THE SPACE OF METRICS OF POSITIVE SCALAR CURVATURE

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ABSTRACT. We study the topology of the space of positive scalar curvature metrics on high dimensional spheres and other spin manifolds. Our main result provides elements in higher homotopy and homology groups of these spaces, which, in contrast to previous approaches, are of infinite order and survive in the (observer) moduli space of such metrics.

Along the way we construct smooth fiber bundles over spheres whose total spaces have non-vanishing $\hat{A}$-genera, thus establishing the non-multiplicativity of the $\hat{A}$-genus in fiber bundles with simply connected base.

1. INTRODUCTION AND SUMMARY

The classification of positive scalar curvature metrics on closed smooth manifolds is a central topic in Riemannian geometry. Whereas the existence question has been resolved in many cases and is governed by the (stable) Gromov-Lawson conjecture (compare e.g. [22, 20]), information on the topological complexity of the space of positive scalar curvature metrics on a given manifold M has been sparse and only recently some progress has been made [3, 6].

We denote by $\mathrm{Riem}^+(M)$ the space of Riemannian metrics of positive scalar curvature, equipped with the C^∞-topology, on a closed smooth manifold M. If it is not empty, we want to give information on the homotopy groups $\pi_k(\mathrm{Riem}^+(M), g_0)$ for $g_0 \in \mathrm{Riem}^+(M)$ in the different path components of $\mathrm{Riem}^+(M)$. One method to construct non-zero elements in these homotopy groups, developed by Hitchin, is to pull back g_0 along a family of diffeomorphisms of M. In [13, Theorem 4.7] this was used to prove existence of non-zero classes of order two in $\pi_1(\mathrm{Riem}^+(M), g_0)$ for certain manifolds M. In [6, Corollary 1.5] this method has been refined to show that there exist non-zero elements of order two in infinitely many degrees of $\pi_*(\mathrm{Riem}^+(M), g_0)$, when M is a spin manifold admitting a metric g_0 of positive scalar curvature.

In our paper we construct non-zero elements of *infinite order* in $\pi_k(\mathrm{Riem}^+(M), g_0)$ for $k \in \mathbb{N}$, among others. Our construction is quite different from Hitchin's in that it does not rely on topological properties of the diffeomorphism group of M.

Our first main result reads as follows.

Theorem 1.1. *Let $k \geq 0$ be a natural number. Then there is a natural number $N(k)$ with the following properties:*

a) *For each $n \geq N(k)$ and each spin manifold M admitting a metric g_0 of positive scalar curvature and of dimension $4n - k - 1$, the homotopy group*

$$\pi_k(\mathrm{Riem}^+(M), g_0)$$

contains elements of infinite order if $k \geq 1$, and infinitely many different elements if $k = 0$. Their images under the Hurewicz homomorphism in $H_k(\mathrm{Riem}^+(M))$ still have infinite order.

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b) *For $M = S^{4n-k-1}$, the images of these elements in the homotopy and homology groups of the observer moduli space* $\mathrm{Riem}^+(S^{4n-k-1})/\operatorname{Diff}_{x_0}(S^{4n-k-1})$*, see Definition 1.3, have infinite order.*

For $k = 0$ these statements are well known with $N(k) = 2$ and $\operatorname{Diff}(S^{4n-1})$ instead of $\operatorname{Diff}_{x_0}(S^{4n-1})$, see [17, Theorem IV.7.7] and [11, Theorem 4.47].

Refined versions of part b) of Theorem 1.1 will be stated in Theorems 1.11 and 1.12 below.

Following a suggestion by one of the referees we remark that Theorem 1.1 implies the following stable statement for manifolds in arbitrary dimension.

Proposition 1.2. *For each $k \geq 1$ and each spin manifold M in dimension $(3-k) \bmod 4$ and admitting a metric of positive scalar curvature, the homotopy group $\pi_k(\mathrm{Riem}^+(M \times B^N))$ contains elements of infinite order. Here, N is a constant depending on k and $\dim M$, B denotes a Bott manifold, an eight dimensional closed simply connected spin manifold satisfying $\hat{A}(B) = 1$, and B^N is the N-fold cartesian product.*

Note that, as a spin manifold with non-vanishing $\hat{A}$-genus, the manifold B does not admit a metric of positive scalar curvature. Inspired by the stable Gromov-Lawson-Rosenberg conjecture, we wonder whether there is a stability pattern in $\pi_k(\mathrm{Riem}^+(M \times B^N))$ for growing N.

Remarkably, at the end of [11, Section 5] Gromov and Lawson write: "*The construction above can be greatly generalized using Browder-Novikov Theory. A similar construction detecting higher homotopy groups of the space of positive scalar curvature metrics can also be made.*" We are not sure what Gromov and Lawson precisely had in mind here. Our paper may be viewed as an attempt to realize the program hinted at by these remarks.

Recall that the diffeomorphism group $\operatorname{Diff}(M)$ acts on $\mathrm{Riem}^+(M)$ via pull-back of metrics. The orbit space $\mathrm{Riem}^+(M)/\operatorname{Diff}(M)$ is the *moduli space* of Riemannian metrics of positive scalar curvature. Because the action is not free (the isotropy group at $g \in \mathrm{Riem}^+(M)$ is equal to the isometry group of g, which is a compact Lie group by the Myers-Steenrod theorem) it is convenient to restrict the action to the following subgroup of $\operatorname{Diff}(M)$.

Definition 1.3. Let M be a connected smooth manifold and let $x_0 \in M$. The *diffeomorphism group with observer* $\operatorname{Diff}_{x_0}(M)$ is the subgroup of $\operatorname{Diff}(M)$ consisting of diffeomorphisms $\phi\colon M \to M$ fixing $x_0 \in M$ and with $D_{x_0}\phi = \mathrm{id}_{T_{x_0}M}$.

This definition first appeared in [1]. It is easy to see that $\operatorname{Diff}_{x_0}(M)$, unlike $\operatorname{Diff}(M)$, acts freely on the space of Riemannian metrics on M (as long as M is connected). The orbit space $\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M)$ is called the *observer moduli space* of positive scalar curvature metrics.

The construction of Theorem 1.1 is based on the following fundamental result, which we regard of independent interest.

Theorem 1.4. *Given $k, l \geq 0$ there is an $N = N(k,l) \in \mathbb{N}_{\geq 0}$ with the following property: For all $n \geq N$, there is a $4n$-dimensional smooth closed spin manifold P with non-vanishing $\hat{A}$-genus and which fits into a smooth fiber bundle*

$$F \hookrightarrow P \to S^k.$$

In addition we can assume that the following conditions are satisfied:

(1) The fiber F is l-connected, and
(2) the bundle $P \to S^k$ has a smooth section $s\colon S^k \to P$ with trivial normal bundle.

Recall that in a fiber bundle $F \to E \to B$ of oriented closed smooth manifolds the Hirzebruch L-genus is multiplicative if B is simply connected, i.e. $L(E) = L(B) \cdot L(F)$, see [5]. Theorem 1.4 shows that a corresponding statement for the $\hat{A}$-genus is wrong, even if B is a sphere.

We remark that our construction, which is based on abstract existence results in differential topology, does not yield an explicit description of the diffeomorphism type of the fiber manifold F.

Remark 1.5. In Theorem 1.4 we can assume in addition that $\alpha(F) = 0$ by taking the fiber connected sum of P with the trivial bundle $S^k \times (-F) \to S^k$ where $-F$ denotes the manifold F with reversed spin structure (so that $-F$ represents the negative of F in the spin bordism group). This can be done along a trivialized normal bundle of a smooth section $s\colon S^k \to P$ as in point (2) of Theorem 1.4.

In this case the fiber F carries a metric of positive scalar curvature by a classical result of Stolz [21] if $l \geq 1$ and $\dim F \geq 5$. But note that there is no global choice of such metrics on the fibers of $E \to S^k$ depending smoothly on the base point, because this would imply that P carries a metric of positive scalar curvature by considering a metric on P whose restriction to the vertical tangent space coincides with the given family of positive scalar curvature metrics, shrinking the fibers and applying the O'Neill formulas [2, Chaper 9.D.]. However, this is impossible as P is spin and satisfies $\hat{A}(P) \neq 0$.

Let us point out an implication pertaining to diffeomorphism groups. For an oriented closed smooth manifold F and a pointed continuous map $\phi\colon (S^k, *) \to (\mathrm{Diff}^+(F), \mathrm{id})$ into the group of orientation preserving diffeomorphisms of F we define

$$M_\phi = \big(S^k \times F \times [0,1]\big) \,/\, (x, f, 0) \sim (\psi(x,f), 1)$$

where $\psi\colon S^k \times F \to S^k \times F$ is a smooth map homotopic to the adjoint of ϕ and inducing diffeomorphisms $\psi(x, -)\colon F \to F$ for each $x \in S^k$ (depending on ϕ there is a canonical isotopy class of such maps ψ). We can regard M_ϕ as the total space of a fiber bundle $F \to M_\phi \to S^k \times S^1$ obtained by a parametrized mapping cylinder construction for $\psi(x, -)$. The diffeomorphism type of this bundle depends only on the homotopy class of ϕ.

For each $k \geq 0$ this leads to a group homomorphism

$$\hat{A}_{\mathrm{Diff}}\colon \pi_k(\mathrm{Diff}^+(F), \mathrm{id}) \to \mathbb{Q}, \quad [\phi] \mapsto \hat{A}(M_\phi).$$

If F admits a spin structure and $k \geq 2$, this invariant takes values in $\mathbb{Z}$, as in this case the manifold M_ϕ carries a spin structure, $\psi\colon S^k \times F \to S^k \times F$ admitting a lift to the spin principal bundle. By Proposition 2.2 below the same conclusion holds in general if F admits both a spin structure and a metric of positive scalar curvature.

Corollary 1.6. *For each $k \geq 0$ there is an oriented closed smooth manifold F such that the homomorphism $\hat{A}_{\mathrm{Diff}}\colon \pi_k(\mathrm{Diff}^+(F), \mathrm{id}) \to \mathbb{Q}$ is non-trivial.*

To our knowledge, this is new for all $k \geq 0$.

Proof. Take a bundle $F \to P \to S^{k+1}$ from Theorem 1.4. The total space can be obtained by a clutching construction

$$P = (D^{k+1} \times F) \cup_\psi (D^{k+1} \times F)$$

for an appropriate smooth map $\psi\colon S^k \times F \to S^k \times F$. Denote by

$$(W; S^k \times I, D^{k+1} \times \partial I) \subset D^{k+1} \times I$$

the standard bordism relative boundary of an index-zero surgery, as depicted in the upper part of Figure 1. Using ψ to identify the left with the right end of $W \times F$ produces a bordism between P and the required bundle over $S^k \times S^1$. The result follows from bordism invariance of the $\hat{A}$-genus. □

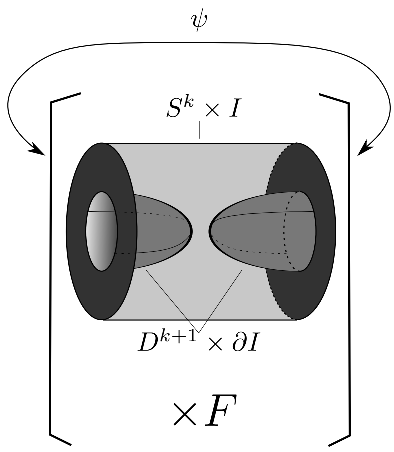

FIGURE 1.

The proof of Theorem 1.4 is based on results from classical differential topology: Surgery theory [7], Casson's theory of pre-fibrations [4] and Hatcher's theory of concordance spaces [12]. Theorem 1.1 for $M = S^{4n-k-1}$ follows from this with the use of Igusa's fiberwise Morse theory [15] and Walsh's generalization of the Gromov-Lawson surgery method to families of generalized Morse functions with critical points of coindex at least three [23].

