Content

Editorial
Tamara Bianco & Volker Ulm........................................................................................... 5

Digital Media – A Catalyst for Innovations in Mathematics Education?
Volker Ulm.......................................................................................................................... 7

The Impact of an In-service Course on New Technologies in Mathematics Education on Teachers and their Students
Paraskevi Sophocleous, Demetra Pitta-Pantazi, George N. Philippou & Nicholas Mousoulides.......................................................................................... 30

Explorations Around the Rotational Solids
Evgenia Sendova & Toni Chehrarova................................................................................ 57

The Design and Implementation of a Pre-service Teacher Training Course on New Technologies in Mathematics Education
Nicholas Mousoulides, George Philippou, Demetra Pitta-Pantazi & Paraskevi Sophocleous.......................................................................................... 67

Narrative Didactics in Mathematical Education: an Innovative Didactical Concept
Matthias Brandl.................................................................................................................. 98

Exploring Geometrical Theorems by using Dynamic Geometry Software
Birgit Brandl & Matthias Brandl......................................................................................... 105

Discovering Proportional Relationships in Everyday Life
Sabrina Asam...................................................................................................................... 116

Interactive Whiteboards – a Pilot Project
Tamara Bianco.................................................................................................................... 126
Editorial:
The European Project “InnoMathEd”

This book results from the project “InnoMathEd – Innovations in Mathematics Education on European Level”. It has been funded with support from the European Commission in the framework of the “Lifelong Learning Programme” between 2008 and 2010.

1 Project Consortium

InnoMathEd was based on the cooperation of ten partners from eight European countries:
- University of Augsburg (DE)
- Bulgarian Academy of Sciences (BG)
- University of South Bohemia (CZ)
- University of Bayreuth (DE)
- Project Education Institute (DE)
- German School Board Bolzano (IT)
- University of Cyprus (CY)
- Tyrolean Educational Service (AT)
- University of Cambridge (UK)
- University of Oslo (NO)

For dissemination processes InnoMathEd was associated to the European projects “Intergeo – Interoperable Interactive Geometry for Europe” (http://i2geo.net) and ”Scientix – The Community for Science Education in Europe” (http://scientix.eu).

2 Aims and Activities of InnoMathEd

In general, the project InnoMathEd aimed at
- developing students’ key competences, their mathematical understanding and their abilities to use ICT for learning processes,
- improving teachers’ competences in creating innovative learning environments with ICT and
- developing mathematics education on European level systematically – at least to some extent.
For that purpose teacher education offers have been arranged according to the theories and strategies described in the first chapter of this book. Teachers have been made acquainted with didactic concepts and pedagogical methodologies for teaching and learning mathematics with ICT. The project consortium as well as the teachers developed innovative learning environments that focus on students’ active, self-responsible and inquiry-based learning. These resources were tested and evaluated in class.

3 Aims of this Book and the CD

This book documents results of InnoMathEd. It provides
- didactic concepts for teaching and learning with ICT promoting pupils’ key competences and their mathematical understanding,
- innovative learning environments with ICT as best practice examples for direct use in class,
- experiences and resources from the activities with teachers and students in the framework of InnoMathEd and
- ideas for the implementation of ICT in the educational system on European level.

The CD provides the text of the book in a PDF file as well as teaching and learning material resulting from the project activities. This material on CD is structured according to the software tools used.

With this book you will get an insight to a European project which brought together various countries united for one central aim: improving teaching and learning in school. The experiences which lie behind these individual contributions go much further. It was a pleasure to learn from the different countries, their culture and the highly motivated people that were willing to come together and work for this important aim. The articles that follow show examples of the educational systems of the participating countries and the solutions and ideas for real-life teaching problems.

Volker Ulm

Digital Media – A Catalyst for Innovations in Mathematics Education?

Abstract

In many contexts the hope has been expressed that ICT (Information and Communication Technology) may serve as a “catalyst” for innovations in mathematics education. By using ICT the way of teaching and learning mathematics is supposed to change in a substantial way. But such innovations do not appear automatically. It is necessary to keep the key role of teachers in mind. For developing mathematics education it is not sufficient to distribute “good” software or “good” learning environments. One should aim at the meta-level of teachers’ notions and beliefs of teaching and learning processes. But for such changes on that meta-level it is necessary to work with teachers for a certain period of time and to relate didactic theories and digital media to their everyday work in class. In the following a pattern for in-service teacher education activities is developed and founded theoretically. It has been realized by the European project “InnoMathEd”.

1 Dynamic Mathematics and Dynamic Worksheets

There are very different digital media for mathematics education, e.g. computer algebra systems, spreadsheets, simulation software, programming tools or learning software focussing on specific curricular topics. In this section we will exemplarily have a short glance at software for dynamic mathematics, since this kind of software is currently of certain importance for mathematics education in Europe and it has been widely used in the project “InnoMathEd”.

1.1 What is Dynamic Mathematics?

Software for dynamic mathematics is a tool for generating mathematical constructions on the screen. In contrast to constructions on paper or on the blackboard, constructions with dynamic mathematics can be changed and varied on the screen dynamically. This offers possibilities of visualisations that are not realisable by means
of traditional media for teaching. One can see mathematical processes and not only single static pictures (see sections 1.3 and 1.4). Furthermore, software for dynamic mathematics provides an integrated computer algebra system to bridge gaps between geometry, algebra and calculus. The tool makes it possible to measure lengths, angles or coordinates of points and to use these measurements for further calculations. It offers working with functions and integrating graphs of functions in dynamic constructions.

Currently, there is quite a wide offer of software for dynamic mathematics. Tools used on European level are e.g.
- GEONEXT (http://geonext.de)
- Geogebra (http://geogebra.org)
- Cinderella (http://cinderella.de)
- Cabri (http://www.cabri.com)
- The Geometer’s Sketchpad (http://www.dynamicgeometry.com)

1.2 What are Dynamic Worksheets?

When designing learning environments with dynamic mathematics for students, it is necessary to relate dynamic constructions to texts, e.g. for explanations or exercises. For this purpose some software for dynamic mathematics can be embedded in web-sites – in HTML structures. Thus, dynamic constructions are combined by the internet browser with texts, pictures, links and other elements of web pages. This kind of digital media is called “dynamic worksheet”. With such tools students may be given dynamic constructions to explore and discover mathematics or they can develop mathematical constructions themselves.

1.3 Example: Parameters in the Sine Function

To see the added value of dynamic mathematics we look exemplarily at the curricular topic “Parameters of the sine function”. In the function \( f(x) = \alpha \sin(x) \) the parameter \( \alpha \) has a certain impact on the graph. To explore this impact students are usually asked to draw some graphs, e.g. of the functions \( y = \sin(x) \), \( y = 2\sin(x) \), \( y = 5\sin(x) \), \( y = -2\sin(x) \) and \( y = -5\sin(x) \). From these concrete examples student should develop insight in what happens when this parameter increases or decreases continuously. Students are expected, to “see” a process in mind: The graph is stretched or compressed in vertical direction. However, the static drawings in the exercise book or on the blackboard cannot depict this functional relation between the parameter and the graph adequately.
The static drawings with traditional media can be seen as snapshots of a movement. Dynamic mathematics makes these movements visible. Here is a first dynamic worksheet:

![Dynamic worksheet for exploring the function $f(x) = a \sin(x)$](image)

In the first exercise the students get the graph of the function $f(x) = a \sin(x)$. They can vary the parameter $a$ by the slider and see the impact on the graph. The continuous functional relation between the parameter and the graph can be explored. The students are asked to note down all their observations in their exercise book which has the character of a study journal (see section 5).

The further exercises deal with a factor not outside but inside the sine expression and with summands instead of factors. In the last exercise all four parameters of the function $f(x) = a \sin(b(x + c)) + d$ can be varied.

This example shows a certain added value of dynamic mathematics. The tool makes movements of graphs and mathematical processes – here the continuous functional impact of the parameters on the graph – visible. This is not possible with static drawings in the book or on the blackboard.

1.4 Example: Circumference of a Circle

The second example refers to a standard subject matter of mathematics education and shows how dynamic mathematics can enrich everyday lessons. With the dynamic worksheet students can explore and discover the relation between the circumference and the diameter of circles. The dynamic construction helps to see the cir-
cumference as a one dimensional line with a length that can be compared with the diameter.

Fig. 5: Dynamic worksheet for exploring the circumference of a circle

The construction shows a circle that can be moved horizontally. Thereby the circumference is unwound. Moreover, the diameter of the circle can be varied. The length of the diameter and the circumference are shown in dynamic texts. By clicking on the arrows in the blue field the students get to six exercises.

First, the students should get familiar with the construction. They note down their observations when varying the construction. This helps to activate basic notions of circles, diameters and circumferences.

The construction should be varied systematically. The students note down the circumference for different diameters in a table.

They are asked to describe observations and discoveries in their exercise book.

Exercise 3
Measure the diameter and the circumference of several circular objects (e.g. bucket, dish, glass, tin, CD, coin,...). Note down the measured values in your table of exercise 2.

The students' activities are extended to exploring circular objects from real life by measuring them. For that, the teacher offers some objects, measuring tape and a string. In this phase the computer has only minor importance. The results are added to the table of exercise 2.

Exercise 4
Depict your measured values in a coordinate system in your exercise book. What do you find about the relation between the the circumference and the diameter of circles? Note down your considerations in your exercise book.

The measured values from the table should be depicted in a coordinate system. The diagram leads to deeper insight: The points lie approximately on a straight line. Thus, the direct proportionality between the diameter and the circumference is visible. The students are asked to note down all their consideration in their exercise book.

Exercise 5
Calculate the quotient \( \frac{c}{d} \) of the pairs of measured values. Note down the results in your table of exercise 2. Describe your observations in your exercise book.

The measurements are evaluated numerically by calculating the quotient \( \frac{c}{d} \). The students find out that this quotient is independent of the size of the circle and that it has a value of approximately 3.14.

Exercise 6
Find a formula for the relation between the circumference and the diameter of a circle. Note down all your considerations in your exercise book.
Combining all results the students get the equation \( u : d = 3,14 \ldots \). The name \( \pi \) for the constant of proportionality may be unknown at this stage. It can be introduced by the teacher afterwards.

1.5 The added value of dynamic mathematics in the examples

In the two examples sketched above the software for dynamic mathematics is only a tool for teaching and learning. But it provides some real added value:
- **Visualization:** The dynamic constructions with the graph of the sine function or with the possibility to unwind the circumference of the circle and to transform it into a line segment help to get clear notions of these mathematical objects.
- **Many examples:** The dynamic constructions provide a lot of graphs and they offer a lot of circles and measured values for the table and the diagram.
- **Precise measuring:** The measured values of the circumference and the diameter are more precise than measurements of circular objects from real life. So the quotient \( u : d \) really equals 3,14. Measurements of real objects give values roughly between 3,0 and 3,3.
- **Continuous variation:** Since the dynamic construction can be varied nearly continuously, students get insight in the continuous functional relation between the parameter and the graph or between the circumference and the diameter. In contrast, measuring real life objects provides only some discrete values of the continuous function \( u(d) = \pi d \).
- **Adjustability:** With the dynamic construction of the circle one can get measured values in regular steps. One sees e.g. that the circumference doubles if the diameter is doubled. Thus, the direct proportionality can easily be checked. With real life objects one can hardly get such equidistant steps of the diameter.

2 Hopes

There are hopes that ICT may improve mathematics education on different levels:
- ICT is supposed to improve students’ understanding of mathematics. Dynamic constructions make mathematical processes visible: Configurations can be varied (nearly) continuously on the screen, functional relationships or invariants can be directly observed. With traditional media and static pictures the teacher can hope at best, that the student “sees” these mathematical processes in his mind’s eye.
- ICT is seen as a tool to make mathematics education more authentic and realistic. Mathematics education often deals with very simplified, pseudo-realistic problems because data from real life are too complex to cope with by traditional media. However, the computer helps to tackle problems that require e.g. extensive numerical or algebraic calculations.
- A further target area, which is often linked with digital media, concerns students’ key competences: By working with ICT students are supposed to increase abilities of autonomous and cooperative learning. Digital media should foster communication and presentation competences as well as “computer literacy”.
- Finally, digital media are supposed to serve as catalysts for innovations in mathematics education. By using ICT the way of teaching and learning mathematics should change in a substantial way. Mathematics education should become more inquiry-based, students should work self-organised, self-responsibly and合作共赢地.

Since these aims are not reached automatically by using computers in mathematics lessons, the main question is: How can these targets be reached?

3 Learning Environments

All efforts to develop mathematics education finally aim at students’ learning. Therefore, a short glance at the nature of learning is quite useful. The following aspects of learning have been formulated by Pedagogical Psychology. They provide a background for the latter.

3.1 Aspects of Learning

Learning is a very complex phenomenon. Initiatives aiming at the development of mathematics education have to take the nature of learning into account. Let us have a quick look at some fundamental aspects of learning (e.g. Reimann-Rothmeier & Mandl, 1998; Haberlandt, 1997):
- Learning is a constructive process. Knowledge and understanding cannot be simply transported from teachers to students. Cognitive psychology describes learning as a...
process of construction and modification of cognitive structures. From the view of neurosciences, learning is the construction of neuronal networks. Connections between neurons develop and change.

- Learning is an individual process. Learning takes place inside the head of each learner. He creates his own knowledge and understanding by interpreting his personal perceptions on the basis of his individual prior knowledge and prior understanding.
- Learning is an active process. Cognitive activity means working with the content in mind, viewing it from different perspectives and relating it to the existing network of knowledge.
- Learning is a self-organized process. The learner is at least partially responsible for the organization of his individual learning. The degree of responsibility may vary in the phases of planning, realizing or reflecting of learning processes.
- Learning is a situative process. It is influenced by the learning situation. A meaningful context or a pleasant atmosphere can foster learning, fear can hamper them.
- Learning is a social process. On the one hand the sociocultural environment has great impact on educational processes. On the other hand learning in school is based on interpersonal cooperation and communication between students and teachers.

3.2 Concept of Learning Environments

Considering the aspects of learning in 3.1 the following model seems quite natural for teaching and learning processes in school (Fig. 6).

According to the constructivist point of view the teacher cannot put knowledge directly in the learners’ heads. The learning environment is the essential link between the teacher and the learner. This notion includes five components: the tasks for the learner’s working with the content, the arrangement of media and the method for teaching and learning as well as the social situation with the teacher and other learners as partners for learning.

It belongs to the teacher’s field of responsibility to design the learning environment. So he offers a basis for the learner’s work. This allows the teacher to get feedback about the learner as well as about the learning environment.

This model is based on and extends the didactical concepts of "substantial learning environments" by Wittmann (1995, 2001) or “strong learning environments” by Dubs (1995).

It makes clear that a learning environment comprises much more than only the media. This reveals a widespread wrong argument: Advocates of ICT in school sometime argue that digital media automatically improve mathematics education. However, the other components of the learning environment are at least as important. It is essential how students work with the media.
4 A Methodical Concept

According to the aspects of learning in section 3.1 we have to organize lessons in a way that students work actively, self-organized, individually and cooperatively. They should experience that mathematics is a field for explorations and discoveries. They also should present and discuss their ideas and results. Thus, the following four phases which structure lessons with students working with ICT are very natural:

<table>
<thead>
<tr>
<th>Table 1: Methodical concept</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(1) Individual working</strong></td>
</tr>
<tr>
<td>Learning is an individual, active and self-organized process. So at first the students work on their own. They are faced with the necessity to explore the content, to activate their prior knowledge, to develop ideas and to make discoveries. E.g. learning environments with dynamic worksheets can offer a framework for such activities and may support them.</td>
</tr>
<tr>
<td><strong>(2) Cooperation with partners</strong></td>
</tr>
<tr>
<td>Learning is a social process. It is very natural that the students discuss their ideas with partners in small groups and that they work on the problems cooperatively. This communication helps to get thoughts in order and to get further ideas. Meanwhile the teacher can remain in the background or turn his attention to individual learners.</td>
</tr>
<tr>
<td><strong>(3) Presentation of ideas</strong></td>
</tr>
<tr>
<td>After having worked individually and in groups the students present their ideas and discuss them in the plenum. The different contributions reveal multiple aspects of the topic and help to view it from varying perspectives. Moreover the students train debating and presentation techniques.</td>
</tr>
<tr>
<td><strong>(4) Summary of results</strong></td>
</tr>
<tr>
<td>Finally the students' results are summarized and possibly extended by the teacher. It is his task to introduce mathematical conventions and to consider curricular regulations. But since the students have already discovered the new content on their own paths, they can more likely integrate the teacher's explanations into their individual cognitive structures.</td>
</tr>
</tbody>
</table>

This methodical concept combines individual learning with working in small groups as well as in the plenum of the class in a very natural way. It is based on existing didactical concepts like “Think – Pair – Share” (Lyman, 1981) or “I – You – We” (Gallin & Ruf, 1998).

5 Writing in the Study Journal

Of course, such a methodical model simplifies real processes in mathematics education and their complexity. In reality these four phases will not always happen in a linear way. They may overlap or there may be cycles. Nevertheless, such a model can help to structure lessons and to make the complex processes of teaching and learning a bit more understandable.

Technology in school should be considered within a wider range of resources for teaching and learning. Even if the students explore mathematics with digital media, the students' exercise books don't lose relevance. Students should regularly be asked to take notes into their exercise books – also called “study journals”. Noting down thoughts helps to make thoughts clear and to order and arrange them. Writing helps to develop understanding for new subject matters. Hence, when students work with computers they should regularly get the task:

- to sketch figures in their study journal,
- to describe observations,
- to note down assumptions,
- to write down proofs or even
- to express individual impressions and comments.

The study journal can even get the character of a personal diary that accompanies students on their individual learning paths.

These individual notes in the study journal ensure that ideas and results are still available when the computer is switched off. They are a basis for further presentations, discussions and summaries in class.

Another effect: When students are regularly asked to make notes, they are prevented from clicking through the digital media rapidly and from playing with it as with a computer game only superficially without getting to the deeper mathematical content.

Till now, we presented some didactical concepts for working with digital media in class. A very natural question is: Which role do these media currently play in everyday mathematics education? In the next paragraph we will have a look at the current situation of ICT in school.
6 Current State

If we think of offers of digital media for mathematics education in Europe we can be quite satisfied – at least to some extent.

