

# $\Gamma$ -STRUCTURES AND SYMMETRIC SPACES

BERNHARD HANKE AND PETER QUAST

ABSTRACT.  $\Gamma$ -structures are weak forms of multiplications on closed oriented manifolds. As shown by HOPF the rational cohomology algebras of manifolds admitting  $\Gamma$ -structures are free over odd degree generators. We prove that this condition is also sufficient for the existence of  $\Gamma$ -structures on manifolds which are nilpotent in the sense of homotopy theory. This includes homogeneous spaces with connected isotropy groups.

Passing to a more geometric perspective we show that on compact oriented Riemannian symmetric spaces with connected isotropy groups and free rational cohomology algebras the canonical products given by geodesic symmetries define  $\Gamma$ -structures. This extends work of ALBERS, FRAUENFELDER and SOLOMON on  $\Gamma$ -structures on Lagrangian Grassmannians.

## INTRODUCTION

In his seminal papers [Ho1, Ho2] on the (co)homological structure of Lie groups HOPF introduced the notion of  $\Gamma$ -*manifolds*. By definition these are closed connected oriented manifolds  $M$  together with continuous maps

$$\psi : M \times M \rightarrow M$$

so that the mapping degrees of the two restrictions  $\psi_p : y \mapsto \psi(p, y)$  and  $\psi^q : x \mapsto \psi(x, q)$  are non-zero for some (and hence for all)  $p, q \in M$ . In some sense such maps  $\psi$ , which we call  $\Gamma$ -*structures*, capture the simplest non-trivial homological information of Lie group multiplications. HOPF proved that the rational cohomology rings of  $\Gamma$ -manifolds are of a surprisingly restricted type: Because they admit compatible comultiplications (and are hence *Hopf algebras* in modern terminology) they are free graded commutative  $\mathbb{Q}$ -algebras, whose generators must be in odd degrees as  $M$  is finite dimensional, see [Ho1, Ho2] and [Di, Chap. VI, §2.A] for further information. As carried out by BOREL in [Bo, Chapitre II, § 7], further divisibility restrictions on the mapping degrees of  $\psi_p$  and  $\psi^q$  have similar implications for the cohomology rings over finite fields.

A closely related and much better known structure is that of an *H-space*, with both restrictions  $\psi_p$  and  $\psi^q$  being homotopic to the identity. While every compact connected oriented manifold that is an H-space is obviously a  $\Gamma$ -manifold, the converse fails: All odd dimensional unit spheres are  $\Gamma$ -manifolds (see [Ho2] and our Theorem 13 for a generalization), but ADAMS celebrated ‘Hopf invariant one theorem’ in [Ad] says that only spheres of dimension 1, 3 or 7 admit H-space structures.

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In the first part of our paper, Section 1, we shall make some general remarks on the existence of  $\Gamma$ -structures on manifolds satisfying the above cohomological condition. As the only non-trivial requirement for a  $\Gamma$ -structure  $\psi : M \times M \rightarrow M$  is the non-vanishing of the mapping degrees of  $\psi_p$  and  $\psi^q$ , a construction of such structures by obstruction theory requires the separation of rational and torsion information in the homotopy type of  $M$ . This is the underlying idea of *rational homotopy theory*, which works best for spaces whose Postnikov decompositions consist of principal fibrations and can hence be described by accessible cohomological invariants. Examples are *simple* spaces, whose fundamental groups are abelian and act trivially on higher homotopy groups, and, more generally, *nilpotent* spaces, whose fundamental groups are nilpotent and act nilpotently on higher homotopy groups.

In Section 1 we will prove the following converse of HOPF'S result. From now on the notion *free algebra* stands for *free graded commutative algebra over the rationals*.

**Theorem 1.** *Let  $M$  be a closed connected oriented manifold which is nilpotent as a topological space. If  $H^*(M; \mathbb{Q})$  is a free algebra - necessarily over odd degree generators - then  $M$  admits a  $\Gamma$ -structure.*

This is in sharp contrast to the existence of H-space structures, which is a much more restrictive property.

**Corollary 2.** *Let  $M = G/H$  be a compact connected homogeneous space where  $G$  is a Lie group and  $H < G$  is a closed connected subgroup. If  $H^*(M; \mathbb{Q})$  is a free algebra, then  $M$  admits a  $\Gamma$ -structure.*

Theorem 1 also implies (see Corollary 11) that nilpotent manifolds with free rational cohomology algebras have virtually abelian fundamental groups.

Our abstract existence results motivate the search for explicit geometric constructions of  $\Gamma$ -structures. Already HOPF [Ho2] used geodesic symmetries on odd dimensional spheres to write down  $\Gamma$ -structures. It is therefore natural to consider Riemannian symmetric spaces  $P$  endowed with their *canonical products* (see e.g. [Lo1, Chap. II, §1])

$$\Theta : P \times P \rightarrow P, \quad (x, y) \mapsto s_x(y), \quad (1)$$

where  $s_x$  denotes the *geodesic symmetry* of  $P$  at the point  $x \in P$ , that is the involutive isometry of  $P$  that fixes  $x$  and that reverses the direction of all geodesics emanating from  $x$ . In Section 2 we investigate in which cases the canonical product  $\Theta$  of a compact oriented symmetric space  $P$  is a  $\Gamma$ -structure. This leads to a proof of the following conclusive result.

**Theorem 3.** *Let  $P$  be a compact symmetric space and let  $G$  be the identity component of its isometry group. Assume that the isotropy subgroup  $K \subset G$  of a base point in  $P$  is connected. Then the following assertions are equivalent:*

- $H^*(P; \mathbb{Q})$  is a free algebra.
- The canonical product on  $P$  is a  $\Gamma$ -structure.

We remark that whenever the canonical product  $\Theta$  defines a  $\Gamma$ -structure on  $P$ , then the mapping degrees of  $\Theta_p$  and  $\Theta^q$  are (up to sign) powers of 2 so that by [Bo] additional

restrictions on the cohomology algebras over finite fields of characteristic different from two are implied ex post by properties of the rational cohomology algebras.

Unfortunately Theorem 3 does not cover all the manifolds from Corollary 2. For example it remains an open problem to provide a geometric construction of  $\Gamma$ -structures on complex and quaternionic Stiefel manifolds (see [MT, Thm. 3.10, p. 119]), which are not symmetric spaces.

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## 1. POSTNIKOV DECOMPOSITIONS AND $\Gamma$ -STRUCTURES

In this section we present a homotopy theoretic construction of  $\Gamma$ -structures. Recall that the homotopy type of a path connected CW complex  $X$  can be analysed by means of its *Postnikov decomposition*, see, for example, [Ha1, p. 410 ff.]: Let  $X_n$  be obtained by killing all homotopy groups of  $X$  above degree  $n$  by attaching cells of dimension at least  $n + 2$ . Up to homotopy equivalence we can assume that each inclusion  $p_{n+1} : X_{n+1} \rightarrow X_n$  is a fibration. The long exact homotopy sequence shows that the fibre is an Eilenberg–MacLane space  $K(\pi_{n+1}, n + 1)$  where  $\pi_{n+1} := \pi_{n+1}(X)$ . In particular,  $X_1$  is the classifying space  $B\pi_1(X)$ .