We will now formulate part b) of Theorem 1.1 in a more general context.

In order to express the fact that, contrary to the constructions in [13] and [6], our families of metrics in Theorem 1.1 are not induced by families of diffeomorphisms of M, we propose the following definition.

Definition 1.7. Let M be an oriented closed smooth manifold and let g_0 be a positive scalar curvature metric on M. A class $c \in \pi_k(\mathrm{Riem}^+(M), g_0)$ is called *not geometrically significant* if c is represented by

$$S^k \to \mathrm{Riem}^+(M), \quad t \mapsto \phi(t)^* g_0$$

for some pointed continuous map $\phi\colon (S^k, *) \to (\mathrm{Diff}^+(M), \mathrm{id})$. Otherwise, c is called *geometrically significant*.

Note that the homotopy classes of [6, 13] are by their very construction not geometrically significant.

Definition 1.8. Let F be an oriented closed smooth manifold. We call F

- an *$\hat{A}$-multiplicative fiber in degree k* if for every oriented fiber bundle $F \to E \to S^{k+1}$ we have $\hat{A}(E) = 0$. (This implies that the map $\hat{A}\colon \pi_k(\mathrm{Diff}^+(F), \mathrm{id}) \to \mathbb{Q}$ defined before Corollary 1.6 is zero).
- a *strongly $\hat{A}$-multiplicative fiber in degree k* if for every oriented smooth fiber bundle $F \to E \to B$ over a closed oriented manifold B of dimension $k+1$ we have $\hat{A}(E) = \hat{A}(B) \cdot \hat{A}(F)$.

Obviously F is a strongly $\hat{A}$-multiplicative fiber in degree k if $\dim F + k + 1$ is not divisible by four.

Proposition 1.9.

a) *If F has vanishing rational Pontryagin classes, in particular if it is stably parallelizable or a rational homology sphere, then F is an $\hat{A}$-multiplicative fiber in any degree.*
b) *Every homotopy sphere is a strongly $\hat{A}$-multiplicative fiber in any degree.*

Proof. For a) we observe that F is framed in E, so that the rational Pontryagin classes $p_i(E)$ restrict to $p_i(F) = 0$. The long exact sequence of the pair (E, F) implies that all $p_i(E)$ pull back from $H^*(E, F)$.

Denote by $D^{k+1}_{\pm}$ the upper and lower hemisphere of S^{k+1}, respectively. By excision and the Künneth isomorphism we obtain isomorphisms of rational cohomology rings

$$H^*(E, F) \cong H^*(E, E|_{D^{k+1}_+}) \cong H^*(E|_{D^{k+1}_-}, E|_{S^k}) \cong H^*(F) \otimes H^*(D^{k+1}_-, S^k).$$

But in $H^*(D^{k+1}_-, S^k)$ all non-trivial products vanish, and hence the same is true for $H^*(E, F)$.

So there are no non-trivial products of Pontryagin classes in E. It follows that the $\hat{A}$-genus of E is a multiple of the signature of E, which is known to vanish in fiber bundles over spheres.

For assertion b), we note that, given an oriented bundle $E \to B$ with fiber S^n and structure group the orientation preserving homeomorphism group of S^n, we obtain an oriented topological disc bundle $W \to B$ with $\partial W = E$ by "fiberwise coning off the spheres". More precisely, the Alexander trick defines an embedding $\mathrm{Homeo}^+(S^n) \to \mathrm{Homeo}^+(D^{n+1})$ by extending homeomorphisms $S^n \to S^n$ to homeomorphisms $D^{n+1} \to D^{n+1}$ radially over the cone over S^n, which is identified with D^{n+1} in the standard way. This embedding splits the restriction homomorphism, and W is obtained by an associated bundle construction.

Now *rational* Pontryagin classes are defined for topological manifolds and the rational Pontryagin numbers are invariants of oriented topological bordism in the usual way. In particular, $\hat{A}(E) = 0$, as E is a topological boundary. Because $\hat{A}(S^n) = 0$ the claim follows. □

One of the referees pointed out that strongly $\hat{A}$-multiplicative fibers different from spheres can be obtained using results of Farrell-Jones on rational homotopy types of automorphism groups of hyperbolic manifolds.

Proposition 1.10. *Let $m \geq 10$ and let F be a connected oriented closed hyperbolic manifold of dimension m. Then for all $0 \leq k \leq \frac{m-4}{3}$ the manifold F is a strongly $\hat{A}$-multiplicative fiber in degree k.*

Proof. Let B be a closed smooth oriented manifold of dimension $k+1$ and $F \to E \to B$ be an oriented smooth fiber bundle classified by a map $B \to B\,\mathrm{Diff}^+(F)$. We pass to the underlying topological bundle, which is classified by the composition $\phi : B \to B\,\mathrm{Diff}^+(F) \to B\,\mathrm{Top}^+(F)$. In the following we abbreviate $\mathrm{Top}^+(F)$ by Top.

By [9, Corollary 10.16.] $\pi_0(\mathrm{Top})$ is a semidirect product of a countable elementary abelian 2-group and the group $\mathrm{Out}(\pi_1(F))$. The last group is finite by the Mostow rigidity theorem. Using the fact that the rational homology of finite and of abelian torsion groups vanishes (as each homology class is carried by a finitely generated subgroup) we get $\widetilde{H}_*(K(\pi_0(\mathrm{Top}), 1); \mathbb{Q}) = 0$. Hence the Leray-Serre spectral sequence for the orientation fibration $\widetilde{B\,\mathrm{Top}} \to B\,\mathrm{Top} \to K(\pi_0(\mathrm{Top}), 1)$ shows that $H_*(\widetilde{B\,\mathrm{Top}}; \mathbb{Q}) \cong H_*(B\,\mathrm{Top}; \mathbb{Q})$.

Now the rational Hurewicz theorem can be applied to the simply connected space $\widetilde{B\,\mathrm{Top}}$. Because $\pi_s(\widetilde{B\,\mathrm{Top}}) \otimes \mathbb{Q} = \pi_s(B\,\mathrm{Top}) \otimes \mathbb{Q} = \pi_{s-1}(\mathrm{Top}) \otimes \mathbb{Q} = 0$ for $2 \leq s \leq \frac{m-4}{3} + 1$ by [9, Corollary 10.16.], we hence obtain $H_s(B\,\mathrm{Top}; \mathbb{Q}) = H_s(\widetilde{B\,\mathrm{Top}}; \mathbb{Q}) = 0$ for $1 \leq s \leq \frac{m-4}{3} + 1$

With this information we go into the Atiyah-Hirzebruch spectral sequence for oriented bordism with rational coefficients and conclude that rationally the map ϕ is oriented bordant to the constant map $B \to B\,\mathrm{Top}$. This implies that rationally the topological manifold E is oriented bordant to the product $B \times F$. Hence $\hat{A}(E) = \hat{A}(B \times F) = \hat{A}(B) \cdot \hat{A}(F)$ as claimed. □

Closed hyperbolic spin manifolds do not admit metrics of positive scalar curvature [11] and so this construction will not be used further in our discussion.

Theorem 1.11. *In the situation of part a) of Theorem 1.1 assume in addition that the manifold M is an $\hat{A}$-multiplicative fiber in degree $k \geq 1$. Then $\pi_k(\mathrm{Riem}^+(M), g_0)$ contains elements all of whose multiples are geometrically significant. The images of these elements under the Hurewicz map have infinite order.*

Theorem 1.12. *In the situation of part a) of Theorem 1.1,*

a) *if in addition M is a connected $\hat{A}$-multiplicative fiber in degree $k \geq 0$, then the map*

$$\pi_k(\mathrm{Riem}^+(M), g_0) \to \pi_k(\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M), [g_0])$$

has infinite image. If $k \geq 1$, the image contains elements of infinite order.

b) *if in addition M is a simply connected strongly $\hat{A}$-multiplicative fiber in degree $k \geq 0$, the image of the map*

$$\pi_k(\mathrm{Riem}^+(M), g_0) \to H_k(\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M))$$

contains elements of infinite order.

Part b) of Theorem 1.1 is the case $M = S^{4n-k-1}$ of the last result.

The elements in $\pi_k(\mathrm{Riem}^+(M), g_0)$ constructed in [6, 13] can in fact be obtained by pulling back g_0 along families in $\operatorname{Diff}_{x_0}(M)$. Hence these elements are not only not geometrically significant, but are even mapped to zero under the map in part a) of Theorem 1.12.

In [3] the authors constructed, for arbitrary $k \geq 1$, non-zero elements in the $4k$-th homotopy groups of the full moduli space $\mathrm{Riem}^+(M)/\mathrm{Diff}(M)$ of certain closed non-spin manifolds M of odd dimension. These elements can be lifted to $\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M)$, but not to $\mathrm{Riem}^+(M)$.

It remains an interesting open problem whether there are examples of manifolds M for which any of the statements of Theorem 1.12 remain valid for the full moduli space $\mathrm{Riem}^+(M)/\operatorname{Diff}(M)$.

Kreck and Stolz [16] constructed closed 7-manifolds M for which the moduli spaces $\mathrm{Sec}^+(M)/\operatorname{Diff}(M)$ of positive sectional curvature metrics are not path connected and closed 7-manifolds for which the moduli spaces $\mathrm{Ric}^+(M)/\operatorname{Diff}(M)$ of positive Ricci curvature metrics have infinitely many path components. It remains a challenging problem to construct non-zero elements in higher homotopy groups of (moduli) spaces of positive sectional and positive Ricci curvature metrics.

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2. APS INDEX AND FAMILIES OF METRICS OF POSITIVE SCALAR CURVATURE

If (W, g_W) is a compact Riemannian spin manifold with boundary and with positive scalar curvature on the boundary (the Riemannian metric always assumed to be of product form near the boundary), then the Dirac operator on W with Atiyah-Patodi-Singer boundary conditions is a Fredholm operator. Equivalently, if one attaches an infinite half-cylinder $\partial W \times [0, \infty)$ to the boundary and extends the metric as a product to obtain W_∞, the Dirac operator on L^2-sections of the spinor bundle on W_∞ is a Fredholm operator. This uses invertibility of the boundary operator due to the positive scalar curvature condition. Both operators have the same index, the APS index $\operatorname{ind}(D_{g_W}) \in \mathbb{Z}$, which depends on the Riemannian metric on the boundary.

We collect some well known properties of this index. A detailed discussion can be found in [18].

(1) If g_W is of positive scalar curvature, we have $\mathrm{ind}(D_{g_W}) = 0$. This follows from the usual Weitzenböck-Lichnerowicz-Schrödinger argument.
(2) $\mathrm{ind}(D_{g_W})$ is invariant under deformations of the metric g_W during which the metrics on the boundary maintain positive scalar curvature. This follows from the homotopy invariance of the index of Fredholm operators.
(3) The index can be computed by the APS index theorem
$$\mathrm{ind}(D_{g_W}) = \int_W \hat{\mathcal{A}}(W) - \frac{1}{2}\eta(\partial W),$$
which involves the $\hat{A}$-form and the η-invariant of Atiyah-Patodi-Singer.
(4) (Gluing formula) If (V, g_V) is another Riemannian spin manifold and there is a spin preserving isometry $\psi\colon \partial V \to -\partial W$, then $\mathrm{ind}(D_{g_V}) + \mathrm{ind}(D_{g_W}) = \hat{A}(V \cup_\psi W)$ where $\hat{A}$ is the usual $\hat{A}$-genus of the closed spin manifold $V \cup_\psi W$.

Definition 2.1. Let $M \to E \to B$ be a smooth fiber bundle of closed manifolds over a compact base manifold B, let g_B be a Riemannian metric on B, let $\mathcal{H}$ be a horizontal distribution on E and $(g_b)_{b\in B}$ be a smooth family of positive scalar curvature metrics on the fibers of E. Combining these data we obtain a Riemannian metric $g_E = g_B \oplus (g_b)$ on the total space E, where g_B is lifted to horizontal subspaces of E using the given distribution $\mathcal{H}$. Note that $(E, g_E) \to (B, g_B)$ is a Riemannian submersion.