6.1 Hard- and Software

Meanwhile the large majority of (secondary) schools is equipped with computer labs that offer technical resources for integrating ICT in mathematics education. There is a wide offer of – free or commercial – software products for students' mathematics learning, e.g. software for dynamic mathematics, computer algebra systems, spreadsheets, simulation software, programming tools or learning software focussing on specific curricular topics (see section 1). Some of these products have been developed specially for the use in school. Moreover, a huge amount of teaching and learning material for direct use in class has been published in the web or by print media, that applies these software tools to the content of the curriculum (e.g. dynamic worksheets provide this).

6.2 Availability of Teaching and Learning Material

However, there are some critical aspects with respect to the availability of the digital media just mentioned. A lot of resources for teaching and learning with ICT have been published. But are they really available to teachers who plan their lessons for the following day? How easy is it for a teacher to find resources that fit his lessons?

Here the current state is not very satisfying. Of course there are positive examples and attempts to change the situation: For example, the European project "Scientix" (see Editorial) tries to provide access to teaching and learning material for mathematics and sciences on the European level. Especially for dynamic mathematics the "Intergeo" platform (see Editorial) offers access to a data base with thousands of resources. But nevertheless, for a usual teacher it is not easy to have all these changing offers in mind and to find resources that fit exactly to his personal needs in class.

6.3 Use in Practice

In our analysis of the current state let's finally look at the use of ICT in everyday mathematics lessons. On the one hand, there are some teachers who are very active in using digital media, who use the computer as others use the blackboard, who participate in ICT projects or who even work as teacher educators in the field of ICT.

But on the other hand, these teachers are – related to all teachers in Europe – only a small minority. The majority of mathematics teachers uses ICT for students' learning only rarely or never (e.g. OECD 2005).

6.4 Improvement of Mathematics Education

Accordingly, the aim of improving mathematics education with digital media is far from being reached. In section 2 the hope was expressed that ICT could serve as "catalyst" for innovations. Of course, there have been many successful projects and attempts to develop mathematics education by using ICT in the last two decades, but in the educational system as a whole their impact has been rather limited. This situation was analyzed by K. Ruthven (University of Cambridge) and M. Artigue (University of Paris):

- "Advocacy for new technology is part of a wider reform pattern which has had limited success in changing well established structures of schooling." (Ruthven, 2007).
- "From the very beginning, digital technologies have been considered as a tool for educational change [...]. Unfortunately, the results are far from being those expected." (Artigue, 2007)

These analyses are quite disillusioning. But they describe the situation in common mathematics education in Europe. In section 2 the hope was mentioned, that if digital media were introduced in school then teaching and learning would change in a substantial way. Maybe, this idea is too simple – at least it hasn't worked in the educational system as a whole until now.

6.5 A Systemic Problem

The crucial question is: Why are "the results far from those being expected" – as M. Artigue described it?

Actually, we have a very comfortable situation: As we discussed above, ICT can have a real added value in mathematics education when students work on certain topics. There is no lack of hard- and software, of teaching and learning resources, of pedagogical concepts and didactic ideas for mathematics education with ICT.

The problem seems to be a structural one that is related to teachers' and students' attitudes towards mathematics and their beliefs of mathematics education:

If a teacher tries to work with his class in a computer lab, the students have to work necessarily self-organised and inquiry-based – at least to some extent – because traditional methods of schooling do not work when the students sit at the screens and are supposed
to learn with the computer. Thus, mathematics education in a computer lab requires teachers’ abilities and attitudes to organise teaching in an inquiry-based way and it requires students’ abilities and attitudes to work individually and cooperatively. If regular mathematics education in the classroom is very different – very teacher-dominated –, then neither teachers nor students have these abilities and attitudes. As a consequence, lessons in the computer lab fail or turn out to be waste of time. One cannot expect that students explore mathematics self-organised in some lessons with ICT if mathematics education in all other lessons is totally different.

A bit metaphorically: There was the hope that ICT may serve as a “catalyst” for innovations in mathematics education. But the potential of ICT seems to be not strong enough to change the big, stable and complex system “mathematics education”.

Thus, the question is: What could be done? Should we resign? Of course not! But we have to take into account the complexity of mathematics education and of the educational system as a whole.

7 Innovations in Complex Systems

There are many efforts to innovate the educational system – on regional, national and international level – aiming at changes of teaching and learning. For understanding the structure of such initiatives a short glance at theories of cybernetics and management theory is quite useful.

7.1 Innovations

The OECD defines an innovation as the implementation of a new or significantly improved product, process or method (OECD & Eurostat, 2005, p. 46). Thus an innovation requires both an invention and the implementation of the new idea.

In the educational system we are in a situation where lots of concepts, methods and tools have been developed for substantial improvements of teaching and learning (see sections 1 – 6). But for real innovations these promising theories and products have to be implemented in the educational system. Here implementation means a good deal more than diffusion or dissemination of material (papers, guidelines, software tools etc.). Implementation should reach the real agents in the school system, i.e. the teachers and students, their thinking and their working. We need changes in teachers’ and students’ notions of educational processes, in their attitudes towards mathematics and in their beliefs concerning teaching and learning at school.

Hence, the crucial question is: How can substantial innovations in the complex system of mathematics education be initiated and maintained successfully?

To get hints for answers a quick look at complex systems in general is valuable.

7.2 Complex Systems

In theories of cybernetics a system is called “complex” if it can potentially be in so many states that nobody can cognitively grasp all possible states of the system and all possible transitions between the states (Malik, 1992; Vester, 1999). Examples are the biosphere, a national park, the economic system, mathematics education in Europe and even mathematics education at a concrete school. Complex systems usually are networks of multiply connected components. One cannot change a component without influencing the character of the whole system. Furthermore real complex systems are in permanent exchange with their environment. Maybe this characterization of complex systems seems a bit fuzzy. But, nevertheless, it is of considerable meaning. Let us regard the opposite: If a system is not complex, someone can overview all possible states of the system and all transitions between the states. So this person should be able to steer the system as an omnipotent monarch leading it to “good” states. In contrast, complex systems do not allow this way of steering.

7.3 Steering of Complex Systems

The fundamental problem of mankind dealing with complex systems is how to manage the complexity, how to steer complex systems successfully and how to find ways to sound states. With reference to theories of cybernetics two dimensions of steering complex systems can be distinguished (Malik, 1992). The first one concerns the manner, the second one the target level of steering activities (Fig. 7).
The method of analytic-directive steering needs a controlling and governing authority that defines objectives for the system and determines ways for reaching the aims. Authoritarian systems with strong hierarchies are founded on this principle. However, fundamental problems are caused just by the complexity of the system. In complex systems no one has the chance to grasp all possible states of the system cognitively. Thus, the analytic-constructive approach postulates the availability of information about the system that cannot be reached in reality.

In contrast incremental-evolutionary steering is based on the assumption that changes in complex systems result from natural growing and developing processes. The steering activities try to influence these systemic processes. They accept the fact that complex systems cannot be steered entirely in all details and they aim at incremental changes in promising directions. The focus on little steps is essential, since revolutionary changes can have unpredictable consequences which may endanger the soundness or even the existence of the whole system.

The second dimension distinguishes between the object and the meta-level. The object level consists of all concrete objects of the system. In the school system such objects are e.g. teachers, students, books, computers, buildings etc. Changes on the object level take place if new books are bought or if a new computer lab equipped. Of course such changes are superficial without reaching the substantial structures of the system.

The meta-level comprehends e.g. organizational structures, social relationships, notions of the functions of the system etc. In the school system e.g. notions of the nature of the different subjects and beliefs concerning teaching and learning (e.g. Pehkonen & Törner, 1996, Leder, Pehkonen & Törner, 2002) are included.

7.4 Innovations in Complex Systems

The pivotal question is: How can substantial innovations in the complex system “mathematics education” be initiated successfully? The theory of cybernetics gives useful hints.

Attempts of analytic-directive steering will fail in the long term, since they ignore the complexity immanent in the system. Changes on the object level do not necessarily cause structural changes of the system. According to the theory of cybernetics it is much more promising to initiate incremental-evolutionary changes on the meta-level (Fig. 8).

They are in accordance with the complexity of the system and do not endanger its existence. Nevertheless, they can cause substantial changes within the system by having effects on the meta-level, especially when they work cumulatively.

8 Incremental-Evolutionary Systemic Innovations in the Educational System

8.1 A Summary with Respect to Teacher Education

Let us summarize three central aspects of our previous discussion:

- Innovations in the educational system are most effective if they reach the meta-level of beliefs and attitudes of all agents involved (see 7.4). Since teachers are the key persons for initiating educational processes, one should aim at their notions of teaching and learning. Thus, we need effective concepts for teacher education.
Since the aspects of learning sketched in 3.1 characterize human learning in general, they should also be taken into account for the learning of teachers. Currently, teacher education often ignores them. In many cases teacher education consists of talks to an audience.

If the model of learning environments described in 3.2 is valid for human learning in general, it can be fruitful for the conceptual design of teacher education. From this point of view we need learning environments for teachers which have effects on their teaching in class.

Summing up, we have a quite strange situation: There are promising concepts and theories for teaching and learning. But we do not apply them on the learning of teachers. Thus, it is not astonishing that lots of efforts in teacher education only have rather superficial effects. Teacher may acquire theoretical knowledge about mathematics education, but do not act accordingly because the meta-level of beliefs and attitudes is not reached. There remains a wide gap between knowledge and action. Everyday teaching and learning at school is not influenced.

8.2 Conclusion: A Pattern for Innovation Projects

Combining the theory of cybernetics and the concept of learning environments we get a pragmatic, but also theory-based way of initiating innovations in school. Activities are most promising, if they focus on incremental-evolutionary changes on the meta-level of beliefs and attitudes of all agents involved. The concept of learning environments may serve as a framework for learning processes of teachers and students. How can this be done concretely?

As a conclusion from all reflections above we sketch and propose a pattern for innovation projects for mathematics education. It has been realized e.g. by the project “InnoMathEd – Innovations in Mathematics Education on European Level” and it is a basic structure of “The Fibonacci Project – Large-scale dissemination of inquiry based science and mathematics education”, see http://fibonacci-project.eu.

(1) Aiming at teachers
The key persons for innovations in school are the teachers. Their beliefs, motivation and abilities are crucial for everyday teaching and learning in school. Thus, projects for innovations in mathematics education should focus on teachers’ professional development.

(2) Networks of schools
Since learning is a social process, regional networks of schools should be established. They form frameworks for teachers’ exchange of experience and for their cooperative learning.

(3) Aiming at the meta-level
It is by far not sufficient only to offer innovative software tools or teaching and learning material. Innovation projects should aim at the meta-level of attitudes towards mathematics and beliefs of mathematics education. This concerns e.g. the role of the teacher, the role of the students and the nature of mathematics.

(4) Development of learning environments
Maybe, “aiming at the meta-level” sounds quite abstract and difficult. But the concept of learning environments can serve as a convenient framework. It may help bridging the gap between theory and practice and relating the teachers’ project activities strongly to their regular work at school. Teachers develop learning environments for their students, they use and test them in class, they optimize them on the basis of all experiences and they finally exchange and discuss them in their school network. Thus, by working with concrete learning environments teachers get acquainted with general pedagogical ideas. In this way developments on the meta-level of attitudes and beliefs are initiated.

(5) Areas of activity
Since systemic innovations are incremental-evolutionary, participating schools should concentrate on one or a few areas of innovation, e.g. autonomous learning with ICT, promoting students’ cooperation or fostering key competences with ICT. It is not promising to aim at total changes of mathematics education – because of the complexity of the system. However, such bounded fields of activity enable teachers to begin with substantial changes without the risk of losing their professional competence in class.

(6) Universities as innovation centres
In these processes teachers get guidance and coaching by Universities. Universities could serve as innovation centres for teacher education. They provide regular and systematic in-service teacher education offers. This teaching and learning is designed according to the aspects of learning and the concept of learning environments described above. Thus, the teachers get acquainted with these theories and concepts by making personal experiences in learning environments designed for them.
(7) (Inter-)National teacher education

Teachers should be given possibilities to exchange experiences with colleagues and to participate in teacher education programs on national and international level. Thus, they understand that problems and necessities for developments have systemic character and concern the fundamentals of mathematics education far beyond their own professional sphere. Moreover, they get ideas for innovation activities from a large community.

(8) Evolutionary processes take time

Finally, such evolutionary processes take time. This seems to be rather trivial. But one cannot expect really substantial innovations in the educational system within the usual project funding periods of 2 to 3 years. A more realistic time-frame covers 10 to 15 years. But for that we need prospective and foresighted political decisions to fund long-term innovation programmes.

This approach may be called “theory based and material driven”. On the basis of the theory of cybernetics and the theories of learning the teachers involved make incremental-evolutionary steps on the meta-level of beliefs and attitudes by designing and working with concrete learning environments for their classes. If, as a consequence, teachers and students get accustomed to seeing mathematics as a field for individual and collaborative exploration, ICT may become a very natural tool for inquiry-based learning.

Summing up, we see that attempts to integrate ICT in mathematics education should very closely be connected to attempts of developing the complex system of mathematics education as a whole. The hope that ICT will “automatically” work as a catalyst for systemic innovations seems to be too simple.

Literature

The Impact of an In-service Course on New Technologies in Mathematics Education on Teachers and their Students

Abstract

In this chapter we present an in-service course on the integration of technological tools in mathematics and discuss its impact on teachers and on their students (after the teachers tested their new knowledge in their classes). More specifically, we discuss the rational of the course and its evaluation, in terms of teachers' and students' practices and emotions. The evaluation of the course draws on the analysis of a group interview and questionnaires completed by teachers and their students at the end of the lessons that these teachers taught after the in-service course. The analysis of the data showed that teachers felt more confident to use technology after their participation in this course, but they also outlined some adverse factors which tend to minimize their success. The primary concern of the teachers focuses on means and methods that would familiarize their students with the software and enhance their mathematical understanding. On the other hand, students were found to enjoy their experience, to use efficiently the mathematical vocabulary, provide sufficient explanations about the assigned tasks and make generalizations and expansions. The exploratory-dynamic nature of the lesson enhanced communication and together with its visualizing characteristics was projected by students as the main factors which lead to better mathematical practices. In conclusion, some implications for further research are drawn.

1 Introduction

Despite the explosive growth in the availability of technology in the mathematics classroom, the outcomes of employing such tools have so far been disappointing. While technology are purported to enhance students’ mathematical understanding, higher order thinking and motivation (Deane, Ruthven, & Hennessy, 2003; NCTM, 2000), the teaching of mathematics remains rather the same and the results continue to be non-satisfactory (Wilson, Notar, & Yunker, 2003). Since the teacher is the key person for the successful integration of technology in the classroom (Kendal & Stacey, 2002; Drijvers, Doorman, Boon, van Gisbergen, & Reed, 2009), it goes without saying that educators should develop in-service courses that will facilitate the successful implementation of technological tools in the classroom.

Along this line, we have developed an in-service course on the use of technological tools in mathematics and investigated its impact on primary teachers and their students. This course was developed in the frame of a European Project entitled Innovations in Mathematics Education on European Level (InnoMathEd). The project aimed at the development of students’ key competences and their ability to use technology for learning processes in mathematics.

In the next section we present an overview of two theoretical frameworks that have been proposed to explain the integration and the impact of new technologies in learning environments; some research findings within these frameworks in mathematics are also presented. In section three we present an outline of the course, the procedure and the setting of the study, the data collection, and the statistical analysis employed. The results are presented in section four, while in section five we discuss the results of the study.

2 Theoretical Background

2.1 Technology in Mathematics Education: Theoretical Frameworks

The description and analysis of the integration of new technologies in mathematics is quite complex (Monaghan, 2004; Drijvers et al., 2009). Two theoretical frameworks that acknowledged this complexity and model the approach effectively are the Instrumental Approach (Artigue, 2002) and the Affordances Constraints Approach (Gibson, 1977). According to the first framework, integration of technology relies on instrumental genesis, i.e., the construction of mental schemes (Drijvers et al., 2009; Goos & Soury-Lavergne, 2009). This process comprises two parts: instrumentalisation which involves the construction of schemes directed towards the technological tool, and instrumentation which is directed towards using the tool in mathematical concepts (Artigue, 2002). In brief, through instrumental genesis, the students become able to know how and why to use technology (instrumentalisation and instrumentation) in the mathematical context.

In this line, Assude (2007) proposed the idea of instrumental integration, which concerns four types of technology integration into
mathematical teaching, consisting of two parts addressing beginner students and students already familiarized with technology, respectively. Instrumental initiation and instrumental exploration refer to beginners; at these stages teachers focus mainly on how to initiate their students to the use of technology and to help them accomplish “some mathematical knowledge”. During instrumental initiation teachers use tasks that refer to the way in which technology is used, while during instrumental exploration teachers use mathematical tasks to help their students explore the possibilities of technology (Assude, 2007). The other two types of technology integration, instrumental reinforcement and instrumental symbiosis, concern students who have already experienced the first two steps. In practice, instrumental reinforcement occurs when students face technical difficulties with the use of technology while they solve mathematical tasks. At this stage teachers who aim to improve students’ mathematical knowledge, provide information and tasks about the specific functionalities, which will allow them to overcome these difficulties. Instrumental symbiosis is the ideal incorporation of new technologies in the mathematics lesson (Assude, 2007; Goos & Soury-Lavergne, 2009). In this stage, students work with mathematical tasks which allow them to improve both their know-how and their mathematical knowledge. This is the stage where the two kinds of knowledge are interconnected.

Several authors argue that mere use of the theoretical background of instrumental genesis to explain the integration of technology is not sufficient and cannot be “the complete solution to everything” (Drijvers, Kieran & Mariotti, 2009, p. 113). On the contrary, it takes the coordination of more than one theoretical framework or perspective.

Affordances and Constraints (Gibson, 1977; Greeno, 1998), the second framework has been employed by several researchers to describe the integration of technology in the classroom (e.g., Kennewell, 2001; Brown, Stillman, & Herbert, 2004). Affordances refer to “the attributes of the setting which provide potential for action of students” (Kennewell, 2001, p. 106), while constraints are these “conditions and relationships amongst attributes which provide structure and guidance for the course of actions” (Kennewell, 2001, p. 106). In other words, affordances refer to potential of an action (for instance, the use of technological tools are an affordance), while constraints concern the structure of that action, which may tend to limit the success (for example, time available to use the technology and the content of curriculum are constraints) (Thomas & Chinnappan, 2008). The role of the teacher is to guide, monitor, and organize affordances and constraints in a technology based classroom (Kennewell, 2001). However, the efficient use of the framework relies on the students’ capability in using technology to work with content and all together reflect their work on activities.