Recall that  $X$  is called *simple*, if  $\pi_1(X)$  is abelian and acts trivially on higher homotopy groups. For simple  $X$  each fibration  $p_{n+1} : X_{n+1} \rightarrow X_n$  is *principal*, see [Ha1, Theorem 4.69], that is the pull back of the path-loop fibration

$$K(\pi_{n+1}, n + 1) \rightarrow PK(\pi_{n+1}, n + 2) \rightarrow K(\pi_{n+1}, n + 2)$$

along a map  $X_n \rightarrow K(\pi_{n+1}, n + 2)$  which defines the  $n$ -th  $k$ -invariant  $k_n \in H^{n+2}(X_n; \pi_{n+1})$ . This class is equal to the image of the fundamental class in  $H^{n+1}(K(\pi_{n+1}, n + 1); \pi_{n+1})$  under the transgressive differential  $d_{n+2}$  in the Leray–Serre spectral sequence for the fibration  $p_{n+1}$ . We denote by  $(k_n)_{\mathbb{Q}} \in H^{n+2}(X_n; \pi_{n+1} \otimes \mathbb{Q})$  the image of  $k_n$  under the coefficient homomorphism  $\pi_{n+1} \rightarrow \pi_{n+1} \otimes \mathbb{Q}$ .

In the following we collect some well known facts on the cohomology of Eilenberg–MacLane spaces.

**Lemma 4.** *Let  $C$  be a cyclic group, and let  $m, n > 0$  be positive natural numbers. Then*

- $H^*(K(C, n); \mathbb{Z})$  is a finitely generated group in each degree.
- for  $C = \mathbb{Z}$  the cohomology algebra  $H^*(K(\mathbb{Z}, n); \mathbb{Q})$  is free with one generator in degree  $n$ .
- for  $|C| < \infty$  the reduced cohomology  $\tilde{H}^*(K(C, n); \mathbb{Q})$  is equal to 0.

Let  $\mu_m : C \rightarrow C$  be the multiplication by  $m$ . Then

- for  $C = \mathbb{Z}$  the induced map

$$\mu_m^* : H^*(K(\mathbb{Z}, n); \mathbb{Q}) \rightarrow H^*(K(\mathbb{Z}, n); \mathbb{Q})$$

is given by multiplication with  $m^k$  on  $H^{kn}(K(\mathbb{Z}, n); \mathbb{Q})$ .

- for all  $C$  the map

$$\mu_m^* : \tilde{H}^*(K(C, n); \mathbb{Z}/m) \rightarrow \tilde{H}^*(K(C, n); \mathbb{Z}/m)$$

is equal to 0.

*Proof.* Apart from the last assertion the lemma can be proven fairly directly by induction on  $n$ . For  $n = 1$  we have  $K(C, 1) = BC$ , the classifying space of  $C$ , so that the assertions are clear for  $C = \mathbb{Z}$  (as  $B\mathbb{Z} = S^1$ ). For  $|C| < \infty$  the classifying space  $BC$  is an infinite dimensional lens space, the cohomology  $\tilde{H}^*(BC; \mathbb{Z})$  is equal to  $\mathbb{Z}/|C|$  in even degrees and 0 in odd degrees, and  $\mu_m^* : H^*(BC; \mathbb{Z}) \rightarrow H^*(BC; \mathbb{Z})$  is given by multiplication with  $m^k$  on  $H^{2k}(BC; \mathbb{Z})$ . Together with the universal coefficient theorem this implies all the assertions of the lemma for  $n = 1$ .

For the induction step we recall that the cohomology with coefficients in some commutative ring  $R$  of base and fibre of the path loop fibration

$$K(C, n) \rightarrow PK(C, n+1) \rightarrow K(C, n+1)$$

appear on the two coordinate axes of the  $E_2$ -term of the Leray-Serre spectral sequence and that this spectral sequence is natural with respect to homomorphisms  $C \rightarrow C$ . The spectral sequence converges to  $H^*(PK(C, n+1); R)$ , which vanishes in positive degrees, because  $PK(C, n+1)$  is contractible.

By induction this shows that  $H^*(K(C, n+1); \mathbb{Z})$  is finitely generated in each degree, that  $\tilde{H}^*(K(C, n+1); \mathbb{Q}) = 0$  for  $|C| < \infty$ , and that  $H^*(K(\mathbb{Z}, n+1); \mathbb{Q})$  is a free algebra in one generator of degree  $n+1$ . The last implication uses the multiplicative structure of the spectral sequence.

Now let  $m > 0$  be a natural number and  $\mu_m : \mathbb{Z} \rightarrow \mathbb{Z}$  be multiplication by  $m$ . Then the naturality of the spectral sequence shows inductively that  $\mu_m^* : H^*(K(\mathbb{Z}, n+1); \mathbb{Q}) \rightarrow H^*(K(\mathbb{Z}, n+1); \mathbb{Q})$  is multiplication by  $m^k$  in degree  $k(n+1)$ .

We are left with showing that for all  $C$  and all  $n$  the map

$$\mu_m^* : \tilde{H}^*(K(C, n); \mathbb{Z}/m) \rightarrow \tilde{H}^*(K(C, n); \mathbb{Z}/m)$$

is equal to 0. First, let us assume that  $m = p$  is a prime number. As  $\tilde{H}^*(K(C, n); \mathbb{Z}/p) = 0$ , if the order of  $C$  is finite and coprime to  $p$ , (this fact can be shown by induction on  $n$ ), we can restrict attention to the case  $C = \mathbb{Z}$  or  $C = p^r$ . It is well known, see [Ca, Se], or [Mc, Theorem 6.19], that  $H^*(K(C, n); \mathbb{Z}/p)$  is a polynomial algebra with free generators of the form  $\mathcal{P}(\iota_n)$  where  $\iota_n \in H^n(K(C, n); \mathbb{Z}/p)$  is the fundamental class and  $\mathcal{P}$  is some mod  $p$  cohomology operation. Because the map  $\mu_p^*$  is zero on  $H^n(K(C, n); \mathbb{Z}/p)$  the assertion is hence implied by the naturality of the operations  $\mathcal{P}$ .

Next, for arbitrary cyclic  $C$  and  $m = p^r$  with a prime number  $p$ , the assertion follows by induction on  $r$  by use of the exact Bockstein sequence

$$\cdots \rightarrow \tilde{H}^*(K(C, n); \mathbb{Z}/p^{r-1}) \rightarrow \tilde{H}^*(K(C, n); \mathbb{Z}/p^r) \rightarrow \tilde{H}^*(K(C, n); \mathbb{Z}/p) \rightarrow \cdots,$$

which is natural with respect to homomorphisms  $C \rightarrow C$ , observing that  $\mu_m = \mu_{p^{r-1}} \circ \mu_p$ .

Finally, for  $m = p_1^{r_1} \cdot \dots \cdot p_k^{r_k}$  with pairwise different primes  $p_i$ , the map  $\mu_m^*$  is zero on  $\tilde{H}^*(K(C, n); \mathbb{Z}/m)$  by splitting this cohomology according to the primary decomposition  $\mathbb{Z}/m \cong \mathbb{Z}/p_1^{r_1} \times \dots \times \mathbb{Z}/p_k^{r_k}$ . □

Let  $X$  be a connected finite simple CW complex whose rational cohomology algebra is free. It must be finitely generated with all generators in odd degrees, because  $X$  is assumed to be finite. Serre's finiteness theorem (or a direct inspection of the Postnikov decomposition in connection with Lemma 4) implies that the homotopy groups of  $X$  are finitely generated in each degree. Hence they are finite products of cyclic groups.

**Lemma 5.** *For each  $n > 0$  the following holds.*

- $H^*(X_n; \mathbb{Q})$  is a free algebra with generators in degrees  $\leq n$  corresponding to the duals of the generators of  $\pi_i(X) \otimes \mathbb{Q}$ ,  $i \leq n$ .
- The canonical map  $X \rightarrow X_n$  induces an injective map in rational cohomology.
- The rationalized  $k$ -invariant  $(k_n)_{\mathbb{Q}} \in H^{n+2}(X_n; \pi_{n+1} \otimes \mathbb{Q})$  vanishes.

