BORDISM OF ELEMENTARY ABELIAN GROUPS VIA INESSENTIAL BROWN-PETERSON HOMOLOGY

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ABSTRACT. We compute the equivariant bordism of free oriented $(\mathbb{Z}/p)^n$-manifolds as a module over Ω_*^{SO}, when p is an odd prime. We show, among others, that this module is canonically isomorphic to a direct sum of suspensions of multiple tensor products of $\Omega_*^{SO}(B\mathbb{Z}/p)$, and that it is generated by products of standard lens spaces. This considerably improves previous calculations by various authors.

Our approach relies on the investigation of the submodule of the Brown-Peterson homology of $B(\mathbb{Z}/p)^n$ generated by elements coming from proper subgroups of $(\mathbb{Z}/p)^n$.

We apply our results to the Gromov-Lawson-Rosenberg conjecture for atoral manifolds whose fundamental groups are elementary abelian of odd order.

1. Overview

Conner and Floyd, in their seminal work [6], introduced and studied bordism groups of free oriented G-manifolds for finite groups G. This is equivalent to the oriented bordism of BG, the classifying space of G. Of fundamental interest are the elementary abelian groups $G = (\mathbb{Z}/p)^n$, where p is a prime.

Recall that $\Omega_*^{SO}(B\mathbb{Z}/p)$, the oriented bordism of free $\mathbb{Z}/p$-manifolds, is generated as a module over $\Omega_*^{SO} = \Omega_*^{SO}(pt.)$ by elements $z_m \in \Omega_{2m+1}^{SO}(B\mathbb{Z}/p)$, $m \geq 0$, represented by classifying maps $L^{2m+1} \to B\mathbb{Z}/p$ of standard lens spaces $L^{2m+1} = S^{2m+1}/(\mathbb{Z}/p)$. These correspond to spheres S^{2m+1} equipped with standard free $\mathbb{Z}/p$-actions of weight $(1, \ldots, 1)$.

For odd p the oriented bordism of (classifying spaces of) elementary abelian p-groups fits into Landweber's exact Künneth sequence [14, Theorem A]

$$
\begin{aligned}
(1) \qquad & 0 \to \Omega_*^{SO}(B(\mathbb{Z}/p)^{n-1}) \otimes_{\Omega_*^{SO}} \Omega_*^{SO}(B\mathbb{Z}/p) \to \Omega_*^{SO}(B(\mathbb{Z}/p)^n) \to \\
& \to \left(\mathrm{Tor}_{\Omega_*^{SO}}(\Omega_*^{SO}(B(\mathbb{Z}/p)^{n-1}), \Omega_*^{SO}(B\mathbb{Z}/p))\right)_{*-1} \to 0.
\end{aligned}
$$

The calculation of the middle term by induction on n requires a splitting of this sequence as Ω_*^{SO}-modules. Indeed, individual elements of the torsion product can be lifted to $\Omega_*^{SO}(B(\mathbb{Z}/p)^n)$ by a matrix Toda bracket construction, see [4, p. 195] and [1], but this involves choices (of zero bordisms) and hence does not give an Ω_*^{SO}-linear splitting. Landweber observed that for $n = 2$ the parity of degrees of elements in $\Omega_*^{SO}(B(\mathbb{Z}/p)^2)$ induces a canonial splitting, see [14, Theorem 7.1]. One referee pointed out that more generally Holzsager's stable splitting of $B\mathbb{Z}/p$ in [10] can be used to split Landweber's Künneth sequence for

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$n \leq 2(p-1)$ by considering degrees modulo $2(p-1)$ of elements in $\Omega_*^{SO}(B(\mathbb{Z}/p)^n)$, rather than modulo 2. This leads to short proofs of Theorems 2.1 and 2.2 below.

Despite a number of related contributions [6, 12, 13, 14, 15, 20, 22] it remained unclear how to construct a splitting of Landweber's exact sequence for all n. The following result settles this problem.

Theorem 1.1. *Landweber's exact Künneth sequence (1) splits Ω_*^{SO}-linearly. There is a preferred splitting, not depending on any choices.*

For a full description of the Ω_*^{SO}-module $\Omega_*^{SO}(B(\mathbb{Z}/p)^n)$ see Theorem 2.2 and the following remarks. Theorem 1.1 will be proven by a combination of algebraic and homotopy theoretic arguments, which do not give a geometric interpretation of the claimed splitting, a priori. However, our proof of Theorem 1.1 still enables us to identify a set of generators of $\Omega_*^{SO}(B(\mathbb{Z}/p)^n)$. For $n = 2$ Botvinnik-Gilkey [3] proved that the bordism classes that are represented by classifying maps $L^{2m_1+1} \times L^{2m_2+1} \to B(\mathbb{Z}/p)^2$ or by compositions $L^{2m+1} \to B\mathbb{Z}/p \xrightarrow{B\phi} B(\mathbb{Z}/p)^2$ for the various group homomorphisms $\phi : \mathbb{Z}/p \to (\mathbb{Z}/p)^2$ span $\Omega_*^{SO}(B(\mathbb{Z}/p)^2)$ as an Ω_*^{SO}-module. We will show that this scheme generalizes to elementary abelian p-groups of arbitrary rank.

Let $1 \leq k \leq n$ and let $\phi : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^n$ be a group homomorphism, inducing a map of Ω_*^{SO}-modules

$$\phi_* : \Omega_*^{SO}(B(\mathbb{Z}/p)^k) \to \Omega_*^{SO}(B(\mathbb{Z}/p)^n).$$

If we represent an element $z \in \Omega_d^{SO}(B(\mathbb{Z}/p)^k)$ by a free oriented $(\mathbb{Z}/p)^k$-manifold M^d, then $\phi_*(z)$ is represented by the free oriented $(\mathbb{Z}/p)^n$-manifold $(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^k} M$, where $(\mathbb{Z}/p)^k$ acts on $(\mathbb{Z}/p)^n$ by the map ϕ. In the special case when M can be taken as a product of k odd dimensional spheres with a product of standard free $\mathbb{Z}/p$-actions, we call the quotient $\left((\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^k} M\right)/(\mathbb{Z}/p)^n$ a *generalized product of lens spaces.*

Theorem 1.2. *Let p be an odd prime. Then the Ω_*^{SO}-module $\Omega_*(B(\mathbb{Z}/p)^n)$ is generated by generalized products of lens spaces (including the empty product, represented by a point).*

Note that the corresponding assertion for singular homology $\mathrm{H}_*(B(\mathbb{Z}/p)^n;\mathbb{Z})$ is true only for $n = 1$.

The work on this paper started with an attempt to understand the proof of [4, Theorem 5.6], which is a weak version of our Theorem 1.2 and represents the crucial step in the proof of the Gromov-Lawson-Rosenberg conjecture about the existence of Riemannian metrics of positive scalar curvature on closed manifolds with elementary abelian fundamental groups given in [4] and [5]. It turned out that the proof of [4, Theorem 5.6] in *loc. cit.* is incorrect, but the theorem itself is true and can be proven with our methods. Theorem 1.2 and its analogue for spin bordism lead to a proof of the Gromov-Lawson-Rosenberg conjecture for elementary abelian groups of odd order, see Section 7, where we also make some remarks on the argument in [4].

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2. Brown-Peterson homology of elementary abelian groups

In this section we summarize our computations in some detail and motivate our approach. For simplicity and in accordance with an analogous convention in group homology we often suppress the letter B in classifying spaces in our computations of generalized homology groups of classifying spaces.

The classifying space $B(\mathbb{Z}/p)^n$ being p-local we can restrict ourselves to a computation of $\Omega_*^{SO}((\mathbb{Z}/p)^n) = \Omega_*^{SO}(B(\mathbb{Z}/p)^n)$ localized at p. We recall [22] that for odd primes p, the p-local oriented bordism spectrum $\mathrm{MSO}_{(p)}$ splits into suspensions of copies of BP, Brown-Peterson theory for the prime p, with coefficients

$$\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],\ \deg(v_m) = 2p^m - 2.$$

Hence, for odd primes p, the computation of $\Omega_*^{SO}((\mathbb{Z}/p)^n)$ reduces to a computation of $\mathrm{BP}_*((\mathbb{Z}/p)^n)$. This is the theme of the paper at hand.

Recall that $B\mathbb{Z}/p$ has a CW structure with one cell in each dimension greater than or equal to zero [2, Example (1.1.2)]. The associated reduced cellular chain complex C_* has one generator $c_d \in C_d$ for each $d \geq 1$ and is equipped with the differential

$$c_{2m} \mapsto p \cdot c_{2m-1}\ ,\ \ c_{2m+1} \mapsto 0.$$

In particular, $\widetilde{\mathrm{H}}_*(\mathbb{Z}/p)$, the reduced integral homology of $B\mathbb{Z}/p$, is an $\mathbb{F}_p$-module with generators $[c_{2m+1}] \in \widetilde{\mathrm{H}}_{2m+1}(\mathbb{Z}/p)$, $m \geq 0$.

The reduced homology $\widetilde{\mathrm{H}}_*(\wedge^n\mathbb{Z}/p)$ of the n-fold smash product of $B\mathbb{Z}/p$ is equal to the homology of the n-fold tensor product of C_*,

$$\widetilde{\mathrm{H}}_*(\wedge^n\mathbb{Z}/p) = \mathrm{H}_*(C_* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} C_*).$$

It follows from this description that the iterated Künneth map

$$\widetilde{\mathrm{H}}_*(\mathbb{Z}/p) \otimes \cdots \otimes \widetilde{\mathrm{H}}_*(\mathbb{Z}/p) \to \widetilde{\mathrm{H}}_*(\wedge^n\mathbb{Z}/p)$$

is injective and has a canonical splitting

$$\Psi_n : \widetilde{\mathrm{H}}_*(\wedge^n\mathbb{Z}/p) \to \widetilde{\mathrm{H}}_*(\mathbb{Z}/p) \otimes \cdots \otimes \widetilde{\mathrm{H}}_*(\mathbb{Z}/p)$$

induced by the chain map

$$C_* \otimes \cdots \otimes C_* \to \widetilde{\mathrm{H}}_*(\mathbb{Z}/p) \otimes \cdots \otimes \widetilde{\mathrm{H}}_*(\mathbb{Z}/p)$$

that sends each tensor product $c_{2m_1+1} \otimes \cdots \otimes c_{2m_n+1}$ to $[c_{2m_1+1}] \otimes \cdots \otimes [c_{2m_n+1}]$, and each tensor product involving an even dimensional generator c_{2m} to zero.

Theorem 2.4 of our paper says that the reduced Brown-Peterson homology $\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)$ has a similar chain model description, if p is odd. The next theorem is our first result in this direction.

From now on we assume that p is an odd prime, if not stated otherwise.

