POINCARÉ DUALITY AND DEFORMATIONS OF ALGEBRAS

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Abstract. Let p be a prime number and X a finite dimensional connected $\mathbb{Z}/p$-CW complex. If $H^*(X;\mathbb{F}_p)$ is a Poincaré duality algebra, then by a basic theorem of Smith theory proven by G.E. Bredon, T. Chang and T. Skjelbred, the same is true for $H^*(F;\mathbb{F}_p)$, where $F \subset X^{\mathbb{Z}/p}$ is an arbitrary component of the fixed point set. We give an elementary proof of this fact using deformations of algebras.

The existing proofs of the cited theorem are based on the localization theorem for $\mathbb{Z}/p$-actions that is combined with an inductive argument within the Leray-Serre spectral sequence for the Borel construction $E\mathbb{Z}/p \times_{\mathbb{Z}/p} X$ ([3, 4]) or is interpreted as an evaluation theorem ([2]). One notorious technical difficulty in these proofs is to show that the obtained Poincaré pairing on each $H^*(F;\mathbb{F}_p)$ is induced by the cup product and an orientation of this cohomology algebra.

In this note we suggest a conceptually more concise treatment of $\mathbb{Z}/p$-actions on Poincaré duality spaces by exploiting the point of view in [2]: to regard $H^*(X^{\mathbb{Z}/p};\mathbb{F}_p)$ as an algebraic deformation of an algebra closely related to $H^*(X;\mathbb{F}_p)$. In this context the above mentioned difficulty disappears and makes the discussion of $\mathbb{Z}/p$-actions on Poincaré duality spaces shorter and perhaps more enlightening than in the existing sources.

Throughout the paper, k denotes a field and t and s denote indeterminates. All algebras are assumed to be (graded) commutative.

1. Deformations of algebras, evaluation theorem

If M is a $k[t]$-module or a $k[t]$-algebra and $\alpha \in k$, we set

$$M_\alpha = M/(t-\alpha)M = M \otimes_{k[t]} k_\alpha,$$

where k_α is the ring k equipped with the $k[t]$-multiplicaton induced by $t \mapsto \alpha$. This construction is functorial in M and called *evaluation* of t at α. Let us recall the following definition.

Definition 1. Let A be a k-algebra. The $k[t]$-algebra B is a *one parameter family of deformations* of A, if $B \cong A \otimes_k k[t]$ as $k[t]$-modules and if $B_0 \cong A$ as k-algebras. In this situation, the k-algebra B_1 is called a *deformation* of A *along* B.

For later reference we state the following basic fact.

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Lemma 2. *Let M be a finitely generated free $k[t]$-module of rank n and let $\alpha \in k$. Then M_α is a k-vector space of dimension n and the k-vector spaces $\mathrm{Hom}_{k[t]}(M,M)_\alpha$ and $\mathrm{Hom}_k(M_\alpha, M_\alpha)$ are canonically isomorphic.*

Let X be a finite dimensional $\mathbb{Z}/p$-CW complex, where p is an odd prime number and let g be a generator of $\mathbb{Z}/p$. Additionally, we assume that $H^*(X;\mathbb{F}_p)$ is a finitely generated vector space (this is automatically satisfied in our later applications). As coefficients we will use the field $\mathbb{F}_p$ throughout. Let $(C^*(X),\delta)$ denote the singular cochain complex of X. It has the structure of a differential graded $\mathbb{F}_p[\mathbb{Z}/p]$-module. The equivariant cohomology $H^*_{\mathbb{Z}/p}(X) = H^*(E\mathbb{Z}/p \times_{\mathbb{Z}/p} X)$, where $E\mathbb{Z}/p$ is a free contractible $\mathbb{Z}/p$-CW complex, can be calculated using the cochain complex $\beta^*(X) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/p]}(\mathcal{E}_*, C^*(X))$, $\mathcal{E}_*$ being a $\mathbb{Z}[\mathbb{Z}/p]$-free resolution of $\mathbb{Z}$ (with the trivial $\mathbb{Z}/p$-operation). Using the special model for $\mathcal{E}_*$ described in [2], p. 61, $\beta^*(X)$ is isomorphic to $C^*(X)[s,t]/(s^2)$ as a graded module, where s and t carry gradings 1 and 2, respectively. The $\mathbb{F}_p[t]$-linear differential and the $\mathbb{F}_p[t]$-bilinear product on $C^*(X)[s,t]/(s^2)$ induced by $\beta^*(X)$ are given by

$$\begin{aligned} c &\mapsto \delta c + (-1)^{|c|+1}(1-g)cs\,,\\ cs &\mapsto (\delta c)s + (-1)^{|c|}(1+\ldots+g^{p-1})ct\,, \end{aligned}$$

and

$$\begin{aligned} (c,d) &\mapsto c\cup d\,, & (c,ds) &\mapsto (c\cup d)s\,,\\ (cs,d) &\mapsto (-1)^{|d|}(c\cup gd)s\,, & (cs,ds) &\mapsto (-1)^{|d|}\sigma(c,d)t\,. \end{aligned}$$