*Proof.* The first assertion implies the second one, because the canonical map  $X \rightarrow X_n$  induces an isomorphism in rational cohomology up to degree  $n$  and the cohomology algebra  $H^*(X; \mathbb{Q})$  is free. The third assertion is then also immediate because  $(k_n)_{\mathbb{Q}}$  is the image of the fundamental class in  $H^{n+1}(K(\pi_{n+1}, n+1); \mathbb{Q})$  under the differential  $d_{n+2}$  and the map  $X \rightarrow X_n$  factors through  $X_{n+1}$  so that  $H^*(X_n; \mathbb{Q}) \rightarrow H^*(X_{n+1}; \mathbb{Q})$  is also injective.

It is hence enough to prove the first assertion by induction on  $n$ .

The first assertion is clear for  $n = 1$  by Lemma 4 and the Künneth theorem. In the inductive step the assumption  $(k_n)_{\mathbb{Q}} = 0$  implies

$$H^*(X_{n+1}; \mathbb{Q}) \cong H^*(X_n; \mathbb{Q}) \otimes H^*(K(\pi_{n+1}, n+1); \mathbb{Q}).$$

Hence the assertion is again implied by Lemma 4 and the Künneth theorem. □

**Proposition 6.** *Let  $n > 0$ . Then for any  $m > 0$  there is a self map  $f_n : X_n \rightarrow X_n$  with the following properties.*

- The induced map  $f_n^* : H^*(X_n; \mathbb{Q}) \rightarrow H^*(X_n; \mathbb{Q})$  is an isomorphism.
- The induced map  $f_n^* : \tilde{H}^*(X_n; \mathbb{Z}/m) \rightarrow \tilde{H}^*(X_n; \mathbb{Z}/m)$  is equal to 0.

*Proof.* Induction on  $n$ , the assertion being immediate for  $n = 1$  by Lemma 4 and the Künneth theorem. By Lemma 5 the  $k$ -invariant  $k_n \in H^{n+2}(X_n; \pi_{n+1})$  is a torsion class of order  $\kappa$ , say. This has two useful implications. First, by the inductive hypothesis we find an  $f_n : X_n \rightarrow X_n$  inducing an isomorphism in rational cohomology and satisfying

$f_n^*(k_n) = 0$ . This results in a commutative diagram

$$\begin{array}{ccc}
K(\pi_{n+1}, n+1) & \xrightarrow{=} & K(\pi_{n+1}, n+1) \\
\downarrow \text{incl.} & & \downarrow \\
X_n \times K(\pi_{n+1}, n+1) & \xrightarrow{\alpha} & X_{n+1} \\
\downarrow \text{proj.} & & \downarrow p_{n+1} \\
X_n & \xrightarrow{f_n} & X_n
\end{array}$$

Second, we claim that multiplication  $h : \pi_{n+1} \rightarrow \pi_{n+1}$  with  $\kappa$  gives rise to a commutative diagram

$$\begin{array}{ccc}
K(\pi_{n+1}, n+1) & \xrightarrow{K(h, n+1)} & K(\pi_{n+1}, n+1) \\
\downarrow & & \downarrow \text{incl.} \\
X_{n+1} & \xrightarrow{\beta} & X_n \times K(\pi_{n+1}, n+1) \\
\downarrow p_{n+1} & & \downarrow \text{proj.} \\
X_n & \xrightarrow{=} & X_n
\end{array}$$

The only non-trivial task is the construction of the map  $X_{n+1} \rightarrow K(\pi_{n+1}, n+1)$  appearing in the middle horizontal line. For this we consider the diagram

$$\begin{array}{ccccc}
K(\pi_{n+1}, n+1) & \xrightarrow{=} & K(\pi_{n+1}, n+1) & \xrightarrow{K(h, n+1)} & K(\pi_{n+1}, n+1) \\
\downarrow & & \downarrow & & \downarrow \\
X_{n+1} & \longrightarrow & PK(\pi_{n+1}, n+2) & \xrightarrow{PK(h, n+1)} & PK(\pi_{n+1}, n+2) \\
\downarrow & & \downarrow & & \downarrow \\
X_n & \xrightarrow{k_n} & K(\pi_{n+1}, n+2) & \xrightarrow{K(h, n+2)} & K(\pi_{n+1}, n+2)
\end{array}$$

By assumption the composition in the lower row is homotopic to a constant map. Hence, by the homotopy lifting property, the composition  $X_{n+1} \rightarrow PK(\pi_{n+1}, n+2)$  in the second row is fibrewise homotopic to a map into just one fibre  $K(\pi_{n+1}, n+1)$  of the path loop fibration appearing in the right column. This concludes the construction of the map  $X_{n+1} \rightarrow K(\pi_{n+1}, n+1)$ .

Note that the maps  $\alpha$  and  $\beta$  appearing in the first two diagrams induce isomorphisms in rational cohomology by a spectral sequence argument.

Now let  $m > 0$  be arbitrary. Using the induction hypothesis, Lemma 4 and the Künneth formula it is easy to construct a self map

$$f' : X_n \times K(\pi_{n+1}, n+1) \rightarrow X_n \times K(\pi_{n+1}, n+1)$$

with the required properties. But then the composition

$$f_{n+1} : X_{n+1} \xrightarrow{\beta} X_n \times K(\pi_{n+1}, n+1) \xrightarrow{f'} X_n \times K(\pi_{n+1}, n+1) \xrightarrow{\alpha} X_{n+1}$$

is as required.  $\square$

**Proposition 7.** *Let  $X$  be a connected finite CW-complex which is a simple topological space and whose rational cohomology algebra is free. Then there is a continuous map*

$$\psi : X \times X \rightarrow X$$

so that for all  $p, q \in X$  the maps  $\psi(p, -) : X \rightarrow X$  and  $\psi(-, q) : X \rightarrow X$  induce isomorphisms in rational cohomology.

*Proof.* It is clear that such a map exists on  $X_1$ . Assume that we have already constructed a map  $\psi_n : X_n \times X_n \rightarrow X_n$  with the required properties.

Taking the existing product on  $X_n$  and the product on  $K(\pi_{n+1}, n+1)$  induced by the addition map on  $\pi_{n+1}$  we obtain a multiplication  $\psi'$  on  $X' := X_n \times K(\pi_{n+1}, n+1)$  with the required properties. Now we define  $\psi_{n+1}$  as the composition

$$X_{n+1} \times X_{n+1} \xrightarrow{\beta \times \beta} X' \times X' \xrightarrow{\psi'} X' \xrightarrow{\alpha} X_{n+1}.$$

where  $\alpha$  and  $\beta$  are taken from the proof of Proposition 6. Because these maps induce isomorphisms in rational cohomology, the map  $\psi_{n+1}$  has the required properties.