Using the O'Neill formulas [2, Chaper 9.D.] there is an $\epsilon_0 > 0$ with the following property: For each $0 < \epsilon \le \epsilon_0$ the metric $g_B \oplus (\epsilon \cdot g_b)$ on E is of positive scalar curvature. We call such an ϵ_0 an *adiabatic constant* for the triple $(E, g_B, \mathcal{H})$.

Assume that B and M are closed spin manifolds and $\phi\colon B \to \mathrm{Riem}^+(M)$ is a continuous map. This induces a family $(g_b)_{b\in B}$ of positive scalar curvature metrics on M, which can be assumed to be smooth after a small perturbation. After picking some metric g_B on B we can consider both the product metric $g_B \oplus g_0$, and the metric $g_B \oplus (g_b)_{b\in B}$ on $B \times M$.

We now multiply both fiber metrics on $B\times M \to B$ with adiabatic constants so that the resulting metrics on $B \times M$ are of positive scalar curvature and denote these new metrics by the same symbols. Let $g = g_{(B\times[0,1])\times M}$ be an arbitrary metric on $(B \times [0,1]) \times M$ interpolating between the metrics $g_B \oplus g_0$ on $(B\times 0)\times M$ and $g_B \oplus (g_b)$ on $(B\times 1)\times M$ (always of product form near the boundaries) and set
$$\hat{A}_\Omega(\phi) = \mathrm{ind}(D_g) \in \mathbb{Z},$$
the APS-index of the Dirac operator D_g for the metric g on the spin manifold $(B\times[0,1])\times M$. Note that $\mathrm{ind}(D_g)$ is equal to the relative index $i(g_B \oplus g_0\, , \, g_B \oplus (g_b))$ of Gromov-Lawson [10, p. 329].

The following facts show that the invariant $\hat{A}_\Omega$ is well defined on $\Omega_k^{\mathrm{Spin}}(\mathrm{Riem}^+(M))$ and hence defines a group homomorphism
$$\hat{A}_\Omega : \Omega_k^{\mathrm{Spin}}(\mathrm{Riem}^+(M)) \to \mathbb{Z}.$$

- If we choose a different metric on B or scale the metrics g_0 and (g_b) by other adiabatic constants, the index $\mathrm{ind}(D_g)$ remains unchanged by property (2) above.
- If we choose another metric g' on $(B \times [0,1]) \times M$ interpolating between the two given metrics on the boundary, the gluing formula implies
$$\mathrm{ind}(D_g) - \mathrm{ind}(D_{g'}) = \hat{A}(M \times B \times S^1) = 0$$
using the fact that the $\hat{A}$-genus is multiplicative in products.

- Now assume that (W, g_W) is a compact Riemannian spin manifold of dimension $k+1$ with boundary B and that $(g_w)_{w\in W}$ is a smooth family of positive scalar curvature metrics on M restricting to a given family $(g_b)_{b\in B}$ over B. We also consider the constant family (g_0) on M. With respect to the given metric g_W on W we scale both fiberwise metrics on $W \times M$ by adiabatic constants.

  By the gluing formula and the fact that the APS-indices of both metrics on $W \times M$ vanish, we obtain (with $g = g_{(B\times[0,1])\times M}$ as before)

  $$\operatorname{ind}(D_g) = \operatorname{ind}(D_{g_W \oplus g_0}) + \operatorname{ind}(D_g) + \operatorname{ind}(D_{g_W \oplus (g_w)_{w\in W}}) = \hat{A}(X),$$

  where

  $$X = (W \cup_{\partial W = B\times\{0\}} B \times [0,1] \cup_{B\times\{1\}=\partial W} W) \times M.$$

  Because X is spin and M admits a metric of positive scalar curvature we have $\hat{A}(X) = 0$ and hence $\operatorname{ind}(D_g) = 0$, as required.

Let $g_0 \in \operatorname{Riem}^+(M)$. Using composition with the canonical map $\pi_k(\operatorname{Riem}^+(M), g_0) \to \Omega_k^{Spin}(\operatorname{Riem}^+(M))$ we also define

$$\hat{A}_\pi \colon \pi_k(\operatorname{Riem}^+(M), g_0) \longrightarrow \Omega_k^{Spin}(\operatorname{Riem}^+(M)) \xrightarrow{\hat{A}_\Omega} \mathbb{Z}$$

for all $k \geq 0$. This is a group homomorphism for $k > 0$ and a map of sets for $k = 0$.

The action of $\operatorname{Diff}(M)$ on $\operatorname{Riem}^+(M)$ by pull-back induces an action of $\pi_k(\operatorname{Diff}(M), \operatorname{id})$ on $\pi_k(\operatorname{Riem}^+(M), g_0)$. For a pointed map $\phi\colon (S^k, *) \to (\operatorname{Diff}(M), \operatorname{id})$ and a family $c = (g_t)_{t\in S^k}$ based at g_0 the pair $([\phi], [c])$ is mapped to the homotopy class represented by the family $(\phi(t)^* g_t)_{t\in S^k}$, which we denote by $\phi^* c$ for short. The neutral element $e \in \pi_k(\operatorname{Riem}^+(M), g_0)$ is given by the constant family with value g_0.

Proposition 2.2. *For a closed spin manifold M and $[\phi] \in \pi_k(\operatorname{Diff}^+(M), \operatorname{id})$ where $k \geq 1$ we have*

$$\hat{A}_\pi(\phi^* e) = \hat{A}_{\operatorname{Diff}}(\phi)$$

with $\hat{A}_{\operatorname{Diff}}(\phi)$ as defined before Corollary 1.6. If $k = 0$, the same equation holds provided $[\phi]$ is represented by a spin-preserving diffeomorphism.

Proof. In the case that the map $S^k \times M \to S^k \times M$ adjoint to ϕ is spin preserving, the assertion follows from the gluing formula for the APS index. This shows in particular the last claim of the proposition.

Moreover we observe that if $k \geq 1$, both sides of the asserted equation define group homomorphisms $\pi_k(\operatorname{Diff}^+(M), \operatorname{id}) \to \mathbb{Q}$ — for the left hand side we use that it is given by the composite of group homomorphisms

$$\pi_k(\operatorname{Diff}^+(M), \operatorname{id}) \to \pi_k(\operatorname{Riem}^+(M), g_0) \xrightarrow{\hat{A}_\pi} \mathbb{Z} \subset \mathbb{Q}$$

where the first map is induced by the action of the diffeomorphism group on the base-point $g_0 \in \operatorname{Riem}^+(M)$. The proof is concluded by the fact that each diffeomorphism $S^k \times M \to S^k \times M$ has a power which is spin preserving. □

We obtain the following corollaries for a closed spin manifold M which is an $\hat{A}$-multiplicative fiber in degree k.

Corollary 2.3. *If, for $k \geq 1$, an element $c \in \pi_k(\operatorname{Riem}^+(M), g_0)$ is not geometrically significant then $\hat{A}_\pi(c) = 0$.*

By considering the long exact sequence in homotopy associated to the fibration

$$\operatorname{Riem}^+(M) \hookrightarrow \operatorname{Riem}^+(M)/\operatorname{Diff}_{x_0}(M) \to B\operatorname{Diff}_{x_0}(M)$$

we also have

Corollary 2.4. *Let M be a connected. Then for $k \geq 1$ the map $\hat{A}_\pi \colon \pi_k(\mathrm{Riem}^+(M), g_0) \to \mathbb{Z}$ factors through the image of the projection-induced map*

$$\pi_k(\mathrm{Riem}^+(M), g_0) \to \pi_k(\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M), [g_0]).$$

In particular, the group $\pi_k(\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M), [g_0])$ contains an element of infinite order if the map $\hat{A}_\pi$ is non-zero in degree $k \geq 1$. As the isotopy classes of spin-preserving diffeomorphisms form a finite-index subgroup of $\pi_0(\operatorname{Diff}(M))$, we also deduce:

Corollary 2.5. For $k = 0$ the set $\pi_0(\mathrm{Riem}^+(M)/\operatorname{Diff}(M))$ is infinite if the image of $\hat{A}_\pi$ in degree $k = 0$ is infinite.

In passing we also note the following consequence.

Corollary 2.6. *Let F be a manifold as in Remark 1.5 admitting a positive scalar curvature metric g_0 and appearing as fiber in $F \to P \to S^{k+1}$. Then $\pi_k(\mathrm{Riem}^+(F), g_0)$ contains elements of infinite order (and infinitely many elements for $k = 0$), which are not geometrically significant.*

3. Fiberwise Morse theory and metrics of positive scalar curvature

Let B be a closed smooth connected manifold and let W be a smooth $n+1$-dimensional compact manifold with two connected boundary components $M_0 = \partial_0 W$ and $M_1 = \partial_1 W$. We assume throughout this section that $\dim W = n + 1 \geq 6$. Let

$$W \hookrightarrow E \xrightarrow{\pi} B$$

be a smooth fiber bundle with structure group $\operatorname{Diff}(W; M_0, M_1)$ consisting of diffeomorphisms $W \to W$ mapping M_i to M_i for $i = 0, 1$ and preserving pointwise some collar neighborhoods of M_0 and M_1 in W. Then the total space E has again two boundary components $\partial_i E$ which are total spaces of fiber bundles $M_i \hookrightarrow \partial_i E \to B$.

In the following discussion, the notion *fiberwise* in connection with a mathematical object defined on E is a shorthand for the fact that this object is defined or constructed on each fiber $E_t = \pi^{-1}(t) \subset E$, $t \in B$, but still defines a global object on E. In this terminology, the *fiberwise tangent bundle of E* is equal to the *vertical tangent bundle*

$$T^{vert}E = \ker(T\pi \colon TE \to TB) \to E$$

and a *fiberwise Morse function* is a smooth function $E \to \mathbb{R}$ which restricts to a Morse function on each fiber E_t.

Theorem 3.7 below states when a fiberwise metric of positive scalar curvature on $\partial_0 E$, i.e. a smooth metric on $T^{vert}(\partial_0 E)$ that restricts to a metric of positive scalar curvature on each fiber $(\partial_0 E)_t$, can be extended to a fiberwise metric of positive scalar curvature on E. This is a fibered version of the well known fact, due to Gromov-Lawson, Schoen-Yau and Gajer, that if $M_1 \hookrightarrow W$ is a 2-equivalence (inducing a bijection on π_0 and π_1 and a surjection on π_2), then a positive scalar curvature metric on M_0 can be extended to W, cf. [22, Theorem 3.3]. This uses a handle decomposition of W induced by a Morse function. In a fibered situation the situation is more complicated, because it is in general not possible to construct a fiberwise Morse function on $E \to B$. We handle this situation by combining Igusa's theory of fiberwise generalized Morse functions [15] with Walsh's generalization of the Gromov-Lawson surgery method to a fibered situation [23].