On the other hand, it was found that the outcome of students’ work with those activities affected their capability in the subject and in the use of technology (Kennewell, 2001). Therefore, learning processes in a technology based classroom seems to be a cyclic process.

2.2 Technology in mathematics education: Research findings

In this subsection, three strands of research findings about the use of technology in mathematics are presented. First, we focus on factors that influence the implementation of technology, next we focus on teachers’ practices in a technological enrichment environment and finally we discuss the role of technology in students’ learning.

Teachers’ and students’ beliefs and views are one of the factors identified by several authors as a major influence in using technology in mathematics classrooms. For example, Drijvers and colleagues (Drijvers, et al., 2009; Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010) found that teachers’ choices that participated in their study were strongly related to their views on mathematics learning and teaching. Furthermore, although teachers could recognize the educative value of technological tools, they were willing to implement them when they believed in their effectiveness in comparison to other approaches (Hennessy, Ruthven, & Brindley, 2005). For instance, Forgasz (2006) found that teachers used the technology in their classroom if they believed that this will increase students’ participation, motivation and interest. Other researchers found that teachers argued that the successful implementation of technology depends on the availability of multiple hardware and software as well as on relevant teaching materials and technical assistance (e.g., Simonsen & Dick, 1997). In addition, some researchers (e.g., Goos & Bennison, 2004) found that teachers argued that the lack of time and professional development programs were obstacles, because teachers felt unsure to implement technological tools in their teaching.

As regards teachers’ practices in a technology based environment, Drijvers and colleagues (2009, 2010) analyzed video tapes of 38 lessons by three teachers in an attempt to define the ways that teachers organized their classroom. Their analysis revealed six orchestration types: technical-demo, explain-the-screen, link-screen-board, discuss-the-screen, spot-and-show and sherpa-at-work. The first three types are teacher-centered approaches, while the last three are more interactive approaches. Technical-demo orchestration which refers to teacher presentation of techniques was used more frequently. The two types of orchestrations less frequently used by teachers were explain-the-screen, which in-

32 33
includes whole-class teacher explanation, and sherpa-at-work, where students use the technology.

In this respect, several case studies report on students’ opportunity to experiment, use visual reasoning, generalize problems and relationships, extent and prove, check the validity of their answers and assess their hypotheses using technological tools (e.g., Sinclair, 2004; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2005; Olive & Makar, 2009). However, research studies found that not all students employ the above practices with the use of technology in the same extent. More specifically, Olivero (2006) found that students who were “expert” with the dynamic geometry environment and had average mathematical abilities, used the explorative and dynamic characteristics of software in more “effective” ways (exploit fully the possibilities offered by software) than students who had higher performance in mathematics and little experience with dynamic geometry software. Other studies identified ways that students used technology while learning mathematics (Doerr & Zangor, 2000; Galbraith, Goos, Renshaw, & Geiger, 2001; Geiger, 2006). For example, Doerr and Zangor (2000) found that in a case study of mathematics classroom graphic calculator was used as a computational tool, transformational tool, visualization tool and checking tool. Galbraith et al. (2001) and Geiger (2006) reported that technology was used as a master, as a servant, as a partner and as an extension of self (Geiger, 2006). It seems, however, that the existing evidence is not sufficient to understand how far the way in which students’ work and learning has changed due to this different, technologically enriched environment (e.g., Aviram, 2001; Christou et al., 2005).

Generally, more research is needed to reveal how teachers and students use technology successfully and not simply whether technology is used in the classroom (Haughland, 2000). In mathematics education, Heid and Blume (2008) argue that future research must focus on how technology influences the mathematics taught and enhances students’ learning. This is what we aspired to address in this study by drawing on teachers’ and their students’ involvement in technology based lessons.

3 The Study

3.1 Purpose

In this chapter we analyze the impact of an in-service mathematics education course on the integration of technological tools on teachers and their students. More specifically, we address the following questions:

(a) What were teachers’ and students’ beliefs about the use of technological tools before and after they were exposed to teaching and learning mathematics with technology?

(b) How did an in-service course on the use of technological tools affect teachers’ and students’ mathematical learning environment?

(c) What were the constraints and obstacles that teachers came across in their effort to use technological tools in their mathematics classrooms?

3.2 Participants and Procedure

Participants were fourteen volunteer teachers from primary and secondary education who taught mathematics in their school classrooms in Cyprus. More specifically, these teachers attended an in-service course on the use of technology in mathematics and then applied the knowledge gained in one or two 80 minutes mathematical lessons. Hundred and seventy seven students (76 boys and 101 girls) ranging from 9 to 14 years of age, participated in these lessons (23 Grade 3 students, 41 Grade 4, 58 Grade 5, 28 Grade 6 and 27 Grade 7). These students were all students of the teachers that participated in the in-service course and were from nine schools of rural and urban areas of Cyprus.

3.3 In-service Course

The report of the National Council for Accreditation of Teacher Education (2001) highlights the role of technology in many of teachers’ professional development programs. It argues that this role is not clearly defined or incorporated in the mathematics teaching. It is therefore important that teacher educators develop in-service courses, based on principles where the role of technology is explicit. In developing the in-service course for the present study (hereafter we use “the course” as an abbreviation of this course) we tried to materialize the principle “learning with technology” and not “learning from technology” (Jonassen, Howland, Moore, & Marra, 2003) and also to use of technology as a means to reform practices and not simply add to existing practices (Papert, 1995). We also adopted Assude’s idea of instrumental integration (Assude, 2007) and emphasized learning which occurs when students know how and why to use the functions of the software in mathematics learning. We therefore presented teachers with tasks where they were learning the functionalities of the software and at the same time develop mathematical concepts. Furthermore, the development of the course was based on the belief that technology is best learned in context rather than in isolated situations (Willis, 2001).
The course objective was to initiate teachers to dynamic geometry softwares (Euclidraw and DALEST/Elica); as a result we were expecting them to become able to use these tools in the classroom to enhance students’ understanding of Euclidean geometry, three-dimensional geometry, and transformational geometry.

We chose to work with dynamic geometry software because they offer a powerful environment for the creation of rich mathematical activities. This software provides tools that can be used by the learners to produce and manipulate geometric objects, conduct explorations, make conjectures, experiment, reach conclusions and prove mathematical statements (Laborde, Kynigos, Hollebrands, & Straßer, 2006).

The course lasted for 18 hours and was organized in six three-hour sessions. These sessions were organized in three parts. In the first part teachers were presented with the specific software and discussed its main features. In the second part the discussion was focused on the development of lesson plans with the use of these new tools. The final part of the program constituted a meeting, which was held after the teachers had organized and conducted a lesson with the use of the new technological tools that they have learned. During this meeting, the teachers had the opportunity to share their experiences and feelings. They talked about the lessons that they organized, what they found useful or difficult in this practice and how they could better use it in the future. All sessions were held at the Mathematics Computer Laboratory of the University of Cyprus.

The course was scheduled as follows:

**Session one – Familiarisation with Dynamic Geometry Software**

During the first session teachers were introduced to the Dynamic Geometry Software Euclidraw Jr (Logismos Inc., 2002) and specifically to the software’s menus, processes and help. All participants had a computer in front of them, installed with the latest version of Euclidraw software. The instructors showed one or two of these functions (depending on their complexity) and then the attendees were asked to repeat the action or in some cases work on a simple task. At any point of the presentation the course participants were able to express their comments or questions. The aim of this session was to help teachers learn how to use some functions of Euclidraw Jr and also see how these functions are linked to the development of various mathematical concepts. This approach is similar to what Assude (2007) calls instrumental initiation and instrumental exploration.

**Session two, three and four – Euclidean Geometry Activities with the use of Dynamic Geometry Software**

In sessions two, three and four, the participants were presented with eleven Euclidraw activities: sum of angles in a triangle, properties of quadrilaterals, construction of quadrilaterals, polygons, radius of a circle, circumference of a circle, Evdoxos method of finding the area of a circle, construction of triangles, area of a parallelogram and area of a triangle. The handouts that the teachers were given had two distinct parts: “Teacher’s Handout” and “Student’s Handout”. The “Teacher’s Handout” included a small description of the activity, a reference to its place in the mathematics curriculum, its teaching duration, its aims and the specific functions of software one should now in order to execute the activities. The “Student’s Handout” included a series of exploration tasks that the students could perform. The first exploration tasks were more guided whereas the later more open. The idea was that students may initially need some guidance to understand the mathematical concept whereas once this is achieved they may be able to explore and expand their ideas.

Teachers were asked to read carefully each activity and execute it as if they were students themselves. Once this was done, teachers were given the opportunity to express their questions, views and ideas about possible modifications of the activities. The teachers were given the opportunity to read and discuss the rest of the activities and decide which ones to apply in their classrooms. This approach gave them the opportunity to learn the functionalities of Euclidraw Jr and also the way in which it can be used for the development of mathematical concepts. Assude (2007) characterises this stage as instrumental reinforcement and instrumental symbiosis.

**Session five – Dynamic Geometry Software for the teaching of stereometry and transformational geometry**

During the fifth session, we followed the same procedure as described above to introduce the teachers to the DALEST-3D menus, processes and help. We also presented teachers with two DALEST activities. The first activity dealt with the exploration of the relationship between the number of sides, vertices and edges of pyramids and the second activity with the formula of the volume of rectangular parallelepiped.

In the rest of the session teachers were introduced to the DALEST/Elica tool library. The DALEST/Elica tool library offers a number of mathematical applets related to three dimensional and transformational geometry: (a) Elica Nets, (b) Cubix Editor, (c) Potter’s Wheel, (d) Origami Nets, (e) Slider, (f) Stuffed toys and (g) Scissor Application. Teachers further explored the Elica tools in two activities dealing with cubes and cubes constructions.
Teachers were presented with the functionalities of DALEST and Elica and the way in which these connected with specific mathematical concepts.

Session six – Discussion about the use of new technologies in the mathematics classroom
Once the teachers had implemented their innovative activities in their classrooms they attended this session and discussed their experiences, difficulties, and suggested improvements.

3.4 Data Collection

Teachers completed a pre-questionnaire regarding their views and use of technology before the commencement of the course and a post-questionnaire after they completed the course and taught at least one lesson with the use of new technologies in their classroom. Both in the pre- and in the post-questionnaire teachers comprised of 23 statements, on a 4-point Likert scale (1 indicating total disagreement and 4 total agreement). Eight of the items referred to teachers previous knowledge (e.g., I am familiar in using computers in mathematics lessons) and perceptions about computers (e.g., I support the use of computers in my mathematics class), ten of the statements concerned students’ learning, beliefs and feelings about technology (e.g., I think my pupils enjoyed the work with the computer) and five of the statements referred to teachers’ beliefs about the work with computer in a mathematics class (e.g., I think that the use of the computer supports weak learners). In addition, the post questionnaire included open ended statements reflecting teachers’ feelings, beliefs and practices, about their new experience from using of technological tools in their mathematical classrooms. A sample of the open ended questions is presented in Table 1.

Table 1: Sample of the open ended questions in teachers’ post-questionnaire

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Describe your feelings after conducting a technology based lesson in mathematics.</td>
</tr>
<tr>
<td>2. Which are the most significant benefits from your participation in this course?</td>
</tr>
<tr>
<td>3. How did you organise the technology based lesson(s) in mathematics?</td>
</tr>
<tr>
<td>4. “The use of computers changes the way in which children learn”. Do you agree or disagree? Justify your answer, providing a description of a special part of your technology based lesson.</td>
</tr>
<tr>
<td>5. Can you use the technological tools in every mathematical lesson?</td>
</tr>
</tbody>
</table>

The teachers also participated in a group interview which was conducted after the completion of the course and after conducting the mathematics lesson with the use of technology in their classroom. The group interview was audio taped with teachers’ permission. The group interview was semi-structured and included questions regarding teachers’ experiences and actions in this lesson. Table 2 provides a sample of the questions that were used in the group interview.

Table 2: Sample of the questions used in the group interview of teachers

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Describe the mathematical context of the lesson that you taught with technological tools and provide an explanation for your choice.</td>
</tr>
<tr>
<td>2. Can you characterize your mathematics lesson with the use of new technologies as successful? Explain why.</td>
</tr>
<tr>
<td>3. Which characteristics of your lesson with the use of new technologies let it to being successful?</td>
</tr>
<tr>
<td>4. What obstacles did you face during the mathematics lesson with the use of new technology?</td>
</tr>
<tr>
<td>5. What were your students’ feelings during this mathematics lesson?</td>
</tr>
<tr>
<td>6. What is the role of technology in a mathematics lesson?</td>
</tr>
<tr>
<td>7. Who benefited the most from the use of new technology in your classroom, the low achievers or the high achievers?</td>
</tr>
</tbody>
</table>

The participating students completed a pre-questionnaire before being taught with the use of new technologies and a post-questionnaire after they participated in a mathematics lesson with the use of technological tools. In the pre-questionnaire students were asked to specify how often they worked with computers at home as well as in their mathematics classroom. In addition to this, students were asked to read 18 statements regarding previous knowledge, experiences and beliefs about the use of technology and indicate their degree of agreement with the statement on a 4-point Likert scale (1 indicating total disagreement and 4 total agreement). Nine of the items referred to previous knowledge and experiences of working with computers (e.g., I have experiences in working with computers) and learning software (e.g., I normally don’t work with learning software). Four items stated their beliefs about the use of learning software in their classroom (e.g., I think we should work with computers in class) and five items referred to their general attitudes towards mathematics lesson (e.g., I think mathematics is difficult). In the post questionnaire students were asked to specify what they have done in their mathematics classroom with the use of computer, their feelings with this lesson and their beliefs about the role of technology in their classroom. Furthermore, they were asked to respond 21 items of 4-point Likert scale. These items referred mainly to the context of the lesson with computer (e.g., The colorful design of the learning material was helpful) and to students’ beliefs about the use of learning software in their mathematical classroom (e.g., I can imagine writing mathematics-tests on the computer). Table 3 provides some examples of the questions used in the students’ post questionnaire.
Table 3: Sample of the items in the post questionnaire of students

1. Complete the sentences below based on your experiences with the technology based mathematical lesson.
   - I found learning mathematics with the use of new technologies more difficult because ____________________.
   - Using new technologies helped me understand the mathematical lesson because ____________________.

2. Read the students’ statements below. Write the name of the student that you agree with and justify your answer.

   - The computer gives me the chance to explore, understand and solve a problem.
   - The computer helps me find the solution to a problem as a good partner could do.

   I agree with _____ because _____.

3.5 Data Analysis

In correspondence to the aims of the study, both quantitative and qualitative techniques were used to analyze the data, since a combination of both types of methods give a clearer picture of the data (Richardson, 2001). To investigate teachers and their students’ views and beliefs about mathematics and technology as well as their experiences about the use of technology descriptive analysis of the responses to the Likert type items was used. Furthermore, to analyze the impact of the course on teachers and their students and the obstacles and the constraints that teachers faced, we used a combination of coding and categorizing as described by Creswell (1998) and open, axial and selective coding as described by grounded theory (Strauss & Corbin, 1990). More specifically, we organized teachers and their students’ responses in the pre- and post-questionnaires as well as from the group interview that followed teachers’ implementation of new technologies in their classrooms into sub-categories based on related themes (open coding) and then we grouped these sub-categories around general categories (axial coding). These general categories were grouped into three specific axes: learning environment with computers, constraints and obstacles (see Fig. 1). These axes were determined from a combination of research findings (e.g., Ruthven & Hennessy, 2002) and theories (e.g., Gibson, 1977) about the impact of technology based course on teachers and their students.

4 Results

A number of researchers (e.g., Simonsen & Dick, 1997; Goos & Bennison, 2004; Henessy et al., 2005; Drijvers et al., 2009, 2010) identified several factors that affect the implementation of new technological tools in the mathematics classroom. The factors that we will discuss and explore are: teachers and their students’ views about technology and mathematics, teachers’ previous knowledge and experiences with the use of technology. Teachers’ and students’ beliefs about these factors were examined with the use of questionnaires before the implementation of the course. We will then proceed to discuss the way that the course affected the teachers’ use of technological tools as well as teachers’ and students’ practices, emotions in mathematics about the lesson(s) that they conducted with the use of new technologies. Finally, we examine the constraints and obstacles that teachers faced while using new technologies in the mathematics lesson(s). The aspects that we explore in regard to the impact of this course are illustrated below (see Fig. 1).

4.1 Factors that Affect the Learning Environment with Technology

4.1.1 Views about Technology and Mathematics

Teachers’ views about the use of technology in classroom: All teachers that participated in the in-service course argued that the use of technology in their teaching is very useful. A teacher pointed out that: «Nobody can underestimate the role of technology in today’s...»
classroom. I think, however, that most teachers need more guidance on how to use the technology.»

The quantitative data indicated that teachers support the use of technology in their mathematics classroom. In the relevant question, the mean was found $\bar{x}=3.35$ (SD=0.74), with 4 reflecting complete agreement. With regard to students’ using technology in learning mathematics, the teachers believe that it is important ($\bar{x}=3.28$, SD=0.61), while on the relevant statement that the use of computer in school has only limited benefits, the teachers strongly disagree ($\bar{x}=1.71$, SD=0.61). The teachers were also found to believe that the use of technology increased their students’ interest in mathematics ($\bar{x}=3.50$, SD=0.52) and enjoyment of the mathematics lesson ($\bar{x}=3.57$, SD=0.51); they were very positive that their students’ would favor work with computers in the mathematics classroom ($\bar{x}=3.30$, SD=0.48).

The following extract from a teacher’s comment is indicative of the view that technology might enhance understanding, particularly of some difficult mathematical concepts: «Some students face difficulties in understanding certain mathematical concepts, if they are taught with traditional pen and paper methods. These concepts can be better understood when taught using technology.»