Once we have constructed a map  $\psi_n$  with  $n > 2 \dim X$ , the construction of  $\psi$  is complete by the cellular approximation theorem.  $\square$

For the next notions and results compare [MP, Sections 3.1. and 3.2]. A path connected CW complex  $X$  is called *nilpotent*, if  $\pi_1(M)$  is a nilpotent group and acts nilpotently on the higher homotopy groups. Note that simple complexes are automatically nilpotent. For a nilpotent complex the fibrations  $p_{n+1} : X_{n+1} \rightarrow X_n$  in the Postnikov decomposition are in general not principal, but they admit *finite principal refinements*

$$X_{n+1} =: Y_{r_n} \xrightarrow{q_{r_n}} Y_{r_n-1} \xrightarrow{q_{r_n-1}} \dots Y_1 \xrightarrow{q_1} Y_0 := X_n,$$

where each  $q_i : Y_i \rightarrow Y_{i-1}$  is a principal fibration with fibre  $K(G_i, n+1)$  for some abelian group  $G_i$ , which is classified by a cohomology class in  $H^{n+2}(Y_{i-1}; G_i)$ . The same argument as in the proof of Proposition 7 then shows:

**Proposition 8.** *Let  $X$  be a connected finite CW complex which is a nilpotent topological space. Assume that the rational cohomology algebra of  $X$  is free. Then there is a map  $\psi : X \times X \rightarrow X$  so that for all  $p, q \in X$  the maps  $\psi(p, -) : X \rightarrow X$  and  $\psi(-, q) : X \rightarrow X$  induce isomorphisms in rational cohomology.*

From this Theorem 1 follows.

**Lemma 9.** *Let  $M = G/H$  be a homogeneous space where  $G$  is a connected Lie group and  $H < G$  be a closed connected subgroup. Then  $M$  is a simple topological space.*

*Proof.* This fact is well known and we include a proof for the readers' convenience. Because  $H$  is connected we get an exact sequence

$$\dots \rightarrow \pi_2(G/H) \rightarrow \pi_1(H) \rightarrow \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow 1.$$

The topological group  $G$  has abelian fundamental group, and hence the same holds for  $G/H$ .

Next, let  $f : S^n \rightarrow G/H$  and  $\mu : S^1 \rightarrow G/H$  be based maps, where we take the south pole of any sphere as base point  $*$ . Recall that  $[\mu]_*([f]) \in \pi_n(G/H)$ , the result of the action of  $[\mu] \in \pi_1(G/H; *)$  on  $[f] \in \pi_n(G/H; *)$ , is represented by the following map  $\mu * f : S^n \rightarrow G/H$ . Consider the one point union  $S^n \vee S^n$  where the second sphere is piled above the first one, identifying the north pole of the first with the south pole of the second. We take the south pole of the first sphere as base point of  $S^n \vee S^n$ . Now consider the composition  $S^n \xrightarrow{\text{height}} [0, 1] \xrightarrow{\mu} G/H$  on the left hand sphere, and the map  $f : S^n \rightarrow G/H$  on the right hand sphere, and compose the resulting map  $S^n \vee S^n \rightarrow G/H$  with the base point preserving coproduct  $S^n \rightarrow S^n \vee S^n$ . This defines  $\mu * f$ . Note that for  $n = 1$  this results in the usual conjugation action of  $\pi_1(G/H; *)$  on itself.

By the above exact sequence the map  $\mu$  lifts to a based map  $\bar{\mu} : S^1 \rightarrow G$ . Using the left multiplication of  $G$  on  $G/H$  and the above explicit description of  $\mu * f$  it is easy to show that  $\mu * f$  is based homotopic to  $f$ .  $\square$

Together with our previous results this implies Corollary 2.

**Remark.** Of course the above argument is modelled along the lines of rational homotopy theory. In particular our Proposition 6 is implied by [Su, Theorem (12.2)] (note that by Lemma 5 our  $X$  is a formal space in the sense of rational homotopy theory). However, a complete proof of Sullivan's theorem has been carried out in the literature only for simply connected spaces [Sh] by obstruction theory based on cellular decompositions. Because finite skeleta of non-simply connected nilpotent spaces are in general not nilpotent [Le], the non-simply connected case requires a different approach based on Postnikov decompositions (as indicated already in [Su]). This issue and the special focus of our paper let us prefer the somewhat ad hoc, but self-contained discussion above over an in-depth exploration of rational homotopy theory.

We are grateful to DIETER KOTSCHICK for pointing out the following lemma and corollary.

**Lemma 10.** *Each  $\Gamma$ -manifold has virtually abelian fundamental group.*

*Proof.* Let  $\psi : M \times M \rightarrow M$  be a  $\Gamma$ -structure. Without loss of generality we can pick points  $p, q \in M$  so that  $\psi_p, \psi^q : M \rightarrow M$  preserve a base point in  $M$ . Let  $H_1, H_2 < \pi_1(M)$  be defined as the images of

$$(\psi_p)_* : \pi_1(M) \rightarrow \pi_1(M), \quad (\psi^q)_* : \pi_1(M) \rightarrow \pi_1(M).$$

The map  $\psi_p, \psi^q : M \rightarrow M$  factor through the connected coverings of  $M$  defined by  $H_1$  and  $H_2$ , respectively. Because  $\psi_p$  and  $\psi^q$  have non-zero mapping degrees, these coverings are finite and hence  $H_1$  and  $H_2$  are of finite index in  $\pi_1(M)$ . This implies that also  $H_1 \cap H_2 < \pi_1(M)$  is of finite index.

The map  $\psi_* : \pi_1(M) \times \pi_1(M) \rightarrow \pi_1(M)$  being a group homomorphism, elements in  $H_1$  commute with elements in  $H_2$ . This implies that the finite index subgroup  $H_1 \cap H_2 < \pi_1(M)$  is abelian.  $\square$



Together with Theorem 1 this implies

**Corollary 11.** *Let  $M$  be a closed connected oriented manifold which is nilpotent as a topological space. If  $H^*(M; \mathbb{Q})$  is a free algebra - necessarily over odd degree generators - then  $\pi_1(M)$  is virtually abelian.*

We do not know whether this conclusion can be drawn without applying Theorem 1.

## 2. CANONICAL PRODUCTS ON SYMMETRIC SPACES

In this section we study in which cases the canonical product  $\Theta$  on a compact oriented symmetric space  $P$  defined in (1) is a  $\Gamma$ -structure. First we observe that the degree of the map  $\Theta_p : P \rightarrow P$ ,  $y \mapsto \Theta(p, y) = s_p(y)$  with  $p \in P$  fixed is

$$\det(\Theta_p) = (-1)^{\dim(P)}.$$

For fixed  $q \in P$  we will examine the map

$$\theta := \Theta^q : P \rightarrow P, \quad x \mapsto \Theta(x, q) = s_x(q). \quad (2)$$

If  $\deg(\theta) \neq 0$ ,  $\Theta$  is a  $\Gamma$ -structure. We use of some features of compact symmetric spaces which can be found in the classical literature such as [He, Lo1, Lo2] or [Wo, Chap. 8].

**2.1. The mapping degree of  $\theta$ .** Our key observation is that  $\theta$  is the *squaring map* w.r.t. the origin  $q$  in the sense of LOOS [Lo1, Chap. II, §1]. Indeed, let  $\gamma$  be a geodesic in  $P$  with  $\gamma(0) = q$ , then  $\theta(\gamma(1)) = s_{\gamma(1)}(\gamma(0)) = \gamma(2)$ . For short

$$\theta \circ \text{Exp}_q = \text{Exp}_q \circ \mu_2, \quad (3)$$

where  $\mu_2 : T_q P \rightarrow T_q P$  denotes the multiplication by the scalar 2 and  $\text{Exp}_q$  the Riemannian exponential map of  $P$  at  $q$ . Now the pre-image of a point  $y \in P$  is

$$\theta^{-1}(\{y\}) = \{x \in P : x \text{ is the midpoint of a geodesic arc in } P \text{ between } q \text{ and } y\}. \quad (4)$$

Differentiating Equation (3) at a non-zero point  $X \in T_q P$  yields

$$d_{\text{Exp}_q(X)} \theta(J(1)) = J(2), \quad (5)$$

whenever  $J$  is a Jacobi field with  $J(0) = 0$  along the geodesic  $t \mapsto \text{Exp}_q(tX)$ .

