ be a non-spin simply connected elliptic surface without multiple fibers, for some $n \geq 1$. Then the positive part of its second Betti number $b_2^+(X) > 1$. Therefore [24], the Seiberg–Witten invariant of X is well-defined and non-zero for the Spin^c-structure induced by the Kähler structure of X. We set $M = N \# X$. This is a smooth closed 4-manifold with fundamental group G which is not spin. The following is an immediate consequence of the gluing result of [9].

Proposition 11 (Kotschick-Morgan-Taubes [9]). *For every* Spin^c*-structure on X there is one on M with the same Seiberg–Witten invariant. In particular, M does not admit a metric of positive scalar curvature.*

By Lemma 10 above, the universal covering of M is diffeomorphic to $(d - 1)(S^2 \times S^2) \# d\, X$. Now, the connected sum of any simply connected elliptic surface with $S^2 \times S^2$ dissolves by an application of Proposition 2, see [5, 13]. As X is not spin, we see that the universal cover of M is diffeomorphic to a connected sum of copies of $\mathbb{C}P^2$ and of $\overline{\mathbb{C}P^2}$.

By [8, 20] the class of manifolds of dimension at least three admitting metrics of positive scalar curvature is closed under forming connected sums. Thus connected sums of copies of $\mathbb{C}P^2$ and of $\overline{\mathbb{C}P^2}$ admit such metrics.

In order to obtain spin examples for M, we need to replace X in the above construction with a spin manifold with non-trivial Seiberg–Witten invariants. Then the connected sum $M = N \# X$ and its universal covering will also be spin. The signature of $\tilde{M}$ will be d times the signature of M, which is the same as the signature of X. Therefore, the Lichnerowicz vanishing theorem forces us to choose X so that it has zero signature. We take for X one of the symplectic spin manifolds with zero signature constructed in the proof of Theorem 3. Because this becomes diffeomorphic to a connected sum of copies of $S^2 \times S^2$ after only one stabilization, the universal covering of M is also a connected sum of copies of $S^2 \times S^2$, and thus admits metrics of positive scalar curvature.

This completes the proof of Theorem 9. $\square$

All our counterexamples, M, to Rosenberg's conjecture have the additional property that they are homeomorphic to manifolds M' which do satisfy the conjecture, in that both M' itself and its universal covering $\tilde{M}'$ admit a metric of positive scalar curvature. In fact, the universal covers $\tilde{M}$ and $\tilde{M}'$ are diffeomorphic, and have standard differentiable structures (connected sums of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$, or of copies of $S^2 \times S^2$). The group G acts freely on these standard manifolds in two essentially different ways: there is the standard action by isometries of a positive scalar curvature metric, with quotient M', and there are exotic actions, which do not fix any positive

scalar curvature metric, with exotic quotients M. That gauge theory detects exotic group actions, though not the application to positive scalar curvature metrics, was observed before, for example by Ue [22].

We can use these group actions to say something about the space of metrics of positive scalar curvature on certain manifolds.

Theorem 12. *Let G be any nontrivial finite group which acts freely on S^3. Then G acts freely by diffeomorphisms in infinitely many ways on infinitely many spin and infinitely many non-spin four-manifolds M of positive scalar curvature, such that in each case all the actions are conjugate by homeomorphisms but are not conjugate by diffeomorphisms of M. These actions give rise to infinitely many actions of G without fixed points on the space of positive scalar curvature metrics on M, which are not conjugate in $\mathrm{Diff}(M)$.*

In each case there is also one action which has a fixed point in the space of metrics of positive scalar curvature, giving rise to a standard quotient. The exotic actions have quotients without positive scalar curvature, and so have no fixed points on the space of metrics of positive scalar curvature. Note that if G is of prime order, the exotic actions on the space of positive scalar curvature metrics are free.

Proof. In the spin case, the infinitely many examples are connected sums of different numbers of copies of $S^2 \times S^2$. The group G has infinitely many non-conjugate actions on a fixed such manifold giving rise to quotients of the form $X_i \# N$, with X_i as in Theorem 3.

For the non-spin case the G-actions are on connected sums of copies of $\mathbb{C}P^2$ and of $\overline{\mathbb{C}P^2}$, with quotients of the form $X_j \# N$, with X_j infinitely many homeomorphic but pairwise non-diffeomorphic elliptic surfaces (obtained from each other by logarithmic transformation).