Students’ views about the use of technology in classroom: All participating students, supported the importance of work with computer in their classroom ($\bar{x}=2.82$, SD=1.07); they also agreed that the use of computer will help them understand the mathematical concepts ($\bar{x}=2.97$, SD=1.93). Furthermore, they found work with computers quite interesting ($\bar{x}=3.32$, SD=0.92), and they expressed the wish to work with computers ($\bar{x}=3.06$, SD=1.07).

Students’ views about mathematics lessons: In general, students claimed that they liked mathematics ($\bar{x}=3.37$, SD=0.90) and felt that they learned important ($\bar{x}=3.70$, SD=0.53) and useful things ($\bar{x}=3.54$, SD=0.74).

4.1.2 Previous Experiences/Knowledge

Teachers’ experiences with the use of technology in classroom: Half of the participating teachers claimed that they used computers in their mathematics classrooms up to five times during the current school year while the rest of the teachers used computers five to ten times. They claimed that they did not use the computers very often neither to do their teaching ($\bar{x}=2.07$, SD=0.47) nor to prepare their lessons ($\bar{x}=2.00$, SD=0.39). In brief, according to their responses, the teachers had limited experiences with technological tools. They had some experience in using mathematical applets, but they felt that some of these mathematical applets did not promote mathematical thinking. One of these teachers said: «I am familiar with some applets and I have access to applets in my school which focus only on the development of mathematical procedures but not of mathematical thinking. This is because both my colleagues and I are not aware of the use and role of technology in our classroom.»

The teachers’ lack of knowledge about computers was also evident from their response to the statement reflecting the level of difficulties when using the computer in their teaching ($\bar{x}=2.78$, SD=0.43) and on the statement reflecting their confidence when working with computer in their class ($\bar{x}=2.25$, SD=0.73).

Students’ experiences with the use of technology: Students, similarly to their teachers claimed that they did not use computers in their classroom very often. Around 41% of the students claimed that they had never used the computer in their classroom during the current school year. It is noteworthy that these students were experienced in the use of computers ($\bar{x}=3.47$, SD=0.73) and they felt confident in working with it ($\bar{x}=3.43$, SD=0.81). This was probably due to the access they had to computers at their home ($\bar{x}=3.44$, SD=0.91); 86% of the students said that they used a computer every day or at least once a week. It appears, therefore that the students use the computer for other purposes, but not for learning in school.

4.2 Learning Environment with Technology

4.2.1 Organisation

This part of the analysis can be characterized by the metaphorical notion “instrumental orchestration”. Trouche (2004) used this notion to describe teachers’ organization and use of the technological tools in a mathematics classroom. In this subsubsection we analyze the way in which the participating teachers organized the mathematics lesson(s) with the use of technology (for the rest of the chapter we will use “the lesson(s)” as an abbreviation for this lesson(s) with technology). We will present the aim of their lesson(s), the mathematical content, the arrangement of the class, the way in which technology was used, the role of the teacher and the role of the students. We drew information about these issues from the questionnaires completed by the teachers and students as well as from the group interview conducted with the teachers.

Aim of the lesson(s): All the teachers aimed at helping students to better understand the use of the software, while they argued that the role of technology is not as a servant but it should be used to expand students’ mathematical thinking. One of the teachers
In-service Course on New Technologies in Mathematics Education

P. Sophocleous, D. Pitta-Pantazi, G. Philippou, N. Mousoulides

that allowed students to generalize and expand their conclusions.

One of the teachers pointed out: “During the lesson, I used the technological tools in an extensive way. Students investigated and discussed their activities. This was done through the use of technology in the classroom, as well as by similarity claims made by students to their mathematical thinking and understanding.»

The choice of content: Eleven out of the 14 teachers organized one mathematics lesson with the use of dynamic geometry software (Euclidraw, DALEST, Elica), while three of them organized two mathematics lessons. Four teachers taught the basic properties of two-dimensional geometrical shapes. Two teachers used the Euclidraw to teach the area of the parallelogram, two worked on the properties of polygons, and one teacher on the properties of solids.

Arrangement of the class: Ten teachers conducted their lessons in the computer laboratory of their school, where they asked students to work in groups of two or three. Each group sat in front of a computer and they collaboratively worked on the software to generate answers to the questions given on a worksheet. The teachers presented the various functionalities of the software, which were necessary for the execution of the task. In these cases, the teachers and the students worked together, while the whole classroom was observing. Once the tasks were done, students were asked to carry out the exploration on the computer screen and discussed the possible solutions. The facilitator role of the teacher and the active role of students were also mentioned by students. Three indicative students commented on the facilitative role of the teacher and the active role of students.
ments were: «Today, my teacher didn’t speak very much and gave us the chance to work with the computer.»; «I liked the fact that the teacher allowed us to work on our own in some of the tasks.»; «I found the lesson with computers very easy, because my teacher explained things very well and helped us with our work on the computer.»

As it arose from teachers’ and their students’ responses, the teachers had integrated technology in their mathematical lesson(s) by employing instrumental exploration at the beginning of their lesson(s) and instrumental reinforcement in the rest of their lesson(s) (Assude, 2007).

4.2.2 Mathematical Practices

To discuss the interactions and learning process in the classrooms during the technology based lesson(s) we use the term “mathematical practices”. According to Ball (2003), “mathematical practices” refer to the things that mathematicians and mathematics users do, which are important in learning and doing mathematics. To achieve this, we analysed teachers’ and students’ responses into two subcategories: students’ participation and collaboration in this lesson and students’ learning. We focused mainly on the changes that either the teachers or the students observed, in regard to students’ participation, collaboration and learning, when comparing this technology based lesson(s) to non-technology based lessons.

Participation of students: All teachers were impressed by the increased interest and participation of their students in comparison with other lessons. More specifically, two teachers pointed out that: «All my students wanted to talk during the lesson. They were disappointed when I didn’t give them the chance to talk in the classroom!»; «I noticed that the oral participation of all my students, irrespective of the level of achievement, had increased.»

Four teachers, when referring to students’ work talked mostly about the active participation of low achieving students which was a surprise to them. One of these teachers said: «My students showed an unexpected interest. I was impressed when I saw low achieving students posing questions and making comments about their explorations. For example, they talked about the relationship of the area of a circle and the area of a parallelogram.»

In addition to this, all teachers argued that the use of technology captured students’ attention and made them more active and involved. Two teachers added that the use of technology made students “more quiet”, in the sense that they were focused and engaged with the lesson. One teacher said: «During the technology integrated lesson, I was impressed with the interest expressed by my students when manipulating the geometrical shapes for the first time on the computer screen. It was a very quiet learning environment. My students were absorbed and focused in the lesson.»

In the same line some of the students supported the above comments and pointed out that: «I was fully absorbed with the lesson with technology.»; «I was concentrated on the activities.»

Communication between students: Half of the teachers claimed that the students’ participation increased, because the students were engaged in meaningful communications with their peers, exchanging views in a creative manner. A teacher highlighted that: «My students expressed their agreement or disagreement to the comments made by their peers and all the students wanted to talk during the lesson.»

Similarly, the students enjoyed working with their peers on the computer; they would even use terms like enjoyable and playful. Some indicative responses follow: «I liked collaborating with my friends. It was like a game.»; «The lesson with the computer was better than other lessons in mathematics because I worked with my friends on the computer; it was fun.»

The use of the mathematical vocabulary: Teachers also claimed that the activities that students were engaged, led to increased communication with their peers or with their teacher. In these communications they used accurate mathematical terms and vocabulary. For instance, one teacher said that her students talked about the shapes and their properties: «I was impressed with my students. They used mathematically correct vocabulary to talk about the circle, its diameter, its radius and also to present their results.»

Conceptual understanding of mathematics: All teachers agreed that their students understood conceptually the mathematical concepts which were presented to them with the use of the technological tools. They also thought that their students will remember these mathematical ideas. Two extracts from teachers’ interview follow: «I felt that using technological tools led students to understand and apply more efficiently the formulae of area and circumference of a circle. They understood these mathematical concepts, better than my last year students whom I taught the same content without technology.»; «I think that students will remember the area of parallelogram and the sum of angles of a triangle that I taught with technology.»

Along the same line, several students argued that they had learned new and difficult mathematical concepts with the use of technology. This caused a positive disposition towards the mathematics lesson: «I didn’t know anything about the radius and diameter of a circle and I learned them with the computer.»; In addition to this, a considera-
A considerable proportion of students argued that some mathematical ideas became more tangible to them. Two students commented: “The theory became action in the mathematics lesson with the use of the computer. This is why I found it easier to understand the mathematics.” “I saw all these things practically on my screen and this is why I understood them.”

**Generalization and expansion:** Ten teachers argued that a considerable proportion of their students could generalize and expand further the ideas that they were presented with during the lesson(s) with technology. Three teachers claimed: “My students noticed the relationship between the radius and the diameter and the infinite number of radii in a circle without any intervention of my own questions. They reached these conclusions from their observations of what was happening on the computer screen.” “My students reached a number of conclusions regarding the properties of quadrilaterals.” “My students progressed in an advanced level without any guidance. They were amazed and proud with their results, as they have reached them on their own.”

On the other hand, students claimed that: “I was very impressed when I saw that the parallelogram could be changed into a rhombus, a square and a rectangle.” “I liked it very much when I realised that the circle has an infinite numbers of radii.”

It appears that the technological tools supported students’ development of mathematical practices. Specifically, the use of technological tools facilitated students’ use of accurate mathematical vocabulary, provided students with sufficient explanations about the mathematical tasks and enhanced students generalizations and expansions.

### 4.2.3 Affective Factors

To analyze teachers’ and their students’ emotions about the mathematical lesson(s) with the use of technological tools, we analyzed teachers’ and students’ feelings once the lesson(s) was completed.

**Teachers’ feelings about the lesson(s):** Many of the teachers (ten out of the 14 teachers) argued that they felt confident and safe to use the technology in their class. A teacher pointed that: “It was the first time that I felt safe to use a computer in my class. This was because I learned very well the functions of the software that I was going to use in my teaching.” Similarly, another teacher argued that: “I felt confident because during the course I understood how I can use the functions of the specific software and why should I use it.”

However, there were four teachers who felt insecure, uncomfortable and uneasy during their lesson(s). They attributed this to the fact that it was the first time that they taught with the use of technological tools. One of these teachers explained that: “I was afraid that I was going to do everything wrong when using the software and I wouldn’t know how to rectify my mistake.”

Irrespective of their initial feelings, all teachers stated that they felt great satisfaction and happiness because their students were very happy and the goals of the lesson were achieved. A teacher said: “I felt quite satisfied to have been able to offer my students a “different” lesson than my regular mathematics lesson.”

It is noteworthy that all the teachers pointed out that: “I felt satisfied with my lesson because my students were happy.” In addition, some teachers expressed their feelings like: “I was pleased to see my fourth grade students constructing shapes on the computer. It was the first time that they expressed the wish to complete the worksheets that I gave them.”

It appears that the interactive and explorative character of the worksheets was well received by the students.

**Teachers’ beliefs about students’ feelings from the lesson(s):** All teachers claimed that their students were very happy and enthusiastic with the new experience. They also stated that there were students who expressed the wish to have more technology based lessons: “My students were both surprised and enthusiastic with the new things that they learned with the computer. They told me that they wanted to have more computer based lessons.”

Furthermore, many teachers said that their students enjoyed their lesson(s) since they were given the opportunity to discover new things. This enjoyment and satisfaction is obvious in the following statement: “My students didn’t want to leave the classroom to have their break, when the bell rang. None of them got up from their seat. They also asked permission to save the software on their memory stick and take it home.”

It is clear that the students were impressed by this lesson(s).

**Students’ feelings about the lesson(s):** All students that participated in the lesson(s) argued that they liked it very much: “I liked very much the mathematics lesson. It was perfect. I didn’t have any difficulties with the mathematics.” “I enjoyed the lesson because I learned things in a different way and I understood the mathematics better and easily.” “The lesson with the computer was different and something completely new for me.”

In general, students enjoyed this lesson(s) because they got engaged with the activities and they found the mathematics easy, pleasant, interesting and fun.
4.3 Constraints

Two main constraints that the participating teachers faced when they had to decide the content of the lesson(s) were: the mathematics curriculum and teaching time. More specifically, five teachers pointed out that one of the reasons that they chose the area of parallelogram and the properties of quadrilaterals was because it was part the mandatory mathematics curriculum in Cyprus. The same teachers complained that due to pressure they chose the context that it helped them “cover the pages of mathematics book.” On the other hand, two teachers mentioned that: “We need more than one or two lessons with technological tools, so our students may become familiar with the software and have more learning benefits.”

In addition to this, all teachers supported that the use of technology in mathematical classroom needs more teaching time.

4.4 Obstacles

Although all teachers found the effect of the use of new technologies beneficial and useful to students’ learning, some of them were hesitant whether they would use technological tools in the future. This is because of possible technical problems that may arise with the machines or even with the computer laboratories (e.g., not being available). More specifically, half of the teachers faced problems with the smooth functioning of the computers. On this issue, a teacher said: “I was disappointed when my students and I tried to show on the computer screen the length of the radius of a circle and it didn’t appear. However, I continued with the rest of the activities and I didn’t lose my students’ attention. The next day I tried the same function on the same computers and everything was okay!”

This teacher as well as all participating teachers that faced similar technical problems continued with the rest of the activities and explained to their students what the software would have shown if the program functioned properly. Some teachers claimed that these technical problems probably would not occur, if the necessary technical assistance was available in schools. In addition to this, almost all the teachers declared the need of computer laboratories in all schools. They argued that each student should be able to work on his/her own computer.

5 Discussion

In this century the use of technology became a necessity. It makes our life easier and simpler. For this, the introduction and the use of technological tools in the classroom is a major aim of educational administrators and policy makers. Policy makers feel that the implementation of technology in schools will prove to be beneficial for the educational systems. However, despite the huge investment for the introduction of technology in education, “there is no clear evidence that technology leads to the improvement of students’ outcomes” (Aviram, 2001, p. 331). Therefore, as Aviram (2001) concluded, it is necessary to use technology in a different way to have the expected educational results. In this line, this chapter described the development and implementation of an in-service course on new technologies in mathematics education and its impact on teachers and their students.

This in-service course aimed to enhance teachers’ knowledge about the use of technological tools in their mathematics classroom. The course aimed to familiarize teachers with the functionalities of the software that were necessary for the various activities and also offer guidance to teachers on the way to improve students’ mathematical knowledge through explorative and dynamic tasks with the use of technology. Teachers appeared to have gained this knowledge from the course and the lesson(s) they conducted. This became apparent from the teachers’ responses about the way they used technology in their classroom. Specifically, they did not simply teach the software functionalities during their lesson(s) (this is what Assude calls instrumental initiation), but they taught the software functionalities through mathematical tasks at the beginning of their lesson(s) (instrumental exploration) and concentrated on the development of mathematical concepts during the rest of the lesson(s) (instrumental reinforcement). These types of integration of technology demonstrate teachers’ appreciation of technology based lessons. Teachers chose to employ two medium degree integration models where instrumental knowledge and mathematical knowledge are interwoven. However, they did not reach instrumental symbiosis which it is the ideal integration of technology in classrooms. It can be hypothesized that teachers need to teach more lessons with technology to achieve a strong degree of technology integration. Of course such a claim needs to be tested in order to be confirmed.

All the teachers argued that they felt more confident to use technology in their classroom after their participation in the course. Similar results were also found by Lin (2008) who developed similar courses. He claimed that participating teachers agreed that the workshops helped them increase their confidence and felt less anxious in using computers to teach mathematics. However, in this study half of the participating teachers expressed their disappointment when they faced some constraints and obstacles during their technology based teaching. They suggested that it is not easy to teach with technological tools every day. It is necessary to have adequate time, suitable equipment and resources (e.g., a computer for each student, suitable learning explorative software).
Furthermore, teachers were very impressed by their students’ increased participation and happiness. They also claimed that their students were in charge of their learning and had an active role in the classroom. Students confirmed these claims with their responses in the questionnaires. In addition to this, a number of teachers observed that low achieving students were more motivated and involved in the lesson with technology rather than with the non-technology based lessons. All the above findings are in line with the results by Ruthven and Hennessey (2002). Moreover, teachers observed that the technology gave their students the opportunity to use mathematical vocabulary efficiently and to make generalizations and expansions. The above observations were also confirmed by the students’ responses. We hypothesize that these mathematical practices, were developed due to the way technology was used. All teachers claimed that the use of technology offered an experimental, explorative, dynamic, communicative and motivational environment.

The results of this study constitute some initial indications from a small number of lessons conducted with the use of new technologies. It will be useful for future research to investigate further new mathematical practices raised with the use of technology in classroom settings. For instance, longitudinal studies may show the long-term effect of these programs.

Finally, on a practical level, the results of this study suggest that the use of technology in the mathematics curriculum is beneficial, since it facilitates students’ mathematical understanding. It is noteworthy that a short scale intervention had these positive results. Curriculum designers and authors of mathematics textbooks should invest on the inclusion of these new technologies in the mathematics classroom.

Literature


Explorations Around the Rotational Solids

Abstract

The paper deals with the importance of involving junior high school students in inquiry-based activities related to rotational solids. A pilot experiment carried out with 5-6 graders is discussed in terms of variety of problems solved by means of appropriate computer applications.

1 Introduction

The importance of introducing an inquiry based style in the mathematics and science classes has been addressed by educators and policy makers (Kok, 2010; Rocard, 2007). In this paper we are focusing on the experience gained when applying such a style with sixth graders in the context of rotational solids. Below we present some ideas for organizing the exploratory activities related to discovering patterns in the context of rotational solids (mainly by paper models), together with the formulae for the surface area and the volume of these solids.

The applications our students have used for studying various properties of rotational solids via virtual experiments were Elica Dalest computer applications (Potter’s Wheel, Math Wheel and Bottle Design) (Boytchev, 2008). Their typical features include an easy start combined with a great potential for explorations. From a didactical point of view the difference is determined mainly by the final objective of the scenario – to model an object from the real world by means of sufficiently close approximations, or to compute the volume and the surface of solids that are a combination of the simplest ones. These specifics will be better demonstrated in the context of several didactic scenarios (presented below in italics).
The structure of the scenarios being developed within the European project DALEST (Christou, 2007) and further enriched and elaborated in the frames of the InnoMathEd project (Kenderov, 2010) includes a challenging problem, introduction of a software for modeling rotational solids, hints, and a chain of problems which solution requires a formulation of conjectures and the use of properties the students already know. Here follows an example of explorations involving the volume of a cylinder.