Theorem 2.1. *There is a* BP_**-linear map*

$$\Psi_n : \widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p) \to \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$$

which splits the iterated Künneth map

$$\Phi_n : \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \to \widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p).$$

The map Ψ_n *is a map of* $\mathrm{BP}_*\,\mathrm{BP}$*-comodules.*

Our construction of Ψ_n is canonical and independent of any choices. The main idea is to study the *inessential Brown-Peterson group homology*, the BP_*-submodule of $\mathrm{BP}_*((\mathbb{Z}/p)^n)$ generated by elements coming from proper subgroups of $(\mathbb{Z}/p)^n$, see Definition 3.2. Our paper is based on the observation that this submodule is a complement of the image of the iterated Künneth map.

Based on Theorem 2.1 we can state our computation of $\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$. By regarding S^∞ both as a free contractible $\mathbb{Z}/p$- and S^1-space we have a canonical map

$$\pi : B\mathbb{Z}/p \to \mathbb{C}\mathrm{P}^\infty$$

induced by the standard inclusion $\mathbb{Z}/p \hookrightarrow S^1$. From this we get a map

$$\gamma_{(X_i)} : \bigwedge_n B\mathbb{Z}/p \to \bigwedge_n X_i$$

for each family $(X_i)_{1\le i\le n}$, where each X_i is equal to either $B\mathbb{Z}/p$ or $\mathbb{C}\mathrm{P}^\infty$, and we apply either π or the identity to each smash product factor $B\mathbb{Z}/p$.

Recall that $\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$ is a free BP-module generated by elements $\beta_m \in \mathrm{BP}_{2m}(\mathbb{C}\mathrm{P}^\infty)$, $m \ge 1$, represented by the standard inclusion $\mathbb{C}\mathrm{P}^m \hookrightarrow \mathbb{C}\mathrm{P}^\infty$. In particular, for any pointed space X, there is a canonical isomorphism

$$\widetilde{\mathrm{BP}}_*(X \wedge \mathbb{C}\mathrm{P}^\infty) \cong \widetilde{\mathrm{BP}}_*(X) \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty).$$

Hence, if k is the number of factors $X_i = B\mathbb{Z}/p$ in a smash product as before, we can compose the induced map in reduced BP-theory (involving a permutation of factors)

$$\widetilde{\mathrm{BP}}_*(\bigwedge_n \mathbb{Z}/p) \to \widetilde{\mathrm{BP}}_*(\bigwedge_n X_i) \cong \widetilde{\mathrm{BP}}_*(\bigwedge_k \mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \bigotimes_{n-k} \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$$

with the splitting Ψ_k from Theorem 2.1, to get a BP_*-linear map

$$\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \to \bigotimes_{i=1}^n \widetilde{\mathrm{BP}}_*(X_i).$$

This is a map of $\mathrm{BP}_*\mathrm{BP}$-comodules.

For $k \ge 1$ let $(L_k)_*$ be the free graded BP_*-module with generators y_m in degree $2m$, $0 < m < p^k$. We have a canonical BP_*-linear projection

$$\begin{aligned} \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty) &\to (L_k)_* \\ \beta_m &\mapsto \begin{cases} y_m & \text{for } 0 < m < p^k, \\ 0 & \text{for } m \ge p^k. \end{cases} \end{aligned}$$

However, $(L_k)_*$ does not carry an induced $\mathrm{BP}_*\mathrm{BP}$-comodule structure.

Theorem 2.2. *These maps introduced so far induce an isomorphism of* BP_**-modules*

$$\Gamma_n : \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \cong \bigoplus J_1 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} J_n.$$

The direct sum is over all tensor products with J_i equal to $\widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$ or to $(L_k)_*$, where k is the number of J_j, $j < i$, with $J_j = \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$.

In [12, Theorem 5.1] it was shown (for any prime) that $\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$ has a BP_*-module filtration whose associated graded module is BP_*-isomorphic to the right hand side of Theorem 2.2. Our result shows that the filtration can be omitted (for odd primes). Theorem 2.2 implies a similar description of $\widetilde{\Omega}^{SO}_*(\wedge^n \mathbb{Z}/p) = \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \otimes_{\mathrm{BP}_*} (\Omega^{SO}_*)_{(p)}$. .

Corollary 2.3. *The Landweber exact Künneth sequence* [14]

$$0 \to \widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \to \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \to$$
$$\to \left(\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p), \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p))\right)_{*-1} \to 0$$

splits BP_**-linearly.*

In fact, with respect to Theorem 2.2, the tensor product on the left corresponds to the tensor product summands with $J_n = \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$, and the torsion product on the right corresponds to the tensor product summands with J_n equal to some $(L_k)_*$. Corollary 2.3 immediately implies Theorem 1.1.

We can conveniently summarize Theorems 2.1 and 2.2 in terms of a chain model description of $\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$ similar to the one for ordinary homology. Recall that $\widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$ is generated by elements $z_m \in \widetilde{\mathrm{BP}}_{2m+1}(\mathbb{Z}/p)$, $m \geq 0$, represented by classifying maps $L^{2m+1} \to B\mathbb{Z}/p$ of standard lens spaces $L^{2m+1} = S^{2m+1}/(\mathbb{Z}/p)$. These generators are subject to the relations

$$\sum_{i=0}^{m} a_i \cdot z_{m-i} = 0$$

where $a_i \in \mathrm{BP}_{2i}$ appear in the formal group law for BP-theory, cf. Section 3.

Motivated by this calculation let C^{BP}_* be the free BP_*-chain complex in one generator c_d in each degree $d \geq 1$ and equipped with the BP_*-linear differential $C^{BP}_* \to C^{BP}_{*-1}$ equal to

$$c_{2m} \mapsto \sum_{i=0}^{m-1} a_i \cdot c_{2(m-i)-1}\,, \quad c_{2m+1} \mapsto 0.$$

It is then clear that $\mathrm{H}_*(C^{BP}_*) = \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$. Note the formal similarity to the chain complex C_*, computing the reduced singular homology $\widetilde{\mathrm{H}}_*(\mathbb{Z}/p)$ considered before.

Theorem 2.4. *There is a canonical* BP_**-linear isomorphism*

$$\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \cong \mathrm{H}_*(C^{BP}_* \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} C^{BP}_*).$$

Again, Holzsager's stable splitting of $B\mathbb{Z}/p$ in [10] leads to an alternative proof of Theorem 2.4 for $n \leq 2(p-1)$.

The above isomorphism is compatible with the isomorphism of Theorem 2.2 in the sense that the right hand side maps isomorphically to the right hand side of Theorem 2.2 by sending each generator c_{2m+1} to $z_m \in \widetilde{\mathrm{BP}}_*(B\mathbb{Z}/p)$ and each generator c_{2m} to $y_m \in (L_k)_*$ (respectively to 0 for $m \geq p^k$).

The BP_*-module on the right of Theorem 2.2 is not invariant under the canonical Sym_n-action permuting the factors of $(\mathbb{Z}/p)^n$, and does not carry an induced $\mathrm{BP}_*\mathrm{BP}$-comodule structure. However, we can equally well consider the induced map

$$\widetilde{BP}_*(\wedge^n \mathbb{Z}/p) \to \bigoplus J_1 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} J_n,$$

where now the sum is over all tensor products with $J_i = \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$ or $J_i = \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$ and at least one occurence of $\widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$. As a corollary to Theorem 2.2 this is an injective BP_*-linear map and a map of $\mathrm{BP}_*\,\mathrm{BP}$-comodules. It is equivariant with respect to the natural Sym_n-actions on both sides.

Since the beginning of equivariant bordism theory [6] the toral element $z_0 \otimes \cdots \otimes z_0 \in \mathrm{BP}_n(\wedge^n \mathbb{Z}/p)$ has played a significant role. The solution of the Conner-Floyed conjecture, stating that the annihilator ideal of this element is equal to $(p, v_1, \ldots, v_{n-1}) \subset \mathrm{BP}_*$, took more than fifteen years and was finally achieved in [20, 15]. This shows that extraordinary group homology is difficult to calculate, even for groups as basic as $(\mathbb{Z}/p)^n$.

For us it is important that Ravenel-Wilson's solution of the Conner-Floyd conjecture not only gives information on the annihilator ideal of the toral class or injectivity of the iterated Künneth map [12], but implies a stronger statement: The image of the iterated Künneth map on the one hand, and the BP_*-submodule of $\mathrm{BP}_*((\mathbb{Z}/p)^n)$ generated by elements coming from proper subgroups of $(\mathbb{Z}/p)^n$, the inessential Brown-Peterson group homology, on the other, intersect trivially.

The largest part of our paper is devoted to showing that the image of the iterated Künneth map and the inessential Brown-Peterson homology span the whole of $\mathrm{BP}_*((\mathbb{Z}/p)^n)$. This involves a detailed examination of the Pontryagin product on $\mathrm{BP}((\mathbb{Z}/p)^n)$ induced by the group structure on $(\mathbb{Z}/p)^n$.

As a consequence of these calculations we get an interesting level structure on $\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$, which nicely complements Theorem 2.4.

Definition 2.5. For $1 \le k \le n$ let

$$\widetilde{\mathrm{BP}}_*^{(k)}(\wedge^n \mathbb{Z}/p) \subset \mathrm{BP}_*(\wedge^n \mathbb{Z}/p)$$

be the BP_*-submodule generated by the images of the compositions

$$\widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \xrightarrow{\Phi_k} \mathrm{BP}_*((\mathbb{Z}/p)^k) \xrightarrow{\phi_*} \mathrm{BP}_*((\mathbb{Z}/p)^n) \to \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$$

where on the left we take a k-fold tensor product and $\phi : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^n$ is some group homomorphism.

Theorem 2.6. *There is a direct sum decomposition*

$$\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) = \bigoplus_{k=1}^{n} \widetilde{\mathrm{BP}}_*^{(k)}(\wedge^n \mathbb{Z}/p)$$

as BP_**-modules and* $\mathrm{BP}_*\,\mathrm{BP}$*-comodules. The summand* $\widetilde{BP}_*^{(k)}(\wedge^n \mathbb{Z}/p)$ *corresponds to the homology classes in Theorem 2.4 which are represented by chains involving exactly* k *odd degree generators* c_{2m+1}.

Theorem 2.6 implies Theorem 1.2 from the beginning of this paper. We remark that the decomposition in Theorem 2.6 is (already for $n = 2$) not induced by a stable splitting of $B(\mathbb{Z}/p)^n$, as constructed [16, 17] for instance, because the $\mathbb{F}_p$-homology of each of the resulting wedge summands must be invariant under the Bockstein homomorphism.

Some of our results, including the computations in Section 5, carry over to the case $p = 2$ and hence have implications for free stably almost complex $(\mathbb{Z}/2)^n$-manifolds. However, we will show at the end of section 6 that the image of the iterated Künneth map and the inessential Brown-Peterson homology do intersect nontrivially for $p = 2$, so that not all of our methods cover the case $p = 2$. In particular Theorem 2.6 requires odd p. This is remarkable, because the Conner-Floyd conjecture eventually holds for all primes, see [15] and [12, Appendix]. We conjecture that Theorem 2.2 remains true for $p = 2$.