The following discussion summarizes [15, §3] in a form needed for our purpose. Let M be a compact smooth manifold. Recall that a smooth function

$$f \colon M \to \mathbb{R}$$

is a *generalized Morse function* if the gradient of f is transverse to ∂M and if each critical point $p \in M$ of f is either of Morse or birth-death type. By definition, in the first case there are local coordinates $(x_1, \dots, x_i, y_1, \dots, y_j)$ around p so that p has coordinates $(x(p), y(p)) = (0, 0)$ and f can locally be written as

$$f(x, y) = f(p) - x_1^2 - \cdots - x_i^2 + y_1^2 + \cdots + y_j^2 .$$

In the second case there are local coordinates $(x, y_1, \dots, y_k, z_1, \dots, z_l)$ around p so that p has coordinates $(x(p), y(p), z(p)) = (0, 0, 0)$ and f can locally be written in the form

$$f(x, y, z) = f(p) + x^3 - y_1^2 - \cdots - y_k^2 + z_1^2 + \cdots + z_l^2 .$$

The type of the critical point as well as the numbers i and k are uniquely determined by p and f. In the first case (of a Morse singularity) we say that p is a critical point of *index* i, in the second case (of a birth-death singularity) we say that p is a critical point of *index* $k + 1/2$. This is motivated by the fact that in a parametrized family of Morse functions birth-death singularities typically arise in the situation when two critical points of index k and $k + 1$ cancel each other along a one-parameter sub-family. The normal form for such a family, parametrized by $t \in (-\epsilon, \epsilon)$, is $f_t(x, y, z) = f_0(p) + x^3 - tx - y^2 + z^2$ with a birth-death singularity at $t = 0$. Note that in [15] the index of a birth-death singularity as above is defined to be k.

Definition 3.1. By a *generic family of generalized Morse functions* on the bundle $E \to B$ we mean a smooth function

$$F \colon E \to [0, 1]$$

with the following properties:

- $\partial_0 E = F^{-1}(0)$, $\partial_1 E = F^{-1}(1)$.
- The function F is fiberwise generalized Morse, i.e. for each $t \in B$ the restriction

  $$f_t = F|_{E_t} \colon E_t \to [0, 1]$$

  is a generalized Morse function.
- The birth-death points in each f_t are *generically unfolded*, i. e. they represent a transversal intersection of the fiberwise 3-jet defined by F and the subspace of fiberwise birth-death 3-jets inside the bundle of fiberwise 3-jets of smooth functions $E \to \mathbb{R}$, compare [15, §2].

This is essentially a global version of [15, Definition 3.1] on a non-trivial (as opposed to a product) bundle E. However, we do not require "linear independence of the birth directions of the different birth-death points of a fixed fiber E_t". Instead we follow [23, Definition 4.8] at this point.

For a smooth function $F \colon E \to [0, 1]$ we denote the subset of fiberwise critical points in E by $\Sigma(F)$. We remark that if F is a generic family of generalized Morse functions then the subsets

$$A_1(F) \subset E, \quad A_2(F) \subset E$$

of fiberwise Morse and birth-death singularities are smooth submanifolds of E of dimension $\dim B$ and $\dim B - 1$, respectively. The manifold $A_2(F)$ is closed and $\Sigma(F) = A_1(F) \cup A_2(F)$. Furthermore, π restricts to an immersion $\pi| \colon A_2(F) \to B$. For these statements, compare [15, Lemma 3.3]. The additional property of self-transversality of $\pi| \colon A_2(F) \to B$ obtained in [15, Lemma 3.3] cannot be assumed in our situation because we do not insist on linear independence of different birth-directions in a fixed fiber. Also we remark that in [23] the sets $A_1(F)$ and $A_2(F)$ are denoted Σ^0 and Σ^1, respectively.

We state the following fundamental existence result.

Proposition 3.2. *Let* $\dim W > \dim B$. *Then the generic families of generalized Morse functions* $F \colon E \to [0, 1]$ *form a nonempty open subset of* $C^\infty(E, [0, 1])$ *equipped with the* C^∞*-topology.*

Proof. The main result of [14] implies that the space of smooth functions $E \to [0,1]$ which restrict to generalized Morse functions on each fiber is nonempty in $C^\infty(E,[0,1])$. The assertion now follows from multijet transversality in the same way the corresponding [15, Lemma 3.2] follows. □

The next proposition shows that we have good control of the index of critical points (of Morse or birth-death type) of f_t, $t \in B$, for a generic family of generalized Morse functions $F\colon E \to [0,1]$. Following [23, Section 4] we call such a family *admissible* if all critical points are of index $\le n-2$ for each t. (Recall that $\dim W = n+1$, which is in accordance with the convention in [15]).

Proposition 3.3. *Assume that* $\dim W \ge 2\dim B + 5$ *and that the inclusion* $M_1 \hookrightarrow W$ *is* 2*-connected. Then the space of admissible generic families of generalized Morse functions* $E \to [0,1]$ *is a nonempty open subset of* $C^\infty(E,[0,1])$.

Proof. This follows from the proof of Proposition 3.2 by applying a variant of Hatcher's two-index theorem, see [15, Corollary VI.1.4] (with $i=0$, $j=n-1$ and $k=\dim B$ in the notation of loc. cit.) before appealing to multijet transversality. Here we note that the subset of generalized Morse functions on W whose Morse critical points have indices at most $n-2$ and birth-death critical points have indices at most $n-2\frac{1}{2}$ (resp. $n-3$ in the notation of [15]) is open in the set of all C^∞-maps. □

We note the following addendum which is proved by the same methods.

Addendum 3.4. *If in the situation of Proposition 3.3 the inclusion* $M_0 \hookrightarrow W$ *is also* 2*-connected, then the space of generic families of generalized Morse functions* $f\colon E \to [0,1]$ *with the property that both* f *and* $1-f$ *are admissible is a nonempty open subset of* $C^\infty(E,[0,1])$.

For our later discussion we need convenient coordinates around the fiberwise singular set $\Sigma(F)$ for generic families of generalized Morse functions F, cf. [23, Definition 4.3]. Because the normal bundles of $A_1(F) \subset E$ and $A_2(F) \subset E$ are in general nontrivial, we need to work in a twisted setting.

Choose a fiberwise Riemannian metric g^{vert} on E, i. e. a smooth section of $(T^{vert}E)^* \otimes (T^{vert}E)^*$ that defines a Riemannian metric on each E_t. At a critical point $p \in E_t$ of f_t we obtain an orthogonal (with respect to g^{vert}) splitting $V_p^+ \oplus V_p^- \oplus V_p^0$ into positive, negative and null eigenspaces of the (fiberwise) Hessian of $f_t\colon E_t \to \mathbb{R}$ on $T_p^{vert}E$. The space V_p^0 is zero if $p \in A_1(F)$ and one-dimensional if $p \in A_2(F)$. Hence we obtain vector bundles

$$V^\pm \to A_1(F), \quad V^\pm \to A_2(F), \quad V^0 \to A_2(F)$$

so that $V^+ \oplus V^-$ has structure group $O(i) \times O(j)$ on path components of $A_1(F)$ and $V^+ \oplus V^- \oplus V^0$ has structure group $O(k) \times O(l) \times SO(1)$ on path components of $A_2(F)$. The numbers i and k refer to indices of critical points as before and V^0 is a one-dimensional real bundle. These indices can of course vary on different path components of $A_1(F)$ and $A_2(F)$. Note that $V^0 \to A_2(F)$ carries a preferred orientation pointing in the direction of increasing f_t for $t \in A_2(F)$.

In order to describe the behavior of critical points around a birth-death singularity $p \in E_t$ in a coordinate-independent way it is useful to choose an additional Riemannian metric g_B on B and a horizontal distribution on TE together with the lift g^{hor} of g_B to this horizontal distribution. Together with the metric g^{vert} we hence obtain a Riemannian submersion

$$(E, g^{vert} \oplus g^{hor}) \to (B, g_B)\,.$$

Fixing this, for small enough $\epsilon > 0$ we obtain canonical embeddings

$$\xi\colon A_2(F) \times (-\epsilon,\epsilon) \to E$$

restricting to the inclusion $A_2(F) \hookrightarrow E$ on $A_2(F) \times \{0\}$ and so that for each $p \in A_2(F)$ the restricted map $\xi\colon \{p\} \times (-\epsilon, \epsilon) \to E$ describes the unique horizontal curve of unit speed which maps to a geodesic in B orthogonal to $D_p\pi(T_pA_2(F)) \subset T_{\pi(p)}B$ and points into the birth direction of the birth-death singularity p.

For such an $\epsilon > 0$ consider the vector bundle of rank $n+1$

$$V^\epsilon = \mathrm{pr}_1^*(V^0 \oplus V^- \oplus V^+) \to A_2(F) \times (-\epsilon, \epsilon)$$

i.e. the pull-back of $T^{vert}E|_{A_2(F)}$ to $A_2(F) \times (-\epsilon, \epsilon)$ along the projection $\mathrm{pr}_1\colon A_2(F) \times (-\epsilon, \epsilon) \to A_2(F)$. The vector bundle V^ϵ splits in a canonical way as

$$V^\epsilon = V^{\epsilon,0} \oplus V^{\epsilon,-} \oplus V^{\epsilon,+}\,.$$

We equip V^ϵ with the pull-back metric of the induced metric on $T^{vert}E|_{A_2(F)}$ and consider for $\delta > 0$ the disc bundle $D_\delta(V^\epsilon) \to A_2(F) \times (-\epsilon, \epsilon)$. An *extended tubular neighborhood* around $A_2(F)$ is a smooth embedding

$$\phi\colon D_\delta(V^\epsilon) \hookrightarrow E$$

so that the diagram

$$\begin{array}{ccc} D_\delta(V^\epsilon) & \xrightarrow{\ \phi\ } & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A_2(f) \times (-\epsilon, \epsilon) & \xrightarrow{\ \pi\circ\xi\ } & B \end{array}$$

commutes. This is exactly the global version (on non-product bundles) of the normal form defined in [15, Appendix, Lemma 3.5].

After choosing some local isometric trivialization

$$E^{vert}|_U \cong U \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^l$$

over an open subset $U \subset A_2(F)$, where the isomorphism on the $\mathbb{R}$-factor is orientation preserving, we obtain a corresponding local isometric trivialization

$$V^\epsilon|_{U\times(-\epsilon,\epsilon)} \cong (U \times (-\epsilon, \epsilon)) \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^l$$

so that we can work with local coordinates $(p, s, x, y, z) \in A_2(F) \times (-\epsilon, \epsilon) \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^l$. For the following definition compare [15, Appendix, Definition 3.9].

Definition 3.5. Let g^{vert} be a fiberwise Riemannian metric on E. We call a generic family of generalized Morse functions $F\colon E \to [0,1]$ in *normal form* with respect to g^{vert}, if there are numbers $\sigma, \tau > 0$, $\delta > 3\sigma$ and $\epsilon > \sigma^2$ and an extended tubular neighborhood

$$\phi\colon D_\delta(V^\epsilon) \hookrightarrow E$$

(with respect to some horizontal metric g^{hor} as before) so that

- ϕ is fiberwise isometric,
- for all $\{(p, s, x, y, z) \in D_{3\sigma}(V^\epsilon) \mid |s| \le \sigma^2\}$ the function $F \circ \phi$ is given by

  $$(s, x, y, z) \mapsto F(p) + g_s(x) - \|y\|^2 + \|z\|^2$$

  where

  $$g_s\colon \mathbb{R} \to \mathbb{R}\,,\ s \in [-\sigma^2, \sigma^2]$$

  is the smooth family of functions of [15, Appendix, Lemma 3.7] (with the chosen σ and τ),
- for each critical point

  $$q \in A_1(F) \setminus \phi\big(\{(p, s, x, y, z) \in D_{3\sigma}(V^\epsilon) \mid |s| \le \frac{2}{3}\sigma^2\}\big)$$

there is a $g^{vert}_{\pi(q)}$ isometric embedding $\mu\colon D^{n+1}_\tau \to E_{\pi(p)}$ mapping the midpoint of the disc $D^{n+1}_\tau \subset \mathbb{R}^{n+1}$ of radius τ (which is equipped with the standard metric) to q and so that F is given by

$$F(\mu(z_1,\dots,z_{n+1})) = F(q) + \sum_{i=1}^{n+1} \pm z_i^2\,,$$

where $(z_1,\dots,z_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$. Furthermore, the embedding μ can be assumed to vary smoothly on a contractible neighborhood of q in $A_1(F)$.

We obtain the following existence result.