**The challenge**

A firm wants to place an advertisement in the shape of a 12x3 rectangle. The rectangle will turn around one of its sides. The firm wants to know which of the two adjacent sides should be the axis of rotation, since the cost depends on the space taken by the ad while it turns.

![Fig. 1:](image1)

We invite the students to solve a sequence of related warm-up problems so as to answer this question with confidence. To verify their answers and further explorations they construct a model of each cylinder in the Math Wheel by selecting the vertices of the rotating rectangle on a Cartesian coordinate system (Fig. 2):

![Fig. 2: Model of cylinders in the Math Wheel](image2)

Then they can observe the rotational solid generated after the constructed figure rotates around the y-axis and use the Math button to check their answers (Fig. 3).

![Fig. 3: Visualizing the computation of the cylinder’s volume and surface](image3)

Next the students are expected to work in inquiry-based style. They fill the first four columns of Table 1 using the Math Wheel software and make a conjecture about the numbers in the last two columns. Then they use the formulae they know to check their conjecture. Finally, they formulate a rule (1) about how the cylinder's surface changes when its height changes, and (2) about how the cylinder’s volume changes when its height changes.

<table>
<thead>
<tr>
<th>Table 1</th>
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<td>S</td>
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<tr>
<td>V</td>
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</tbody>
</table>

Similarly they explore how the volume changes if h is constant. Their observations and conclusions enable them to compare the volumes of different cylinders without calculating them explicitly:

**Task**

To increase the cylinder's volume 100 times, how many times do you have to increase:

a) its height?

b) its radius?
3 Playing, Conjecturing and Verifying with Potter’s Wheel

When working with the Potter’s Wheel program the students:

- use different figures to be rotated – a segment, a triangle, a square, a circle, a sine fragment, a free curve (Fig. 4); then they explore various positions of the rotated object (Fig. 5) in a rather chaotic manner and eventually – verify their conjectures about the solid to be obtained.

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**Fig. 4:** The Potter’s Wheel objects under rotation

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**Fig. 5:** Exploring the effect of changing the square’s positions under rotation

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- use the options for showing consecutively the contour, the rotational solid and the cross-section (Fig. 6).

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**Fig. 6:** The objects under rotation, the solids they generate and their cross-sections

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- observe the solids from different perspectives (Fig. 7);

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**Fig. 7:** A rotational solid from different perspectives

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- model objects chosen among a collection of the kind (Fig. 8).

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**Fig. 8:** A collection of real-world objects used for modeling with Potter’s Wheel
Here are some of the computer models of objects being chosen by the kids (Fig. 9):

![Images of computer models]

**Fig. 9: Modeling real-world objects by means of Potter's Wheel**

Those who chose to model the glass shared: This one is the easiest to do. I'll make it in no time…

However, at the moment the teacher posed a problem in a contest-like form (Who will get the best model of the hat for a fixed time - 3 min), all the participants demonstrated a high ability to concentrate and achieved good results.

When playing with Potter's Wheel the students explore a rich variety of rotational objects for a relatively short time in terms of their generating figures, different perspectives, etc. The problems posed directed their thinking to figuring out how to find the volume of more complex solids (e.g. by adding or removing certain parts similarly to what they had done with 2D figures). The goals of the activities included modeling and transformation of a rotational solid by given parameters – a contour, intersections, volume, etc. In addition, the rotational solids obtained were a good source for artistic ideas. In view of the user-friendliness of the program it was possible to acquire the technical details during the process of problem solving.

4 "Raising the Bar" – Computing the Volume of More Complex Rotational Solids

A specific feature of Math Wheel and Bottle Design is that together with the moving points which form the rotating figure, these programs provide a grid of square units to facilitate the measurement of the needed quantities. Below we are presenting a system of problems on computing the volume of rotational solids we have offered to our pilot students.

**Task**

Find the volume of the rotational solids generated by Math Wheel:

(a) ![Diagram](a.png)  
(b) ![Diagram](b.png)  
(c) ![Diagram](c.png)  
(d) ![Diagram](d.png)  
(e) ![Diagram](e.png)

**Fig. 10: Tasks concerning the volume**

The rotational solids could be considered as a difference of the volumes of two cylinders (a), a cylinder of which a cone has been cut out (b) and as a cylinder of which two cones have been cut out (c). For the case (c) some of the students needed help with the shape of the rotational solid whose volume they were expected to find. The reason was that they hadn’t figured out that the object under rotation was the triangle formed by the three movable points rather than the polygon passing through them starting at the rotational axis and ending at the axis again (as it is within Bottle Design) (Fig. 11).

**Fig. 11: The difference when rotating a triangle (Math Wheel) and a polygon (Bottle Design)**

Being able to identify the most appropriate among several computer applications is an important soft skill (Stefanova et al., 2007) which our students were given the chance to develop while working on the topic of rotational solids. Here are examples experienced with students and teachers (Chehlarova & Sendova, 2010; Kenderov & Sendova, 2010):
Task
Do you remember the fable of La Fontaine about the fox and the stork? (The fox offered the stork the soup in a very shallow dish, and the stork took his revenge by serving the dinner in a very long-necked jar with a narrow mouth.) Choose the most appropriate Elica-DALEST application to make computer models of the containers offered by the stork and the fox. Is it possible that they could contain the same quantity of food?

Fig. 12: Models of the containers used by the fox and the stork by means of two points

Now you will use Bottle Design to create models of vases under specific requirements for the volume. It is sufficient to construct the profile of the vase.

Next the students use the Bottle Design to create models of vases under specific requirements for the volume.

Task
The vase obtained by rotating the profile below is of volume $200\pi$ cubic units. Change the profile so that the volume of the vase is greater than $215\pi$ cubic units and smaller than $225\pi$ cubic units by

a) adding a dot,
b) removing a dot,
c) moving a dot,
d) moving two dots.

Fig. 13: Contours for modeling a vase

After having warmed up with similar problems the students attack with easiness the following problem.

Task
Which of the bottles obtained by rotating the profiles below will have equal volumes?

Fig. 14: “Profiles” of vases

5 Concluding Remarks
The experience gained so far makes us optimists since it has shown that when working within the computer application developed for studying rotational solids the sixth graders managed to perform
various explorations with solids more complex than the ones present in the traditional curriculum. As far as the in-service teachers are concerned they could apply their previous expertise and interests in designing and developing projects in the context of rotational solids and IT. Furthermore, the work never stopped with solving a specific problem (or a chain of problems) given by the teacher or the lecturer – the participants were excited with their achievements and enthusiastically would continue with trying out new ideas on their own.

**Literature**

teacher education in order to enhance pupils' key competencies, especially with regard to modern technology, has been given political priority (see, e.g., Teacher Education, Aug. 2007). Due to the variety of new and rapidly changing tasks and roles that teachers are expected to fulfill, substantial change and reform of teacher education programs seem imperative. Among the central methodologies as well as innovative learning environments (BLEs), and Self Regulated Learning (SRL) as the core of this project was to promote teachers' theoretical background and skills in teaching mathematics education is far from being a field of inquiry, creativity, and of free thinking. Therefore changes in teaching and learning processes, to help pupils acquire key competencies (especially mathematical literacy, digital competence, learning to learn, social and communicative competences). The primary target of this project was to promote teachers' theoretical background and skills in teaching mathematics for themselves. On the contrary, many teachers feel and give into the urge to tell students exactly what to do, to control the direction of a student's thought, or to attempt to make a student's understanding of mathematics exactly like their own (Tyminski, 2010, p. 296); in other words, they are giving into a temptation defined as "teacher lust." The notion teacher lust defines the desire of teachers to make students conform to their own way of thinking, that is, they mean to "proselytize" convince, control, and arrest students spontaneous action. The damage however is always the same. When teachers fail to make conscious choices in the act of teaching, the resulting outcomes can be potentially detrimental to their students; teachers are expected to carefully consider other options and the consequences.
In the past few decades, the exploitation of Computer-Based Learning Environments (CBLEs) in mathematics teaching and learning has been widely studied. A rather unanimous feeling is that despite the existence of certain imperatives of policy, CBLEs are considered as an organic part of the organization of the opportunities to learn mathematics and knowledge. The teachers who generate these deficiencies, and finally to explore means to cure them. In this effort, the role of modern technology is highly recommended.

### 2.2 Computer-Based Learning Environments

In the past few decades, the exploitation of Computer-Based Learning Environments (CBLEs) in mathematics teaching and learning has been widely studied. A rather unanimous feeling is that despite the existence of certain imperatives of policy, CBLEs are considered as an organic part of the organization of the opportunities to learn mathematics and knowledge. The teachers who generate these deficiencies, and finally to explore means to cure them. In this effort, the role of modern technology is highly recommended.

### 2.3 Self Regulated Learning (SRL)

The concept of Self Regulated Learning derives from the constructivist view of learning and the urge to recognize that part of the responsibility for learning remains with the student. Self regulation might be defined as consisting of “self generated thoughts, feelings and actions that are planned and cyclically adapted to the attainment of personal goals (Amal, 2004).” In this respect, CBLEs provide for a high degree of learner control and cooperation in order to explore and to discover mathematics as well as to develop fundamental competencies.
Two major categories of SRL models can be distinguished: Operant Models and Social Cognitive Models (Whipp & Chiarelli, 2004, p. 6). Operant Models focus on the behavioral processes of SRL. Based on the premise that behavior and learning is influenced by observable records and evaluable progress in practice, SRL occurs in four stages: planning, enactment, and adaptation. Planning involves setting goals and selecting strategies. Enactment involves the execution of the selected strategies. Adaptation involves the monitoring of the outcomes and adjusting the strategies accordingly. Social Cognitive Models, on the other hand, focus on the cognitive processes (e.g., metacognitive strategies) that work in conjunction with the behavioral processes to influence SRL. These models are based on the premise that individuals who can effectively plan, monitor, and control their own learning are capable of taking advantage of the resources available to them. Social Cognitive Models are based on the premise that learning and behavior is influenced by external factors such as the nature of the task, as well as to internal factors such as learners' metacognitive processes (e.g., self-efficacy, self-regulation) and their impact on students' performance. Although mostCBLEs can potentially impact on students' motivational beliefs and self-regulation processes and their impact on students' performance, they often lack systematic evaluation and comparison of their immediate products against standards. In case of discrepancies, the student is able to perform fix-up operations (Pieschl et al., 2008, p. 19).

Models of SRL have received a great deal of attention in CBLEs research because they can help students develop the skills necessary to succeed in complex learning environments. The teacher's role is to enhance students' key SRL competences, which include the ability to plan, monitor, and control their own learning. The question is to what extent students will be able to apply SRL strategies effectively in CBLEs. The specific didactic concepts and pedagogical principles that we have developed for CBLEs are based on the premise that SRL processes play a vital role in explaining students' self-assessment and self-reflection. As we have already discussed, there is a very powerful tool in educational contexts to promote students' SRL processes.
that CBLEs is able to facilitate such transformation in mathematics education is elusive. Researchers point out that to overcome the barriers to this end requires a pedagogical shift (Linn, 1999; Bagott & Nichol, 1998). Working within CBLEs may enhance pupils' learning requires investigation. The next question is how the above pedagogical principles are achieved in CBLEs and particularly using different types of software, such as simulations, data logging, dynamic geometry environments, and spreadsheets. Simulation is considered to support science learning through encouraging students to pose and investigate explanatory models (What If?) (Bagott & Nichol, 1998). Data logging automates the recording and handling of experimental data and allows students to focus on over‐arching or salient issues without distraction (Osborne & Hennessy, 1993). The focus of our work was to provide answers to one primary question: How can teachers support students in using interactive technologies to solve real world based problems and to access the 'theory world' of mathematics (Whitelock & Jelfs, 2005)? Further, our work was informed by the complex mathematical concepts and the ability to interact with models which CBLEs enable, can assist children in developing understanding: this allows them to recognize the relevance of that experience with a concrete manipulative and further, these virtual manipulatives are often called virtual manipulatives in mathematics.

Learners commonly experience difficulty in applying appropriate knowledge to solve novel problems. A transformation strategy is needed to supplement and transform their existing knowledge base (Friedler & McFarlane, 1997). There are indications that the dynamic visual representations of concrete manipulatives are essentially different from their real counterparts. These dynamic visual representations are manipulatives that students can construct using mathematical concepts in designing and implementing activities within CBLEs. Great efforts have been placed in examining opportunities and well‐established literature derived from that experience and the complex methods mean that opportunities for building knowledge are important for knowledge building. Learners commonly experience difficulty in applying appropriate knowledge to solve novel problems. A transformation strategy is needed to supplement and transform their existing knowledge base (Friedler & McFarlane, 1997). There are indications that the dynamic visual representations of concrete manipulatives are essentially different from their real counterparts. These dynamic visual representations are manipulatives that students can construct using mathematical concepts in designing and implementing activities within CBLEs. Great efforts have been placed in examining opportunities and well‐established literature derived from that experience and the complex methods mean that opportunities for building knowledge are important for knowledge building.
The availability of dynamic geometry applications gave a new impetus on the teaching of geometry based on students' investigations and exploration. Polya (1957) emphasized the connection between deductive reasoning with exploration. He pointed out that solving a problem amounts to finding the connection between the given data and the unknown, and to do it, one must use a kind of reasoning based on deduction. In the dynamic geometry verifying further curiosity to explain why a particular result is true. Students working in these environments are able to produce numerous corresponding configurations of the problem, and in turn, this understanding solicits further curiosity to explain why a particular result is true (De Villiers, 1996, 2003). Students quickly admit that the inductive verification merely confirms and the "why" questions never cease to challenge them by asking why they think a particular result is true. Students working in these environments are able to produce numerous corresponding configurations of the problem, and in turn, this understanding solicits further curiosity to explain why a particular result is true (De Villiers, 1996, 2003). Students quickly admit that the inductive verification merely confirms and the "why" questions never cease to challenge them by asking why they think a particular result is true (De Villiers, 1996, 2003). Students quickly admit that the inductive verification merely confirms and the "why" questions never cease to challenge them by asking why they think a particular result is true (De Villiers, 1996, 2003).

The participants in this study were one class of 23 prospective teachers in the Department of Education at the University of Cyprus. As part of their undergraduate studies, elementary school prospective teachers have to attend one course (EP 472) that focuses on teaching the learning of mathematics. The prospective teachers were given the opportunity at the end of their in-service training to attend a course in the use of the computer (EP 473), to be taught by an educational technologist who is also a prospective teacher participant. The participants of the study are the prospective teachers who attended the course in the use of the computer (EP 473), the participants were asked to participate in the study and give their consent.

3.2 Setting and Participants

The participants in this study were one class of 23 prospective teachers in the Department of Education at the University of Cyprus. As part of their undergraduate studies, elementary school prospective teachers have to attend one course (EP 472) that focuses on teaching the learning of mathematics. The prospective teachers were given the opportunity at the end of their in-service training to attend a course in the use of the computer (EP 473), to be taught by an educational technologist who is also a prospective teacher participant. The participants of the study are the prospective teachers who attended the course in the use of the computer (EP 473), the participants were asked to participate in the study and give their consent.
of the new course that was developed during the autumn semester in 2009. The course was delivered during the autumn semester in 2009 in 26, 90-minute sessions by the first author. The sessions took place at a computer lab at the Department of Education, equipped with 30 computers and an Interactive Whiteboard. The majority of the participating prospective teachers were familiar with computers, accessing course materials through the course's Wiki, and preparing course materials in small groups.

The aim of the course was to provide prospective teachers with opportunities to work with various technological tools in their mathematics lessons, (c) to develop the competences of design and implementation of technology-based problem solving in everyday mathematics teaching, (d) to provide arguments in support of a wider implementation of technology-based problem solving in everyday mathematics teaching, and (e) to provide arguments in support of a wider implementation of technology-based problem solving in everyday mathematics teaching.

3.3 Pre-Service Training Course Modules

Towards addressing the needs of prospective teachers, the Problem Solving and Modelling module was developed as part of the undergraduate curriculum for an elementary school teacher bachelor degree. The aims of the course were: (a) to develop prospective teachers' understanding of the various forms of problem solving and the role of technology in their implementation, (b) to provide prospective teachers with opportunities to work with various technological tools in their mathematics lessons, (c) to develop the competences of design and implementation of technology-based problem solving in everyday mathematics teaching, (d) to provide arguments in support of a wider implementation of technology-based problem solving in everyday mathematics teaching, and (e) to provide arguments in support of a wider implementation of technology-based problem solving in everyday mathematics teaching.

The course consisted of five modules: The Problem Solving and Modelling module, the Dynamic Geometry module, the Math Applications module, the Programming module, and the Problem Solving and Modeling module. The rationale of these modules is presented in the next section.
During the course, prospective teachers had to prepare individual and group projects, using the various software presented during the course. More specifically, prospective teachers had to prepare a technology rich lesson plan of a concept in mathematics, using one or more math applets. The second assignment focused on developing a set of dynamic geometry based activities, following Van Hiele’s model. In their third assignment prospective teachers had to develop a set of learning environments for using a real world interdisciplinary problem approach in mathematics. More specifically, prospective teachers, following a project based approach, developed a set of learning environments, using at least two of the types of software they learnt during the course. After each assignment, prospective teachers had to make presentations to their peers for discussion and constructive feedback. Unstructured group interviews were held during these presentations, focused on the use of the various types of software on designing appropriate learning environments for prospective teachers and on how these learning environments could facilitate the construction of mathematical ideas and processes.

3.5 Data Analysis

In correspondence to the aims of the study, both quantitative and qualitative techniques were used to analyze the data. To investigate prospective teachers’ attitudes towards mathematics and ICT and their views about the developed course, descriptive statistics were used. Moreover, to examine the impact of the course on their attitudes and views, pair sample t-test analysis was conducted. To further analyze the impact of the course on prospective teachers’ views and teaching practices and to analyze their developed learning environments, interpretative techniques were used (Miles & Huberman, 1994). The prospective teachers’ learning environments were analyzed to identify and trace developments with respect to: (a) the ways in which the prospective teachers adopted and appropriate use the available software for developing learning environments, and (b) the ways in which their learning environments could effectively be used for teaching specific mathematical concepts and problem solving.