3. Outline of proof

For more detailed information on the following material we refer to [12] and [22]. We have $\mathrm{BP}^*(\mathbb{C}\mathrm{P}^\infty) = \mathrm{BP}^*[[x]]$ with a generator $x \in \mathrm{BP}^2(\mathbb{C}\mathrm{P}^\infty)$ equal to the first Conner-Floyd characteristic class. The group homomorphism $S^1 \to S^1$, $t \mapsto t^p$, induces a map $p : BS^1 \to BS^1$ and hence, via the formal group law for Brown-Peterson theory and the model $BS^1 = \mathbb{C}\mathrm{P}^\infty$, leads to an equation

$$(p)^* x = \sum_{i=0}^{\infty} a_i \cdot x^{1+i} \in \mathrm{BP}^*(\mathbb{C}\mathrm{P}^\infty)$$

with $a_i \in \mathrm{BP}_{2i}$, $i \geq 0$. It is well known that the generators $v_m \in \mathrm{BP}_{2p^m-2}$ can be chosen such that

$$a_{p^m-1} \equiv v_m \mod (v_0, \ldots, v_{m-1})$$

for $m \geq 0$. Here we set $v_0 := p$ so that in particular $a_0 = p$. We have elements $\beta_m \in \mathrm{BP}_{2m}(\mathbb{C}\mathrm{P}^\infty)$ dual to x^m. These are represented by the standard inclusions $\mathbb{C}\mathrm{P}^m \hookrightarrow \mathbb{C}\mathrm{P}^\infty$, and satisfy the equation

$$\beta_m \cap x = \beta_{m-1}$$

for $m \geq 1$, where we set $\beta_0 := 1 \in \mathrm{BP}_0(\mathbb{C}\mathrm{P}^\infty)$. The Gysin sequence associated to the fibration

$$S^1 \hookrightarrow B\mathbb{Z}/p \xrightarrow{\pi} \mathbb{C}\mathrm{P}^\infty$$

shows that

$$\mathrm{BP}_{2m+1}(\mathbb{Z}/p) = \mathrm{coker}(- \cap (p)^* x : \mathrm{BP}_{2m+2}(\mathbb{C}\mathrm{P}^\infty) \to \mathrm{BP}_{2m}(\mathbb{C}\mathrm{P}^\infty)).$$

Hence the generators $\beta_{2m} \in \mathrm{BP}_m(\mathbb{C}\mathrm{P}^\infty)$ induce generators $z_m \in \mathrm{BP}_{2m+1}(\mathbb{Z}/p)$ for $m \geq 0$, that are subject to the relations

$$\sum_{i=0}^{m} a_i \cdot z_{m-i} = 0.$$

Note that in [12] a slightly different notation is used, where z_m denotes the generator in $\mathrm{BP}_{2m-1}(\mathbb{Z}/p)$. We have $z_m = t_*(\beta_m)$ with a stable transfer $t : \Sigma\mathbb{C}\mathrm{P}^\infty \to B\mathbb{Z}/p$, see [12, (2.12.)]. From this we obtain relations

$$z_m \cap \pi^*(x) = z_{m-1}$$

for $m \geq 1$. Also the stable transfer map allows us to compute the $\mathrm{BP}_*\,\mathrm{BP}$-comodule structure on $\widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$ from the one on $\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$, whis is well known, see for example [22, Theorem 1.48]. In accordance with Section 2 the element z_m is represented by the classifying map $L^{2m+1} \to B\mathbb{Z}/p$ of the standard lens space L^{2m+1}.

Following [12] we set

$$N_* := \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$$

and use the shorthand

$$N_*^k := N_* \otimes_{BP_*} \cdots \otimes_{BP_*} N_*$$

with k tensor factors.

From the above description of $\mathrm{BP}_{2m+1}(\mathbb{Z}/p)$ we get a free resolution

$$(F_1)_* \xrightarrow{f_1} (F_0)_* \xrightarrow{f_0} N_{*-1} \to 0$$

where F_1 and F_0 are free graded BP_*-modules in generators y_m of degree $2m$, $m > 0$, and

$$f_1(y_m) = \sum a_i \cdot y_{m-i}\,,\ f_0(y_m) = z_{m-1}.$$

Hence for any graded BP_*-module M_* we have

$$(\mathrm{Tor}_{\mathrm{BP}_*}(M_*, N_*))_{m-1} = \ker\big(\mathrm{id} \otimes_{\mathrm{BP}_*} f_1 : M_* \otimes_{\mathrm{BP}_*} F_1 \to M_* \otimes_{\mathrm{BP}_*} F_0\big)_m \subset (M_* \otimes_{\mathrm{BP}_*} F_1)_m.$$

The BP_*-homology of $\wedge^n B\mathbb{Z}/p$ fits into Landweber's exact Künneth sequence [14]

$$0 \to \widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} N_* \to \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \to \big(\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p), N_*)\big)_{*-1} \to 0.$$

Using the above free resolution of N_* the map $\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \to \big(\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p), N_*)\big)_{*-1}$ has the following geometric interpretation, cf. [12, Section 5]. We define C as the stable cofibre in

$$B\mathbb{Z}/p \xrightarrow{\pi} \mathbb{C}\mathrm{P}^\infty \to C.$$

After taking the smash product with $\wedge^{n-1} B\mathbb{Z}/p$ on the left, the induced long exact sequence in BP_*-theory induces a short exact sequence

$$0 \to \mathrm{coker}(\mathrm{id} \otimes_{\mathrm{BP}_*} f_1) \to \widetilde{BP}_*(\wedge^n \mathbb{Z}/p) \to \ker(\mathrm{id} \otimes_{\mathrm{BP}_*} f_1) \to 0$$

which can be identified with the exact Künneth sequence (with a degree shift in the torsion product). In this description the map in the Künneth sequence is induced on the space level by

$$\gamma_{(B\mathbb{Z}/p,\dots,B\mathbb{Z}/p,\mathbb{C}\mathrm{P}^\infty)} = \mathrm{id} \wedge \pi : \bigwedge_n B\mathbb{Z}/p \to \bigwedge_{n-1} B\mathbb{Z}/p \wedge \mathbb{C}\mathrm{P}^\infty.$$

The proofs of Theorems 2.1 and 2.2 are parallel and by induction on n. We hence assume that Theorem 2.2 holds for $n-1$. According to Corollary 5.5 below (also see [12, Theorem 4.1]) the composition

$$(\mathrm{Tor}_{\mathrm{BP}_*}(N_*^k, N_*))_{*-1} = \ker\big(\mathrm{id} \otimes_{\mathrm{BP}_*} f_1 : N_*^k \otimes_{\mathrm{BP}_*} F_1 \to N_*^k \otimes_{\mathrm{BP}_*} F_0\big)_* \to N_*^k \otimes_{\mathrm{BP}_*} L_k$$

is an isomorphism for $1 \le k \le n-1$. We hence get a commutative diagram

$$\begin{array}{ccc} \big(\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p), N_*)\big)_{*-1} & \hookrightarrow & \widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty) \\ \cong \downarrow \mathrm{Tor}_{\mathrm{BP}_*}(\Gamma_{n-1},\mathrm{id}) & & \downarrow \Gamma_{n-1}\otimes\mathrm{proj.} \\ \big(\mathrm{Tor}_{\mathrm{BP}_*}(\bigoplus J_1 \otimes \cdots \otimes J_{n-1}, N_*)\big)_{*-1} & \xrightarrow{\cong} & \bigoplus_{J_n = L_k} J_1 \otimes \cdots \otimes J_n \end{array}$$

where Γ_{n-1} is taken from Theorem 2.2 and the subscript $J_n = L_k$ indicates that we are just considering those summands from Theorem 2.2 with $J_n = (L_k)_*$ (with appropriate k).

As in Theorem 2.2 we have a canonical map

$$\Gamma_n : \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \to \bigoplus_{J_1 \otimes \cdots \otimes J_n \neq N_*^n} J_1 \otimes \cdots \otimes J_n,$$

where the subscript means that we just sum over those tensor factors in Theorem 2.2 that are not equal to N_*^n. This is achieved by first applying $\pi : B\mathbb{Z}/p \to \mathbb{C}\mathrm{P}^\infty$ to at least one factor of $\wedge^n B\mathbb{Z}/p$ and then applying the splitting of Theorem 2.2 to the BP-homology of the smash product of the remaining copies of $B\mathbb{Z}/p$. Note that Theorem 2.2 for $n-1$ (or less) is sufficient for this construction.

Proposition 3.1. *Let*

$$K_* \subset \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$$

be a BP_**-submodule and* $\mathrm{BP}_*\mathrm{BP}$*-sub-comodule which maps surjectively onto the image of the map*

$$\Gamma_n : \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \to \bigoplus_{J_1 \otimes \cdots \otimes J_n \neq N_*^n} J_1 \otimes \cdots \otimes J_n$$

defined above.

Let $\Phi_n : N_*^n \to \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$ *be the iterated Künneth map and assume that*

$$K_* \cap \operatorname{im} \Phi_n = 0.$$

Then both Theorems 2.1 and 2.2 hold for n*.*

Proof. Using the Künneth sequence and the above commutative diagram the assumption on K_* leads to

$$\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) = K_* + \widetilde{\mathrm{BP}}_*(\wedge^{n-1} \mathbb{Z}/p) \otimes_{\mathrm{BP}_*} N_*. \tag{2}$$

Because Theorem 2.2 holds for $n-1$ we have a canonical isomorphism

$$\widetilde{\mathrm{BP}}_*(\wedge^{n-1} \mathbb{Z}/p) \otimes_{\mathrm{BP}_*} N_* = \bigoplus_{J_n = N_*} J_1 \otimes_{\mathrm{BP}_*} \otimes \cdots \otimes_{\mathrm{BP}_*} J_n$$

where the subscript $J_n = N_*$ means that we restrict ourselves to the tensor product summands with $J_n = N_*$. Now consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{BP}}_*(\wedge^{n-1} \mathbb{Z}/p) \otimes_{\mathrm{BP}_*} N_* & \hookrightarrow & \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \\ \downarrow \cong & & \downarrow \mathrm{proj.} \circ \Gamma_n \\ \bigoplus_{J_n = N_*} J_1 \otimes \cdots \otimes J_n & \longrightarrow & \bigoplus_{J_1 \otimes \cdots \otimes J_n \neq N_*^n, J_n = N_*} J_1 \otimes \cdots \otimes J_n \end{array}$$

where the horizontal map on the bottom is a projection.

By assumption the right hand vertical map is surjective after restriction to $K_* \subset \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p)$. Together with equation (2) this implies

$$\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) = K_* + \operatorname{im} \Phi_n$$

and with the assumption $K_* \cap \operatorname{im} \Phi_n = 0$ we conclude

$$\widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \cong K_* \oplus \operatorname{im} \Phi_n$$

as BP_*-modules and $\mathrm{BP}_*\mathrm{BP}$-comodules.