Proposition 3.6. *Assume that* $\dim W \geq 2\dim B + 5$ *and that the inclusion* $M_1 \hookrightarrow W$ *is* 2*-connected. Then there is a fiberwise Riemannian metric* g^{vert} *on* E *and a generic family of generalized Morse functions* $F\colon E \to [0,1]$ *which is admissible and in normal form with respect to* g^{vert}.

Proof. The proof is along the lines of the proof of [15, Appendix, Theorem 3.10]. Here we notice that in Igusa's work this proof is given for the case when the bundle $E = W \times B \to B$ is trivial, but under the general assumption that $E^{vert}|_{A_1(F)}$ and $E^{vert}|_{A_2(F)}$ are nontrivial. Because the deformations of f and g^{vert} are carried out loc. cit. in a coordinate-independent way, the same proof can be used to treat the general case of the nontrivial bundle $W \hookrightarrow E \to B$ appearing in our discussion. □

Combining this result with [23, Theorem 1.4] we obtain

Theorem 3.7. *Assume that* $\dim W \geq 2\dim B + 5$ *and that the inclusion* $M_1 \hookrightarrow W$ *is* 2*-connected. Also assume that there exists a fiberwise metric of positive scalar curvature on a fiberwise collar neighborhood of* $\partial_0 E \subset E$ *which is fiberwise of product form on this collar neighborhood. Then this metric can be extended to a fiberwise metric of positive scalar curvature on* $E \to B$ *which is fiberwise of product form on a fiberwise collar neighborhood of* $\partial_1 E$ *in* W .

4. HOMOTOPY CLASSES OF POSITIVE SCALAR CURVATURE METRICS

Using the invariant $\hat{A}_\Omega$ from Section 2 it is now rather straightforward to prove Theorems 1.1, 1.11 and 1.12, assuming Theorem 1.4.

We fix $k \geq 0$ and pick a bundle $P \to S^k$ whose total space is of non-zero $\hat{A}$-genus and of dimension $4n$, where we also fix $l = 2$ and require $4n \geq 3k+5$ (i. e. the fiber dimension satisfies $4n - k \geq 2k + 5$). Note that this and Theorem 1.4 determines the lower bound $N(k)$ on n for our main theorems. We assume conditions (1) and (2) of Theorem 1.4.

Let $s\colon S^k \to P$ be a smooth section with trivialized normal bundle. Inside the resulting embedding of $S^k \times D^{4n-k}$ we construct an embedding

$$\rho\colon S^k \times \left(D^{4n-k} \cup_{D^{4n-k-1}\times 0} D^{4n-k-1} \times [0,1] \cup_{D^{4n-k-1}\times 1} D^{4n-k}\right) \hookrightarrow P.$$

Removing the interiors of the two copies of $S^k \times D^{4n-k}$ yields a fibration

$$W \hookrightarrow E \to S^k$$

where E has two boundary components $\partial_0 E$ and $\partial_1 E$ each of which may be identified (by the map ρ) with the total space of the trivial fibration $S^k \times S^{4n-k-1} \to S^k$. Furthermore the bundle E comes with a fiberwise embedding of $S^k \times D^{4n-k-1} \times [0,1]$ meeting $\partial_0 E$ and $\partial_1 E$ in $S^k \times D^{4n-k-1} \times 0$ and $S^k \times D^{4n-k-1} \times 1$, respectively. A typical fiber W is displayed in Figure 2.

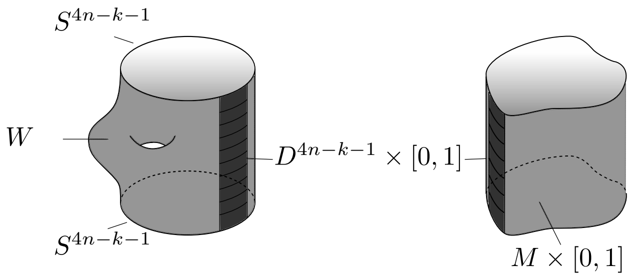

FIGURE 2.

We use this embedding to form the fiberwise connected sum of $E \to S^k$ with the trivial bundle $S^k \times (M \times [0,1]) \to S^k$ (the interval $[0,1]$ being embedded vertically in each fiber of E) to obtain a new fiber bundle

$$W_M \hookrightarrow E_M \to S^k$$

where each fiber W_M is a bordism from M to M. Figure 2 illustrates how a typical fiber of this bundle emerges from the connected sum of W and $M \times [0,1]$.

To keep our notation short we drop the index M from now on and call this new bundle $W \to E \to S^k$ again.

If $l = 2$ and the fiber dimension satisfies $4n - k \geq 2k + 5$, we can apply Theorem 3.7 so as to extend the constant fiberwise positive scalar curvature metric g_0 on $\partial_0 E$ to a fiberwise positive scalar curvature metric on E which is fiberwise of product form near $\partial_1 E$.

In view of the fact that $\partial_1 W$ is identified with the trivial bundle $S^k \times M \to S^k$ we obtain a new family

$$\phi\colon S^k \to \mathrm{Riem}^+(M)$$

of positive scalar curvature metrics on M.

Unfortunately this need not be in the path component of g_0 so that we modify our construction as follows.

Let $F\colon E \to [0,1]$ be the generic family of generalized Morse functions in normal form that was used for the construction of the fiberwise metric of positive scalar curvature on E. Then the image of the set of birth-death singularities

$$\pi(A_2(F)) \subset S^k$$

is an immersed submanifold of dimension $k-1$ and hence not equal to S^k. Let $t \in S^k$ be a point not lying in this image. The singularities of the restriction $f_t\colon E_t \to [0,1]$ are only of Morse type. In view of Addendum 3.4 we can assume that they have not only coindex at least 3 but also index at least 3. We set $W = E_t$, consider the constant family of Morse functions

$$F^{const}\colon S^k \times W \to [1,2]\,, \qquad (c,w) \mapsto 2 - f_t(w)$$

and the resulting family of Morse functions

$$F \cup F^{const}\colon E \cup_{\partial_1 E = S^k \times \partial_1 W} (S^k \times W) \to [0,2]$$

on the bundle E together with an upside down copy of the trivial bundle $S^k \times W \to S^k$. Using [23, Theorem 1.4], we can extend the family of positive scalar curvature metrics on E to the new fiber bundle

$$E' = E \cup_{\partial_1 E = S^k \times \partial_1 W} (S^k \times W) \to S^k\,.$$

Note that the restriction of the Morse function $F \cup F^{const}$ to the fiber E'_t over t is the smooth function

$$W \cup_{\partial_1 W = \partial_1 W} W \to [0,2]$$

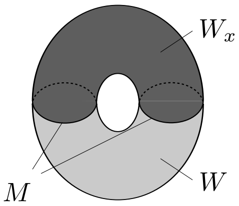

FIGURE 3.

given by f_t on the first and by $2 - f_t$ on the second copy of W. It therefore induces a handle decomposition in which each handle of index i in the first copy of W corresponds to a handle of coindex i in the second copy (in a canonical way). By the results of Walsh the positive scalar curvature metric obtained on E'_t by the method from [23], which on the single fiber E'_t restricts to the classical construction from [10], can be assumed to coincide on the two copies of W (glued together at $\partial_1 W$). In particular we can assume that the resulting metric on E'_t restricts to g_0 on both ends.

This means that the family of positive scalar curvature metrics

$$\phi' \colon S^k \to \mathrm{Riem}^+(M)$$

on the upper boundary component $\partial_1 E'$ is in the path component of g_0.

Proposition 4.1. *For this family of metrics ϕ' we get*

$$\hat{A}_\pi(\phi') = \hat{A}(P) \neq 0,$$

where P is the total space of the bundle from Theorem 1.4.

Proof. By definition

$$\hat{A}_\pi(\phi') = \mathrm{ind}(D_g),$$

where g is a metric on $(S^k \times [0,1]) \times M$ interpolating between the constant family (g_0) on $(S^k \times 0) \times M$ and the family ϕ on $(S^k \times 1) \times M$, both families being scaled by an adiabatic constant with respect to some metric on S^k.

We can isometrically glue $(S^k \times [0,1]) \times M$ along top and bottom to E' to obtain a new spin manifold P'. Because E' carries a metric of positive scalar curvature, the gluing formula for the APS index yields

$$\mathrm{ind}(D_g) = \hat{A}(P').$$

Notice that the underlying smooth manifold P' is just obtained from E' by identifying the two boundary components. A fiber P'_x over $x \in S^k$ is depicted in Figure 3.

Let us now write $P' = P'_M$ to denote the dependency on M. By construction P'_M is obtained as the fibered connected sum of $P'_{S^{4n-k-1}}$ and the trivial bundle $S^k \times M \times S^1$, which has trivial $\hat{A}$-genus. By bordism invariance we conclude that $\hat{A}(P'_M)$ is independent of M, so we may assume that $M = S^{4n-k-1}$, i. e. that no connected sum construction has been performed.

In this case we carry out fiberwise two coindex-zero surgeries on the two copies of $M = S^{4n-k-1}$ inside the fibers of $P'_{S^{4n-k-1}}$ as shown in Figure 3. We get a bordism from $P'_{S^{4n-k-1}}$ to the disjoint union of the manifolds P and $S^k \times F$, where F is the fiber of the bundle $F \to P \to S^k$ we started with. The claim follows since the manifold $S^k \times F$ has again vanishing $\hat{A}$-genus. □

Because $\hat{A}_\pi$ is a homomorphism we can conclude that $\pi_k(\mathrm{Riem}^+(M), g_0)$ contains elements of infinite order if $k > 0$. For $k = 0$ we use the fact that $\hat{A}(P)$ can assume infinitely many integer values for different bundles $F \to P \to S^0$ so that $\pi_0(\mathrm{Riem}^+(M))$ is infinite.

Now consider the chain of group homomorphisms (for $k = 0$ the first map is just a map of sets)

$$\pi_k(\mathrm{Riem}^+(M), g_0) \to \Omega_k^{fr}(\mathrm{Riem}^+(M)) \to \Omega_k^{Spin}(\mathrm{Riem}^+(M)) \to H_k(\mathrm{Riem}^+(M))$$

where the superscript "fr" stands for framed bordism. Because the invariant $\hat{A}$ is integer-valued and defined on Spin bordism, it follows that the images of our non-trivial homotopy classes remain non-zero in rational framed bordism. The composed map between rational framed bordism and rational singular homology being an isomorphism we conclude the proof of part a) of Theorem 1.1.

Together with Corollary 2.3 we also obtain Theorem 1.11. Part a) of Theorem 1.12 follows with Corollaries 2.4 and 2.5 and part b) of Theorem 1.12 is a consequence of the following result.

Proposition 4.2. *Let M be a simply connected spin manifold which is a* strongly $\hat{A}$-*multiplicative fiber in degree k and admits a metric of positive scalar curvature g_0. Then the invariant $\hat{A}_\Omega$ factors through the image of the canonical map*

$$\Omega_k^{\mathrm{fr}}(\mathrm{Riem}^+(M)) \to \Omega_k^{\mathrm{fr}}(\mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M)).$$

Proof. Precomposing the canonical map from framed to spin bordism yields a map

$$\hat{A}_{\mathrm{fr}} \colon \Omega_k^{\mathrm{fr}}(\mathrm{Riem}^+(M)) \to \mathbb{Z}.$$

We note that each diffeomorphism in $\operatorname{Diff}_{x_0}(M)$ canonically lifts to a spin diffeomorphism, because M is simply connected by assumption and the differential is the identity at x_0.

Let B be a closed stably parallelizable manifold of dimension k, let $\phi\colon B \to \mathrm{Riem}^+(M)$ be a continuous map and assume that there is a compact stably parallelizable manifold Y with boundary B together with a map $\Phi\colon Y \to \mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M)$ so that

$$\Phi|_{\partial Y} = \pi \circ \phi$$

where $\pi\colon \mathrm{Riem}^+(M) \to \mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M)$ is the canonical projection. We need to show that $\hat{A}_{\mathrm{fr}}(\phi) = 0$.