4 Results

Results are presented on two strands. The first strand focuses on participants’ learning environments. Prospective teachers developed their own learning environments using a variety of technological tools (e.g., Math Applets, Dynamic Geometry Packages, Programming, and Spreadsheets). Due to space limitations, examples from prospective teachers’ work in employing Virtual Manipulatives and Dynamic Geometry tools in designing their own environments are presented next. The second strand of the results focuses on prospective teachers’ answers in the provided pre and post questionnaire.

4.1 Virtual Learning Environments to Teach Mathematics

4.1.1 Web-Based Virtual Manipulatives

The learning environments developed by two prospective teachers are presented here. Both prospective teachers used applets from the National Library of Virtual Manipulatives (www.mattl.usu.edu/nlvm) to develop a set of activities for teaching a concept from mathematics. The National Library of Virtual Manipulatives (NLVM) is a NSF supported project that began in 1999 to develop a library of uniquely interactive, web-based virtual manipulatives, mostly in the form of Java applets, for mathematics instruction (K-12 emphasis). Prospective teachers used these Virtual Manipulatives to develop their own learning environments for assisting students’ visualization relationships and engage them in the learning process.

John, the first prospective teacher decided to work with the GeoBoard virtual manipulative. He decided to start his activity by asking students to draw in their Geoboards a garden with a surface area of 60 cm². This prospective teacher commenced his activity using a more open-ended task in order to familiarize students with the applet and to take advantage of applet’s functions (e.g., perimeter and area calculations). Two examples of answers in this activity are presented in Fig. 1.

Fig. 1: Two figure examples in the GeoBoard Environment

In the second activity students are asked to calculate the surface area of more complex forms. More specifically, the assigned prob-
A farmer decided to grow strawberries in the blue area and tomatoes in the green area. He also grows watermelons in the navy blue area. Find out the peppers area, without using applet's functions, considering that the whole garden area is 60 cm². According to John, the teacher who designed the activity, the purpose of this activity was to use the virtual manipulative to find different ways to calculate the area of a non-rectangular area (see for example, the yellow area in Fig. 2).

In the third activity students are given the following problem: “Find the maximum area of the garden, taking into consideration that the farmer can only use 24 m of fence for it.” The activity, although quite common in textbooks, allows further exploration, since students could draw a great variety of figures. In contrast to what students usually do in a more traditional setting (e.g., draw rectangles and squares), in the environment of a virtual manipulative (e.g., geoboard) students can draw a variety of figures (e.g. rectangles, rhombi, triangles, etc) and explore the relationship between their surface area and their perimeter.

A whole class discussion and an informal group interview followed the presentation of the prospective teacher’s learning environment. Like in all other teachers’ presentations, discussion focused on key mathematical ideas and processes and the way ICT tools were used to enhance student explorations and visualization processes. John reported that designing the activities was not a straightforward process, but it was rather a cyclic process; he had to set a target for the tools and the mathematical concept, then design and apply a procedure for developing the activities. During the whole class discussion, he reported that he re-evaluated his proposed activities and reflected on how he could better use the Geoboard virtual manipulative to enrich his activities.

Mary, the second prospective teacher used the Base Blocks virtual manipulative (www.matti.usu.edu/nlvm). The Base Blocks is a virtual representation of the Dienes material. Specifically, Base Blocks consist of individual units, longs (containing 10 units), flats (containing 10 longs), and blocks (containing 10 flats) which are used to represent place value for numbers and to increase understanding of addition and subtraction algorithms (see Fig. 3).

In the first activity Mary asked students to find at least two ways to represent number 274. According to her notes, she decided to use this activity, as it allows freedom for students to make different
constructions and to enhance their explorations. It also allows students, during class discussion to compare their different solutions. In the second activity students are asked to construct the number 230, using exactly 14 pieces of the Dienes blocks. According to Mary, there were two different ways to solve the problem. She also documented in her activity that the immediate feedback from the virtual manipulative could assist students in solving the solution. She also moved a step further, claiming that applet’s features (e.g., feedback, use of different types of blocks) could improve students’ explorations and solutions.

In the third activity, students are asked to represent in the manipulative’s environment 3-digit numbers, using digits 2, 5, and 6. According to the prospective teacher’s notes, although this is a closed activity, students have plenty of opportunities to explore the virtual manipulative’s environment in order to construct all six possible numbers (256, 265, 526, 562, 625, and 652) (see for example, the Base Block representation of the number 256 in Fig. 4).

![Fig. 4: One of the six 3-digit numbers in a base block representation](image)

In the last activity students are provided with the following problem:

“Five pieces of Dienes blocks were used to construct a number in the Base Blocks environment. Please answer the following questions:

(a) Find at least five possible numbers
(b) Which is the hidden number, if no flats (100s) were used?
(c) Which is the smallest 3-digit number that could be constructed? (The answer is presented in Fig. 5)

Pose a similar problem for your classmates. Try to make it quite challenging!”

Like with John, a whole class discussion followed the presentation of Mary’s learning environment. Mary explicitly mentioned that her initial thoughts and reactions when asked to design her own learning ICT based learning environment were not that positive. However, according to her words, she really enjoyed the process and she was convinced that ICT tools could definitely assisted her in improving her lesson plans. During the discussion, Mary among other prospective teachers reported that they were optimistic that similar to virtual manipulatives (first module implemented during the course) other software would also be interesting and productive in designing their own activities.

4.1.2 Dynamic Geometry Learning Environments

While the second strand for constructing the pre-service course was dynamic geometry environments, prospective teachers were also asked to use one of the proposed Dynamic Geometry packages (Euclidraw, Sketchpad, GEONExt) to develop their own learning activities for their students. Due to space limitations, the learning environments developed by two prospective teachers are presented next.

The first prospective teacher, Georgia, decided to work with Euclidraw. Additional to common tools and functions that appear in all Dynamic Geometry packages, it has the Divide/Join function. Using this function, the user can split a polygon in two or more polygons, or join two or more polygons into a new one. This specific feature can be used in transforming a parallelogram into a rectangle for measuring surface area. The prospective teacher proposed the specific environment as a means for finding the formula for paralle-
logram surface area. She further documented that the core aim of her activity was to transform the parallelogram into a rectangle, in order to use the well known formula for the rectangle surface area in finding the corresponding one for the parallelogram.

In the first activity students are prompted to draw a parallelogram in Euclidraw’s environment, using the Parallelogram tool. Students are then guided to draw a height of the parallelogram. An example of the construction is presented in Fig. 6.

Fig. 6: A parallelogram and its height in Euclidraw environment

In the second activity students are asked to use the Divide tool in order to transform the parallelogram into a figure for which they can calculate the surface area. Students are expected to use the Divide tool and split the parallelogram into a triangle and a trapezoid (see Fig. 7). Students can also use the Measure tool, to make sure that the surface area of the parallelogram equals the surface area of the trapezoid and the triangle (see Fig. 7).

Fig. 7: The Parallelogram Before and After Using the Divide Tool

Following, students can move the triangle on the other side and then use the Join tool to join the trapezoid and the triangle into a rectangle. Students can also use the Measure tool (see Fig. 8), to confirm that the surface area of the new rectangle equals the surface area of the trapezoid and the triangle, and therefore the surface area of the initial parallelogram.

Fig. 8: Joining the Trapezoid and the Triangle into a Rectangle

In the last activity students are encouraged to identify how the two sides of the rectangle are linked to the base and height of the parallelogram. Students are expected to conclude that the surface area \( A \) of a parallelogram is \( A = bh \), where \( b \) is the base of the parallelogram and \( h \) is the height.

The whole class discussion that followed focused on two dimensions. First, prospective teachers extensively discussed the Divide/Join function and how the use of this function can dramatically change the way students can work in solving the problem of the parallelogram surface area. Second, similar to Georgia, who presented the activities, other students successfully used the aforementioned tool (among other tools) to develop challenging and interesting learning activities. All those teachers claimed that their activities are totally different than those they used to develop without the use of ICT tools. A number of participating teachers continued documenting that their designed activities provide students with opportunities to explore the related mathematical ideas and construct their own knowledge.

The second example of the prospective teachers’ learning environments using dynamic geometry tools focused on the teaching of the circle (circumference, surface area, and \( \pi \)). It should be noted here that the prospective teacher who developed the Circle activity also used some spreadsheet features. The prospective teacher commenced the activity by asking students to draw five different circles and then use the provided spreadsheet to complete the table presented in Fig. 9. The provided table required students to measure circle’s radius, diameter, circumference, and in the last column to divide circumference by diameter.
Students are expected to conclude that the result of the division (circumference/diameter) is a constant number, and teacher can introduce \( \pi \). Prospective teacher also noted that it was a great opportunity to introduce students to the history of \( \pi \). In concluding the activity, students are encouraged to write a formula for calculating the circumference of any circle, using the \( \pi \) constant.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CIRCLE</td>
<td>RADIUS</td>
<td>DIAMETER</td>
<td>CIRCUMFERENCE</td>
</tr>
<tr>
<td>2</td>
<td>RED</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>YELLOW</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>BLUE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>PINK</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 9: Exploring Circle’s Properties to find \( \pi \)

Similar to the previous activity, students are asked to use the Measure tool of the software to measure radius and surface area of the circle, in order to find a formula for calculating the surface area of a circle. More specifically, in the provided spreadsheet table, students completed circle radius, the square of the radius and by dividing the area by the square of the radius to find again \( \pi \).

The third activity is also focused on finding the area of a circle. First, students are encouraged to use the Divide a Circle tool in order to split a circle into sectors (see Fig. 10). This tool can split a circle into a predefined number of sectors. Students are then prompted to move and then join the sectors in a way as to form a known figure.

Fig. 10: Splitting a Circle into Six Sectors

Through guided discovery, students are encouraged to join the sectors in a way as to form a well known figure. Students are expected to form a parallelogram and then identify that the area of the circle equals the area of the parallelogram (see Fig. 11). Finally, students are prompted to “link” the base and the height of the parallelogram to the half of the circumference and the circle radius respectively. Students could also use a different number of sectors (e.g., eight or ten) for their explorations and work with a rectangle instead of a parallelogram. Finally, the prospective teacher concluded that the activity is quite different than those which usually appear in mathematics textbooks.

During the whole class discussion, the prospective teacher shared with her peers that she runs the activity, during her school field experience and she was more than enthusiastic with students’ positive reactions and fruitful explorations.
**4.2 Prospective Teachers’ Beliefs Knowledge and Concerns Towards ICT Based Mathematics Teaching**

The first set of items in the provided questionnaire focused on prospective teachers’ practices and knowledge towards using ICT in mathematics and their fluency with computers in general. To examine the possible impact of the provided course, the questionnaire was administered before and right after the completion of the course. In order to investigate whether prospective teachers’ practices and knowledge towards using ICT in mathematics changed as a result of their participation in the course, t-test pair samples analysis was conducted. The results of the analysis showed that there were statistically significant differences in all statements/questions (see Table 1). More specifically, after completing the course, prospective teachers appeared to be comfortable \( \bar{x}=3.36, SD=0.49 \) and confident \( \bar{x}=3.57, SD=0.59 \) in using computers in mathematics, while the respective scores in pre questionnaire were only \( \bar{x}=2.18, SD=0.80 \) and \( \bar{x}=3.00, SD=0.67 \). Similarly, after the course they reported that they would regularly use the computers to solve mathematical problems \( \bar{x}=2.79, SD=0.59 \). A more detailed analysis of prospective teachers’ answers is presented in Table 1.

<table>
<thead>
<tr>
<th>Question</th>
<th>Before Course ( x ) (SD)</th>
<th>After Course ( x ) (SD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>How often have you used mathematical software so far?*</td>
<td>2.04 (0.77)</td>
<td>4.96 (0.20)***</td>
</tr>
<tr>
<td>I am familiar in using computers in mathematics.**</td>
<td>2.18 (0.80)</td>
<td>3.36 (0.49)***</td>
</tr>
<tr>
<td>I regularly use the computer to solve mathematical problems.**</td>
<td>2.00 (0.76)</td>
<td>2.79 (0.59)***</td>
</tr>
<tr>
<td>I feel confident in working with the computer in general.**</td>
<td>3.00 (0.67)</td>
<td>3.57 (0.59)***</td>
</tr>
</tbody>
</table>

*Five point Likert scale: 0=Never – 5=More than 20 times  
**Four point Likert scale: 1=Strongly disagree – 4=Strongly agree  
***Statistical significant difference between the mean in pretest and in posttest (t-test paired – samples).

The second set of statements focused on prospective teachers’ acceptance of ICT in mathematics education and the corresponding advantages from ICT infusion in mathematics teaching and learning. Results indicate that prospective teachers felt quite positive in working with computers in their mathematics classes before the course. As a consequence, their answers related to the benefits of using a computer and the necessity of having ICT related skills did not differ due to their participation in the course. However, it appeared that the course helped prospective teachers realize how important it is for students to use ICT tools in exploring and discovering mathematical concepts and in problem solving. Finally, the course appeared to be effective in minimizing prospective teachers’ concerns in working with a computer. Mean scores and standard deviations for the aforementioned statements are presented in Table 2.
Table 2: Prospective teachers’ acceptance of ICT in mathematics teaching and learning

<table>
<thead>
<tr>
<th>Question</th>
<th>Before Course</th>
<th>After Course</th>
</tr>
</thead>
<tbody>
<tr>
<td>I think the use of computers in school has only limited benefits.</td>
<td>1.55 (0.51)</td>
<td>1.46 (0.66)</td>
</tr>
<tr>
<td>I think for me as a future teacher it is necessary to have skills in using the computer in mathematics.</td>
<td>3.52 (0.59)</td>
<td>3.79 (0.41)</td>
</tr>
<tr>
<td>I think it is important that my future students also use a computer in mathematics.</td>
<td>3.43 (0.51)</td>
<td>3.79 (0.41)*</td>
</tr>
<tr>
<td>I have concerns in working with a computer myself.</td>
<td>2.29 (0.86)</td>
<td>1.96 (0.86)*</td>
</tr>
</tbody>
</table>

*Statistical significant difference between the mean in pretest and in posttest (t-test paired – samples).

The third set of statements explicitly asked prospective teachers to comment on their participation in the developed course. Prospective teachers reported that the design of the course encouraged them to work together with their peers (\( \bar{X} = 3.67, SD = 0.56 \)) in designing their own learning environments and in further improving their designed activities. Prospective teachers also commented that the covered subjects and activities were useful not only for their studies (\( \bar{X} = 3.83, SD = 0.48 \)), but also for their future work as teachers (\( \bar{X} = 3.91, SD = 0.28 \)). Finally, differences between prospective teachers’ answers in pretest and posttest were statistically significant in a number of the aforementioned statements. Mean scores and standard deviations are presented in Table 3.

Table 3: Prospective teachers’ comments on the pre-service course

<table>
<thead>
<tr>
<th>Question</th>
<th>Before Course</th>
<th>After Course</th>
</tr>
</thead>
<tbody>
<tr>
<td>I was encouraged to work together with others.</td>
<td>3.17 (0.49)</td>
<td>3.67 (0.56)*</td>
</tr>
<tr>
<td>I think the covered subjects in the course were interesting.</td>
<td>3.50 (0.51)</td>
<td>3.62 (0.58)</td>
</tr>
<tr>
<td>I think the covered subjects in the course were useful for my studies.</td>
<td>3.52 (0.51)</td>
<td>3.83 (0.48)*</td>
</tr>
<tr>
<td>I think the covered subjects in the course will be useful for my future work as a teacher.</td>
<td>3.52 (0.51)</td>
<td>3.91 (0.28)*</td>
</tr>
</tbody>
</table>

*Statistical significant difference between the mean in pretest and in posttest (t-test paired – samples).

The last set of statements were administered only in the posttest and focused on the impact of the designed course on three dimensions, namely: (a) prospective teachers’ knowledge of software tools and functions, (b) their knowledge and fluency in using various types of software in their mathematics teaching, and (c) their willingness to use ICT tools in their future mathematics courses. All mean scores were above 3 (on a 4-point Likert scale), indicating that the great majority of prospective teachers well perceived the provided course. Further, their scores were above 3 in all four different types of software environments provided throughout the course (see Table 4). These results underline that all suggested learning environments (Virtual Manipulatives, Dynamic Geometry, Programming, and Spreadsheets) were well adopted by prospective teachers. The latter was not only evident in prospective teachers’ answers in the provided questionnaire, but was also evident in their own developed learning environments. Finally, of interest is prospective teachers’ willingness to use all types of software in their future teaching. Again, all mean scores were above three (3), indicating that to a great majority prospective teachers’ not only felt comfortable in using the software, but they were also convinced that integrating ICT tools in their mathematics teaching would have a positive impact on their students’ learning.
Table 4: Prospective teachers’ comments on the mathematical software used in the course

<table>
<thead>
<tr>
<th>Question</th>
<th>Virtual Manipulatives</th>
<th>Dynamic Geometry</th>
<th>Spreadsheets</th>
<th>Programming</th>
</tr>
</thead>
<tbody>
<tr>
<td>I know how to use software’s basic tools and functions.</td>
<td>3.75 (0.44)</td>
<td>3.42 (0.58)</td>
<td>3.42 (0.65)</td>
<td>3.46 (0.59)</td>
</tr>
<tr>
<td>I know how to use the software in my mathematics teaching.</td>
<td>3.75 (0.44)</td>
<td>3.13 (0.61)</td>
<td>3.13 (0.68)</td>
<td>3.29 (0.62)</td>
</tr>
<tr>
<td>I will definitely use the software in my teaching.</td>
<td>3.79 (0.42)</td>
<td>3.54 (0.59)</td>
<td>3.32 (0.72)</td>
<td>3.38 (0.65)</td>
</tr>
</tbody>
</table>

5 Discussion

Preparing prospective teachers to effectively use ICT in their mathematics classes presents a challenge in mathematics education. There is so much to cover given the scope of the elementary and middle school mathematics curriculum. Prospective teachers need to become competent in the breadth of mathematics content and pedagogical methods, and become familiar with a variety of technological tools. Following a constructivist approach, prospective teachers can learn best when given the opportunity to construct their knowledge from their own experience. Consequently, teacher educators must provide opportunities for prospective teachers to do mathematics and to use technology; not merely, provide lectures and readings about what constitutes best practice in elementary mathematics teaching methods. Prospective teachers need to become informed about not only the technology available, but also how to use these technological tools in well designed and delivered mathematics lessons.