Proposition 11 shows that the distinct differentiable structures on X_i or X_j, which are detected by Seiberg–Witten invariants, remain distinct after connected sum with N.   □

Remark 13. Beyond the non-spin case with G of order two of the above Theorem 9, LeBrun's paper [11] contains a result (Theorem 2) on manifolds with infinite fundamental groups, where he considers finite coverings of high degree. The claimed coverings do not exist, as they would violate the multiplicativity of the Euler characteristic in finite coverings. However, the argument can be salvaged by considering a correctly chosen sequence of coverings of arbitrarily large degree. These coverings will always have even degrees.

REFERENCES

[1] L. Bérard Bergery, *Scalar curvature and isometry groups*, in *Spectra of Riemannian manifolds*, eds. M. Berger, S. Murakami, T. Ochiai, Kagai, Kagai (1983), 9–28.

[2] S. K. Donaldson, *Yang–Mills invariants of four-manifolds*, in "Geometry of Low-Dimensional Manifolds: 1", LMS Lecture Notes Series vol. 150, Cambridge UP 1990.

[3] R. Fintushel and R. J. Stern, 2-*torsion instanton invariants*, Jour. Amer. Math. Soc. **6** (1993), 299–339.

[4] M. H. Freedman, *The topology of four–dimensional manifolds*, J. Differential Geometry **17** (1982), 357–453.

[5] R. E. Gompf, *Sums of elliptic surfaces*, J. Differential Geometry **34** (1991), 93–114.

[6] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. Math. **142** (1995), 527–595.

[7] R. E. Gompf and A. I. Stipsicz, 4-*manifolds and Kirby calculus*, Graduate Studies in Mathematics, Vol. 20, Amer. Math. Soc. Providence (1999).

[8] M. Gromov and H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. Math. **111** (1980), 423–434.

[9] D. Kotschick, J. W. Morgan and C. H. Taubes, *Four-manifolds without symplectic structure, but with nontrivial Seiberg–Witten invariants*, Math. Res. Lett. **2** (1995), 119–124.

[10] S. Kwasik and R. Schultz, *Positive scalar curvature and periodic fundamental groups*, Comm. Math. Helv. **65** (1990), 271–286.

[11] C. LeBrun, *Curvature, covering spaces and Seiberg–Witten theory*, preprint arXiv:math.DG/0110329.

[12] R. Mandelbaum, *Decomposing analytic surfaces*, in "Geometric Topology", ed. J. C. Cantrell, Academic Press 1979.

[13] R. Mandelbaum, *Four-dimensional topology: an introduction*. Bull. Amer. Math. Society **2** (1980), 1–159.

[14] J. W. Morgan, T. S. Mrówka and Z. Szabó, *Product formulas along T^3 for Seiberg–Witten invariants*, Math. Res. Lett. **4** (1997), 915–929.

[15] J. W. Morgan, Z. Szabó and C. H. Taubes, *A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture*, J. Differential Geometry **44** (1996), 706–788.

[16] J. Park, *The geography of spin symplectic 4-manifolds*, Math. Z. **240** (2002), 405–421.

[17] U. Persson, C. Peters and G. Xiao, *Geography of spin surfaces*, Topology **35** (1996), 845–862.

[18] J. Rosenberg, C^*-*algebras, positive scalar curvature, and the Novikov conjecture - II*, in *Geometric methods in operator algebras*, Kyoto, 1983, eds. H. Araki, E. Effros, Pitman research notes in mathematics 123 (1986), 341–374.

[19] J. Rosenberg, C^*-*algebras, positive scalar curvature, and the Novikov conjecture - III*, Topology **25** (1986), 319–336.

[20] R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159–183.

[21] C. H. Taubes, *SW $\Rightarrow$ Gr, From the Seiberg–Witten equations to pseudo–holomorphic curves*, Jour. Amer. Math. Soc. **9** (1996), 845–918.

[22] M. Ue, *A remark on the exotic free actions in dimension* 4, J. Math. Soc. Japan **48** (1996), 333–350.

[23] C. T. C. Wall, *On simply-connected* 4-*manifolds*, J. London Math. Soc. **39** (1964), 141-149.

[24] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), 809–822.

[25] J. A. Wolf, *Spaces of constant curvature*, Publish or Perish 1984.

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