The iterated Künneth map Φ_n is injective [12, Corollary 3.3] (this also follows from the injectivity of the usual Künneth map and our assumption that Theorem 2.2 holds for $n-1$). From this Theorem 2.1 follows for n.

Once this has been established Theorem 2.2 for n follows by induction using the splitting Ψ_n and the canonical isomorphisms $\mathrm{Tor}_{\mathrm{BP}_*}(N^k_*, N_*)_{*-1} \cong N^k_* \otimes_{\mathrm{BP}_*} L_k$ from Corollary 5.5 for $k \leq n-1$. Indeed, using Theorem 2.2 for $n-1$, we then obtain the isomorphism

$$\big(\,\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p), N_*)\big)_{*-1} \cong \big(\,\mathrm{Tor}_{\mathrm{BP}_*}(\bigoplus J_1 \otimes \cdots \otimes J_{n-1}, N_*)\big)_{*-1} \cong \bigoplus_{J_n = L_k} J_1 \otimes \cdots \otimes J_n.$$

And this, together with the fact that the map

$$\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p) \to \big(\,\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^{n-1}\mathbb{Z}/p), N_*)\big)_{*-1}$$

in the Künneth exact sequence is induced by $\mathrm{id}\wedge\pi : \wedge^n B\mathbb{Z}/p \to \wedge^{n-1}B\mathbb{Z}/p \wedge \mathbb{C}\mathrm{P}^\infty$, concludes the proof of Theorem 2.2 for n. □

For the proof of Theorem 2.4 observe that the homology $\mathrm{H}_*(C_* \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} C_*)$ (we now write C_* instead of C^{BP}_*) can be computed by induction in a completely analogous fashion as $\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)$. The only adjustment consists in replacing the topological map $\pi : B\mathbb{Z}/p \to \mathbb{C}\mathrm{P}^\infty$ by the chain map

$$\begin{aligned} \epsilon : C_* &\to F_1 \\ c_d &\mapsto \begin{cases} y_m \text{ for } d = 2m, \\ 0 \text{ for } d \text{ odd}, \end{cases} \end{aligned}$$

where F_1 is equipped with the zero differential. We hence get a Künneth sequence (setting $C^k_* := C_* \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} C_*$)

$$0 \to \mathrm{H}_*(C^{n-1}_*) \otimes_{\mathrm{BP}_*} \mathrm{H}_*(C_*) \to \mathrm{H}_*(C^n_*) \to \big(\,\mathrm{Tor}_{\mathrm{BP}_*}(\mathrm{H}_*(C^{n-1}_*), \mathrm{H}_*(C_*))\big)_{*-1} \to 0$$

induced by the map

$$\mathrm{H}_*(C^n_*) \to \mathrm{H}_*(C^{n-1}_*) \otimes_{\mathrm{BP}_*} F_1$$

which on the chain level is given by $\mathrm{id} \otimes \epsilon : C^n_* \to C^{n-1}_* \otimes F_1$.

This and an iterated use of Corollary 5.5 show that there is a canonical isomorphism of BP_*-modules

$$\mathrm{H}_*(C^n_*) \cong \bigoplus J_1 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} J_n$$

much as in Theorem 2.2.

It remains to find a submodule $K_* \subset \widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)$ enjoying the properties described in Proposition 3.1.

Definition 3.2. Consider the BP_*-submodule of $\mathrm{BP}_*((\mathbb{Z}/p)^n)$which is generated by the images $\phi_*(\mathrm{BP}_*((\mathbb{Z}/p)^k))$, where $\phi : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^n$ is a group homomorpism and $k < n$. The image of this submodule in $\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)$ is called the *(reduced) inessential Brown-Peterson homology* of $(\mathbb{Z}/p)^n$. We denote this submodule by K_*. This is a $\mathrm{BP}_*\mathrm{BP}$-subcomodule of $\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)$.

Recall that the *essential cohomology* of a group G is defined as the ideal in $\mathrm{H}^*(G)$ consisting of those classes that restrict to 0 under all inclusions of proper subgroups $H < G$. Definition 3.2 can be viewed as a dual notion for Brown-Peterson group homology.

Theorem 3.3. *The module K_* fulfills the requirements of Proposition 3.1.*

This will be proved in Section 6 below.

4. Preliminaries on N^k_*

We need some preparation concerning the structure of $N^k_* = \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p) \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$ for $k \geq 1$, extending results in [12, Section 3], from which we borrow our notation (with the exception that $z_m \in N_{2m+1}$ for $m \geq 0$).

For $I = (i_1, \ldots, i_k) \in \mathbb{N}^k$ we write $z_I := z_{i_1} \otimes \cdots \otimes z_{i_k} \in N^k_*$. The BP_*-module N^k_* is generated by the elements z_I. It is worthwhile to consider the increasing BP_*-module filtration of N^k_* given by

$$\mathcal{F}_J(N^k) := \langle z_I | I \leq J \rangle_{\mathrm{BP}_*} \subset N^k_*$$

using the lexicographic order on $\mathbb{N}^k$. Let $E_*(N^k_*)$ be the associated graded BP_*-module.

We know from [12, Theorem 3.2] that $E_*(N^k_*)$ is a free module over $\mathrm{BP}_*/(v_0, \ldots, v_{k-1})$. This implies the following non-squeezing result.

Lemma 4.1. *Let $k > \ell$ and let*

$$\phi : N^k_* \to N^\ell_*$$

be a not necessarily grading preserving BP_**-linear map. Then ϕ is equal to zero.*

Proof. We consider multiplication by powers of the generator $v_\ell \in \mathrm{BP}_*$. On the one hand, every element in N^k_* is v_ℓ-torsion because multiplication by v_ℓ is zero on $E_*(N^k_*)$. On the other hand, if $c \neq 0 \in N^k_*$, then for all $\nu \geq 0$ we have $(v_\ell)^\nu \cdot c \neq 0$ because $E_*(N^\ell_*)$ is free over $\mathrm{BP}_*/(v_0, \ldots, v_{\ell-1})$. From this our assertion follows. □

Some of our later computations will first be carried out in singular homology and then translated to BP-homology. For this transition we need the following preliminaries.

Let $H_* := \widetilde{\mathrm{H}}_*(\mathbb{Z}/p)$ and let $u_* : N_* \to H_*$ be the homological orientation. The module H_* is generated by

$$h_m := u(z_m) \in H_{2m+1},$$

where $m \geq 0$. As for BP-homology we have a filtration

$$\mathcal{F}_J(H^k_*) := \langle h_I | I \leq J \rangle \subset H^k_* := H_* \otimes \cdots \otimes H_*$$

and an associated graded $\mathbb{F}_p$-module $E_*(H^k_*)$, which is canonically isomorphic to H^k_*.

In the following we fix k and consider the ideal

$$R := (v_k, v_{k+1}, \ldots) \subset \mathrm{BP}_* .$$

Lemma 4.2. *The kernel of the (obviously surjective) map*

$$u^k_* := u_* \otimes \cdots \otimes u_* : N^k_* \to H^k_*$$

is equal to $R \cdot N^k_*$.

Proof. It is clear that $R \cdot N^k_*$ is contained in the kernel of u^k_*.

For the reverse inclusion we observe that the map u^k_* induces a canonical map

$$\omega : E_*(N^k_*) \to E_*(H^k_*).$$

Let $c \in \ker u^k_*$ and let J be minimal with $c \in \mathcal{F}_J(N^k_*)$. Then the class $[c] \in (E_J)_*(N^k_*)$ lies in the kernel of ω. Because $(E_J)_*(N^k_*)$ is free over $\mathrm{BP}_*/(v_0, \ldots, v_{k-1})$ with basis $([z_I])$ and $(E_J)_*(H^k_*)$ is free over $\mathbb{F}_p = \mathrm{BP}_*/(v_0, v_1, \ldots)$ with basis $([h_I])$ we get

$$c \in R \cdot N^k_* + \mathcal{F}_{J-1}(N^k_*).$$

As $R \cdot N^k_* \subset \ker u^k_*$, the assertion follows by induction on the filtration degree J. □

Proposition 4.3. *Assume that I is an index set and that*

$$\phi : \bigoplus_I N_*^k \to \bigoplus_I N_*^k$$

is a (not necessarily grading preserving) BP_**-linear map which induces a surjection*

$$\bigoplus_I H_*^k \to \bigoplus_I H_*^k$$

after dividing out $R \cdot \bigoplus_I N_*^k$. Then ϕ itself is surjective.

Proof. For $j \geq 0$ consider the decreasing filtration

$$\mathcal{F}_j \left(\bigoplus_I N_*^k \right) = \bigoplus_I \mathcal{F}_j(N_*^k)$$

where

$$\mathcal{F}_j(N_*^k) := R^j \cdot N_*^k \subset N_*^k$$

and $R^j = R \cdot \ldots \cdot R$ with j factors. Note that in each single degree of $\oplus_I N_*^k$ this filtration is finite. The associated graded BP_*-modules is denoted $E_*(\oplus_I N_*^k)$.

By Lemma 4.2 and our assumption the map ϕ induces a surjective map $E_*(\oplus_I N_*^k) \to E_*(\oplus_I N_*^k)$. From this the assertion of the proposition follows by a (finite in each degree) induction on filtration degrees.

□

5. Calculations of inessential Brown-Peterson homology

In this section we perform some explicit calculations that will be employed in the proof of Theorem 3.3. Throughout we assume that Theorems 2.1 and 2.2 have been proven for $n-1$.

Let $k, \ell \geq 1$ with $k + \ell = n$. Furthermore let $1 \leq \delta_1 \leq \cdots \leq \delta_\ell \leq k$. We choose integers

$$\begin{array}{rcl} 0 & \leq & \lambda_{1,1}, \ldots, \lambda_{1,\delta_1} < p \\ 0 & \leq & \lambda_{2,1}, \ldots, \lambda_{2,\delta_2} < p \\ & \vdots & \\ 0 & \leq & \lambda_{\ell,1}, \ldots, \lambda_{\ell,\delta_\ell} < p \end{array}$$

and collect them to a vector $\Lambda = (\lambda_{1,1}, \ldots, \lambda_{1,\delta_1}, \ldots, \lambda_{\ell,1}, \ldots, \lambda_{\ell,\delta_\ell})$ of length $\delta_1 + \cdots + \delta_\ell$. For $(x_1, \ldots, x_k) \in (\mathbb{Z}/p)^k$ and $1 \leq j \leq \ell$ we set

$$y_j := \sum_{i=1}^{\delta_j} \lambda_{j,i} \cdot x_i \in \mathbb{Z}/p$$

and use this to define the group homomorphism $\phi_\Lambda : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^{k+\ell}$ by the formula

$$\phi_\Lambda(x_1, \ldots, x_k) := (x_1, \ldots, x_{\delta_1}, \mathbf{y_1}, x_{\delta_1+1}, \ldots, x_{\delta_2}, \mathbf{y_2}, \ldots, \mathbf{y}_\ell, x_{\delta_\ell+1}, \ldots, x_k).$$

We print y_j in boldface to make the definition of ϕ_Λ more transparent: We just plug in $y_1, \ldots y_\ell$ at positions $\omega_j := \delta_j + j$ for $j = 1, \ldots, \ell$.

