POINCARÉ DUALITY IN P.A. SMITH THEORY

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Abstract. Let $G = S^1$, $G = \mathbb{Z}/p$ or more generally G be a finite p-group, where p is an odd prime. If G acts on a space whose cohomology ring fulfills Poincaré duality (with appropriate coefficients k), we prove a mod 4 congruence between the total Betti number of X^G and a number which depends only on the $k[G]$-module structure of $H^*(X;k)$. This improves the well known mod 2 congruences that hold for actions on general spaces.

1. Introduction and statement of results

Let X be a finite dimensional connected CW complex and let k be a commutative ring with unit. We say that X is a *Poincaré duality space* over k (we will often write *k-PD space* instead) of *formal dimension n*, if $H^*(X;k)$ is a finitely generated k-module and if there is a class $\nu \in H_n(X;k)$ such that

$$H^*(X;k) \rightarrow H_{n-*}(X;k), \quad c \rightarrow c \cap \nu$$

is an isomorphism. Note that if k is a field, this is equivalent to requiring that

$$H^*(X;k) \times H^*(X;k) \xrightarrow{\cup} H^*(X;k) \xrightarrow{\nu} k$$

is a nonsingular pairing (viewing ν as an element in $\mathrm{Hom}(H^*(X;k),k)$).

Let G denote the group S^1 (with its usual topology) or $\mathbb{Z}/p$, where p is an odd prime number. Let $k = \mathbb{Q}$, if $G = S^1$, and $k = \mathbb{F}_p$, if $k = \mathbb{Z}/p$. By a well known result proven independently by Chang-Skjelbred in [2] and Bredon in [3], each component of the fixed point set of a finite dimensional G-CW complex X fulfills Poincaré duality over k, if this is property holds for X. By now there are several further versions and variants of proofs of this result ([1, 5, 8]). In this paper, we will use certain consequences of Poincaré duality to deduce relations between the total Betti number of X^G and the $k[G]$-module $H^*(X;k)$.

Theorem 1. *Let $G = S^1$ and let X be a finite dimensional connected G-CW complex such that X is a $\mathbb{Q}$-PD space of formal dimension n. If*

- *n is even or*

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- $n = 2m+1$, $X^G \neq \emptyset$, $H^i(X;\mathbb{Q}) = 0$ *for* $0 < i \leq m$, i *even*,

the following congruence holds.

$$\dim_{\mathbb{Q}} H^*(X^G;\mathbb{Q}) \equiv \dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \mod 4 .$$

If $G = S^1 \times \ldots \times S^1$ and if X is a finite dimensional connected G-CW complex with finitely many orbit types that fulfills Poincaré duality over $\mathbb{Q}$, then an analogue of Theorem 1 holds. This is true, because in this case we can choose a subcircle $S^1 \subset G$, such that $X^{S^1} = X^G$.

A version of Theorem 1 for actions of $G = \mathbb{Z}/p$, where p is an odd prime, can be proven under additional assumptions on the space X. It turns out that one also has to take into account the fact that now the induced G-action on $H^*(X;\mathbb{F}_p)$ might be nontrivial. Using a spectral sequence argument together with results from [6], we show:

Theorem 2. *Let* $G = \mathbb{Z}/p$ *(where* p *is an odd prime) and let* X *be a finite dimensional connected* G*-CW complex such that* X *is an* $\mathbb{F}_p$*-PD space of formal dimension* n*. Furthermore, assume that* $H^*(X;\mathbb{Z}_{(p)})$ *does not contain* $\mathbb{Z}/p$ *as a direct summand. Then, we get a decomposition as graded* $\mathbb{F}_p[G]$*-modules*

$$H^*(X;\mathbb{F}_p) = F^* \oplus T^* \oplus R^* ,$$

where F^* *is a free* $\mathbb{F}_p[G]$*-module,* T^* *is a trivial* $\mathbb{F}_p[G]$*-module and* R^* *is a direct sum of* $\mathbb{F}_p[G]$*-modules of the form* $\ker \epsilon$*, where* $\epsilon : \mathbb{F}_p[G] \to \mathbb{F}_p$ *is the augmentation map. If*