Here, $\cup$ is the product on $C^*(X)$ induced by a diagonal and $\sigma(c,d) = \sum_{0\le i<j\le p-1}(g^i c)\cup(g^j d)$. We obtain an $\mathbb{F}_p[s,t]/(s^2)$-algebra structure on $H^*_{\mathbb{Z}/p}(X)$ that may be identified with the usual $H^*(B\mathbb{Z}/p;\mathbb{F}_p)$-algebra structure on $H^*_{\mathbb{Z}/p}(X)$. The following theorem is well known (cf. [2], Theorem 1.4.5).

Theorem 3 (Evaluation theorem)**.** *The inclusion $i : X^G \hookrightarrow X$ induces an isomorphism of $\mathbb{F}_p[s]/(s^2)$-algebras*

$$H^{(*)}_{\mathbb{Z}/p}(X)_1 \cong H^{(*)}_{\mathbb{Z}/p}(X^{\mathbb{Z}/p})_1 = H^{(*)}(X^{\mathbb{Z}/p})[s]/(s^2).$$

The stars in brackets indicate that degrees are only preserved mod 2. For the last equality, note that $\mathbb{Z}/p$ acts trivially on $X^{\mathbb{Z}/p}$. In the following, let Tor denote $\mathbb{F}_p[t]$-torsion. We set $F^* = H^*_{\mathbb{Z}/p}(X)/\,\mathrm{Tor}$.

Corollary 4. *$H^{(*)}(X^{\mathbb{Z}/p})[s]/(s^2)$ is a deformation of $F_0^{(*)}$ along $F^{(*)}$.*

Proof. Note that F^* is a free $\mathbb{F}_p[t]$-module [1], hence $F^* \cong F_0^* \otimes \mathbb{F}_p[t]$ as $\mathbb{F}_p[t]$-modules. Furthermore, the map i in Theorem 3 induces an isomorphism $F_1^{(*)} \cong H^{(*)}(X^{\mathbb{Z}/p})[s]/(s^2)$, because $\mathrm{Tor}\, H^*_{\mathbb{Z}/p}(X^{\mathbb{Z}/p}) = 0$. □

[1] Our assumption on $H^*(X)$ ensures that $H^*_{\mathbb{Z}/p}(X)$ is a finitely generated $\mathbb{F}_p[t]$-module. A refined argument shows that Corollary 4 holds without this assumption.

2. Poincaré duality

Definition 5. Let $n \in \mathbb{N}$. We call an $\mathbb{N}$-graded k-algebra A^* a *Poincaré duality algebra* of *formal dimension* n, if $A^0 \cong k$ (i.e. if A^* is *connected*), $A^n \cong k$ and if the map

$$A^* \xrightarrow{\mu_{A^*}} \mathrm{Hom}_k(A^*, A^*) \longrightarrow \mathrm{Hom}_k(A^*, A^n) \cong \mathrm{Hom}(A^*, k)$$

is an isomorphism (μ_{A^*} is adjoint to the multiplication on A^*, i.e. $\mu_{A^*}(a)(x) = a \cdot x$; the second map is induced by the projection $A^* \to A^n$).

In this situation, A^* is a finite dimensional k-vector space and $A^{>n} = 0$. Let $\mu_{A^{>0}} : A^{>0} \to \mathrm{Hom}_k(A^{>0}, A^{>0})$ be adjoint to the multiplication on $A^{>0}$. The following is an immediate consequence of [2], Proposition 5.1.4.

Lemma 6. *Let A^* be a connected $\mathbb{N}$-graded k-algebra with $\dim_k A^* < \infty$. Then the following statements are equivalent.*

i. *A^* is a Poincaré duality algebra.*
ii. $\dim_k \mathrm{Ker}\, \mu_{A^{>0}} \leq 1$.

The next proposition is the main algebraic ingredient for a deformation theoretic proof of the theorem of Bredon, Chang and Skjelbred in the following section. Roughly speaking, it says that Poincaré duality is preserved under deformations.