Following the recommendation of Browning and Klepsis (2000), the provided course aimed at providing teachers with opportunities to be engaged in more activities that are designed for their level of understanding, present interesting mathematical investigations, and are facilitated by the use of technology in their initial constructions, so the prospective teachers can determine the impact of technology on their own learning and teaching. Further, as presented throughout the chapter, course activities and prospective teachers’ designed learning environments were used to foster discussion on good pedagogy when working with technology. Results indicate that the course was quite successful towards this aspect. The prospective teachers in general participated in the whole class discussions and further reflected on their own designed environments. Further, the prospective teachers’ evaluations of the course were generally very positive both in regards to the use of technology and the suggested learning activities. All the prospective teachers expressed that the course had been beneficial to them and began to recognize how technology can be appropriately used in learning and teaching mathematics. Although the prospective teachers improved in attitudes towards technology in mathematics, some were still concerned of issues like time constrains and computer availability.

The technology-enhanced mathematics courses proved to provide a truly transformational experience for the prospective teachers. The greatest transformation was occurring in the affective domain. Results presented here provide a number of good examples of the above statement. A number of prospective teachers throughout the course underlined how their attitudes towards technology improved and that their concerns towards ICT integration in mathematics teaching changed. The use of multiple representations, the use of various tools (e.g., Virtual Manipulatives, Internet, Dynamic Geometry) appear to convince prospective teachers that ICT can be effectively used for a more conceptual teaching and learning of mathematics. The latter also has consequences for prospective teachers’ self-regulated learning. Results from the study presented in this chapter indicate that well designed activities along with prospective teachers’ own constructed learning environments have the potential to have enhance their self-regulated learning and thus promote their attitudes towards lifelong learning. Throughout the course, the open-ended activities, prospective teachers’ project-based work and their designed learning environments appeared to be beneficial for prospective teachers’ self-regulated strategies and self-confidence towards ICT in mathematics teaching and learning.

Clearly, one course exposing prospective teachers to a number of technological tools and appropriate pedagogical methods for integrating these tools in mathematics teaching is not sufficient to provide them with all necessary knowledge and skills to successfully incorporate technology into the teaching and learning of their students. It could, however, act as a catalyst for them if they continue to explore and adopt effective uses of technology in mathematics class. Finally, research has shown that it is also important for the prospective teachers to have follow-up professional development.
opportunities such as workshops and seminars to refresh their knowledge and skills of previously learnt technological tools and to expand on new ways to effectively incorporate technology in their mathematics classes for the benefit of their students.

**Literature**


Narrative Didactics in Mathematics Education

Matthias Brandl

Abstract

Innovation in Mathematical Education mostly focuses on the use of narrative didactics as a methodological concept to speak of is called “Narrative Didactics” and has been provided with a theoretical background recently by several authors (e.g., Kruteskii, 2001; Kubli, 2006; Norris et al., 2005). The way of how to make up good narratives in regular school lessons was investigated and described by linking the theoretical science and narrative (Kubli, 2001; 2002; 2006) or learning (Klincksen, 2006; 2009) theory. Those attempts determine the essential role of success in performing a mathematical activity requires a certain combination of personal traits, these permanent dispositions Kruteskii (1976), Bartkovich & George (1980), Greenes (1981), König (1986), Myasishchev (1960, 1962) Kruteskii hereby means the necessity of an aptitude and interest for mathematics in order to be successful. A first part emphasizing the nature and the modern scientific impact of narrative didactics is followed by an example picking out the emblem of the Pythagorean theorem.
aspects of narratives (Norris et al., 2005) that make up a good science story: event-token, the narrator, narrative appetite, past time, the structure, agency, the purpose, the role of the reader or listener, the effect of the untold and irony. Science stories can be analyzed using these ten elements of narrative as a case study in Klassen (2009) illustrates.

2.2 A Justification for Narratives in Math Lessons

However, mathematics, sometimes attributed to science, sometimes attributed to humanities, sometimes attributed to itself, seldom appears as the subject matter “to be told” – although there might be, especially within mathematics lessons, a need for some kind of “softer” teaching styles. Just as Kubli (2002, p. 128) points out that one of the presuppositions of a narrative didactic is that rational thinking cannot happen independently from the affective zone, the peculiar, but also precious ability of mathematics to be very abstract can be accompanied successfully by the converse teaching style of sophisticated “story telling”.

There are some rare, mostly example-driven, attempts of bringing mathematics into the discussion of using narrative methodology (Brandl, 2009; Kubli, 2005a; 2005b; Sinclair et al., 2009) that must be distinguished from approaches using narrative structures in order to analyze classroom practice (Wake et al., 2007, Williams et al., 2007). While the latter ones ascribe an immanent narrative structure to the teaching process, the first ones provide concrete advice to the teacher suggesting a different type of “serving” the subject. Using good narratives like little anecdotes or brief sketches of the mathematician’s (here the Pythagoreans’) life can help the student to embed the perfectly finished mathematical results into a mental surrounding that is made up by deeply human characteristics. As the Nobel Prize winner I. I. Rabi puts it: “I propose that science should be taught at whatever level, from the lowest to the highest, in the humanistic way. It should be taught with a certain philosophical understanding and a human understanding in the sense of biography, the nature of the people who made this construction, the triumphs, the trials, the tribulations.” (Holton et al., 1970) But this holds for the special case mathematics as well as for science at all. There is this aspect that especially mathematics shows, namely the fact that in its end product (i.e. theorems, definitions, ...) the long way of discovering or inventing has been erased completely. However, there has often been a simple playful access to the problem accompanied by random, joy and frustration. In Ruelle (2007) the French mathematician D. Ruelle points out to the concept of “tinkering” (going back to Kantorovich, 1993) that is inspired by biological evolution processes in order to describe the way of mathematical discovery by generalizing ideas from the famous mathematicians Henri Poincaré and Jaques Hadamard (Hadamard, 1945; Poincaré, 1973), who reflected their own process of doing mathematical research in a similar way.

3 Example: the Pythagoreans

A very fruitful context for the extraction of interesting and the learning process supporting narratives is the School of the Pythagoreans. This – in modern terms – sect was founded by the famous Pythagoras from Samos (572 – 497 BC) who, unlike Aristotle’s thought of him as a personification of the Pythagoreans ex post, probably was a real person as antic biographies from Diogenes, Laertios, Porphyrios and Iamblichos exist and therefore prove it. After traveling widely for his studies, Pythagoras founded his school in the house of his patron Mylon in Kroton. To this secret philosophic-political society after long and hard examinations not only men were admitted but also women. It was Theano, the beautiful daughter of Mylon, who despite of the big difference in age should become Pythagoras’ wife. The goal of the school was to explain life in ethical categories. Life as the Pythagoreans saw it consists of unexplainable inner contradictions and chaos. However, there must be some kind of hidden harmony or structure that guarantees stability and order for the world, mankind and the gods. So the highest task for men must be the investigation of this hidden harmony. As the clearest occurrence of this harmony seemed to be located in music, the soon detected whole-numbered harmonic ratios were transferred to other sciences like geometry, for example. Just by these few sentences about this famous and cultural influential group it becomes clear that there are lots of interesting little stories to tell. Those narrative elements may then serve as the scaffold for the mathematical content in the student’s mind.

3.1 Pythagorean Knowledge at School

One of the standard content in school curriculums is the Theorem of Pythagoras. As the algebraic solution of the well-known formula \(a^2 + b^2 = c^2\) requires the square root, in German schools the – in its geometric core rather easy – statement is placed rather late in 5th grade of secondary school. But there are other contents that are related to the Pythagoreans and can be discussed in lower grades.

The following drafted example of the Pythagorean emblem was performed in the 3rd grade at secondary school when introducing rotational and point symmetry, respectively. The use of narrative elements in form of stories about Pythagoras’ life, the sect and the
disaster with the emblem, paid off when only some weeks later the text book (Bortolazzi et al., 2005, p. 59) offered a task picking the Pythagorean emblem once again and asking for its angular sum. At first, the students could not remember any characteristics of the emblem (like its symmetry) at all. But just after delivering a word about the illustrious sect all the stories they had been told came to their minds again followed up only some minutes later by the mathematical facts concerning the emblem, its rotational symmetry and the abstract facts about the latter. This little teaching experience may serve as a motivational kick for the teacher and the students, too, to think of non-symbolic learning contents in form of narrative elements first so that the learners can put the hard mathematical details onto them just like a jacket to a wardrobe hook.

3.2 The Emblem

The emblem of the Pythagoreans got famous because of its disastrous part in the decline of the antic school. Having found the harmonic ratios of whole numbers that described the whole Pythagorean world, it was the new sect member Hippasus from Metapontum (c. 450 BC), who discovered inconsumable segment lengths just within the emblem. His penalty consisted in the banishment from the society; a symbolic tomb for him was built. Just after this event the Pythagoreans split up into the Acusmaticians and the Mathematicians. While the first ones remained in meditation over the obviously unsolvable, the latter ones came to the conclusion that they had to face the challenge and look for a solution. So for the first time in history, mathematicians appeared on the stage. The Pythagorean emblem itself is also called pentagram and shows a star that can be inscribed into a pentagon.

For the lesson concerning its symmetry the pentagram can be constructed by a dynamic geometry software (DGS) so that the rotational symmetry of (360° : 5 =) 72° can first be explored in an enactive way moving the star at one point around the (hidden) circle. Of course, a cut out piece of paper (or plastic) serves the purpose as well. However, when it comes to questions regarding the measurement of rotation angles or the illustration of the circumscribing pentagon and its characteristics – a DGS is clearly advantageous. In fact, it is this mixture of rigid symbolic mathematics, the affect-orientated humanistic narrative and the modern dynamic geometry software that allows for an innovative teaching and learning process in class.

3.3 For Further Reading

The story of Pythagoras and his school has been described in innumerable books and articles. Just in order to give a very short selection of German books that also addresses didactic issues related to mathematics lessons at school there are Affolter et al. (2003, p. 26-31), Baptist (1998) and Kubli (2005a, p. 137-141).

Literature

Exploring Geometrical Theorems by using Dynamic Geometry Software

Abstract

We present some appropriate contents from (slightly advanced) elementary geometry that benefit from the use of dynamic geometry software in mathematics lessons in secondary school. In the first part we deal with the law of cosine which is illustrated by GEONExT in such a way that both the proof idea and the theorem of Pythagoras as a special case can be observed. In the second part the students use an arranged GEONExT file to reconstruct the nine points of Feuerbach’s circle and enunciate the according theorem. Afterwards they construct the Euler line. Some more interesting aspects of Feuerbach’s circle follow (tangent circles, harmonic divisions).

1 Trigonometry: The “Coronation” of School Geometry

Trigonometry can be seen as the completion or even as “coronation” of mathematics lessons at secondary schools. There are lots of aspects referring to application, modeling or the didactical principle of integration. The latter one means the linking between the mathematical fields of algebra, analysis and, of course, geometry. Missing parts of figures like triangles now can be calculated by trigonometric approaches easily instead of executing time-consuming constructions that may only lead to approximations. Another aspect of the didactical principle of integration can be seen in the combination of standard teaching methods like drawing static chalk figures on the board and new media like dynamic geometry software, which emphasizes the functional or - as the name implies - dynamical aspect of a geometric construction. This can also mean that the construction connects the new content of a theorem (like the law of cosine) to the already well learned knowledge of another theorem (like the theorem of Pythagoras). This connection can be observed by deforming the (hopefully well-constructed) figure into or from the status of the well-known context that can appear to be just a special case of the new one. In this sense the didactical procedure may be interpreted as one kind of fruitful variation of a task...
or situation namely generalization (see Leuders, T. & Ulm, V. 2007, for example).

1.1 The Law of Cosine

We look at one mathematical rich example from the trigonometric treasure chest: the law of cosine (also known as the cosine formula or the cosine rule). What’s the wording of the theorem? Look at the notation in Fig. 1:

![Fig. 1: General triangle](image)

Then the law of cosine says:

\[ c^2 = a^2 + b^2 - 2ab \cos \gamma \]

At the moment of discussion in class the students already have learned the theorem of Pythagoras. The abbreviating formula for the relation of the triangle sides (or the squares above them, respectively) is

\[ c^2 = a^2 + b^2. \]

But this only holds for (at the corner C) rectangular triangles. So, this is a special case of the law of cosine, where the cosine disappears for \( \gamma = 90^\circ \) just algebraically.

The task for the teacher – or the gifted student – is to construct a dynamic figure in form of a general triangle that

- first illustrates the law of cosine geometrically and
- second leads to the theorem of Pythagoras by creating a right angle at one corner.

Furthermore, there might be another aspect that is important when it comes to the proof of the law and that again stresses on the aspect of variation. Just thinking of the theorem of Pythagoras Loomis (1972) lists about 370 different proofs. Looking at a problem from several perspectives can help to understand the crux of the matter.

1.2 The Construction

We draw an acute-angled triangle and put squares onto each side (just as it is done when illustrating the theorem of Pythagoras). Then we add the altitudes of the triangle and extend them so that the squares on the triangle sides in each case are cut into two rectangles (Fig. 2).

![Fig. 2: Dynamic construction of an acute-angled triangle with squares over the sides and extended squares-dividing altitudes](image)
Theorem 1
The extended altitudes of a general triangle divide the three squares on the sides of the triangle into rectangles such that each two rectangles that come together at one corner of the triangle have the same area.

Where is the connection to the law of cosine? Well, it is just hidden behind the words of the above theorem. Looking at Fig. 2, which first of all is the illustration of the theorem, one can see the law of cosine, too: for example, subtract the two upper rectangles that come together at C (i.e. CSTO and CHWV) from the two lateral squares (i.e. CBJO and CHMA); then – if Theorem 1 is true – the rest (i.e. the lower rectangles BJTS and AVWM) gives the area of the lower square (i.e. ALNB):

\[ A_{BJTS} = A_{CBJO} - A_{CSTO} = BC^2 - \overline{CS} \cdot \overline{BC} = a^2 - (b \cos \gamma) a \]
\[ A_{AVWM} = A_{CHMA} - A_{CHWV} = AC^2 - \overline{CV} \cdot \overline{AC} = b^2 - (a \cos \gamma) c \]

and by Theorem 1 we get
\[ c^2 = A_{ALNB} = A_{ALRF} + A_{BFNR} = A_{AVWM} + A_{BJTS} = \]
\[ = a^2 - ab \cos \gamma + b^2 - ab \cos \gamma = a^2 + b^2 - 2ab \cos \gamma \]

So the aim is to proof Theorem 1 which implies the law of cosine. How can this be done? For this we specialize in the case for rectangles coming together at point C (i.e. CSTO and CHWV in Fig. 2).

Way 1 (similar triangles)
This is a rather algebraic way. If we look at the triangles ASC and BCV in Fig. 2, we detect their similarity in their common right angle and the common angle \(\gamma\) at the corner C. Consequently there are corresponding ratios of side lengths:
\[ \frac{CS}{CA} = \frac{CV}{CB} \Rightarrow \frac{CS \cdot CB}{\overline{ACSTO}} = \frac{CV \cdot CA}{\overline{ACHWV}} \]

Exploring Geometrical Theorems using Dynamical Geometry Software

Way 2 (shearing of parallelograms)
This way that is inspired by Haag (2003, p. 34), uses the dynamic aspect of the construction. We look at the two rectangles CSTO and CHWV in Fig. 2 and shear them into congruent parallelograms. For this purpose we grab the points S and V and move them along the altitudes into the points A and B. Thereby the lengths of the sides [ST] and [VW] stay fix as the points T and W also move on the altitudes in fixed distance (see Fig. 3).

Fig. 3: Shearing of the rectangles into parallelograms with constant area

The shearing keeps the area of the sheared figures constant and the congruency of the resulting parallelograms CATO and CHWB results from
- the common side \(a = \overline{CO} = \overline{CB}\)
- the common side \(b = \overline{CH} = \overline{CA}\)
- the angle \(\gamma\) between \(a\) and \(b\).

Another way for a proof that uses the Intersecting Secants Theorem (and for obtuse-angled triangles the Intersecting Chords Theorem) is, for example, given in Baptist (1998, p. 135).

1.3 The Connection to Pythagoras
The construction in the dynamic geometry software was carried out in such way that the illustrated facts are not restricted to the original kind of triangle but to every general triangle (even for obtuse-angled). This is the great advantage of this method compared to the chalk board: by pulling at any corner point of the triangle...
Theorem 1 can be observed for any kind of triangle form as well as the success of the shear within the second proving way.

Now, there is a special way of moving point C that is for the case that $\gamma=90^\circ$. Then in the dynamic construction one can see the two upper rectangles vanish and the remaining figure illustrates a way that is similar to the way that EUKLID used in his “Elements” when proving the Theorem of Pythagoras: in Fig. 4 we see the orthographic projection of point C onto the hypotenuse c of the now rectangular triangle ABC by drawing (and extending) altitude $h_c$. The lower square results from the sum of the two rectangles that are equivalent in area to the squares over the triangle legs.

Fig. 4: The special case $\gamma=90^\circ$

2 Feuerbach’s Circle and Euler Line

Euler line and Feuerbach’s circle are usually not part of the curricula but open a rich field of mathematical research using only fundamental theorems that are part of the geometry curricula. Dealing with the learning environment described below, the students make fascinating discoveries and draw connections to familiar theorems.

According to international efforts into improving students’ mathematical literacy the tasks should always include requests to write down the discoveries, assumptions and results in whole sentences in the exercise books.

2.1 Feuerbach’s Circle

Opening the GEONExT data file „Feuerbach.gxt“ (cf. Fig. 5) the nine given points $H_a$, $H_b$, $H_c$, $M_a$, $M_b$, $M_c$, $X$, $Y$ and $Z$ (i.e. the three midpoints of each side of the triangle $M_a$, $M_b$ and $M_c$, the three feet of the altitude $H_a$, $H_b$ and $H_c$ and the three midpoints of the line segment from each vertex of the triangle to the orthocenter where the three altitudes meet $X$, $Y$ and $Z$) attract the students’ attentions resulting in the assumption that these points lie on a circle. By constructing the circle with midpoint F in the applet the assumption seems to be affirmed (but still needs to get proved of course).