On the one hand, because $\mathrm{Riem}^+(M) \to \mathrm{Riem}^+(M)/\operatorname{Diff}_{x_0}(M)$ is a fiber bundle projection, the map Φ gives rise to a (non-trivial) smooth bundle $E \to Y$ with fiber M and structure group $\operatorname{Diff}_{x_0}(M)$ equipped with a fiberwise metric of positive scalar curvature. As the map Φ restricts to $\pi \circ \phi$ on the boundary, the bundle $E|_{\partial Y}$ admits a trivialization $E|_{\partial Y} \cong S^k \times M$ such that the family of metrics coincides with the one given by ϕ.

On the other hand we consider the trivial bundle $Y \times M \to Y$ equipped with the constant fiberwise metric g_0 of positive scalar curvature.

After choosing a Riemannian metric on Y, a horizontal distribution on the bundle $E \to Y$ and scaling the fiberwise metrics on $E \to Y$ and $Y \times M \to Y$ by an adiabatic constant, we get positive scalar curvature metrics g_E and $g_{Y\times M}$ on the total spaces E and $Y \times M$. Both of these total spaces admit canonical spin structures, so that the APS indices of E and $Y \times M$ equipped with these metrics vanish.

Choose a metric g on $(B \times [0,1]) \times M$ inducing the restriction of $g_{Y\times M}$ on $(B\times 0)\times M$ and the restriction of g_E on $(B \times 1) \times M$.

From this we obtain a fiber bundle with fiber M, total space

$$X = Y \times M \bigcup_{\partial Y\times M = B\times 0\times M} B \times [0,1] \times M \bigcup_{B\times 1\times M = \partial E} E$$

and base

$$Y \bigcup_{\partial Y = B\times 0} B \times [0,1] \bigcup_{B\times 1 = \partial Y} Y.$$

We have $\hat{A}(X) = 0$, because M is assumed to be a strongly $\hat{A}$-multiplicative fiber in degree k and the base manifold of this bundle is parallelizable. Hence

$$\hat{A}_{\mathrm{fr}}(\phi) = \mathrm{ind}(D_{g_{Y\times M}}) + \mathrm{ind}(D_g) + \mathrm{ind}(D_{g_E}) = \hat{A}(X) = 0\,.$$

as required. □

5. Proof of Theorem 1.4

Let us first assume $k \geq 1$, the case $k = 0$ being postponed to the end of the proof. We construct P is in several steps. Let $\alpha \geq k/4 + 2$ be a natural number, let either $\beta = \alpha + 1$ or $\beta = \alpha + 2$ and $n = \alpha + \beta$. We consider the trivial fibration

$$\phi_0\colon P_0 = S^k \times S^{4\alpha-k} \times S^{4\beta} \to S^k$$

with (path connected) total space of dimension $4n$. By choosing α appropriately we can assume in addition that the fiber $S^{4\alpha-k} \times S^{4\beta}$ of ϕ_0 is l-connected.

We first apply the following result of surgery theory, in which $\tau\colon P_0 \to B\pi_1(P_0)$ denotes the classifying map of the universal covering.

Theorem 5.1 ([7, Theorem 6.5])**.** *Let $\mathcal{L} \in \bigoplus_{j>1} H^{4j}(P_0;\mathbb{Q})$ be a class such that*

$$\tau_*(\mathcal{L} \cap [P_0]) = 0 \in H_{4n-4*}(B\pi_1(P_0);\mathbb{Q}).$$

Then there is some non-zero integer R and a homotopy equivalence $f\colon P_1 \to P_0$ of closed smooth manifolds such that the L-polynomials of P_0 and P_1 satisfy the equation

$$L(P_1) = f^*\big(L(P_0) + R\mathcal{L}\big).$$

Sketch of proof. As we need a slight modification of the argument later, we sketch the proof. For more details, proofs or references concerning some of the statements below we refer the reader to [7].

At first we identify the set of topological normal invariants on P_0 with $[P_0, G/\mathrm{Top}]$, which we equip with the Abelian group structure so that the surgery obstruction is a group homomorphism. There is an isomorphism

$$\ell\colon [P_0, G/\mathrm{Top}] \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{j>0} H^{4j}(P_0,\mathbb{Q})$$

sending the class of a degree one normal map $f\colon (P_1, TP_1) \to (P_0, \xi)$ to the class $\ell(f) = \frac{1}{8}(L(\xi)L(P_0)^{-1} - 1)$. Thus there exists a non-zero integer R_1 and a normal invariant $f \in [P_0, G/\mathrm{Top}]$ such that $\ell(f) = R_1\mathcal{L}L(P_0)^{-1}$. We note that for this normal invariant the desired equation of L-polynomials for $L(P_1) = f^*L(\xi)$ holds with $8R_1$ instead of R.

We need to compute the surgery obstruction for this normal invariant f. The surgery obstruction

$$[P_0, G/\mathrm{Top}] \otimes \mathbb{Q} \to L_{4n}(\mathbb{Z}[\pi_1(P_0)]) \otimes \mathbb{Q}$$

factors along

$$[P_0, G/\mathrm{Top}] \otimes \mathbb{Q} \xrightarrow[\ell]{\cong} \bigoplus_{k>0} H^{4k}(P_0;\mathbb{Q}) \xrightarrow{(-\cup L(P_0))\cap[P_0]}$$

$$H_{4n-4*}(P_0;\mathbb{Q}) \xrightarrow{\tau_*} H_{4n-4*}(B\pi_1(P_0);\mathbb{Q}).$$

Hence by assumption the surgery obstruction of f is zero in $L_{4n}(\mathbb{Z}[\pi_1(P_0)]) \otimes \mathbb{Q}$. This implies that there is a non-zero integer R_2 and a normal invariant $f \in [P_0, G/\mathrm{Top}]$ so that $\ell(f) = R_2R_1\mathcal{L}L(P_0)^{-1}$ and so that the surgery obstruction of this f vanishes. Because $\pi_1(P_0)$ is either trivial or equal to $\mathbb{Z}$ in our case, we have $L_{4n}(\mathbb{Z}[\pi_1(P_0)]) \cong \mathbb{Z}$ by the Bass-Heller-Swan splitting theorem, so that we can in fact choose $R_2 = 1$.

Finally it follows from [24] that for each normal invariant $f \in [P_0, G/\mathrm{Top}]$ some multiple of f lies in the image of the canonical map $[P_0, G/O] \to [P_0, G/\mathrm{Top}]$. Hence we find a non-zero multiple R_3 of R_2R_1 and a smooth normal invariant $f \in [P_0, G/O]$

which satisfies $\ell(f) = R_3 \mathcal{L} L(P_0)^{-1}$ and whose surgery obstruction vanishes. Performing surgery along this normal invariant yields a homotopy equivalence $f : P_1 \to P_0$ so that the stated equation holds with $R = 8R_3$. □

We apply this result to the following situation: Let

$$e_k \in H^k(S^k;\mathbb{Z}), \quad e_{4\alpha-k} \in H^{4\alpha-k}(S^{4\alpha-k};\mathbb{Z}), \quad e_{4\beta} \in H^{4\beta}(S^{4\beta};\mathbb{Z})$$

be generators. As P_0 is stably parallelizable, $L(P_0) = 1$. Moreover, $\pi_1(P_0)$ is trivial if $k \geq 2$ or infinite cyclic if $k = 1$. We conclude from Theorem 5.1 that we can find $R \neq 0$ and a homotopy equivalence

$$f\colon P_1 \to P_0$$

of smooth closed manifolds so that all homogeneous components of the Hirzebruch L-class of P_1 vanish except

$$L_\alpha = R \cdot (e_k \times e_{4\alpha-k}), \quad L_\beta = R \cdot e_{4\beta}.$$

Here we use the identification

$$f^*\colon H^*(S^k \times S^{4\alpha-k} \times S^{4\beta};\mathbb{Q}) \cong H^*(P_1;\mathbb{Q}).$$

We will show that a manifold with the above properties automatically has non-zero $\hat{A}$-genus. For $j \geq 1$ we denote by $\lambda_j^L \in \mathbb{Q}$ the coefficient of the degree $4j$ Pontryagin class p_j in $L_j(p_1, ..., p_j)$, and by λ_j^A the corresponding coefficient of the $\hat{A}$-polynomial $\hat{A}_j(p_1, \ldots, p_j)$.

Lemma 5.2. *We have*

$$\lambda_j^L = \frac{2^{2j}(2^{2j-1}-1)}{(2j)!} \cdot B_j,$$
$$\lambda_j^A = -\frac{1}{2(2j)!} \cdot B_j$$

where B_j denotes the j-th Bernoulli number, defined by the relation (cf. [19, p. 281]*)*

$$\frac{x}{\tanh x} = 1 + \sum_{j=1}^{\infty} (-1)^{j-1} \frac{B_j}{(2j)!}(2x)^{2j}.$$

Proof. According to [19, Problem 19-C] we have equations

$$L(t)\frac{d(t/L(t))}{dt} = 1 + \sum_{j=1}^{\infty}(-1)^j \lambda_j^L \cdot t^j$$
$$\hat{A}(t)\frac{d(t/\hat{A}(t))}{dt} = 1 + \sum_{j=1}^{\infty}(-1)^j \lambda_j^{\hat{A}} \cdot t^j$$

where

$$L(t) = \frac{\sqrt{t}}{\tanh(\sqrt{t})}, \qquad \hat{A}(t) = \frac{\frac{1}{2}\sqrt{t}}{\sinh(\frac{1}{2}\sqrt{t})}$$

are the formal power series for the multiplicative sequences $\{L_n\}$ and $\{\hat{A}_n\}$.

For the $\hat{A}$-polynomial we compute

$$\hat{A}(t)\frac{d(t/\hat{A}(t))}{dt} = \frac{1}{2} + \frac{1}{2}\frac{\frac{1}{2}\sqrt{t}}{\tanh(\frac{1}{2}\sqrt{t})}.$$

This gives (setting $x = \frac{1}{2}\sqrt{t}$ in the relation defining the Bernoulli numbers)

$$\hat{A}(t)\frac{d(t/\hat{A}(t))}{dt} = 1 + \frac{1}{2} \cdot \sum_{j=1}^{\infty}(-1)^{j-1}\frac{B_j}{(2j)!}t^j.$$

This implies the second equation. The first equation appears explicitly in [19, Problem 19-C] or can be obtained by similar methods. □

This lemma implies that all λ_j^L and λ_j^A are different from zero. By our choice of $\beta = \alpha+1$ or $\beta = \alpha+2$, it follows recursively that all the Pontryagin classes of P_1 vanish except possibly p_α, p_β, and p_n, and that p_α and p_β as well as $p_\alpha \cdot p_\beta$ are non-zero.

Let us write the degree $4n$-components of the L- and the $\hat{A}$-polynomial of an arbitrary vector bundle with vanishing p_j for $j \neq \alpha, \beta, n$ as

$$\begin{aligned} L_n(p_\alpha, p_\beta, p_n) &= \lambda_n^L \cdot p_n + \mu^L \cdot p_\alpha p_\beta \\ \hat{A}_n(p_\alpha, p_\beta, p_n) &= \lambda_n^A \cdot p_n + \mu^A \cdot p_\alpha p_\beta \,. \end{aligned}$$

with rational numbers μ^L, μ^A.