Note that we get an induced map $\wedge^k B\mathbb{Z}/p \to \wedge^{k+\ell} B\mathbb{Z}/p$. In the following we study this map in BP-theory. At first we prove a vanishing result.

For each $1 \le i \le k+\ell$ we choose a space X_i equal to $B\mathbb{Z}/p$ or $\mathbb{CP}^\infty$. Applying the maps $\pi : B\mathbb{Z}/p \to \mathbb{CP}^\infty$ at the positions i with $X_i = \mathbb{CP}^\infty$ we obtain a map

$$\gamma_{(X_1,\dots,X_n)} : \bigwedge_n B\mathbb{Z}/p \to \bigwedge_{i=1}^{k+\ell} X_i$$

as before Theorem 2.2.

We now assume that we have at least one $X_i = \mathbb{CP}^\infty$. Then the number of $X_i = B\mathbb{Z}/p$ is at most $n-1$, and we get an induced map

$$\Theta_\Lambda : N_*^k \to \widetilde{\mathrm{BP}}_*(\wedge^k \mathbb{Z}/p) \xrightarrow{(\phi_\Lambda)_*} \widetilde{\mathrm{BP}}_*(\wedge^{k+\ell}\mathbb{Z}/p) \to \bigotimes_{i=1}^{k+\ell} \widetilde{\mathrm{BP}}_*(X_i) \tag{3}$$

using Theorem 2.1.

Now choose $1 \le \alpha \le k+\ell = n$ and define

- $r(\alpha)$ as the minimal index $1 \le j \le \ell$ with $\omega_j (= \delta_j + j) \ge \alpha$,
- $n(\alpha)$ as $\ell - r(\alpha) + 1$,
- $m(\alpha)$ as the number of indices $i \ge \alpha$ with $X_i = \mathbb{CP}^\infty$.

In other words, $n(\alpha)$ is is the number of components y_j that appear at a position at least α in $\phi_\Lambda(x_1,\dots,x_k)$, and $m(\alpha)$ is the number of $\mathbb{CP}^\infty$ at a position at least α in the product $\wedge_{i=1}^n X_i$.

Recall that the map $\pi : B\mathbb{Z}/p \to \mathbb{CP}^\infty$ induces the zero map in reduced BP-homology (for degree reasons). The following is a generalization of this fact.

Proposition 5.1. *If there is some $1 \le \alpha \le k+\ell$ with $m(\alpha) > n(\alpha)$, then the map Θ_Λ is equal to zero.*

We have the following important consequence, where we equip the set of ℓ-tuples $1 \le j_1 < \dots < j_\ell \le n$ with the lexicographic order.

Corollary 5.2. *Let the number of $X_i = \mathbb{CP}^\infty$ be equal to ℓ and assume that*

$$(\omega_1,\dots,\omega_\ell) < (j_1,\dots,j_\ell)$$

where $1 \le j_1 < \dots < j_\ell \le n$ are those indices with $X_{j_i} = \mathbb{CP}^\infty$. Then the map Θ_Λ is equal to zero.

Proof of Proposition 5.1. We assume that $X_i = B\mathbb{Z}/p$ for all $i < \alpha$, which is no loss of generality, because the maps $B\mathbb{Z}/p \to X_i = \mathbb{CP}^\infty$ with $i < \alpha$ can be applied later.

We have

$$\phi_\Lambda = \phi_2 \circ \phi_1$$

where

$$\begin{array}{rcl} \phi_1 : (\mathbb{Z}/p)^k & \to & (\mathbb{Z}/p)^{k+n(\alpha)} \\ (x_1,\dots,x_k) & \mapsto & (x_1,\dots,x_{\delta_{r(\alpha)}}, \mathbf{y}_{r(\alpha)},\dots,\mathbf{y}_\ell, x_{\delta_\ell+1},\dots,x_k) \end{array}$$

and

$$\begin{array}{rcl} \phi_2 : (\mathbb{Z}/p)^{k+n(\alpha)} & \to & (\mathbb{Z}/p)^{k+\ell} \\ (x_1,\dots,x_{k+n(\alpha)}) & \mapsto & (x_1,\dots,x_{\delta_1}, \mathbf{y}_1,\dots,\mathbf{y}_{r(\alpha)-1}, x_{\delta_{r(\alpha)-1}+1},\dots,x_{k+n(\alpha)}). \end{array}$$

In other words: At first we plug in only $y_{r(\alpha)}, \ldots, y_\ell$, and then, in a second step, the remaining $y_1, \ldots, y_{r(\alpha)-1}$.

For $1 \leq i \leq k + n(\alpha)$ we define

$$Y_i := \begin{cases} B\mathbb{Z}/p \text{ for } i \leq \alpha - r(\alpha), \\ X_{i+r(\alpha)-1} \text{ for } i > \alpha - r(\alpha). \end{cases}$$

We then have a commutative diagram

$$\begin{array}{ccc} \bigwedge^{k+n(\alpha)} B\mathbb{Z}/p & \longrightarrow & \bigwedge_{i=1}^{k+n(\alpha)} Y_i \\ \Big\downarrow {\scriptstyle B\phi_2} & & \Big\downarrow {\scriptstyle \chi} \\ \bigwedge^{k+\ell} B\mathbb{Z}/p & \xrightarrow{\gamma(X_i)} & \bigwedge_{i=1}^{n} X_i. \end{array}$$

Here the map χ is induced by

$$\phi_2|_{(\mathbb{Z}/p)^{\alpha-r(\alpha)}\times 1} : (\mathbb{Z}/p)^{\alpha-r(\alpha)} \to (\mathbb{Z}/p)^{\alpha-1}.$$

The assumption that $X_i = B\mathbb{Z}/p$ for all $i < \alpha$ is used here.

It is hence enough to prove that the composition

$$N_*^k \to \widetilde{\mathrm{BP}}_*(\wedge^k \mathbb{Z}/p) \xrightarrow{(\phi_1)_*} \widetilde{\mathrm{BP}}_*(\wedge^{k+n(\alpha)}\mathbb{Z}/p) \longrightarrow \bigotimes_{i=1}^{k+n(\alpha)} \widetilde{\mathrm{BP}}_*(Y_i)$$

is equal to zero. But this holds by the non-squeezing Lemma 4.1, because the number of indices $i = 1, \ldots, k + n(\alpha)$ with $Y_i = B\mathbb{Z}/p$ is equal to $k + n(\alpha) - m(\alpha) < k$ and $\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$ is a free BP_*-module.

□

In the following we consider the maps Θ_Λ for the special case $k = n - 1$, $\ell = 1$ and $\delta_1 = k$. This means we are given $0 \leq \lambda_1, \ldots, \lambda_k < p$ and a group homomorphism

$$\begin{array}{rcl} \phi = \phi_{(\lambda_1,\ldots,\lambda_k)} : (\mathbb{Z}/p)^k & \to & (\mathbb{Z}/p)^{k+1} \\ (x_1, \ldots, x_k) & \mapsto & (x_1, \ldots, x_k, \lambda_1 x_1 + \cdots + \lambda_k x_k). \end{array}$$

We obtain induced maps

$$\begin{array}{rcl} \phi_* : N_*^k \to \widetilde{\mathrm{BP}}_*(\wedge^k \mathbb{Z}/p) & \to & \widetilde{\mathrm{BP}}_*(\wedge^{k+1}\mathbb{Z}/p), \\ \phi_* : H_*^k \to \widetilde{\mathrm{H}}_*(\wedge^k \mathbb{Z}/p) & \to & \widetilde{\mathrm{H}}_*(\wedge^{k+1}\mathbb{Z}/p), \end{array}$$

and by composition with the map

$$\gamma_{(B\mathbb{Z}/p,\ldots,B\mathbb{Z}/p,\mathbb{C}\mathrm{P}^\infty)} : \bigwedge_{k+1} B\mathbb{Z}/p \to \bigwedge_{k} B\mathbb{Z}/p \wedge \mathbb{C}\mathrm{P}^\infty$$

we obtain the maps

$$\Theta_{(\lambda_1,\ldots,\lambda_k)} : N_*^k \xrightarrow{\phi_*} \widetilde{\mathrm{BP}}_*(\wedge^{k+1}\mathbb{Z}/p) \to \widetilde{\mathrm{BP}}_*(\wedge^k \mathbb{Z}/p \wedge \mathbb{C}\mathrm{P}^\infty) \xrightarrow{\Psi_k \otimes \mathrm{id}} N_*^k \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$$

$$\Theta_{(\lambda_1,\ldots,\lambda_k)} : H_*^k \xrightarrow{\phi_*} \widetilde{\mathrm{H}}_*(\wedge^{k+1}\mathbb{Z}/p) \to \widetilde{\mathrm{H}}_*(\wedge^k \mathbb{Z}/p \wedge \mathbb{C}\mathrm{P}^\infty) \xrightarrow{\Psi_k \otimes \mathrm{id}} H_*^k \otimes \widetilde{\mathrm{H}}_*(\mathbb{C}\mathrm{P}^\infty).$$

Here we use the splittings of the iterated Künneth map for BP_*-theory (see Theorem 2.1) and ordinary homology (recall that $k = n - 1 < n$).

The generators $\beta_m \in \widetilde{\mathrm{BP}}_{2m}(\mathbb{C}\mathrm{P}^\infty)$ induce generators of $\mathrm{H}_{2m}(\mathbb{C}\mathrm{P}^\infty)$ which we denote by the same symbol. Similarly to $(L_k)_*$ let $(M_k)_*$ be the free graded $\mathbb{F}_p$-module on generators y_m of degree $2m$, $0 < m < p^k$. In addition to the canonical projection $\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty) \to (L_k)_*$ we obtain a canonical projection

$$\widetilde{\mathrm{H}}_*(\mathbb{C}\mathrm{P}^\infty) \to (M_k)_*.$$

by sending $\beta_m \mapsto y_m$ similar as before.

We now consider the compositions

$$\Theta_{(\lambda_1,\ldots,\lambda_k)} : N_*^k \stackrel{\Theta_{(\lambda_1,\ldots,\lambda_k)}}{\longrightarrow} N_*^k \otimes_{\mathrm{BP}_*} \widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty) \to N_*^k \otimes_{\mathrm{BP}_*} L_k$$
$$\Theta_{(\lambda_1,\ldots,\lambda_k)} : H_*^k \stackrel{\Theta_{(\lambda_1,\ldots,\lambda_k)}}{\longrightarrow} H_*^k \otimes \widetilde{\mathrm{H}}_*(\mathbb{C}\mathrm{P}^\infty) \to H_*^k \otimes M_k$$

for various $0 \leq \lambda_1, \ldots, \lambda_k < p$. The first map is graded BP_*-linear and the second map is graded $\mathbb{F}_p$-linear. Both maps are compatible with the orientation $\mathrm{BP} \to \mathrm{H}$.