- n *is even or*
- $n = 2m+1$, $X^G \neq \emptyset$, $T^i = 0$ *for* $0 < i \leq m$, i *even,* $R^i = 0$ *for* $0 < i \leq m$, i *odd,*

the following congruence holds.

$$\dim_{\mathbb{F}_p} H^*(X^G;\mathbb{F}_p) \equiv \dim_{\mathbb{F}_p} T^* + \frac{1}{p-1} \dim_{\mathbb{F}_p} R^* \mod 4 .$$

Theorem 1 was first proved by A. Sikora in his PhD thesis. This and Theorem 2 for the case that $R^* = 0$ and $H^*(X;\mathbb{Z})$ does not contain any p-torsion are also contained in his paper [9], where a somewhat different line of argument is used.

It is well known that the cited result by Bredon, Chang and Skjelbred immediately generalizes to actions of finite p-groups G. This is achieved by applying induction on the order of G and using the fact that every finite (nontrivial) p-group contains a normal subgroup of order p. This procedure can be applied in our situation as well and shows the following result.

Theorem 3. *Let* G *be a finite* p*-group (where* p *is an odd prime) and let* X *be a finite dimensional connected* G*-CW complex that is an* $\mathbb{F}_p$*-PD space of even formal dimension. If* $p > \dim H^*(X;\mathbb{F}_p)$*, then*

$$\dim_{\mathbb{F}_p} H^*(X^G;\mathbb{F}_p) \equiv \dim_{\mathbb{F}_p} H^*(X;\mathbb{F}_p) \mod 4 .$$

If we impose more restrictions on X and on the group operation, then using a result from [7], we can use a similar induction argument for comparing rational Betti numbers:

Theorem 4. *Let G be a finite p-group (where p is an odd prime) and let X be a finite connected simplicial complex with G acting simplicially such that X is an even dimensional orientable $\mathbb{Z}_{(p)}$-homology manifold. Furthermore, assume that $p > \dim H^*(X;\mathbb{F}_p)$. Then*

$$\dim_{\mathbb{Q}} H^*(X^G;\mathbb{Q}) \equiv \dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \mod 4\,.$$

In the case of smooth actions, this result can be proven without making use of [7].

In Section 5, we will show by examples that none of the additional assumptions in the second part of Theorem 1 and Theorem 2 can be dropped.

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2. Algebraic preliminaries

For our later applications to spectral sequences, it will be useful to provide a notion of Poincaré duality for bigraded algebras.

Definition 1. Let k be a field and let $A^{*,*}$ be a $(\mathbb{Z}/2\times\mathbb{N})$-bigraded associative and graded commutative k-algebra with unit that is finitely generated as k-vector space. (The graded commutativity is required with respect to the total $\mathbb{Z}/2$-grading, where $A^{i,j}$ has total degree $i+j \mod 2$.)

i. We call $A^{*,*}$ *connected*, if $A^{0,0}\cong k$.
ii. $A^{*,*}$ is a *k-Poincaré duality algebra* of *formal dimension* n, if there is a surjective linear map $\phi : A^{*,*}\to k$ (called *orientation*) with $\phi(A^{i,j}) = 0$ for $(i,j)\neq(0,n)$ such that the bilinear form

$$\zeta : A^{*,*}\times A^{*,*} \overset{\text{mult.}}{\longrightarrow} A^{*,*} \overset{\phi}{\longrightarrow} k$$

is nondegenerate.

Note that in a connected Poincaré duality algebra of formal dimension n, we have $A^{0,n}\cong k$. The following elementary fact provides a connection between the Euler characteristic and the total Betti number of Poincaré duality algebras of even formal dimension.

Lemma 1. *Let $A^{*,*}$ be a Poincaré duality algebra of even formal dimension over a field k. If* char $k\neq 2$, *then*

$$\dim_k A^{*,*} \equiv \chi(A^{*,*}) \mod 4\,,$$

where the Euler characteristic is calculated using the total $\mathbb{Z}/2$-grading of A.