To fix the notations and for the sake of completeness we first determine some Jacobi fields on  $P$  explicitly. Our approach is standard. For convenience we fix a *flat*  $A \subset P$ , that is a maximal dimensional connected totally geodesic submanifold of  $P$  whose sectional curvature vanishes, with  $q \in A$ . Flats in compact symmetric spaces are embedded flat tori. Since the curvature tensor  $R$  of  $P$  is parallel and  $A$  is flat, the self-adjoint operators

$$T_q P \rightarrow T_q P, \quad V \mapsto R(V, X)X,$$

with  $X \in T_q A$  commute. Thus, if  $P \neq A$ , there exists a collection

$$\text{Spec} := \{\lambda_1, \dots, \lambda_s\}$$

of distinct quadratic forms  $\lambda_j : T_q A \rightarrow \mathbb{R}$  (the squares of the roots of  $P$ ) so that the spaces

$$E_{\lambda_j} := \{V \in T_q P : R(V, X)X = \lambda_j(X) \cdot V \text{ for all } X \in T_q A\}, \quad j = 1, 2, \dots, s,$$

have positive dimension, and  $T_qP$  is the orthogonal direct sum

$$T_qP = T_qA \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_s}. \quad (6)$$

Since  $A$  is a flat, no  $\lambda \in \mathbf{Spec}$  vanishes identically. Moreover  $\lambda \geq 0$  for all  $\lambda \in \mathbf{Spec}$ , because  $P$  has non-negative sectional curvature. Recall that a Jacobi field  $J$  along a non-constant geodesic  $\gamma$  in  $P$  is a solution of  $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}J) + R(J, \dot{\gamma})\dot{\gamma} = 0$ , where  $\nabla$  is the Levi-Civita connection of  $P$ . As  $R$  is parallel, the Jacobi field  $J$  with  $J(0) = 0$  and  $(\nabla_{\dot{\gamma}}J)(0) = V \in T_qP$  along a non-constant geodesic  $\gamma$  in  $A$  starting in  $q$  satisfies

$$J(t) = \begin{cases} \frac{1}{\sqrt{\lambda(\dot{\gamma}(0))}} \sin(t\sqrt{\lambda(\dot{\gamma}(0))}) \cdot V^{\parallel}(t), & \text{if } V \in E_{\lambda}, \lambda \in \mathbf{Spec}, \lambda(\dot{\gamma}(0)) \neq 0 \\ t \cdot V^{\parallel}(t), & \text{if } V \in T_qA \text{ or } V \in E_{\lambda}, \lambda(\dot{\gamma}(0)) = 0 \end{cases}, \quad (7)$$

where  $V^{\parallel}$  denotes the parallel vector field along  $\gamma$  with  $V^{\parallel}(0) = V$ .

We are now able to find a regular value of  $\theta$ . A point  $a \in A$  is called *regular* w.r.t.  $q$ , if  $a$  is not an element of the cut locus of  $P$  w.r.t.  $q$  and if  $A$  is the unique flat in  $P$  that contains both points  $a$  and  $q$ . Otherwise  $a$  is called *singular* w.r.t.  $q$ . The set  $\mathbf{Sing}$  of all singular points in  $A$  w.r.t.  $q$  is a closed subset of measure zero in  $A$  (see e.g. [Lo2, Chap. VI]). For  $A$  is totally geodesic,  $\theta$  leaves  $A$  invariant. Since  $\theta$  is smooth,  $\theta(\mathbf{Sing})$  is again a compact null set of  $A$ . We take a point

$$y \in A \setminus (\mathbf{Sing} \cup \theta(\mathbf{Sing})).$$

We will see that (up to a slight modification)  $y$  is a regular value of  $\theta$ . As every geodesic in  $P$  lies in some flat and  $A$  is the only flat containing  $q$  and  $y$ , we get by Equation (4)

$$\theta^{-1}(\{y\}) = \{x \in A : x \text{ is the midpoint of a geodesic arc in } A \text{ between } q \text{ and } y\}. \quad (8)$$

Let  $\gamma$  be the shortest geodesic in  $P$  with  $\gamma(0) = q$  and  $\gamma(1) = y$  and set  $Y := \dot{\gamma}(0)$ . The restriction of the Riemannian exponential to  $T_qA$  is a covering of  $A$ . By Equation (3) the lift of  $\theta|_A$  to  $T_qA$  is the multiplication by 2. So we get

$$\theta^{-1}(\{y\}) = \text{Exp}_q\left(\frac{1}{2}Y + \frac{1}{2}\Lambda\right) \quad \text{with} \quad \Lambda := \{X \in T_qA : \text{Exp}_q(X) = q\}.$$

Let  $r = \text{rank}(P) := \dim(A)$  be the *rank* of  $P$ . Taking a basis  $\{B_1, \dots, B_r\}$  of  $T_qA$  that generates  $\Lambda$ , i. e.  $\Lambda = \text{span}_{\mathbb{Z}}(B_1, \dots, B_r)$ , we get

$$\theta^{-1}(\{y\}) = \left\{ \text{Exp}_q\left(\frac{1}{2}Y + \frac{1}{2} \sum_{j=1}^r \delta_j B_j\right) : \delta_j \in \{0, 1\} \text{ for } j = 1, \dots, r \right\}.$$

Since  $\text{Exp}_q$  is injective on  $\left\{ \frac{1}{2}Y + \frac{1}{2} \sum_{j=1}^r \delta_j B_j : \delta_j \in \{0, 1\} \text{ for } j = 1, \dots, r \right\}$ , the pre-image  $\theta^{-1}(\{y\})$  has cardinality  $2^r$  (cf. [AFS, p. 933]). We set

$$\theta^{-1}(\{y\}) = \{x_1, \dots, x_{2^r}\}.$$

Since  $y \notin \mathbf{Sing} \cup \theta(\mathbf{Sing})$ , each point in  $\theta^{-1}(\{y\})$  is regular w.r.t.  $q$ . Let  $\gamma_j$  be the unique shortest geodesic with  $\gamma_j(0) = q$  and  $\gamma_j(1) = x_j$ . Since the first conjugate point along a geodesic never occurs before the cut point, we have  $\lambda(\dot{\gamma}_j(0)) < \pi^2$  for all  $\lambda \in \mathbf{Spec}$  and all

$j \in \{1, \dots, 2^r\}$ . Recall that  $\text{Sing} \cup \theta(\text{Sing})$  is closed in  $A$ . In view of Equation (7) we want that  $y = \text{Exp}_q(Y) \in A \setminus (\text{Sing} \cup \theta(\text{Sing}))$  satisfies

$$\lambda(\dot{\gamma}_j(0)) \in ]0, \pi^2[ \setminus \{\pi^2/4\} \quad \text{for all } \lambda \in \text{Spec} \text{ and any } j \in \{1, \dots, 2^r\}. \quad (9)$$

This can be achieved by replacing  $Y$  with  $\tau Y$  for some  $\tau \in \mathbb{R}$  close to 1 if necessary.

The decomposition of  $T_q P$  in (6) allows us to choose a positively oriented basis  $V_1, \dots, V_d$  of  $T_q P$ ,  $d = \dim(P)$ , such that

- $V_1, \dots, V_r \in T_q A$ ;
- $V_j \in E_\lambda$  for some  $\lambda \in \text{Spec}$  if  $j \in \{r+1, \dots, d\}$ .