In the following we use the indexing set

$$\Lambda_k := \{\Lambda \in \{0, \ldots, p-1\}^k \mid (\lambda_1, \ldots, \lambda_k) \neq (0, \ldots, 0)\}.$$

The next calculation is central for the present paper.

Proposition 5.3. *The map*

$$\begin{array}{rcl} \bigoplus_{\Lambda \in \Lambda_k} H_*^k & \to & H_*^k \otimes M_k \\ (x_\Lambda) & \mapsto & \sum_{\Lambda \in \Lambda_k} \Theta_\Lambda(x_\Lambda) \end{array}$$

is surjective.

Proof. We work in unreduced cohomology with $\mathbb{F}_p$-coefficients. Identifying

$$\mathrm{H}^*((\mathbb{Z}/p)^k; \mathbb{F}_p) \cong \mathbb{F}_p[t_1, \ldots, t_k] \otimes \Lambda(s_1, \ldots, s_k) \text{ and } \mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{F}_p) \cong \mathbb{F}_p[t]$$

where $t_1, \ldots, t_k, t$ are indeterminates of degree 2 and $s_1, \ldots, s_k$ indeterminates of degree 1, the map induced in $\mathbb{F}_p$-cohomology by

$$B(\mathbb{Z}/p)^k \stackrel{\phi_{(\lambda_1,\ldots,\lambda_k)}}{\longrightarrow} B(\mathbb{Z}/p)^{k+1} \stackrel{\mathrm{id}\times\pi}{\longrightarrow} B(\mathbb{Z}/p)^k \times \mathbb{C}\mathrm{P}^\infty$$

satisfies

$$(t_1^{m_1} s_1 \cdot \ldots \cdot t_k^{m_k} s_k) \cdot t^\nu \mapsto (t_1^{m_1} s_1 \cdot \ldots \cdot t_k^{m_k} s_k) \cdot (\lambda_1 t_1 + \ldots + \lambda_k t_k)^\nu. \tag{4}$$

The $(p^k \times p^k)$-Vandermonde-matrix

$$X := \begin{pmatrix} 1 & (\lambda_1 t_1 + \cdots + \lambda_k t_k) & \cdots & (\lambda_1 t_1 + \cdots + \lambda_k t_k)^{p^k - 1} \end{pmatrix}_{0 \leq \lambda_1, \ldots, \lambda_k < p}$$

(where the subscript parametrizes the rows) with entries in $\mathbb{F}_p[t_1, \ldots, t_k]$ has determinant

$$\prod_{(\lambda_1,\ldots,\lambda_k)<(\mu_1,\ldots,\mu_k)} ((\lambda_1 - \mu_1)t_1 + \cdots + (\lambda_k - \mu_k)t_k) \neq 0,$$

where we use the lexicographic order in the index set. Hence the column vectors of X are linearly independent over $\mathbb{F}_p[t_1, \ldots, t_k]$.

In view of formula (4) this means that the map

$$\bigoplus_{0\le\lambda_1,\dots,\lambda_k<p}\phi^*_{(\lambda_1,\dots,\lambda_k)}:\bigotimes_k \mathrm{H}^{odd}(\mathbb{Z}/p;\mathbb{F}_p)\otimes\mathrm{H}^{0\le 2m<2p^k}(\mathbb{C}\mathrm{P}^\infty;\mathbb{F}_p)\to\bigoplus_{0\le\lambda_1,\dots,\lambda_k<p}\bigotimes_k\mathrm{H}^{odd}(\mathbb{Z}/p;\mathbb{F}_p)$$

is injective. Dualizing this statement over $\mathbb{F}_p$ and using the identification $H_*=\mathrm{H}_{odd}(\mathbb{Z}/p;\mathbb{F}_p)$ we conclude that the map

$$\begin{aligned}\bigoplus_{0\le\lambda_1,\dots,\lambda_k<p} H^k_* &\to H^k_*\otimes\mathrm{span}_{\mathbb{F}_p}\{\beta_0,\beta_1,\dots,\beta_{p^k-1}\}\\ (x_\Lambda) &\mapsto \sum_{\Lambda\in\{0,\dots,p-1\}^k}\Theta_\Lambda(x_\Lambda)\end{aligned}$$

is surjective. This implies our claim, because the component $(\lambda_1,\dots,\lambda_k)=(0,\dots,0)$ maps isomorphically to $H^k_*\otimes\beta_0$. □

Together with Proposition 4.3 (where I is an index set with p^k-1 elements) this implies

Proposition 5.4. *The* BP_**-linear map*

$$\begin{aligned}\bigoplus_{\Lambda_k} N^k_* &\to N^k_*\otimes_{\mathrm{BP}_*}L_k\\ (x_\Lambda) &\mapsto \sum_{\Lambda\in\Lambda_k}\Theta_\Lambda(x_\Lambda)\end{aligned}$$

is surjective.

As a useful corollary we have (cf. [12, Theorem 4.1])

Corollary 5.5. *The composition*

$$(\mathrm{Tor}_{\mathrm{BP}_*}(N^k_*,N_*))_{*-1}\subset N^k_*\otimes_{\mathrm{BP}_*}F_1=N^k_*\otimes_{\mathrm{BP}_*}\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)\to N^k_*\otimes_{\mathrm{BP}_*}L_k.$$

is a BP_**-linear isomorphism.*

Proof. The algebraic Conner-Floyd conjecture implies that the left and right hand sides have the same cardinalities in each degree, see [12, Proof of Theorem 4.1]. By construction, for each $\Lambda\in\Lambda_k$, the map Θ_Λ factors as

$$\begin{aligned}N^k_*\xrightarrow{\phi_*}\widetilde{\mathrm{BP}}_*(\wedge^{k+1}\mathbb{Z}/p)\to\big(\mathrm{Tor}_{\mathrm{BP}_*}(\widetilde{\mathrm{BP}}_*(\wedge^k\mathbb{Z}/p),N_*)\big)_{*-1}\xrightarrow{\mathrm{Tor}_{\mathrm{BP}_*}(\Psi_k,\mathrm{id})}\\ \to(\mathrm{Tor}_{\mathrm{BP}_*}(N^k_*,N_*))_{*-1}\subset N^k_*\otimes_{\mathrm{BP}_*}F_1\to N^k_*\otimes_{\mathrm{BP}_*}L_k.\end{aligned}$$

Hence, by Proposition 5.4, the module $(\mathrm{Tor}_{\mathrm{BP}_*}(N^k_*,N_*))_{*-1}$ has at least as many elements as $N^k_*\otimes_{\mathrm{BP}_*}L_k$ (in each degree). This completes the proof. □

We remark that this proof of Corollary 5.5 is different from the proof of [12, Theorem 4.1].

Now we return to the situation at the beginning of this section for a fixed choice of $1\le\delta_1\le\dots\le\delta_\ell\le k$, and set

$$X_i:=\begin{cases}B\mathbb{Z}/p, & \text{if } i\neq\omega_j \text{ for all } 1\le j\le\ell,\\ \mathbb{C}\mathrm{P}^\infty, & \text{if } i=\omega_j \text{ for some } 1\le j\le\ell.\end{cases}$$

This means that the components $X_i=\mathbb{C}\mathrm{P}^\infty$ exactly match the $y_1,\dots,y_\ell$ in $\phi_\Lambda(x_1,\dots,x_k)$.

For each choice of $\Lambda \in \{0, \dots, p-1\}^{\delta_1+\cdots+\delta_\ell}$ we compose the map

$$\Theta_\Lambda : N_*^k \to \bigotimes_{i=1}^{k+\ell} \widetilde{\mathrm{BP}}_*(X_i)$$

with projections $\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty) \to L_{\delta_j}$ to obtain a map

$$\Theta_\Lambda : N_*^k \to N_*^{\delta_1} \otimes L_{\delta_1} \otimes N_*^{\delta_2-\delta_1} \otimes L_{\delta_2} \otimes \cdots \otimes N_*^{\delta_l-\delta_{l-1}} \otimes L_{\delta_l} \otimes N_*^{k-\delta_l}.$$

In the following we use the indexing set

$$\Lambda_{\delta_1,\dots,\delta_\ell} := \{\Lambda \in \{0,\dots,p-1\}^{\delta_1+\cdots+\delta_\ell} \mid (\lambda_{j,1},\dots,\lambda_{j,\delta_j}) \neq (0,\dots,0) \text{ for } 1 \le j \le \ell\}.$$

Proposition 5.6. *The* BP_**-linear map (with tensor products over* BP_**)*

$$\begin{aligned} \bigoplus_{\Lambda\in\Lambda_{\delta_1,\dots,\delta_\ell}} N_*^k &\to N_*^{\delta_1} \otimes L_{\delta_1} \otimes N_*^{\delta_2-\delta_1} \otimes L_{\delta_2} \otimes \cdots \otimes N_*^{\delta_l-\delta_{l-1}} \otimes L_{\delta_l} \otimes N_*^{k-\delta_l} \\ (x_\Lambda) &\mapsto \sum_{\Lambda\in\Lambda_{\delta_1,\dots,\delta_\ell}} \Theta_\Lambda(x_\Lambda) \end{aligned}$$

is surjective.

Proof. The corresponding fact in homology follows from an application of Proposition 5.3 separately for each tensor factor L_{δ_j}, $j = 1, \dots, \ell$. Proposition 4.3 then implies the claim. □

6. Proof of Theorem 3.3

We still assume that Theorem 2.2 holds for $n-1$. In the following proposition we apply the map Θ_Λ from formula (3) on page 13 to different choices of $(X_1, \dots, X_n)$ and collect the results into a direct sum. For the target of the following map we refer to the conventions of Theorem 2.2 (with additional restrictions, indicated by a subscript under the direct sum sign). In addition, for each tensor product $J_1 \otimes_{\mathrm{BP}_*} \cdots \otimes_{\mathrm{BP}_*} J_n$ appearing in this theorem, let $\sharp_{\mathbb{C}\mathrm{P}^\infty}(J_1 \otimes \cdots \otimes J_n)$ be the number of factors equal to some $(L_\gamma)_*$. (Note that the number k in Theorem 2.2 has a different meaning than the number k in the next Proposition).