Proof. Let $2m$ be the formal dimension of $A^{*,*}$. Using the induced total $\mathbb{Z}/2$ grading of $A^{*,*}$, we claim that the dimension of A^{odd} is even, which obviously proves the lemma. For all i, we get isomorphisms

$$\begin{aligned} A^{0,2i+1} &\cong \mathrm{Hom}(A^{0,2m-2i-1}, k) \\ A^{1,2i} &\cong \mathrm{Hom}(A^{1,2m-2i}, k) \end{aligned}$$

by Poincaré duality. Hence, $A^{0,2i+1}$ and $A^{0,2m-2i-1}$, respectively $A^{1,2i}$ and $A^{1,2m-2i}$ have the same dimension. Furthermore, if m is even, the module $A^{1,m}$, carries a skew nonsingular form by Poincaré duality and therefore has even dimension, as $\mathrm{char}\, k \neq 2$. The same is true for $A^{0,m}$, if m is odd. □

Proposition 1. *Let $(A^{*,*}, \delta)$ be a connected $(\mathbb{Z}/2 \times \mathbb{N})$-graded differential algebra over the field k with a differential δ of arbitrary bidegree, but lowering the second grading parameter and acting as a derivation. Furthermore, assume that $A^{*,*}$ is a Poincaré duality algebra of formal dimension n. Then, after taking homology, we get either $H(A^{*,*}, \delta) = 0$ or $H(A^{*,*}, \delta)$ is again a connected Poincaré duality algebra of formal dimension n.*

Proof. Assume that $H(A^{*,*}) \neq 0$. This implies that the unit of A (that generates $A^{0,0}$) is not in the image of δ. Let $c \in A^{0,n} \cong k$ be a generator. Assume that $\delta(c) = x \neq 0$. By Poincaré duality, we then find an element $y \in A^{*,*}$ with $c = yx \in A^{0,n}$. We therefore get

$$0 \neq x = \delta(c) = \delta(yx) = \delta(y)x \,,$$

which implies that $\delta(y)$ is a generator of $A^{0,0}$, contrary to what we said before. Hence, because c is not hit by δ, any orientation ϕ of A induces a surjective linear map $H^{*,*}(A) \rightarrow k$. To complete the proof, observe that by the derivation property of δ, we have

$$\zeta(\ker \delta, \operatorname{im} \delta) = 0 \,,$$

using the bilinear form ζ from Definition 1. As

$$\dim_k \ker \delta + \dim_k \operatorname{im} \delta = \dim_k A^{*,*} \,,$$

we may conclude that $\ker \delta$ is exactly the orthogonal complement of $\operatorname{im} \delta$ with respect to ζ. This proves that the induced bilinear form on $H^{*,*}(A)$ is nonsingular. □

Proposition 2. *Let $(A^{*,*}, \delta)$ be as in the last Proposition. Additionally, we assume that* $\mathrm{char}\, k \neq 2$, *the formal dimension of $A^{*,*}$ is an odd number $2m+1$, the differential δ has odd (total) degree, lowering the second grading paramter in $A^{*,*}$ and $A^{0,i} = 0$ for $0 < i \leq m$, i even, $A^{1,i} = 0$ for $0 < i \leq m$, i odd. If $H(A^{*,*}) \neq 0$. Then*

$$\dim_k A^{*,*} \equiv \dim_k H(A^{*,*}, \delta) \mod 4 \,.$$

Proof. Let $Z^* = \ker \delta \subset A^*$ denote the cycles in A^* with respect to δ (here we use the total $\mathbb{Z}/2$ grading, again). By our assumption on A^*, the even dimensional part $H^{ev}(A^*, \delta)$ of the homology of A^* with respect to δ, coincides with Z^{ev}, cf. the proof of Proposition 1. Now we define a bilinear form

$$\gamma : A^{ev} \times A^{ev} \to k, \quad (x, y) \to \phi(x \cdot \delta(y)) ,$$

using an orientation ϕ of A. It is easy to check, that γ induces a well defined, nonsingular and skew symmetric form on A^{ev}/Z^{ev}, hence this vector space has even dimension over k (using the fact that char $k \neq 2$). As the Euler characteristic of A^* and of $H(A^*)$ are equal, we get the equation

$$A^{ev} - H^{ev}(A^*) = A^{odd} - H^{odd}(A^*) .$$

Therefore, the number $\dim A^* - \dim H^*(A^*)$ is divisible by 4. □

3. Proof of Theorems 1, 3 and 4

Here, when applying the results from the last section, we usually forget about the first grading parameter of $A^{*,*}$ and use the $\mathbb{N}$-grading by the second parameter. Part of the argument is based on the following well known fact (cf. [1], Exercise (3.29)):