Proposition 7. *Let A^*, B^* and E^* be $\mathbb{N}$-graded k-algebras where A^* and B^* are connected. Let $\mathcal{T}$ be a $k[t]$-algebra such that $B \times E$ is a deformation of A along $\mathcal{T}$ (ignoring gradings). Suppose we have a $k[t]$-algebra map $\epsilon : \mathcal{T} \to k[t]$ with $\mathrm{Ker}\, \epsilon_0 = A^{>0}$ and $\mathrm{Ker}\, \epsilon_1 = B^{>0} \times E^*$. Then, if A^* is a Poincaré duality algebra, B^* is a Poincaré duality algebra.*

Proof. By Lemma 6 is suffices to show that $\dim_k \mathrm{Ker}\, \mu_{B^{>0}} \leq 1$, if the corresponding statement holds for A^*. We have an exact sequence

$$0 \longrightarrow \mathrm{Ker}\, \epsilon \longrightarrow \mathcal{T} \xrightarrow{\epsilon} \mathrm{Im}\, \epsilon \to 0$$

that consists of free $k[t]$-modules and therefore remains exact after applying the functors $(-)_0$ and $(-)_1$. Hence $(\mathrm{Ker}\, \epsilon)_0 = A^{>0}$ and $(\mathrm{Ker}\, \epsilon)_1 = B^{>0} \times E^*$ (including the multiplicative structures). Using Lemma 2 and 6, we get

$$\dim_k \mathrm{Ker}\, \mu_{B^{>0} \times E^*} = \dim_k \mathrm{Ker}\, (\mu_{\mathrm{Ker}\, \epsilon})_1 = \dim_k \mathrm{Ker}\, (\mu_{\mathrm{Ker}\, \epsilon})_0 = \dim_k \mathrm{Ker}\, \mu_{A^{>0}}$$

which is less than or equal to 1 by assumption. It is easy to see that the inequality $\dim_k \mathrm{Ker}\, \mu_{B^{>0}} \leq \dim_k \mathrm{Ker}\, \mu_{B^{>0} \times E^*}$ holds, thereby proving our assertion. □

3. Actions on Poincaré duality spaces

Now we can use our techniques to reprove the following important theorem. We will use $\mathbb{F}_p$ as coefficients as before, where p is an odd prime number.

Theorem 8 ([3], [4]). *Let X be a finite dimensional connected $\mathbb{Z}/p$-CW complex such that $H^*(X)$ is a Poincaré duality algebra of formal dimension n. Let $F \subset X^G$ be a component of the fixed point set. Then $H^*(F)$ is a Poincaré duality algebra of formal dimension less than or equal to n.*

Proof. Note that $H^*(S^1 \times_{\mathbb{Z}/p} X) \cong H^*(\beta^*(X)_0)$, where $\mathbb{Z}/p \subset \mathbb{C}$ acts on $S^1 \subset \mathbb{C}$, is a Poincaré duality algebra of formal dimension $n+1$. This follows easily from the above explicit description of the differential on $\beta^*(X)$ which leads to the observation that $H^*(\beta^*(X)_0)$ is the cohomology of the cochain complex $H^*(X)[s]/(s^2)$ with differential $h \mapsto (-1)^{|h|+1}(1-g)hs$, $hs \mapsto 0$, and a bilinear product induced by the explicit formulas from above (a simple spectral sequence argument can be used just as well). We set

$$\begin{aligned} U &= H^*(j)(H^*_{\mathbb{Z}/p}(X)) \subset H^*(S^1 \times_{\mathbb{Z}/p} X)\,, \\ V &= H^*(j)(\operatorname{Tor} H^*_{\mathbb{Z}/p}(X)) \subset U\,, \end{aligned}$$

where $j : \beta^*(X) \to \beta^*(X)_0$ is the projection. Note that $U \cong H^*_{\mathbb{Z}/p}(X)_0$ and $V \cong (\operatorname{Tor} H^*_{\mathbb{Z}/p}(X))_0$. Using Corollary 4,

$$(H^*(F) \times H^*(X^{\mathbb{Z}/p} \setminus F)) \otimes \mathbb{F}_p[s]/(s^2)$$

is a deformation (not respecting the gradings) of U/V along the one parameter family of deformations $H^{(*)}_{\mathbb{Z}/p}(X)/\operatorname{Tor}$. We want to apply Proposition 7 with $A^* = U/V$, $B^* = H^*(F)[s]/(s^2)$, $E^* = H^*(X^{\mathbb{Z}/p} \setminus F)[s]/(s^2)$, $\mathcal{T} = H^*_{\mathbb{Z}/p}(X)/\operatorname{Tor}$ and $\epsilon : \mathcal{T} \to \mathbb{F}_p[t]$ induced by the inclusion of a point $* \hookrightarrow F \hookrightarrow X$ and the map $H^*_{\mathbb{Z}/p}(*) = \mathbb{F}_p[s,t]/(s^2) \to \mathbb{F}_p[t]$, $s \mapsto 0$. Then the Poincaré duality of $H^*(F)[s]/(s^2)$ implies the Poincaré duality of $H^*(F)$. For Proposition 7 being applicable it remains to show that $V = U^\perp$ with respect to the nonsingular bilinear form on $H^*(S^1 \times_{\mathbb{Z}/p} X)$ induced by the cup product and an isomorphism $H^{n+1}(S^1 \times_{\mathbb{Z}/p} X) \cong \mathbb{F}_p$. Note that the singular cochain complex $C^*(X)$ is $\mathbb{F}_p$-cochain homotopy equivalent to $Z^n(X) \oplus C^{<n}(X)$, where $Z^n(X)$ are the cocycles in $C^n(X)$. We consider the following commutative diagram of cochain complexes (with trivial differential in the third column)