Proposition 6.1. *Let* $k, \ell \ge 1$ *with* $k + \ell = n$ *and let* $1 \le \kappa \le n-1$*. Then the map*

$$\begin{aligned} \bigoplus_{\Lambda\in\Lambda_{\delta_1,\dots,\delta_\ell}} N_*^k &\to \bigoplus_{\sharp_{\mathbb{C}\mathrm{P}^\infty}(J_1\otimes\cdots\otimes J_n)=\kappa} J_1 \otimes \cdots \otimes J_n \\ (x_\Lambda) &\mapsto \left(\sum_{\Lambda\in\Lambda_{\delta_1,\dots,\delta_\ell}} \Theta_\Lambda(x_\Lambda) \right)_{\sharp_{\mathbb{C}\mathrm{P}^\infty}(J_1\otimes\cdots\otimes J_n)=\kappa} \end{aligned}$$

is surjective for $\kappa = \ell$ *and is equal to zero for* $\kappa > \ell$*.*

Proof. The case $\kappa = \ell$ follows by induction on the set of indices $1 \le j_1 < \cdots < j_\ell \le n$ with J_{i_i} equal to some $(L_\gamma)_*$, equipped with the lexicographic order, from Corollary 5.2 and Proposition 5.6. The case $\kappa > \ell$ follows from the non-squeezing Lemma 4.1 and the fact that $\widetilde{\mathrm{BP}}_*(\mathbb{C}\mathrm{P}^\infty)$ is a free BP_*-module. □

This proposition shows that the map

$$K_* \subset \widetilde{\mathrm{BP}}_*(\wedge^n \mathbb{Z}/p) \xrightarrow{\Gamma_n} \bigoplus_{J_1\otimes\cdots\otimes J_n \neq N_*^n} J_1 \otimes \cdots \otimes J_n$$

is surjective. Hence K_* satisfies the first property stated in Proposition 3.1.

It remains to show that $K_* \cap \operatorname{im} \Phi_n = 0$ where Φ_n is the iterated Künneth map. Similarly as in [12, Proof of Cor. 3.3] we can reduce this claim to the behavior of the toral element in $\mathrm{BP}_*((\mathbb{Z}/p)^n)$. In the next proposition recall that $z_0 \in \widetilde{\mathrm{BP}}_*(\mathbb{Z}/p)$ denotes the class $[L^1 \to B\mathbb{Z}/p]$.

Proposition 6.2. *Assume that*

$$K_* \cap \Phi_n(\mathrm{BP}_* \cdot (z_0 \otimes \cdots \otimes z_0)) = 0.$$

Then $K_* \cap \operatorname{im} \Phi_n = 0$.

Proof. Assume

$$\Phi_n(\sum c_I z_I) \in K_*$$

where each $c_I \neq 0 \mod (v_0, \ldots, v_{n-1})$. Let J such that the degree of z_J is maximal with nonzero c_J. Applying cap products with elements $t_i := 1 \otimes \cdots \otimes t \otimes \cdots \otimes 1 \in \mathrm{BP}^2((\mathbb{Z}/p)^n)$ (unreduced BP-cohomology) where $t = \pi^*(x) \in \mathrm{BP}^2(\mathbb{Z}/p)$ and the subscript i refers to the i-th tensor factor, we see

$$\Phi_n(c_J \cdot (z_0 \otimes \cdots \otimes z_0)) \in K_*.$$

Here we observe that taking cap products with t_i restricts to maps

$$K_* \to K_{*-2}$$

by the naturality of the cap product.

Hence, assuming that $K_* \cap \Phi_n(\mathrm{BP}_* \cdot (z_0 \otimes \cdots \otimes z_0)) = 0$, we conclude $\Phi_n(c_J \cdot (z_0 \otimes \cdots \otimes z_0)) = 0$, and by the injectivity of the iterated Künneth map Φ_n [12, Corollary 3.3] we conclude $c_J = 0 \mod (v_0, \ldots, v_{n-1})$. This is a contradiction and hence an element $\sum c_I z_I$ as above does not exist.

□

The proof that $K_* \cap \Phi_n(\mathrm{BP}_* \cdot (z_0 \otimes \cdots \otimes z_0)) = 0$ is based on the following fact, which led to a solution of the Conner-Floyd conjecture.

Theorem 6.3. [20, Theorem 10.3] *Let p be an odd prime. For the canonical element $\iota_n \in \mathrm{BP}_n(K(\mathbb{Z}/p, n))$ the annihilator ideal is equal to $(v_0, \ldots, v_{n-1})$.*

This result remains valid for $p = 2$, see [12, Appendix].

Corollary 6.4. *Let*

$$\mu_n : B(\mathbb{Z}/p)^n = B\mathbb{Z}/p \times \cdots \times B\mathbb{Z}/p \to K(\mathbb{Z}/p, n)$$

be the canonical map induced by the ring structure on the Eilenberg-MacLane spectrum.

Then the induced map

$$(\mu_n)_* : \mathrm{BP}_*((\mathbb{Z}/p)^n) \to \mathrm{BP}_*(K(\mathbb{Z}/p, n))$$

is injective on $\Phi_n(\mathrm{BP}_* \cdot (z_0 \otimes \cdots \otimes z_0)) \subset \mathrm{BP}_*((\mathbb{Z}/p)^n)$.

Writing $\mathrm{H}^*((\mathbb{Z}/p)^n;\mathbb{F}_p) = \mathbb{F}_p[t_1,\dots,t_n]\otimes\Lambda(s_1,\dots,s_n)$ note that μ_n represents the element $s_1\cdot\ldots\cdot s_n \in \mathrm{H}^n((\mathbb{Z}/p)^n;\mathbb{F}_p)$.

Corollary 6.4 combined with the following proposition finishes the proof that

$$K_*\cap\Phi_n(\mathrm{BP}_*\cdot(z_0\otimes\cdots\otimes z_0)) = 0.$$

Proposition 6.5. *Let $k<n$ and let $\phi:(\mathbb{Z}/p)^k\to(\mathbb{Z}/p)^n$ be a group homomorphism. Then the induced map*

$$B(\mathbb{Z}/p)^k \xrightarrow{B\phi} B(\mathbb{Z}/p)^n \xrightarrow{\mu_n} K(\mathbb{Z}/p,n)$$

is null homotopic.

Proof. The given map defines a class $c\in\mathrm{H}^n((\mathbb{Z}/p)^k;\mathbb{F}_p)$, and it is null homotopic, if and only if $c=0$. We compute

$$c = \phi^*(s_1)\cup\cdots\cup\phi^*(s_n).$$

Now

$$\mathrm{H}^1((\mathbb{Z}/p)^k;\mathbb{F}_p) = \bigoplus_{i=1}^{k}\mathrm{H}^0(\mathbb{Z}/p;\mathbb{F}_p)\otimes\cdots\otimes\mathrm{H}^1(\mathbb{Z}/p;\mathbb{F}_p)\otimes\cdots\otimes\mathrm{H}^0(\mathbb{Z}/p;\mathbb{F}_p)$$

where the tensor factor $\mathrm{H}^1(\mathbb{Z}/p;\mathbb{F}_p)$ sits at position i. This and the fact that $s_i\cup s_i = 0\in\mathrm{H}^2(\mathbb{Z}/p;\mathbb{F}_p)$ for all i (for p odd!) implies that for any $c_1,\dots,c_n\in\mathrm{H}^1((\mathbb{Z}/p)^k;\mathbb{F}_p)$ we have $c_1\cup\cdots\cup c_n = 0$, because $n>k$.

This implies $c=0$ and the proof of the proposition is complete. □

The proof of Theorem 2.6 is now rather easy. We already know from Proposition 6.1 that $\widetilde{\mathrm{BP}}^{(k)}(\wedge^n\mathbb{Z}/p)$ for $1\le k\le n$ span the whole of $\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)$. It remains to show that for fixed k and for $0\le\kappa\le n-1$ the composition

$$\widetilde{\mathrm{BP}}^{(k)}(\wedge^n\mathbb{Z}/p)\hookrightarrow\widetilde{\mathrm{BP}}_*(\wedge^n\mathbb{Z}/p)\xrightarrow{\Gamma_n}\bigoplus_{\sharp_{\mathbb{C}\mathrm{P}^\infty}(J_1\otimes\cdots\otimes J_n)=\kappa} J_1\otimes\cdots\otimes J_n$$

is zero, if $\kappa\ne n-k$. For $\kappa>n-k$ this follows from Lemma 4.1, compare the proof of Proposition 6.1. If $\kappa<n-k$, we argue as follows. Let $X_i = B\mathbb{Z}/p$ or $X_i=\mathbb{C}\mathrm{P}^\infty$ for all $1\le i\le n$ with exactly κ copies of $\mathbb{C}\mathrm{P}^\infty$. Modulo a permutation of factors this happens for $1\le i\le\kappa$. Now the composition

$$\mathrm{BP}_*((\mathbb{Z}/p)^k)\xrightarrow{\phi_*}\mathrm{BP}_*((\mathbb{Z}/p)^n)\to\widetilde{\mathrm{BP}}_*(\wedge^\kappa\mathbb{C}\mathrm{P}^\infty)\otimes_{\mathrm{BP}_*}\widetilde{\mathrm{BP}}_*(\wedge^{n-\kappa}\mathbb{Z}/p)$$

intersects the image of

$$\widetilde{\mathrm{BP}_*}(\wedge^\kappa\mathbb{C}\mathrm{P}^\infty)\otimes_{\mathrm{BP}_*}N_*^{n-\kappa}\to\widetilde{\mathrm{BP}_*}(\wedge^\kappa\mathbb{C}\mathrm{P}^\infty)\otimes_{\mathrm{BP}_*}\widetilde{\mathrm{BP}_*}(\wedge^{n-\kappa}\mathbb{Z}/p)$$

only in the zero element as $k<n-\kappa$, by an argument similar as for Proposition 6.5. This implies the claim for $\kappa<n-k$, and therefore the proof of Theorem 2.6 is complete.

We conclude this section with an example showing that $K_*\cap\operatorname{im}\Phi_n\ne 0$ can occur for $p=2$. Let

$$\alpha: L^3\to B\mathbb{Z}/2\to\wedge^3 B\mathbb{Z}/2$$

be the composition of the classifying map $L^3\to B\mathbb{Z}/2$ with the diagonal map $\Delta: B\mathbb{Z}/2\to B(\mathbb{Z}/2)^3$ and the canonical projection $B(\mathbb{Z}/2)^3\to\wedge^3 B\mathbb{Z}/2$.

We have

$$\Delta^*(s_1\cup s_2\cup s_3) = s^3\ne 0\in\mathrm{H}^3(\mathbb{Z}/2;\mathbb{F}_2)=\mathbb{F}_2$$

where we use $\mathrm{H}^*(\mathbb{Z}/2;\mathbb{F}_2) = \mathbb{F}_2[s]$ with a polynomial generator $s \in \mathrm{H}^1(\mathbb{Z}/2;\mathbb{F}_2) = \mathbb{F}_2$.

Hence, considering the toral element

$$\beta : L^1 \times L^1 \times L^1 \to B(\mathbb{Z}/2)^3 \to \wedge^3 B\mathbb{Z}/2,$$

both of the maps α and β induce the canonical map $\mathbb{Z} \to \mathbb{Z}/2$ in the third integral homology. Because the forgetful map

$$\widetilde{\mathrm{BP}}_3(\wedge^3\mathbb{Z}/2) \to \mathrm{H}_3(\wedge^3\mathbb{Z}/2;\mathbb{Z}) = \mathbb{Z}/2$$

is an isomorphism, this implies that

$$[\alpha] = [\beta] \in \widetilde{BP}_3(\wedge^3\mathbb{Z}/2)$$

and the inessential class $[\alpha]$ is equal to the toral class $[\beta]$.