The cardinality of  $\{V_1, \dots, V_d\} \cap E_\lambda$  is obviously the dimension of  $E_\lambda$ . Let us fix  $j \in \{1, \dots, 2^r\}$ , and let  $V_k^\parallel$  be the parallel vector field along  $\gamma_j$  with  $V_k^\parallel(0) = V_k$ . Since parallel translation preserves the orientation,  $V_1^\parallel, \dots, V_d^\parallel$  is a positively oriented frame field along  $\gamma_j$ . Let  $J_k$  be the Jacobi field along  $\gamma_j$  with  $J_k(0) = 0$  and  $(\nabla_{\dot{\gamma}_j} J_k)(0) = V_k$ . As  $\lambda(\dot{\gamma}_j(0)) < \pi^2$  for each  $\lambda \in \text{Spec}$ , Equation (7) shows that  $J_1(1), \dots, J_d(1)$  is a positively oriented basis of  $T_{x_j} P$ . Equations (5) and (7) yield

$$d_{x_j} \theta(J_k(1)) = J_k(2) = \begin{cases} \frac{1}{\sqrt{\lambda(\dot{\gamma}_j(0))}} \sin(2\sqrt{\lambda(\dot{\gamma}_j(0))}) \cdot V_k^\parallel(2), & \text{if } k > r, V_k \in E_\lambda \\ 2 \cdot V_k^\parallel(2), & \text{if } k \in \{1, \dots, r\} \end{cases} \quad (10)$$

for  $k \in \{1, \dots, d\}$ . From (9) and (10) we now deduce:

**Theorem 12.** *The point  $y$  chosen above is a regular value of  $\theta$ , and the mapping degree of  $\theta$  is*

$$\deg(\theta) = \sum_{j=1}^{2^r} (-1)^{\varepsilon_j},$$

with

$$\varepsilon_j := \sum_{\substack{\lambda \in \text{Spec} \\ \lambda(\dot{\gamma}_j(0)) > \pi^2/4}} \dim(E_\lambda) \equiv \sum_{\substack{\lambda \in \text{Spec} \\ \lambda(\dot{\gamma}_j(0)) > \pi^2/4 \\ \dim(E_\lambda) \text{ odd}}} \dim(E_\lambda) \pmod{2}.$$

**Remark.** The elements of  $\text{Spec}$  are actually the squares of the *roots* of the symmetric space  $P$  and the dimensions of the spaces  $E_\lambda$  are the *multiplicities* of these roots. So Theorem 12 provides an expression of  $\deg(\theta)$  in terms of the root system with multiplicities of  $P$ .

**2.2. Examples.** Let  $P = G/K$  be of *splitting rank*, i. e.  $\text{rank}(G) = \text{rank}(K) + \text{rank}(P)$ , or, equivalently,  $\dim(E_\lambda)$  is even for any  $\lambda \in \text{Spec}$  (see e.g. [He, Thm. 6.1, p. 429] or [Lo2, Thm. 4.3, p. 81]). Then  $\varepsilon_j$  defined in Theorem 12 vanishes for any  $j \in \{1, \dots, 2^r\}$ . Therefore  $\deg(\theta) = 2^r$ .

**Theorem 13.** *The canonical product of an oriented compact symmetric space of splitting rank is a  $\Gamma$ -structure.*

- Remark.** (i) If  $P$  is a compact Lie group with a bi-invariant metric, then  $P$  is orientable and all roots of the symmetric space  $P$  have multiplicity equal to two (see e.g. [Lo2, Thm. 4.4., p. 82]). Thus the canonical product on  $P$  (as a symmetric space) is a  $\Gamma$ -structure different from its Lie theoretic product.
- (ii) By ÉLIE CARTAN'S classification of symmetric spaces (see e.g. [He, Chap. X]) compact symmetric spaces of splitting rank are locally isomorphic to finite products of
- compact Lie groups (including flat tori),
  - odd dimensional round spheres,
  - $SU_{2n}/Sp_n$  with  $n \geq 3$
  - $E_6/F_4$ .
- (iii) Theorem 13 explains the form of the rational cohomology of  $SU_{2n}/Sp_n$  given in [MT, Thm. 6.7(1), p. 149] and of  $E_6/F_4$  determined by ARAKI in [Ar, Prop. 2.5].

Next we recover the following result of ALBERS, FRAUENFELDER and SOLOMON:

**Theorem 14** ([AFS]). *Let  $P := U_n/O_n$  with  $n = 2m+1$  and  $m \in \mathbb{N}$ . Then  $\deg(\theta) = 2^{m+1}$ , and the canonical product on  $P$  is a  $\Gamma$ -structure.*

**Remark.** The space  $P = U_n/O_n$  can be canonically identified with the space of all unoriented real Lagrangian subspaces of the canonical symplectic space  $\mathbb{C}^n$ . Let  $I$  be the identity on  $\mathbb{C}^n$ . Since  $-I$  acts trivially on  $P$ ,  $P$  can be written as  $(U_n/\{\pm I\})/(O_n/\{\pm I\})$ . For  $n$  odd  $O_n/\{\pm I\} \cong SO_n$  is connected, and  $P$  satisfies the assumptions of Theorem 3.

*Proof.* Since  $U_1/O_1$  is isomorphic to  $S^1$ , we may assume that  $m \geq 1$ . Since the Dynkin diagram of  $SU_n/SO_n$  has type  $\mathfrak{a}_{n-1}$  (see e.g. [He, Chap. X]), the tangent space  $T_q A$  of a flat  $A \subset U_n/O_n$  can be identified with  $\mathbb{R}^n$  such that

$$\text{Spec} \cong \{(e_j^* - e_k^*)^2 : 1 \leq j < k \leq n\},$$

where  $e_1^*, \dots, e_n^*$  denote the duals of the standard basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$  (see e.g. [He, p. 462]) and

$$\Lambda \cong \text{span}_{\mathbb{Z}}(\pi e_1, \pi e_2, \dots, \pi e_n).$$

For  $\frac{\pi}{8} \gg \xi_1 > \xi_2 > \dots > \xi_n > 0$  we set  $Y := -2(\xi_1, \dots, \xi_n) \in \mathbb{R}^n \cong T_q A$  and  $y := \text{Exp}_q(Y)$ . Recall that  $\theta^{-1}(\{y\}) = \{\text{Exp}_q(X_\delta) : \delta \in \{0, 1\}^n\}$  with  $X_\delta := \frac{1}{2}Y + \frac{\pi}{2}\delta$ . For  $1 \leq j < k \leq n$  we get

$$(e_j^* - e_k^*)^2(X_\delta) = \underbrace{(\xi_k - \xi_j)}_{\in ]-\pi/2, 0[} + \underbrace{(\delta_j - \delta_k)}_{\in \{-1, 0, 1\}} \cdot \pi/2)^2. \quad (11)$$

This shows that  $y$  is a regular value of  $\theta$ . Moreover, we deduce from Equation (11) that  $(e_j^* - e_k^*)^2(X_\delta) > \pi^2/4$  if and only if  $\delta_j = 0$  and  $\delta_k = 1$ . Following [AFS, p. 934] we define  $\varepsilon_\delta$  for  $\delta \in \{0, 1\}^n$  to be the number modulo 2 of all pairs  $(j, k)$  with  $1 \leq j < k \leq n$  such that  $\delta_j = 0$  and  $\delta_k = 1$ . Since all roots of  $P = U_n/O_n$  have multiplicity 1 (see e.g. [He, p. 532]), i. e.  $\dim(E_\lambda) = 1$  for all  $\lambda \in \text{Spec}$ , we get

$$\varepsilon_\delta \equiv \sum_{\substack{\lambda \in \text{Spec} \\ \lambda(X_\delta) > \pi^2/4}} \dim(E_\lambda) \pmod{2}$$

Thus  $\deg(\theta) = \sum_{\delta \in \{0,1\}^n} (-1)^{\varepsilon_\delta}$  by Theorem 12. These equations also appear in [AFS, p. 934].