Proposition 5.3. *We have*

$$\frac{\mu^L}{\lambda_n^L} \neq \frac{\mu^A}{\lambda_n^A}$$

and hence the following implication for a $4n$-dimensional connected oriented manifold M with $p_j(M) = 0$ for $j \neq \alpha, \beta, n$:

$$\text{If} \quad p_\alpha(M)\cdot p_\beta(M) \neq 0 \quad \textit{and} \quad L(M) = 0, \quad \text{then} \quad \hat{A}(M) \neq 0\,.$$

Proof. This is a calculation in universal characteristic classes. To keep the notation transparent, assume that E_α, E_β are vector bundles with total Pontryagin classes $1 + p_\alpha$ and $1 + p_\beta$, in particular $L_n(E_\alpha) = 0 = L_n(E_\beta)$. By the multiplicativity of the total Pontryagin class and the universal L- and $\hat{A}$-polynomials we obtain

$$p_n(E_\alpha \oplus E_\beta) = p_\alpha \cdot p_\beta, \quad p_\alpha(E_\alpha \oplus E_\beta) = p_\alpha, \quad p_\beta(E_\alpha \oplus E_\beta) = p_\beta.$$

This implies

$$\begin{aligned} (\lambda_n^L + \mu^L)\cdot(p_\alpha \cdot p_\beta) &= L_n(E_\alpha \oplus E_\beta) \\ = L_n(E_\alpha) + L_n(E_\beta) + L_\alpha(E_\alpha)\cdot L_\beta(E_\beta) &= \lambda_\alpha^L \cdot \lambda_\beta^l \cdot (p_\alpha \cdot p_\beta), \end{aligned}$$

and hence

$$\lambda_n^L + \mu^L = \lambda_\alpha^L \lambda_\beta^L.$$

An analogous computation shows

$$\lambda_n^A + \mu^A = \lambda_\alpha^A \lambda_\beta^A,$$

so that altogether we obtain

$$1 + \frac{\mu^L}{\lambda_n^L} = \frac{\lambda_\alpha^L\lambda_\beta^L}{\lambda_n^L}, \quad 1 + \frac{\mu^A}{\lambda_n^A} = \frac{\lambda_\alpha^A\lambda_\beta^A}{\lambda_n^A}.$$

From Lemma 5.2 it follows that

$$1+\frac{\mu^L}{\lambda_n^L} = -\frac{2(2^{2\alpha-1}-1)(2^{2\beta-1}-1)}{2^{2n-1}-1}\frac{\lambda_\alpha^A\lambda_\beta^A}{\lambda_n^A} = -\frac{2(2^{2\alpha-1}-1)(2^{2\beta-1}-1)}{2^{2n-1}-1}\left(\frac{\mu^A}{\lambda_n^A}+1\right).$$

Because $-\frac{2(2^{2\alpha-1}-1)(2^{2\beta-1}-1)}{2^{2n-1}-1} \neq 1$ the conclusion follows. □

If $k = 1$ then the map

$$\phi_0 \circ f \colon P_1 \to S^1$$

is homotopic to the projection map of a smooth fiber bundle. This follows from Farrell's obstruction theory over the circle [8, Theorem 6.4]. Note that in the case at hand $\phi_0 \circ f$ induces an isomorphism of fundamental groups, so that the kernel of the π_1-homomorphism is the trivial group. By [8, Remarks on p. 316], the fibering obstructions vanish and one only has to check that the universal covering of P_1 is homotopy equivalent to a finite CW-complex. But this is homotopy equivalent to the universal covering of P_0 and therefore has this property.

If $k \geq 2$ the map $\phi_0 \circ f$ will usually not be homotopic to a fiber bundle projection and a more complicated construction is needed. The theory which is relevant for the following discussion was developed by Casson [4] and Hatcher [12].

Recall that for any continuous map $f\colon X \to Y$ of topological spaces, the *homotopy fiber* L of f is the fiber of the map

$$\begin{array}{rcl} E_f & \to & Y \\ (x,\gamma) & \mapsto & \gamma(1) \end{array}$$

where

$$E_f = \{(x,\gamma) \mid x \in X, \gamma\colon [0,1] \to Y, \gamma(0) = f(x)\}.$$

Let $b \in S^m$ be the north pole, viewed as base point in S^m. Let V be a closed smooth manifold V equipped with a smooth map $f\colon V \to S^m$ such that the homotopy fiber is simply-connected. Casson [4, Section 1] defines such a map to be a *pre-fibration* if the point $b \in S^m$ is a regular value, $V \setminus f^{-1}(b)$ is simply-connected and the canonical inclusion

$$f^{-1}(b) \to L\,,\ v \mapsto (v, \gamma\colon t \mapsto f(v))$$

of the point-preimage into the homotopy fiber of f is a weak homotopy equivalence. In this case the smooth manifold $F = f^{-1}(b)$ is called the *fiber* of (V, f).

Proposition 5.4. *Let $R \neq 0$ be the number appearing in the construction of $f : P_1 \to P_0$ after the proof of Theorem 5.1. If we construct the homotopy equivalence $f : P_1 \to P_0$ using the number $2R$ instead of R, then the map $\phi_0 \circ f\colon P_1 \to S^k$ constructed above is homotopic to a pre-fibration $\phi_1\colon P_1 \to S^k$.*

Proof. After applying a homotopy to $\phi_0 \circ f$ we obtain a map g for which the value $b \in S^{4k}$ is regular with some fiber F. By [4, Lemma 4] we can assume that F and $P_1 \setminus F$ are simply connected. By [4, Lemma 2 and the proof of Theorem 1] the inclusion from F into the homotopy fiber of g is a degree one normal map. From [4, Theorem 1 and page 497] we know that the obstruction for g to be homotopic to a pre-fibration is given by the (simply-connected) surgery obstruction of this degree one normal map. The surgery obstruction groups in odd dimensions being trivial, we have to distinguish between the cases where k is divisible by 4 and where k is congruent 2 modulo 4.

In the case where k is divisible by 4, the surgery obstruction is an integer which (up to a multiple) is given by the difference of signatures. The homotopy fiber L' of g is homotopy equivalent to $S^{4\alpha-k} \times S^{4\beta}$, so that its signature is zero. On the other hand, by Hirzebruch's signature formula we obtain the signature of F (which is framed in P_1) by

$$\sigma(F) = \langle L(P_1), [F] \rangle\,,$$

which is zero since by our choices of α and β there is no L-class in the relevant degree.

In the case where k is congruent 2 modulo 4, the surgery obstruction group is cyclic of order 2. In the notation of the proof of Theorem 5.1, this surgery obstruction is given by the composite group homomorphism

$$[P_0, G/\operatorname{Top}] \to [S^{4\alpha-k} \times S^{4\beta}, G/\operatorname{Top}] \to L_{4n-k}(\mathbb{Z}) \cong \mathbb{Z}/2$$

where the first map is given by restriction. It follows that this obstruction becomes zero if the number R appearing in Theorem 5.1 is multiplied by 2. □

In general the map ϕ_1 is not homotopic to an actual (smooth) fiber bundle: By the theory developed in [4] there is an obstruction lying in a homotopy group of a certain concordance space. Our goal is to show that (P_1, ϕ_1) can be replaced by some (P_2, ϕ_2) which fibers over S^k.

Casson [4, Section 4] calls two pre-fibrations $f\colon V \to S^m$ and $f'\colon V' \to S^m$ *equivalent* if there is a diffeomorphism $V \to V'$ compatible with f and f' in a neighborhood of $f^{-1}(b)$. This implies that the fibers of f and f' are diffeomorphic. Let F denote this common fiber. Following Casson, equivalence classes of pre-fibrations are classified as follows: Let $\tilde{A}(F)$ be the simplicial group of block diffeomorphisms of F. Recall from [12, p. 5] that the k-simplices in $\tilde{A}(F)$ are given by diffeomorphisms of $F \times \Delta^k$ restricting to diffeomorphisms of $F \times \tau$ for each face of Δ^k. An element in $\pi_k(\tilde{A}(F), \mathrm{id})$ is represented by a self-diffeomorphism of $F \times D^k$ which is the identity in a neighborhood of $F \times S^{k-1}$, and two elements agree if and only if they are concordant by a concordance which keeps a neighborhood of $F \times S^{k-1}$ fixed. Thinking of D^k as the northern hemisphere $D^k_+ \subset S^k$, such an f extends by the identity map to a self-diffeomorphism f' of $F \times S^k$. We may use f' to glue two copies of $F \times D^{k+1}$ along their common boundaries to obtain a closed manifold V which comes with a canonical projection to S^{k+1}, collapsing $F \times D^k_+$ to a point, which is easily seen to be a pre-fibration which we call $V(x)$.

In the following we will suppress the obvious base point id in the notation of homotopy groups.

Proposition 5.5. *Let F be simply connected and of dimension at least 6, and let $k \geq 2$. Then the rule $x \mapsto V(x)$ defines a one-to-one correspondence between $\pi_{k-1}(\tilde{A}(F))$ and equivalence classes of pre-fibrations over S^k with fiber F.*

Proof. This is [4, Lemma 6], once one has identified $\pi_k(\tilde{A}(F))$ with what Casson calls $D_k(F)$. The latter group is given by the diffeomorphisms of $F \times S^k$ keeping a neighborhood of $F \times D^k_-$ pointwise fixed, modulo concordance keeping $F \times D^k_-$ pointwise fixed. (Here D^k_- is the lower hemisphere.) There is a canonical map $D_k(F) \to \pi_k(\tilde{A}(F))$ given by restriction of a diffeomorphism to $F \times D^k_+$. Its inverse is given by extending a diffeomorphism of $F \times D^k_+$ by the identity. □

For a pre-fibration $f\colon V \to S^k$ with fiber F, which we assume to be simply connected and of dimension at least 6, we denote by $h(V, f) \in \pi_{k-1}(\tilde{A}(F))$ the corresponding element. We call this the *characteristic element* of the pre-fibration. We now formulate a condition when a pre-fibration is equivalent to a fiber bundle.

Let $A(F)$ be the simplicial group of diffeomorphisms of F, where the k-simplices are given by diffeomorphisms of $F \times \Delta^k$ which are compatible with the projection to Δ^k. Note that $A(F)$ is a simplicial subgroup of $\tilde{A}(F)$, which is in general not normal.

If $x \in \pi_k(\tilde{A}(F))$ lies in the image of $\pi_k(A(F))$, then $V(x)$ is obtained (up to equivalence) by gluing two copies of $V \times D^k$ along a diffeomorphism $V \times S^{k-1}$ which acts fiberwise over S^{k-1}. It follows that the projection $V(x) \to S^k$ is a smooth fiber bundle. Using the exact sequence

$$\pi_k(A(F)) \to \pi_k(\tilde{A}(F)) \xrightarrow{\psi} \pi_k(\tilde{A}(F), A(F))$$

we obtain:

Proposition 5.6. *Let $f\colon V \to S^k$ be a pre-fibration with simply connected fiber F of dimension at least 6 with characteristic element $h(V, f) \in \pi_{k-1}(\tilde{A}(F))$. If $\psi(h) = 0 \in \pi_{k-1}(\tilde{A}(F), A(F))$, then $f\colon V \to S^k$ is (as a pre-fibration) equivalent to a smooth fiber bundle.*

For a closed smooth manifold K we consider the pre-fibration

$$\phi_1 \times \mathrm{id}\colon P_1 \times K \to S^k \qquad (p, x) \mapsto \phi_1(p)$$

with fiber $F \times K$. Then $\psi(h(\phi_1 \times \mathrm{id}))$ is the image of $\psi(h(\phi_1))$ under the map of homotopy groups induced by the map

$$(\widetilde{A}(F), A(F)) \to (\widetilde{A}(F \times K), A(F \times K))$$

which sends a diffeomorphism $\omega\colon F \times \Delta^k \to F \times \Delta^k$ to $\omega \times \mathrm{id}\colon F \times \Delta^k \times K \to F \times \Delta^k \times K$.