A similar observation was made in connection with the Gromov-Lawson-Rosenberg conjecture for elementary abelian 2-groups in [11].

7. Gromov-Lawson-Rosenberg conjecture for elementary abelian groups

The research leading to this paper was inspired by [4, Theorem 5.6], which claims that the image of the forgetful map (induced by the homological orientation $\mathrm{BP} \to \mathrm{H}$)

$$h : \mathrm{BP}_*((\mathbb{Z}/p)^n) \to \mathrm{H}_*((\mathbb{Z}/p)^n)$$

has the following description: Let $1 \le k \le n$ and consider the classifying map of a product of lens spaces

$$L^{2m_1+1} \times \cdots \times L^{2m_k+1} \to B\mathbb{Z}/p \times \cdots \times B\mathbb{Z}/p.$$

The image of the fundamental class defines an element in $\mathrm{H}_*((\mathbb{Z}/k)^k)$ that can be mapped to $\mathrm{H}_*((\mathbb{Z}/p)^n)$ by some group homomorphisms $\phi : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^n$. It is claimed that the image of h is (in positive degrees) additively generated by elements of this special kind.

The homological version of our Theorem 1.2 implies that this is indeed the case.

However, the proof given in [4] is incorrect. More precisely, at the top of page 204 in *loc. cit.* it is claimed that each element in $\mathrm{Tor}_{\mathrm{BP}_*}(N_*^{r-1}, N_*)$, respectively the image of such an element in $\mathrm{Tor}_{\mathbb{Z}}(H_*^{r-1}, H_*)$, can be realized as a sum of matrix Toda brackets

$$\left\langle z_{m_1} \otimes \cdots \otimes z_{m_{k-1}} \otimes (z_{m_k}, z_{m_k-4}, \ldots) \otimes z_{m_k+1} \otimes \cdots \otimes z_{m_{r-1}}, A, \begin{pmatrix} \vdots \\ z_{2j-5} \\ z_{2j-1} \end{pmatrix} \right\rangle.$$

This amounts to the assertion that the image of the map

$$\mathrm{Tor}_{\mathrm{BP}_*}(N_*^{r-1}, N_*) \to \mathrm{Tor}_{\mathbb{Z}}(H_*^{r-1}, H_*)$$

is generated by the images of the maps

$$N_*^{k-1} \otimes_{\mathrm{BP}_*} \mathrm{Tor}_{\mathrm{BP}_*}(N_*, N_*) \otimes_{\mathrm{BP}_*} N_*^{r-k-1} \to \mathrm{Tor}_{\mathrm{BP}_*}(N_*^{r-1}, N_*) \to \mathrm{Tor}_{\mathbb{Z}}(H_*^{r-1}, H_*)$$

where $1 \le k \le r-1$.

This statement is wrong. To show this we use the commutative diagram

$$\begin{array}{ccc} \bigoplus_{1\le k\le r-1} N_*^{k-1} \otimes_{\mathrm{BP}_*} \mathrm{Tor}_{\mathrm{BP}_*}(N_*, N_*) \otimes_{\mathrm{BP}_*} N_*^{r-k-1} & \longrightarrow & \mathrm{Tor}_{\mathrm{BP}_*}(N_*^{r-1}, N_*) \\ \downarrow & & \downarrow \\ \bigoplus_{1\le k\le r-1} H_*^{k-1} \otimes \mathrm{Tor}_{\mathbb{Z}}(H_*, H_*) \otimes H_*^{r-k-1} & \longrightarrow & \mathrm{Tor}_{\mathbb{Z}}(H_*^{r-1}, H_*) \end{array}$$

and recall that according to Corollary 5.5, for each $1 \le k \le r-1$ the image of the composition of the right vertical with the upper horizontal map is isomorphic to $H_*^{r-1} \otimes M_1$, whereas the image of the right vertical map is isomorphic to $H_*^{r-1} \otimes M_{r-1}$. Here we recall that $(M_k)_*$ denotes the free graded $\mathbb{F}_p$-module with one generator in each even degree $2, \ldots, 2p^k - 2$.

Now we observe that

$$(r-1) \cdot \dim_{\mathbb{F}_p}(H_*^{r-1} \otimes M_1) < \dim_{\mathbb{F}_p}(H_*^{r-1} \otimes M_{r-1})$$

in large degrees, if $r - 1 \ge 2$. From this we conclude that the lower horizontal map in the above diagram cannot contain the whole of $H_*^{r-1} \otimes M_{r-1}$. This shows that a simple reduction to the case $r = 2$ as envisaged in [4] is impossible. A similar problem occurs in the proof of the analogous result [5, Theorem 4.2].

Because the methods developed in our paper, in particular Theorem 1.2, directly apply to bordism theory they can be used to verify the Gromov-Lawson-Rosenberg conjecture for elementary abelian groups of odd order right away, avoiding a further reduction to singular homology or connective K-homology as in [4, 5], which requires additional arguments [19] based on manifolds with Baas-Sullivan singularities.

We shall explain this proof. In the following we denote by $f : X \to B\pi_1(X)$ the classifying map of the universal cover of some path connected topological space X .

Definition 7.1. Let M be a closed oriented manifold of dimension d and let p be a prime. The manifold M is called *p-atoral*, if

$$f^*(c_1) \cup \cdots \cup f^*(c_d) = 0 \in \mathrm{H}^d(M; \mathbb{F}_p)$$

for all one dimensional classes $c_1, \ldots, c_d \in \mathrm{H}^1(B\pi_1(M); \mathbb{F}_p)$.

We can now prove the Gromov-Lawson-Rosenberg conjecture for atoral manifolds with elementary abelian fundamental groups of odd order, compare [4, Theorem 5.8] and [5, Theorem 2.3].

Theorem 7.2. *Assume that M is a p-atoral manifold of dimension $d \ge 5$ with fundamental group $(\mathbb{Z}/p)^n$, where p is an odd prime. Then the following assertions hold.*

- *If M admits a spin structure, then M admits a Riemannian metric of positive scalar curvature, if and only if $\alpha(M) = 0 \in \mathrm{KO}_d$, where α is the index invariant introduced by Hitchin [9] with values in the coefficients of real K-homology.*
- *If M does not admit a spin structure, then M admits a Riemannian metric of positive scalar curvature.*

Note that M in this theorem is automatically p-atoral, if $d > n$. This again uses the fact that for odd p the one dimensional generator $s \in \mathrm{H}^1(\mathbb{Z}/p; \mathbb{F}_p)$ squares to 0.

Proof. First recall [9] that $\alpha(M) = 0$, if M is spin and can be equipped with a Riemannian metric of positive scalar curvature. Furthermore [19] a closed oriented smooth manifold M of dimension $d \ge 5$ admits a Riemannian metric of positive scalar curvature, if and only if

- $[f : M \to B\pi_1(M)] \in \Omega_d^{Spin,+}(B\pi_1(M))$, in the case when M admits a spin structure,
- $[f : M \to B\pi_1(M)] \in \Omega_d^{SO,+}(B\pi_1(M))$, in the case when the universal cover $\widetilde{M}$ does not admit a spin structure.

The superscript + denotes bordism classes that can be represented by singular manifolds $X \to B\pi_1(M)$ (where X is spin, respectively oriented) so that X carries a Riemannian metric of positive scalar curvature.

Now let M be as in the theorem. Because $\pi_1(M) = (\mathbb{Z}/p)^n$ has odd order, the manifold M admits a spin structure, if and only if its universal cover $\widetilde{M}$ does. Oriented and spin bordism are equivalent after localization at an odd prime p. This allows us to treat both cases in parallel to some extent, which we indicate by dropping the superscript *SO* or *Spin*.

Using Theorem 1.2 we can write

$$[f : M \to B(\mathbb{Z}/p)^n] = L_0 + \cdots + L_n \in \Omega_d((\mathbb{Z}/p)^n)$$

where each L_k is the sum of bordism classes each of which is equal to the image of a class

$$[X \times (L^{2m_1+1} \to B\mathbb{Z}/p) \times \cdots \times (L^{2m_k+1} \to B\mathbb{Z}/p)] \in \Omega_d((\mathbb{Z}/p)^k)$$

under the map $\Omega_d((\mathbb{Z}/p)^k) \xrightarrow{\phi_*} \Omega_d((\mathbb{Z}/p)^n)$ induced by some group homomorphism $\phi : (\mathbb{Z}/p)^k \to (\mathbb{Z}/p)^n$. Here $L^{2m_i+1} \to B\mathbb{Z}/p$ are classifying maps of standard lens spaces and X is some conected closed manifold (spin or oriented, respectively). In particular $L_0 \in \Omega_d \subset \Omega_d((\mathbb{Z}/p)^n)$. Without loss of generality we can assume that each group homomorphism ϕ is injective.

For dimension reasons $L_k = 0$ for $k > d$. Because M is p-atoral, we also have $L_d = 0$, because for $k = d$ each of the above summands is equal to $[(L^1 \to B\mathbb{Z}/p) \times \cdots \times (L^1 \to B\mathbb{Z}/p)]$, and the corresponding ϕ is injective. Now

- $\widetilde{\Omega}^{Spin}_{>1}(\mathbb{Z}/p) \subset \Omega^{Spin,+}_*(\mathbb{Z}/p)$, see [18, Theorem 1.3], and
- $\widetilde{\Omega}^{SO}_{>1}(\mathbb{Z}/p) \subset \Omega^{SO,+}_*(\mathbb{Z}/p)$, because the Ω^{SO}_*-module $\widetilde{\Omega}^{SO}_*(\mathbb{Z}/p)$ is generated by lens spaces $[L^{2m+1} \to B\mathbb{Z}/p]$, $m \geq 0$, and $\Omega^{SO}_i = \Omega^{SO,+}_i$ for $i > 0$, see [8].

This implies that each of the bordism classes $L_1, \ldots, L_{d-1}$ lies in $\Omega^+_d((\mathbb{Z}/p)^n)$. In particular, in the spin case, we have $\alpha(M) = \alpha(L_0)$.

Hence our assertion holds, if $L_0 \in \Omega^+_d$, where we assume $\alpha(L_0) = 0$ in the spin case. But this follows from [8] (in the non-spin case) and [21] (in the spin case). □

The Gromov-Lawson-Rosenberg conjecture for toral manifolds with elementary abelian fundamental groups of odd order remains open. We also remark that Sven Führing, in his Augsburg dissertation [7, Corollary 5.2.2], gave a different proof of the second part of Theorem 7.2, using the notion of Riemannian metrics of positive scalar curvature on manifolds with Baas-Sullivan singularities.

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