Lemma 2. *Let $G = S^1$ or $G = \mathbb{Z}/p$ and let X be a finite dimensional G-CW complex such that $H^*(X;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. Then*

i. $\chi(X^G) = \chi(X)$, *if* $G = S^1$.
ii. $\chi(X^G) = \Lambda(g)$, *if* $G = \mathbb{Z}/p$ *and* $g \in G$, $g \neq 1$.

Here, $\Lambda(g)$ denotes the Lefschetz number

$$\Lambda(g) = \sum_i (-1)^i \mathrm{trace}(g_* \colon H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})) ,$$

regarding g as a map $X \to X$.

If we set $A^{0,1} = 0$ and $A^{*,0} = H^*(X;\mathbb{Q})$, respectively $A^{*,0} = H^*(X^G;\mathbb{Q})$, then Lemma 1 and Lemma 2 give the following sequence of equations:

$$\dim H^*(X^G;\mathbb{Q}) \equiv \chi(X^G) = \chi(X) \equiv \dim H^*(X;\mathbb{Q}) \bmod 4 .$$

This shows the first part of Theorem 1.

The proof of Theorem 3 proceeds by induction on the order of G. Assume $|G| \neq 1$ and choose a normal subgroup $H \subset G$, $H \cong \mathbb{Z}/p$ that exists by group theory. Each component of X^H is an $\mathbb{F}_p$-PD space of even formal dimension by the Theorem of Bredon-Chang-Skjelbred. As $\dim H^*(X;\mathbb{Q}) < p$ by our assumption on p, the induced action of H on $H^*(X;\mathbb{Q})$ is trivial: Let V be a rational vector space of dimension smaller than $p-1$ and let f be a linear endomorphism of V with $f^p = \mathrm{id}$. Then the minimal polynomial of f in $\mathbb{Q}[x]$ must divide $x^p - 1$. The last polynomial splits over $\mathbb{Q}[x]$ into $(x-1)$ and an irreducible factor of degree $p-1$. Because the minimal polynomial of f has degree at most $\dim V$, it must therefore be equal to $x-1$. This covers all cases, where $H^*(X;\mathbb{Q})$ is not concentrated in degree 0. In the remaining case, the assertion follows from the fact that $H^0(X;\mathbb{Q})$ is a permutation module.

Altogether, for $1 \neq h \in H$, we obtain

$$\Lambda(h) = \chi(X) .$$

By Lemma 1 and Lemma 2 above, we get

$$\dim H^*(X^H; \mathbb{F}_p) \equiv \chi(X^H) = \chi(X) \equiv \dim H^*(X; \mathbb{F}_p) \bmod 4 .$$

(Note that the Euler characteristic does not depend on the coefficient field used). For the induction step, observe that by Smith theory

$$\dim H^*(F; \mathbb{F}_p) \leq \dim H^*(X; \mathbb{F}_p) < p$$

for each component F of X^G. Furthermore, the group $G' = G/H$ has order less than the order of G and each component of X^H is invariant under the induced G'-action by our assumption on p. Using the Theorem of Bredon-Chang-Skjelbred, each component of X^H is again an $\mathbb{F}_p$-PD space of even formal dimension. Hence the induction hypothesis applies to each component of X^H. The proof of Theorem 3 is now complete.

For the proof of Theorem 4, we recall the following fact.

Proposition 3. *([7], Proposition 13) Let $G = \mathbb{Z}/p$ act simplicially on a finite simplicial complex X that is an orientable $\mathbb{Z}_{(p)}$-homology manifold. If $F \subset X$ is a component of the fixed point set X^G, then F is an orientable $\mathbb{Z}_{(p)}$-homology manifold of even codimension in X.*

Using this fact, Theorem 4 follows from Theorem 3 by observing that the total Betti numbers with either $\mathbb{F}_p$ or $\mathbb{Q}$ coefficients of a space are congruent modulo 4, if this space fulfills Poincaré duality both over $\mathbb{Q}$ and over $\mathbb{F}_p$ and has even formal dimension for both fields of coefficients. This follows by the independence of the Euler characteristic of the coefficient field and by using Lemma 1 twice.