The combinatorial problem to calculate explicitly  $\deg(\theta)$  from these equations is solved in [AFS, Lem. 2.2], where one finds that  $\deg(\theta) = 2^{m+1}$  for  $n = 2m + 1$ ,  $m \in \mathbb{N}$ .  $\square$

**2.3. Proof of Theorem 3.** Just like compact Lie groups (see e. g. [Kn, Thm. 4.29, p. 198]) compact symmetric spaces admit finite covers that split off flat factors:

**Lemma 15.** *Every compact symmetric space  $P$  is finitely covered by a product  $T \times \tilde{Q}$  of a flat torus  $T$  and a simply connected compact symmetric space  $\tilde{Q}$ .*

*Proof.* The deck transformation group  $\Delta$  of the universal cover  $\tilde{P}$  of  $P$  is a finitely generated discrete subgroup of the abelian centralizer  $C_{\text{Iso}(\tilde{P})}(\text{Trans}(\tilde{P}))$  of the transvection group  $\text{Trans}(\tilde{P})$  of  $\tilde{P}$  in its isometry group (see [Sa, Lem. 1.2, p. 194] and [Wo, Thm. 8.3.11, p. 244]). Since  $\tilde{P} \cong \mathbb{R}^k \times \tilde{Q}$ , where  $\tilde{Q}$  is a simply connected compact symmetric space, the isometry group of  $\tilde{P}$  splits as  $\text{Iso}(\tilde{P}) = \text{Iso}(\mathbb{R}^k) \times \text{Iso}(\tilde{Q})$ .

Thus any element of  $\Delta$  has the form  $f \times g$  for some  $f \in C_{\text{Iso}(\mathbb{R}^k)}(\text{Trans}(\mathbb{R}^k)) \cong \mathbb{R}^k$  and some  $g \in C_{\text{Iso}(\tilde{Q})}(\text{Trans}(\tilde{Q}))$ . Since  $\tilde{Q}$  is a symmetric space of compact type  $C_{\text{Iso}(\tilde{Q})}(\text{Trans}(\tilde{Q}))$  is finite. Let  $N$  be a common multiple of the orders of elements of  $C_{\text{Iso}(\tilde{Q})}(\text{Trans}(\tilde{Q}))$ , then  $g^N = e$  for all  $g \in C_{\text{Iso}(\tilde{Q})}(\text{Trans}(\tilde{Q}))$ . Since  $\Delta$  is abelian, the  $N$ -th power is an endomorphism of  $\Delta$  whose image  $\Delta^N$  acts trivially on the  $\tilde{Q}$  factor of  $\tilde{P}$ . As  $\Delta^N$  has finite index in  $\Delta$ , the space  $\tilde{P}/\Delta^N \cong T \times \tilde{Q}$ , which is the desired cover of  $P$ , is compact.  $\square$

**Lemma 16.** *Let  $P_1$  and  $P_2$  be two compact oriented symmetric spaces. Then the canonical product on the Riemannian product  $P_1 \times P_2$  is a  $\Gamma$ -structure if and only if the canonical products on  $P_1$  and on  $P_2$  are both  $\Gamma$ -structures.*

*Proof.* The Riemannian product  $P_1 \times P_2$  is again a compact oriented symmetric space and the geodesic symmetries of  $P_1 \times P_2$  are products of geodesic symmetries of  $P_1$  and of  $P_2$ . Therefore the canonical product  $\Theta_{1 \times 2}$  on  $P_1 \times P_2$  is the product of the canonical products  $\Theta_1$  and  $\Theta_2$  on  $P_1$  and  $P_2$ ,

$$\begin{aligned} \Theta_{1 \times 2} : (P_1 \times P_2) \times (P_1 \times P_2) &\rightarrow P_1 \times P_2, \\ ((x_1, x_2), (y_1, y_2)) &\mapsto s_{(x_1, x_2)}(y_1, y_2) = (\Theta_1(x_1, y_1), \Theta_2(x_2, y_2)). \end{aligned}$$

With a fixed origin  $q = (q_1, q_2) \in P_1 \times P_2$  we get

$$\theta_{1 \times 2}(x_1, x_2) := \Theta_{1 \times 2}((x_1, x_2), (q_1, q_2)) = (\Theta_1(x_1, q_1), \Theta_2(x_2, q_2)) = (\theta_1 \times \theta_2)(x_1, x_2).$$

The claim follows, since the mapping degree is multiplicative, that is

$$\deg(\theta_{1 \times 2}) = \deg(\theta_1 \times \theta_2) = \deg(\theta_1) \deg(\theta_2).$$

$\square$

**Lemma 17.** *Let  $p : \hat{P} \rightarrow P$  be an orientation preserving Riemannian covering between two compact oriented symmetric spaces. Then the canonical product on  $\hat{P}$  is a  $\Gamma$ -structure if and only if the canonical product on  $P$  is a  $\Gamma$ -structure.*

*Proof.* The canonical products  $\hat{\Theta}$  on  $\hat{P}$  and  $\Theta$  on  $P$  are related by

$$\Theta \circ (p \times p) = p \circ \hat{\Theta}.$$

Let  $\hat{q} \in \hat{P}$  be a chosen origin and  $q := p(\hat{q})$ . Then the maps  $\hat{\theta} = \hat{\Theta}^{\hat{q}} : \hat{x} \mapsto \hat{s}_{\hat{x}}(\hat{q})$  and  $\theta = \Theta^q : x \mapsto s_x(q)$  satisfy  $p \circ \hat{\theta} = \theta \circ p$ . Since  $p$  is a covering, we get

$$\deg(p) \deg(\hat{\theta}) = \deg(\theta) \deg(p),$$

where  $\deg(p)$  coincides with the number of sheets of  $p$ . Division by  $\deg(p)$  yields  $\deg(\hat{\theta}) = \deg(\theta)$ .  $\square$

The rational cohomology of a  $d$ -dimensional connected closed manifold  $M$  endowed with the cup product is a graded algebra

$$H^*(M; \mathbb{Q}) = \bigoplus_{k=0}^d H^k(M; \mathbb{Q}).$$

The minimal number of homogeneous generators of  $H^*(M; \mathbb{Q})$  in degree  $k \in \{1, \dots, d\}$  is

$$\dim(H^k(M; \mathbb{Q})) - \dim\left(H^k(M; \mathbb{Q}) \cap \left\langle \bigoplus_{j=0}^{k-1} H^j(M; \mathbb{Q}) \right\rangle_{\text{alg}}\right),$$

where  $\left\langle \bigoplus_{j=0}^{k-1} H^j(M; \mathbb{Q}) \right\rangle_{\text{alg}}$  is the smallest sub-algebra of  $H^*(M; \mathbb{Q})$  containing  $\bigoplus_{j=0}^{k-1} H^j(M; \mathbb{Q})$ .