Proposition 5.7. *For a closed smooth manifold K with vanishing Euler characteristic $\chi(K)$ and $r > 0$, the induced map*

$$\pi_r(\tilde{A}(F), A(F)) \to \pi_r(\tilde{A}(F \times K^r), A(F \times K^r))$$

is equal to 0. *Here K^r denotes the r-fold Cartesian product of K with itself.*

Proof. We consider the Hatcher spectral sequence [12, Proposition 2.1]

$$E^1_{pq} = \pi_q(C(F \times I^p)) \Longrightarrow \pi_{p+q+1}(\widetilde{A}(F), A(F))\,.$$

By [12, Proposition in Appendix I on p. 18], multiplication with K induces the zero map

$$\pi_j(C(F \times I^p)) \to \pi_j(C(F \times I^p \times K))$$

for each j, because $\chi(K) = 0$. It follows that in the filtration of $\pi_r(\tilde{A}(F), A(F))$ induced by the E^∞-term of the Hatcher spectral sequence the map induced by the product with K reduces the filtration degree of each element by one. This implies the assertion of Proposition 5.7. □

Thus let K be a closed $\max(l, 1)$-connected smooth manifold with $\hat{A}(K) \neq 0$ and $\chi(K) = 0$. (For instance, we may use the construction at the beginning of this section for appropriate values of k, α and β). Applying Lemma 5.7 to our previous construction we conclude that the pre-fibration

$$\phi_1 \times \mathrm{id}\colon P_1 \times K^{k-1} \to S^k$$

is equivalent to a smooth fiber bundle $\phi_2\colon P_2 \to S^k$. Because $\hat{A}(K) \neq 0$ we still have $\hat{A}(P_2) \neq 0$.

It remains to prove that we can assume that we have a smooth section $s\colon S^k \to P_2$ with trivial normal bundle. First notice that by choosing α and l large enough, there is a smooth section $s\colon S^k \to P_2$, of which we would like to show that its normal bundle is trivial.

Again the case $k = 1$ is the simplest one. In fact a real bundle over the circle is trivial if and only if its first Stiefel-Whitney class vanishes, and the first Stiefel-Whitney class of the normal bundle agrees with $w_1(P_0) = 0$.

In the case where $k \geq 2$ we argue as follows. Since we are in the stable range, the normal bundle is classified by an element in $\pi_k(BO)$. If $k \equiv 3, 5, 6, 7$ modulo 8, the group $\pi_k(BO)$ is zero, so that the normal bundle is automatically trivial. In the cases where $k = 4l$ is divisible by 4, we have $\pi_k BO = \mathbb{Z}$ and non-trivial bundles over S^k may be detected by their l-th rational Pontryagin classes. The l-th rational Pontryagin class of the normal bundle is equal to $s^*(p_l(P_2)) \in H^{4l}(S^{4l}; \mathbb{Q})$. But by the above construction of P_2 this class is equal to 0, so that the normal bundle is trivial in this case, too.

The remaining cases are $k \equiv 1$ or 2 modulo 8, in which case $\pi_k(BO) = \mathbb{Z}/2$. We do not see a general reason why the normal bundle should be trivial in this case, but we describe a procedure how to change the pre-fibration $\phi_1\colon P_1 \to S^k$ so that the normal bundle of the embedded S^k in P_1 becomes trivial. Since P_2 is diffeomorphic to $P_1 \times K^{k-1}$, this will imply that the normal bundle of the embedding $S^k \to P_2$ is also trivial.

Recall that by Casson's classification result, if $\dim F \geq 6$ and F is simply conneced any pre-fibration over S^k with fiber F is equivalent to one of the form $V(x)$, with $x \in \pi_{k-1}(\tilde{A}(F))$. If F is k-connected and at least $(k+2)$-dimensional, then there is an embedding $S^k \to V(x)$ such that the composite map $S^k \to V(x) \to S^k$ is homotopic to the identity, and any two such embeddings are isotopic. We call such an embedding simply the embedding of S^k into $V(x)$ and denote by $\nu_x\colon S^k \to BO$ the classifying map of its normal bundle.

Lemma 5.8. *Suppose that F is k-connected and at least $(k+2)$-dimensional (where $k \geq 2$), and let $x, y \in \pi_{k-1}(\tilde{A}(F))$. Then the normal bundle of the embedding of S^k into*

$V(x+y)$ is classified by $\nu_x + \nu_y$. Moreover

$$\hat{A}(V(x+y)) = \hat{A}(V(x)) + \hat{A}(V(y)).$$

Proof. Let x and y be represented by automorphisms α and β of $F \times D^{k-1}$, fixing a neighborhood of $F \times S^{k-2}$, and denote by α' and β' the corresponding automorphisms of $F \times S^{k-1}$. Because the element $x+y$ is represented by the composition $\beta \circ \alpha$, the space $V(x+y)$ can be written as a "fibered connected sum"

$$V(x+y) \cong \left(F \times D^k_+\right) \cup_{\alpha'} \left(F \times S^{k-1} \times [0,1]\right) \cup_{\beta'} \left(F \times D^k_-\right).$$

We use this identification to describe the embedding of S^k into $V(x+y)$. To do that, let us first describe the embedding i_x of S^k into $V(x)$. Choose a base point $* \in F$. On the lower hemisphere D^k_- the embedding i_x is given by the inclusion $i_-\colon D^k_- \to D^k_- \times F$ at the base point. The composite

$$S^{k-1} \xrightarrow{i_-} S^{k-1} \times F \xrightarrow{\alpha'} S^k \times F$$

extends continuously to a map $D^k_+ \to D^k_+ \times F$ (coning off and using a null homotopy of the map $\mathrm{pr}_F \circ \alpha' \circ i_-\colon S^k \to F$). The extension may be approximated by a smooth embedding i_+ in such a way that i_+ and i_- together define the embedding of S^k into $V(x)$.

Similarly one obtains a choice of the embedding i_y of S^k into $V(y)$ which is the inclusion at the base point on the upper hemisphere. Denoting the embedding at the base point by $j\colon S^{k-1} \times [0,1] \to F \times S^{k-1} \times [0,1]$, it follows that

$$\begin{aligned} &i_x|_{D^k_+} \cup j \cup i_y|_{D^k_-}\colon \\ &D^k_+ \cup \left(S^{k-1} \times [0,1]\right) \cup D^k_- \to \left(F \times D^k_+\right) \cup_{\alpha'} \left(F \times S^{k-1} \times [0,1]\right) \cup_{\beta'} \left(F \times D^k_-\right) \end{aligned}$$

defines the embedding of S^k into $V(x+y)$.

The normal bundle of this embedding is given by $\nu_x|_{D^k_+}$ on D^k_+ and by $\nu_y|_{D^k_-}$ on D^k_-, while on $S^{k-1} \times [0,1]$ the normal bundle is trivialized. It follows that the classifying map ν_{x+y} factors as

$$S^k \to S^k \vee S^k \xrightarrow{\nu_x \vee \nu_y} BO$$

where the first map is the pinch map. But this composite defines the sum $\nu_x + \nu_y$ in the homotopy group $\pi_k(BO)$ so that the first statement follows.

Finally notice that $V(x+y)$ is obtained from $V(x) \amalg V(y)$ by cutting out two copies of $F \times D^k$ and gluing along the resulting $F \times S^{k-1}$ boundaries, defining a parametrized connected sum. Hence there is a standard bordism between these $V(x+y)$ and the disjoint union $V(x) \cup V(y)$. This implies the second statement. $\square$

Suppose now that in our situation, the normal bundle of S^k in $P_1 = V(x)$ is non-trivial. As $\pi_k(BO)$ has order 2, replacing P_1 by $P_1' = V(2x)$ yields a pre-fibration with the same fiber F, whose section has trivial normal bundle and such that $2\hat{A}(P_1) = \hat{A}(P_1') \neq 0$. This completes the proof of Theorem 1.4 in the case $k \geq 1$.

If $k = 0$ we can use the same construction as for case $k \geq 1$, but starting with $P_0 = S^{4\alpha} \times S^{4\beta}$. With Theorem 5.1, Lemma 5.2 and Proposition 5.3 we find an $N(0,l)$ so that for all $n \geq N(0,l)$ there is a $4n$-dimensional l-connected closed spin manifold F with $\hat{A}(F) \neq 0$. Now we define E as the disjoint union of two copies of the spin manifold F. The surjective map to S^0 is constant on each component.

REFERENCES

[1] Kazuo Akutagawa, Boris Botvinnik, *The relative Yamabe invariant*, Comm. Anal. Geom. **10** (2002), 925–954. 1.3

[2] Arthur Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete **10**, Springer 1.5, 2.1

[3] Boris Botvinnik, Bernhard Hanke, Thomas Schick, Mark Walsh, *Homotopy groups of the moduli space of metrics of positive scalar curvature*, Geom. Topol. **14** (2010), 2047–2076. 1, 1
[4] Andrew J. Casson, *Fibrations over spheres*, Topology **6** (1967), 489–499. 1, 5, 5, 5
[5] Shiing-Shen Chern, Friedrich Hirzebruch, Jean-Pierre Serre, *On the index of a fibred manifold*, Proc. Amer. Math. Soc. **8** (1957), 587–596. 1
[6] Diarmuid Crowley, Thomas Schick, *The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature.*, Geom. Topol. **17** (2013), 1773–1790. . 1, 1, 1, 1
[7] James F. Davis, *Manifold aspects of the Novikov Conjecture*, in: Surveys of Surgery Theory, Vol. 1, S. Cappell, A. Ranicki, J. Rosenberg (eds.), 195–224. 1, 5.1, 5
[8] Tom Farrell, *The obstruction to fibering a manifold over a circle*, Indiana Univ. Math. J. **21** (1971/1972), 315–346. 5
[9] Tom Farrell, Lowell Jones, *A topological analogue of Mostow's rigidity thorem* , J. Amer. Math. Soc. **2**, no. 2 (1989), 257–370. 1
[10] Mikhael Gromov, H. Blaine Lawson, Jr., *The Classification of Simply Connected Manifolds of Positive Scalar Curvature*, Ann. Math., Second Series, Vol. **111**, 423–434. 2, 4
[11] Mikhael Gromov, H. Blaine Lawson, Jr., *Positive Scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. IHES **58** (1983), 295–408. 1, 1, 1
[12] Allen Hatcher, *Concordance spaces, higher simple homotopy theory, and applications*, Algebraic and geometric topology, Stanford 1976. Proceedings of Symposia in Pure Mathematics, Vol 32, pp. 3–21 (1978). 1, 5, 5, 5
[13] Nigel Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55. 1, 1, 1, 1
[14] Kiyoshi Igusa, *The space of framed functions*, Trans. Amer. Math. Soc. **301** (1987), no. 2, 431–477. 3
[15] Kiyoshi Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), 1–355. 1, 3, 3.1, 3, 3, 3, 3, 3.5, 3
[16] Matthias Kreck, Stephan Stolz, *Nonconnected moduli spaces of positive sectional curvature metrics*, J. Amer. Math. Soc. **6** (1993), 825-850. 1
[17] H. Blaine Lawson, Jr. , Marie-Louise Michelsohn, *Spin geometry*, Princeton University Press 1989. 1
[18] Richard Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, Volume 4, A K Peters Wellesley, Massachusetts, 1993. 2
[19] John Milnor, James Stasheff, *Characteristic Classes*, Annals of Mathematics Studies no. 76, Princeton University Press 1974. 5.2, 5
[20] Thomas Schick, *A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture*, Topology **37** (1998), 1165–1168. 1
[21] Stephan Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. Math. **136** (1992), 511–540. 1.5
[22] Stephan Stolz, *Positive Scalar Curvature Metrics - Existence and Classification Questions*, Proc. Int. Cong. Math., Zürich 1994, Birkhäuser, pp. 625–636. 1, 3
[23] Mark Walsh, *Metrics of positive scalar curvature and generalized Morse functions, part II*, Trans. Am. Math. Soc. **366** (2014), 1–50. 1, 3, 3, 3, 3, 3, 4
[24] Shmuel Weinberger, *On smooth surgery*, Comm. Pure Appl. Math. **43** (1990), 695–696. 5

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