For the proof of the second part of Theorem 1 we consider the cohomological Leray-Serre spectral sequence (E_r, δ_r) for the Borel fibration

$$X \rightarrow X_G = EG \times_G X \rightarrow BG$$

with coefficients in $\mathbb{Q}$. For the E_2-term of this spectral sequence as a module over $H^*(BG; \mathbb{Q}) \cong \mathbb{Q}[t]$, where $t \in H^2(BG; \mathbb{Q})$ is a generator, we get

$$E_2^{*,\mu} \cong H^\mu(X; \mathbb{Q}) \otimes \mathbb{Q}[t] .$$

Because all the differentials in the spectral sequence are $\mathbb{Q}[t]$-linear, we can evaluate the terms E_r, $r \geq 2$ at $t = 1$ and because evaluation at $t = 1$ commutes with taking homology with respect to each δ_r (see [1], Lemma (A.7.2)), we get a converging spectral sequence $(\overline{E}_r, \overline{\delta}_r)$, where

$$\overline{E}_r = (E_r)_{t=1}$$

and $\overline{\delta}_r$ is induced by δ_r. This spectral sequence is concentrated in the first column and $\overline{E}_\infty^{*,*}$ is (noncanonically and not preserving the grading) isomorphic to $H^*(X_G; \mathbb{Q})_{t=1}$ as a $\mathbb{Q}$-vector space. By the evaluation theorem (cf. [1], Theorem (3.5.1)), we have a canonical isomorphism (of ungraded modules)

$$H(X_G;\mathbb{Q})_{t=1} \cong H(X^G;\mathbb{Q})\,.$$

Because X fulfills Poincaré duality over $\mathbb{Q}$, the term $\overline{E}_2^{*,*} = \overline{E}_2^{0,*}$ is an $\mathbb{N}$-graded $\mathbb{Q}$-Poincaré duality algebra of formal dimension $2m+1$, where we use the grading induced by E_2. As $X^G \neq \emptyset$, we have $\overline{E}_r \neq 0$ for all r. Now the claim follows from Propositions 1 and 2 in the second section of this paper.

4. Proof of Theorem 2

Let p be an odd prime number, let $G = \mathbb{Z}/p$ and let X be a finite dimensional connected G-CW complex that is a $\mathbb{F}_p$-PD space of formal dimension n. Furthermore, we assume that $\beta(H^*(X;\mathbb{F}_p)) = 0$, where

$$\beta\colon H^*(X;\mathbb{F}_p)\to H^{*+1}(X;\mathbb{F}_p)$$

is the Bockstein operator associated to the exact sequence of coefficients

$$0\to\mathbb{Z}/p\to\mathbb{Z}/p^2\to\mathbb{Z}/p\to 0\,.$$

This condition is equivalent to the requirement, that each cohomology class in $H^*(X;\mathbb{F}_p)$ can be lifted to a cohomology class in $H^*(X;\mathbb{Z}/p^2)$, which is the case, if and only if $H^*(X;\mathbb{Z}_{(p)})$ does not contain a direct summand of the form $\mathbb{Z}/p$. Note that the G-action on X induces a $\mathbb{Z}/p^2[G]$-module structrue on $H^*(X;\mathbb{Z}/p^2)$. By our assumption on β, $H^*(X;\mathbb{Z}/p^2)$ is a free $\mathbb{Z}/p^2$-module, so the following proposition is an immediate consequence of [6], Proposition 6, which is shown using a little representation theory over $\mathbb{Z}/p^2$.

Proposition 4. *For each i there is a $\mathbb{F}_p[G]$-linear decomposition*

$$H^i(X;\mathbb{F}_p) = H^i(X;\mathbb{Z}/p^2)\otimes\mathbb{F}_p \cong T^i \oplus F^i \oplus R^i\,,$$

where T^i is a trivial $\mathbb{F}_p[G]$-module, F^i is a direct sum of free $\mathbb{F}_p[G]$-modules and R^i is a direct sum of modules of the form $\ker\epsilon$, where $\epsilon : \mathbb{F}_p[G]\to\mathbb{F}_p$ is the augmentation map.