Künneth's formula tells us that  $H^*(M_1 \times M_2; \mathbb{Q}) \cong H^*(M_1; \mathbb{Q}) \otimes H^*(M_2; \mathbb{Q})$  for connected closed manifolds  $M_1$  and  $M_2$ . Let  $\{\alpha_1, \dots, \alpha_m\}$  be a minimal system of homogeneous generators of the algebra  $H^*(M_1; \mathbb{Q})$  and let  $\{\beta_1, \dots, \beta_s\}$  be a minimal system of homogeneous generators of  $H^*(M_2; \mathbb{Q})$ . Then  $\{\alpha_1 \otimes 1, \dots, \alpha_m \otimes 1, 1 \otimes \beta_1, \dots, 1 \otimes \beta_s\}$  is a minimal system of homogeneous generators of the algebra  $H^*(M_1 \times M_2; \mathbb{Q})$ . Therefore the algebra  $H^*(M_1 \times M_2; \mathbb{Q})$  is generated by homogeneous elements of odd degrees, if and only if this holds for both  $H^*(M_1; \mathbb{Q})$  and  $H^*(M_2; \mathbb{Q})$ .

We are now able to prove Theorem 3. Note that compact symmetric spaces that satisfy the hypotheses of Theorem 3 are orientable. Furthermore simply connected compact symmetric spaces and their Riemannian products with flat tori meet the assumptions of Theorem 3. Let  $P$  be a symmetric space as in Theorem 3, and suppose that  $H^*(P; \mathbb{Q})$  is an exterior algebra generated by homogeneous elements in odd degrees. By Lemma 15 there is a Riemannian product

$$\hat{P} := T \times \tilde{P}_1 \times \dots \times \tilde{P}_m$$

of a flat torus  $T$  (points and circles are considered zero and one dimensional tori) and irreducible simply connected compact symmetric spaces  $\tilde{P}_1, \dots, \tilde{P}_m$  that finitely covers  $P$ .

**Lemma 18.** *The covering  $p : \hat{P} \rightarrow P$  induces an isomorphism between the graded  $\mathbb{Q}$ -algebras  $H^*(P; \mathbb{Q})$  and  $H^*(\hat{P}; \mathbb{Q})$ .*

*Proof.* We prove this claim in two different ways. First, let  $\hat{q} \in \hat{P}$  and  $q := p(\hat{q})$ . Let  $\hat{G}$  and  $G$  denote the identity components of the isometry groups of  $\hat{P}$  and  $P$ . Since the isotropy subgroups  $\hat{K} \subset \hat{G}$  of  $\hat{q}$  and  $K \subset G$  of  $q$  are both connected, the projection  $p$  identifies their linear isotropy actions. By [Wo, Thm. 8.5.8] the real cohomology algebra of any compact symmetric space  $S$  is isomorphic to the algebra of those elements in  $\bigwedge^* T_x S$ ,  $x \in S$ , which are invariant under the action of the isotropy subgroup of  $x$  in the identity component of the isometry group of  $S$ . Thus  $p$  induces an isomorphism  $H^*(\hat{P}; \mathbb{R}) \cong H^*(P; \mathbb{R})$  and also an isomorphism of the rational cohomologies.

Second, the fundamental group  $\pi_1(\hat{P}) < \pi_1(P)$  is abelian and acts trivially on  $\pi_n(\hat{P}) = \pi_n(P)$  for  $n > 1$ , because  $P = G/H$  with connected  $H$  and by Lemma 9. Hence  $\hat{P}$  is a simple space. Furthermore, because  $p$  is a finite covering, the induced map  $\pi_*(\hat{P}) \otimes \mathbb{Q} \rightarrow \pi_*(P) \otimes \mathbb{Q}$  is an isomorphism. By the Whitehead–Serre theorem for simple spaces (or inspecting the induced map between Postnikov decompositions of  $\hat{P}$  and  $P$ ) the induced map in rational cohomology is an isomorphism as well.  $\square$

Lemma 18 and Künneth’s formula imply

$$H^*(P; \mathbb{Q}) \cong H^*(T; \mathbb{Q}) \otimes H^*(\tilde{P}_1; \mathbb{Q}) \otimes \cdots \otimes H^*(\tilde{P}_m; \mathbb{Q}).$$

By our previous discussion all the algebras  $H^*(\tilde{P}_j; \mathbb{Q})$ ,  $j \in \{1, \dots, m\}$ , are generated by homogeneous elements in odd degrees. To prove Theorem 3 it now suffices by Lemmas 16 and 17 to show that the canonical products on  $T$  and on  $\tilde{P}_1, \dots, \tilde{P}_m$  are all  $\Gamma$ -structures. As flat tori are symmetric spaces of splitting rank, we are left to verify Theorem 3 only for irreducible simply connected compact symmetric spaces  $P = G/K$ , where  $G$  is the identity component of the isometry group of  $P$ .

Since the rational cohomology of an *inner* symmetric space, this is a space all of whose geodesic symmetries belong to the identity component  $G$  of the isometry group, has only contributions in even degrees (see e.g. [Wo, proof of Thm. 8.6.7] and [GHV, Thm. VII, p. 467]), we may assume that  $P = G/K$  is an *outer* symmetric space. Using the classification of symmetric spaces one could at this point determine all outer symmetric spaces that satisfy the assumptions of Theorem 3 by checking case-by-case the rational cohomology of such spaces given in [Ta] (see also [MT, Sp, GHV]). But we prefer a more conceptual approach that we essentially learned from OLIVER GOERTSCHES.

Since  $(\mathfrak{g}, \mathfrak{k})$  is a *Cartan pair* (see [GHV, pp. 448 & 465]) with  $\rho := \text{rank}(\mathfrak{g}) - \text{rank}(\mathfrak{k}) > 0$ ,  $H^*(G/K; \mathbb{Q})$  is isomorphic to a tensor product of a  $2^\rho$  dimensional exterior algebra and a quotient of a symmetric algebra (see [GHV, Thm. IV, p. 463] and [KT, Thm. 3]). Therefore  $G/K$  satisfies the hypotheses of Theorem 3 if and only if  $\dim(H^*(G/K; \mathbb{Q})) = 2^\rho$ . By [Go] this happens precisely if the number of Weyl chambers of  $\mathfrak{g}$  that intersect a given Weyl chamber of  $\mathfrak{k}$  is equal to one. Non-trivial intersections of Weyl chambers of  $\mathfrak{g}$  with Weyl chambers of  $\mathfrak{k}$  are called *compartments* in [EMQ].

It turns out (see [Mu, Sect. 3] and [BR, Prop. 3.20(iii) & Thm. 3.25] and [EMQ, pp. 1128 & 1129]) that  $P$  only meets the assumptions of Theorem 3, if the involution of  $\mathfrak{g}$  associated with  $P$  is a canonical extension  $\tau_\sigma$  of an order two automorphism  $\sigma$  of the Dynkin diagram of  $\mathfrak{g}$  as described in [Mu, Sect. 3] and [BR, p. 33]. The outer symmetric spaces of this

kind are of splitting rank or  $SU_{2m+1}/SO_{2m+1}$  with  $m \in \mathbb{N}$  (see [Mu, Table p. 305]). In the first case Theorem 3 follows directly from Theorem 13. For the second case we note that  $U_{2m+1}/O_{2m+1}$  is finitely covered by  $S^1 \times (SU_{2m+1}/SO_{2m+1})$ . Theorem 14 together with Lemmas 16 and 17 yield  $\deg(\theta) = 2^m$  for the canonical product of  $SU_{2m+1}/SO_{2m+1}$ . Hence the canonical product on  $SU_{2m+1}/SO_{2m+1}$  is a  $\Gamma$ -structure.

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT AUGSBURG, 86135 AUGSBURG, GERMANY  
*E-mail address:* hanke@math.uni-augsburg.de, peter.quast@math.uni-augsburg.de