$$\begin{array}{ccccc} \beta^*(X) & \xleftarrow{\iota} & (Z^n(X) \oplus C^{<n}(X))[s,t]/(s^2) & \xrightarrow{\pi} & \mathbb{F}_p[s,t]/(s^2) \\ {\scriptstyle t\mapsto 0}\downarrow & & {\scriptstyle t\mapsto 0}\downarrow & & {\scriptstyle t\mapsto 0}\downarrow \\ \beta^*(X)_0 & \xleftarrow{\iota} & (Z^n(X) \oplus C^{<n}(X))[s]/(s^2) & \xrightarrow{\phi} & \mathbb{F}_p[s]/(s^2), \end{array}$$

where π and ϕ are induced by a (fixed) isomorphism $H^n(X) \to \mathbb{F}_p$. The differentials in the second column are induced by the differential on $\beta^*(X)$. Applying the cohomology functor to this diagram, the inclusion maps ι become isomorphisms (this is true already on the E_2-terms of the corresponing (Leray-Serre) spectral sequences). Hence $V \subset H^{<n+1}(X \times_{\mathbb{Z}/p} S^1)$, because $\mathbb{F}_p[s,t]/(s^2)$ is $\mathbb{F}_p[t]$-torsion free. This implies $V \subset U^\perp$, as V is an ideal in U. We now consider the universal coefficient sequence for the principal ideal domain $\mathbb{F}_p[t]$

$$0 \to H^*_{\mathbb{Z}/p}(X)_0 \to H^*(\beta^*(X)_0) \to \operatorname{Tor}_{\mathbb{F}_p[t]}(H^*_{\mathbb{Z}/p}(X), (\mathbb{F}_p)_0) \to 0\,.$$

Using the $\mathbb{F}_p[t]$-free resolution $0 \to \mathbb{F}_p[t] \hookrightarrow \mathbb{F}_p[t] \to (\mathbb{F}_p)_0 \to 0$ for calculating the Tor-term on the right and using the structure theorem for the finitely generated $\mathbb{F}_p[t]$-module $H^*_{\mathbb{Z}/p}(X)$ we obtain

$$\dim_{\mathbb{F}_p} U + \dim_{\mathbb{F}_p} V = \dim_{\mathbb{F}_p} H^*(S^1 \times_{\mathbb{Z}/p} X),$$

hence $V = U^{\perp}$. The last assertion in the theorem follows, because the deformation parameter t from U/V to $H^{(*)}(X^{\mathbb{Z}/p})[s]/(s^2)$ is positively graded. □

4. FURTHER REMARKS

If $p = 2$, the above argument simplifies considerably ([7]), because $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$ does not contain an exterior algebra. In Theorem 8, we did not state the well known fact that (for p odd) the formal dimension of the $\mathbb{F}_p$-cohomology of each component of $X^{\mathbb{Z}/p}$ has the same parity as n. This can be shown using deformation theoretic methods, too, but needs a little more machinery than presented in this note. Because each finite p-group contains a normal subgroup of order p, one can generalize Theorem 8 to actions of finite p-groups on $\mathbb{F}_p$-Poincaré duality spaces. For such actions one can ask about further relations between the algebra structures of $H^*(X;\mathbb{F}_p)$ and $H^*(F;\mathbb{F}_p)$ for the different components $F \subset X^{\mathbb{Z}/p}$. If the formal dimension of $H^*(X;\mathbb{F}_p)$ is even, J.P. Alexander and G.C. Hamrick in [1] prove a relation between the Witt classes of $H^{ev\cdot}(X;\mathbb{F}_p)$ and $\sum_{F\subset X^{\mathbb{Z}/p}} H^{ev\cdot}(F;\mathbb{F}_p)$, if X is an integral Poincaré duality space equipped with a $\mathbb{Z}/p$-action. In [5] we prove a relation of the Witt classes of total space and fixed point set for $\mathbb{Z}/p$-actions on $\mathbb{F}_p$-Poincaré duality spaces on which the Bockstein operator associated to the exact sequence of coefficients $0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$ acts trivially. These theorems do not generalize to actions of finite p-groups in an obvious way, because neither the property of being an integral Poincaré duality space nor having a trivial Bockstein operator need to be preserved by passing from a $\mathbb{Z}/p$-space to its fixed point set components. However, in [6] we get a relation between the Witt classes of total space and fixed point set in the case of finite p-group actions on $\mathbb{Z}_{(p)}$-homology manifolds.

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