Let (E_r, δ_r) denote the cohomological Leray-Serre spectral sequence for the Borel fibration

$$X\to X_G = EG\times_G X\to BG$$

with coefficients in $\mathbb{F}_p$. For $r \geq 2$, each E_r is a differential bigraded algebra over $H^*(BG;\mathbb{F}_p)\cong\mathbb{F}_p[t]\otimes\Lambda(s)$, where $t\in H^2(BG;\mathbb{F}_p)$ and $s\in H^1(BG;\mathbb{F}_p)$ are generators and $\beta(s) = t$. By the evaluation theorem (cf. [1], Theorem (1.4.5)), the inclusion $X^G \hookrightarrow X$ induces an isomorphism of (ungraded) $\Lambda(s)$-algebras

$$H(X_G;\mathbb{F}_p)_{t=1} \cong H(X^G;\mathbb{F}_p)\otimes\Lambda(s)\,,$$

hence, after evaluating at $s = 0$, we get an induced isomorphism

$$H(X_G;\mathbb{F}_p)_{t=1,s=0} \cong H(X^G;\mathbb{F}_p)\,.$$

Now we encounter the following difficulty that did not arise in the consideration of S^1-actions above: In order to calculate the dimension of the left hand side of the second isomorphism one has to get around the difficulty that evaluation at $s = 0$ does not commute with taking homology in general, and so a spectral sequence argument as in Section 3 does not seem to be feasible in this case. However, under the additional assumption $\beta = 0$, we can get around this difficulty by using the following fact (cf. [6], Proposition 9, and the last part of the proof of Proposition 10).

Proposition 5. *For all $r \geq 2$, the localized terms $E_r[t^{-1}]$ are finitely generated free $(\mathbb{Z}\times\mathbb{N})$-bigraded differential $\Lambda(s)\otimes\mathbb{F}_p[t,t^{-1}]$-algebras. Furthermore, evaluation at $t = 1$ and $s = 0$ on $E_r[t^{-1}]$ commutes with taking homology with respect to the differentials induced by δ_r.*

Now, we set
$$\overline{E}_r = (E_r)_{t=1,s=0}$$
and use the induced differentials $\overline{\delta}_r$. Each term $\overline{E}_r$ has an induced $(\mathbb{Z}/2\times\mathbb{N})$-grading and for $r = 2$ we obtain
$$\overline{E}_2^{\gamma,\mu} \cong H^\gamma(G; H^\mu(X;\mathbb{F}_p))_{t=1,s=0} \cong \begin{cases} \mathbb{F}_p^{\dim T^\mu}, & \text{if } \gamma \text{ is even,} \\ \mathbb{F}_p^{\frac{\dim R^\mu}{p-1}}, & \text{if } \gamma \text{ is odd}. \end{cases}$$
Here, we used Proposition 4 and the fact (which follows from usual dimension shifting) that $H^i(\mathbb{F}_p;\ker\epsilon)[t^{-1}]$ is isomorphic to $\mathbb{F}_p$ in odd degrees and is equal to zero in even degrees. Further, we get from Proposition 5 and the evaluation theorem
$$\dim_{\mathbb{F}_p} \overline{E}_\infty = \frac{1}{2}\dim_{\mathbb{F}_p}(E_\infty)_{t=1} = \dim_{\mathbb{F}_p} H^*(X^G;\mathbb{F}_p)\,.$$
Now, the proof of Theorem 3 is completed by using induction in the spectral sequence. Notice that $\overline{E}_2^{*,*}$ is a $(\mathbb{Z}/2\times\mathbb{N})$-bigraded connected Poincaré duality algebra over $\mathbb{F}_p$ of formal dimension n in the sense of Definition 1 (cf. [6], proof of Proposition 9). For the induction step, if n is even, we use Lemma 1 and the fact that the Euler characteristic of a $\mathbb{Z}/2$-graded differential complex does not change after taking homology with respect to a differential of odd degree. If n is odd, we use Proposition 1 and Proposition 2.

5. Examples, applications and concluding remarks

The following examples illustrate the significance of the conditions on the Betti numbers in the second part of Theorem 1 and 2. The example of free S^1 or $\mathbb{Z}/p$-actions on spheres of odd dimension shows that the requirement $X^G \neq \emptyset$ in the second part of Theorem 1 and in the second part of Theorem 2 is necessary. In [4], p. 425, an example of an S^1-action on $X = S^3\times S^5\times S^9$ is constructed whose fixed point set is an S^7-bundle over $S^3\times S^5$ and has total Betti number 6 (with coefficients in $\mathbb{Q}$). As the total Betti number of X is 8, one sees that even if fixed points exist, the assumption on the vanishing of certain Betti numbers in the second part of Theorem 1 is needed. Restricting this S^1-action to $\mathbb{Z}/p \subset S^1$, we similarly may conclude that all additional assumptions in the second part of Theorem 2 are necessary.

Now, we will construct examples of Poincaré duality spaces of odd formal dimension that fulfill the additional requirement on Betti numbers in the second part of Theorem 1 and 2. Let X be an arbitrary connected finite simplicial complex with the property that its even dimensional integral cohomology is concentrated in degree 0. Now embed X in a Euclidean space $\mathbb{R}^{2m+1}$, where $2m+1 \geq 2\dim X + 1$, and take a regular neighbourhood R of X inside $\mathbb{R}^{2m+1}$ which can be assumed to be a compact oriented smooth manifold with boundary. Gluing two copies of this manifold (the orientation of one of which had been reversed) along their boundaries yields a $(2m+1)$-dimensional connected oriented smooth manifold Y, whose even dimensional integral cohomology below dimension $m+1$ is concentrated in degree 0. This follows from the Mayer-Vietoris sequence and the fact that by general position, the inclusion $\partial R \hookrightarrow R$ induces isomorphisms of homotopy groups up to degree $2m+1-\dim X - 2 \geq m-1$ and a surjection in degree $2m+1-\dim X - 1 \geq m$. In particular, the connecting homomorphism $H^i(\partial R;\mathbb{Z}) \to H^{i+1}(Y;\mathbb{Z})$ in the Mayer-Vietoris sequence is 0, if $i \leq m-1$. Hence, for actions on Y, the second part of Theorem 1 and 2 can be applied (assuming that the induced action on $H^*(Y;\mathbb{F}_p)$ is trivial in the case of $G = \mathbb{Z}/p$).

Another example illustrating the second part of Theorem 1 and Theorem 2 can be constructed as follows. Consider the space $X = S^1 \times S^{2m}$ equipped with the S^1-action that acts trivially on the first factor and is the usual rotation action (fixing north and south pole) on the second factor. The fixed point set of this S^1-action is the union of two circles. Next we choose one point in each fixed point component and remove small S^1-invariant neighborhoods equivariantly diffeomorphic to D^{2m+1} (with a linear S^1-action) around each of these two fixed points which gives an S^1-manifold Z with two boundary components each of which is equivariantly diffeomorphic to S^{2m} with the rotation action by S^1. Now we form the equivariant connected sum of Z and $[0,1] \times S^{2m}$, where on the last space S^1 acts trivially on the first factor and by a rotation action on the second. In this way, we obtain a $(2m+1)$-dimensional oriented S^1-manifold which fulfills the requirement of the second part of Theorem 1 and 2 (using the induced action by $\mathbb{Z}/p \subset S^1$). It is easy to check that the integral cohomology of this space has rank 1 in degrees 0 and $2m+1$, rank 2 in degrees 1 and $2m$ and is 0 in all other degrees. The total Betti number of the fixed set (which is just a single copy of S^1) is two. So the Leray-Serre spectral sequence for the Borel construction does not collapse at the E_2-level in this case.

In this paper we have been working in the category of G-CW complexes. But using Čech cohomology and the usual somewhat more technical machinery (see, e.g. [1]), one can extend all the results to general G-spaces which fulfill the hypothesis (LT) for the localization theorem (see [1], p. 208). As this generalization is straightforward, we leave it to the interested reader.

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