Fig. 5: Screenshot from Feuerbach.gxt showing a triangle with the nine significant points $H_a$, $H_b$, $H_c$, $M_a$, $M_b$, $M_c$, $X$, $Y$ and $Z$

For it is not explained how the nine points on the circle have been constructed one task might be to reconstruct these points – as suggested in Piechatzek (2008) – using GEONExT. While the construction of the points $M_a$, $M_b$, $M_c$, $X$, $Y$ and $Z$ should not be a problem the teacher might give some hints for the reconstruction of the remaining points $H_a$, $H_b$ und $H_c$ since they do not belong to the group of familiar points in triangles.

By moving the points A, B and C the students can notice that the circle increases or decreases but still passes through the nine constructed points $H_a$, $H_b$, $H_c$, $M_a$, $M_b$, $M_c$, $X$, $Y$ and $Z$.

These first discoveries may lead into
Theorem 2 (Feuerbach’s circle)
For any given triangle the three midpoints of each side of the triangle, the three feet of the altitude and the three midpoints of the line segment from each vertex of the triangle to the orthocenter where the three altitudes meet lie on a circle called Feuerbach’s circle.

Still the question how the circle’s midpoint F was constructed is open. The teacher may decide to let the students try to construct this point using Thales’ theorem or any other familiar method. Another interesting method uses the Euler line opening a new field of research for the students.

2.2 Euler Line
The teacher could hand out the following tasks to his/her students:
(1) Add to your applet “Feuerbach.gxt” the following points: the orthocenter H, the circumcenter M and the centroid S. Write down your observations into your exercise book.
(2) The points F, H, M and S seem to lie on the same straight line. Construct the appropriate line and change the triangle by pulling the points A, B and C to confirm your guess.
(3) Using the GEONExT tool “Measure Distance”, try to find as many coherences between the points F, H, M and S as possible (cf. Fig. 6).

The following results can be found:
- The radius of the Feuerbach circle is half as long as the circumcircle’s radius: $2 \cdot FX = MA$.
- It is $2 \cdot SM = HS$.
- The point F bisects the line segment [MH]: $MF = FH$.

These explorations may lead to

Theorem 2 (Euler line)
For any triangle except for the equilateral triangle the orthocenter H, the circumcenter M, the centroid S and the midpoint of Feuerbach’s circle lie on a line called Euler line.

2.3 Proofs
The proofs can be taken from any standard geometry book. A recommendable proof is the proof being based on the mid triangle $M_aM_bM_c$ arising from the original triangle ABC by a centrical dilation with the factor of dilation -1/2 and center S as for example Breitfeld (2010) suggests (cf. Fig. 7).
Finally the advantages of the dynamical geometry software should not be missed. An appropriate task to investigate some special cases might be the following one:

Have a closer look at your dynamical construction. Observe what happens to Feuerbach’s circle and the Euler line in the following cases:

- The triangle ABC being:
  a) rectangular.
  b) isosceles.
  c) equilateral.

Write your observations down in your exercise book.

The following results can be found:

1. Feuerbach’s circle and Euler line pass through the triangle’s corner where the right angle is situated.
2. The Euler line is identical to the triangle’s axis of symmetry. Feuerbach’s circle is tangent to the triangle’s basis.
3. Because the points H, F, M and S unite there does not exist an Euler line in this case. Feuerbach’s circle is identical to the triangle’s incircle.

Feuerbach’s circle is tangent to that triangle’s three excircles and internally tangent to its incircle.

- F, H, S and M are harmonic points: \( \frac{FS}{SM} = \frac{SM}{MF} \) and
  \[ \frac{MS}{SH} = \frac{SH}{HM} \].

### Literature

Discovering Proportional Relationships in Everyday Life

Abstract

Understanding proportional relationships is an essential cornerstone for comprehending functional relations in mathematics education. This teaching unit wants to connect the internalisation of these relationships with an exploration of students' environment by the use of a spreadsheet program.

1 Curricular Basis

While creating graphics and charts in traditional learning is very time consuming, the usage of computers in mathematics education can save just this time for experimental discovering of mathematical situations. The pupils' workload thus will reduce. As the student makes the computer carry out the stubborn calculation of values as well as the occasionally monotonous plot of graphs into a coordinate system, he may concentrate on other things. Changing values and parameters quickly allows the user to discover their effects fast and effectively. In this way the students will get a better view behind the object.

These days the computer frequently finds its way into the classrooms of primary and secondary schools. The pupils' exposure to new media matters increasingly in every form's mathematics teaching. Having a look at the specifications of the subject in the respective types of school shows a more or less noticeable relation to the use of computers. Thus you can read in primary school's curriculum that "according to the course content different methods and varied media are applied (e.g. individualized instructions, weekly schedule, carousel activity, educational games, computer" (Curriculum for Bavarian Primary Schools, 2000, p. 31) while in Secondary General schools computers are merely "another working equipment for students" (Curriculum for Bavarian Secondary General Schools, 2004, p. 38). Using computers is explicitly mentioned in Intermediate Schools' curriculum. There it is said that "the subject maths accounts for IT-education. The students cognize, that the computer is an outstanding medium for visualisation and solution of mathematical problems. With the topics numeral systems, numeric and algorithms the basics for the subject Information Technology are prepared." (Curriculum for Bavarian Intermediate Schools, 2007, p. 43) The Grammar Schools' curriculum however slashes this detailedness and simply brings up the reasonable use of media as an educational objective of maths teaching (Curriculum for Bavarian Grammar Schools, 2004). Therefore the following remarks particularly follow the Intermediate Schools curriculum, but can also be transferred into different school types.

Mainly the spreadsheet program 'Microsoft Excel' represents an auxiliary tool for many different calculations in school lessons. Especially the recursive data determination is greatly facilitated by the use of 'Microsoft Excel'. But even for locating extremal points of a curve or a graph's zero such software provides real benefit. Whereas the students reduce handling with the pocket calculator and at the same time the results' error rate is enormously decreased, pupils may concentrate to the essential quintessence of the mathematical problems.

In the range of the treatment of proportional relationships in 6th and 7th grade initially the students are being adducted to direct proportionality and its attributes via ordinary examples. Thereby pupils learn to quantify and interpret problems of their field of experience. The hereby acquired knowledge can be improved by different tasks about percentage calculation. In 7th class the idea of proportionality is being expanded. Now the students discover the attributes of indirect proportionality and again search for the proportionality's occurrence in their surroundings.

2 Didactic Considerations

2.1 Necessary Previous Knowledge

In order to solve problems students are confronted with, in an optimal way they should have basic knowledge of working with spreadsheet programs. A previous introduction might be necessary and lasts about two lessons. By the way, it is necessary for the children to know the notion of direct proportionality and its characteristics as well as they have accomplished first calculations in connection with proportionality and per centum. Further previous knowledge is basic understanding of the relation between graphic and calculative solutions of proportionalities and percentages, as well as the illustration of functional relationships via adequate graphs. The following unit rather acts as a lesson of differentiated practice and the learning matter's consolidation, than as an introduction of proportional correlations.
2.2 Variation of Problems

The variation of mathematical problems is a valuable strategy for exploring mathematical situations. Variation means that based on an initial problem, students try to find new problems by varying single parts of the problem. This happens in very different dimensions. Whether in rather small dimensions, where children just juggle the problems, which means they change single parameters, data and terms, or if momentous modifications are made, e.g. the complexity factor varies or new parameters are added. But even a comparison with the students’ environment can be a variation by trying to find more possibilities for applying their acquired knowledge (Schupp, 2000, p. 31ff). A special combination follows up by the use of computers. Variations may be easily carried out by means of a spreadsheet program and they may be undone just as fast. Computing the values will be reduced to a minimum by the use of software, thus trial and error plus modification itself come to the fore. Even if a computer is just the tool to fulfill the variation technically, it is nevertheless an effective instrument for working with variations, because it takes over the monotonous calculation for the students and thus focuses on the essentials of an arithmetic problem (Schupp, 2002, p. 28).

2.3 Knowledge Gains

The single problems are from the category groups ‘direct proportionality’ and ‘percentage calculation’. Students shall develop their knowledge in those categories and tighten them. Because of the free assignment of tasks a good internal differentiation within the class can take place, thus every student is able to balance his/her deficiencies and stabilize his/her strengths. Furthermore, the relationship between tabular and graphic design can be internalized, which again is vitally important to the further school career. Especially for the introduction of indirect proportionality it is essential to have the basics deep-seated. But even for the treatment of different functions students will have an enormous benefit, when they already have developed a certain intuition for attributions and correlations. Furthermore, the pupils should see that proportional relationship appears all over their environment and makes a part of it. Admittedly the students will also cognize, that not each problem can be solved via proportionality.

In addition to working instructions it contains all essential data and information. For example, students will have an enormous benefit, when they already have developed a certain intuition for attributions and correlations. Furthermore, the pupils should see that proportional relationship appears all over their environment and makes a part of it. Admittedly the students will also cognize, that not each problem can be solved via proportionality.

In the course of the lesson pupils can decide on their own which problems they like to solve. By the use of different levels and variations both skilled students and even those with difficulties will be assisted in a differentiated way. In terms of ‘Think-Pare-Share’ the students are firstly required to handle the problems in pairs and then to discuss one of those problems in a group of 5 or 6 before one selected from each group presents its compositions to the class.

Table 1: Flowchart and instructions

<table>
<thead>
<tr>
<th>A. Teamwork (approx. 45 – 60 min.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Work with your partner on at least three of the six problems.</td>
</tr>
<tr>
<td>• Use a new spreadsheet for each problem.</td>
</tr>
<tr>
<td>• Form the tables attractively and clearly.</td>
</tr>
<tr>
<td>• Try to vary the single problems, if you think it is expedient. What do you discover in doing so? Create graphic and tabular illustrations for these variations, too.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. Panel of experts (approx. 30 – 45 min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Choose one of the problems you have been working on and join the accordant panel of experts.</td>
</tr>
<tr>
<td>• Compare your results within the group and discuss about what you like and don’t like.</td>
</tr>
<tr>
<td>• Design a common presentation for your problem based on this discussion. Use the spreadsheet program.</td>
</tr>
<tr>
<td>• Some students of your group will present your results subsequently.</td>
</tr>
</tbody>
</table>
Due to this the teacher takes a back seat in the whole unit. His main task is to support the single teams and to moderate the overall process. The first problem (building a house) is rather an introduction for lower level pupils. Here they exercise themselves in illustrating data via different charts. Thereby the students should especially concentrate on the pros and cons of different types of charts and thus cognize the particular significance of every single diagram.

**Problem 1: Building a house** (Gierse, 2002, p. 124)

<table>
<thead>
<tr>
<th>Position</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Project</strong></td>
<td>bare brick-work &amp; architect</td>
<td>interior work</td>
<td>sanitary &amp; heating installation</td>
<td>interior &amp; exterior plaster</td>
<td>gardening</td>
</tr>
<tr>
<td><strong>Costs</strong></td>
<td>100,000</td>
<td>75,000</td>
<td>40,000</td>
<td>15,000</td>
<td>20,000</td>
</tr>
</tbody>
</table>

These costs should be illustrated via circle, bar, vertical-bar and strip chart, which beside the single values also state their relative proportion of the total costs (250 000 €). Which type of diagram is significant, which isn't? Give reasons for your answer!

Similar to Problem 1 the second one (economy) is also for lower levels, but this time backwards. On the basis of a strip chart (Fig. 1) the students practice the deduction of data from graphs and diagrams. Now they can chart the data again and thus on the one hand control their work and otherwise compare the given chart with other types of charts.

**Problem 2: Economy** (Gierse, 2002, p. 128)

The following strip chart shows the export of bananas from the seven most important exporting countries in the year 2006.

![Strip chart of banana export](image)

a) Gather the percentage from the chart for each country, granted, that the whole chart (100%) means the whole export of bananas. Create a table, in which the 1st column is for the exporting country and the 2nd for the percentage.

b) Calculate the 2006 export for each country in tons, if Costa Rica's export has been 1750 tons in 2006. Now you can add the 3rd column and charge the export in tons for each country. How many tons have been exported in 2006 altogether?

c) Make use of the internet for a possible variation.

Problem 3 (physics) can be solved via diagrams or via calculation. The values follow up from physical problems the students will be going through in future but they are easy enough to understand without consolidated physical knowledge.

**Problem 3: Physics**

The following values result from different experimental series. Analyse them and try to find out whether there is direct proportionality between the values or not.

a) A vehicle is driving on a plane road (Gierse, 2002, p. 122)

<table>
<thead>
<tr>
<th>Time in sec</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance in m</td>
<td>95</td>
<td>190</td>
<td>285</td>
<td>380</td>
<td>475</td>
</tr>
</tbody>
</table>

b) A vehicle slows down (Gierse, 2002, p. 121)

<table>
<thead>
<tr>
<th>Velocity in km/h</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Braking distance in m</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
</tr>
</tbody>
</table>
c) A stone is falling from the television tower (Gierse, 2002, p. 122)

<table>
<thead>
<tr>
<th>Time in sec</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fall distance in m</td>
<td>4,9</td>
<td>19,6</td>
<td>44,1</td>
<td>78,5</td>
<td>122,6</td>
</tr>
</tbody>
</table>

d) Stretching a steel spring (Gierse, 2002, p. 122)

<table>
<thead>
<tr>
<th>Force in N</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extension in mm</td>
<td>28,8</td>
<td>71,4</td>
<td>119</td>
<td>142,9</td>
<td>214,3</td>
</tr>
</tbody>
</table>

e) Heating 400 g of water (Gierse, 2001, p. 213)

<table>
<thead>
<tr>
<th>Time in min</th>
<th>0.5</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
<th>4.5</th>
<th>5.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature in °C</td>
<td>0.6</td>
<td>1.8</td>
<td>3</td>
<td>4.2</td>
<td>5.4</td>
<td>6.6</td>
</tr>
</tbody>
</table>

Problem 4 (mathematics) is a bit more complex because the students don't have any values. Additionally, there is some geometric knowledge necessary for solving these exercises. This problem is for students, who don't want to work exclusively with the spreadsheet. They can use the internet, dynamic geometry or even a piece of paper. Just as they prefer.

Problem 4: Mathematics

Even in geometry there are many proportional correlations. Analyse the following examples.

a) Compare circumference and radius of different circles.
b) The perimeter of an equilateral triangle against its side length. (Gierse, 2001, p. 208)
c) Explore and analyze different rectangles of the area 120 cm².

The 5th problem (nutrition & energy) refers to a contemporary issue. Today about one third of the children below the age of fifteen is overweight. The preoccupation with nutrition facts and consequences of malnutrition may help the students to get a certain sense of alimentation. Already in subtask a) the pupils can decide whether they create a simple ‘energy-calculator’ or rather a complex one. Just depending on their skills and will.

Problem 5: Nutrition & energy (Dlugosch, 2002, p. 102)

<table>
<thead>
<tr>
<th>Food</th>
<th>Fat content</th>
<th>Energy in kJ per 100 g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisps</td>
<td>30.0%</td>
<td>2311</td>
</tr>
<tr>
<td>Chips</td>
<td>9.0%</td>
<td>921</td>
</tr>
<tr>
<td>Milk chocolate</td>
<td>30.0%</td>
<td>2202</td>
</tr>
<tr>
<td>Sausage</td>
<td>31.0%</td>
<td>1524</td>
</tr>
<tr>
<td>Banana</td>
<td>0.8%</td>
<td>293</td>
</tr>
</tbody>
</table>

Recommended maximum nutrient amount per day and kg weight

<table>
<thead>
<tr>
<th>Nutrient</th>
<th>Weight in g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbohydrate</td>
<td>6</td>
</tr>
<tr>
<td>Fat</td>
<td>1.3</td>
</tr>
<tr>
<td>Protein</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Energy required per hour and kg weight

<table>
<thead>
<tr>
<th>Activity</th>
<th>Energy demand in kJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Writing</td>
<td>2</td>
</tr>
<tr>
<td>Breast stroke</td>
<td>20</td>
</tr>
<tr>
<td>Cycling 10km/h</td>
<td>9</td>
</tr>
<tr>
<td>Climbing a stair</td>
<td>33</td>
</tr>
</tbody>
</table>

a) Create an ‘energy-calculator’. Use the data specified above.
b) Susanne weighs 48 kg. How many grams of fat should she eat at most every day?
c) After breakfast and lunch, she has already eaten 25 g of fat. Moreover, playing with her friends Susanne has had half a bag of potato crisps (75 g) and a quarter of a chocolate bar (25 g). May she eat one more serving of chips (300 g) without crossing the recommended fat amount?
d) Thomas prefers bananas instead of potato crisps. How many bananas might he eat in order to ingest the same amount of fat that Susanne ate with the potato crisps?
e) How long does Susanne have to do breast stroke in order to diminish the energy she has affiliated by crisps and chocolate?  
f) For a possible variation make use of the internet. There you will find a lot of information about the energy content of food and statements about the energy consumption of sports and activities.

The last problem (mobile phone-calculator) is the most difficult one. It enables pupils to find different ways of solution. Since there is no "master solution", they will find many different approaches and explanations.

**Problem 6: Mobile phone-calculator**

Tyler wishes for a mobile phone for his birthday. His parents give him three mobile phone rates of which he can choose the most favourable one. Since Tyler has never had a mobile phone yet, he doesn't exactly know how many minutes he is going to phone or how many short messages he is going to send. Can you help him to compare the rates nevertheless?

<table>
<thead>
<tr>
<th>Prepaid-card</th>
<th>Tariff 1</th>
<th>Tariff 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly basic rate</td>
<td>–</td>
<td>7,95 €</td>
</tr>
</tbody>
</table>

**Telephoning**

<table>
<thead>
<tr>
<th></th>
<th>on-net calls</th>
<th>landline (Germany)</th>
<th>off-net calls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Telephoning</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>on-net calls</td>
<td>0,05 €</td>
<td>0,29 €</td>
<td>–</td>
</tr>
<tr>
<td>landline (Germany)</td>
<td>0,15 €</td>
<td>0,29 €</td>
<td>–</td>
</tr>
<tr>
<td>off-net calls</td>
<td>0,15 €</td>
<td>0,29 €</td>
<td>0,29 €</td>
</tr>
</tbody>
</table>

**SMS**

<table>
<thead>
<tr>
<th></th>
<th>on-net</th>
<th>landline (Germany)</th>
<th>off-net</th>
</tr>
</thead>
<tbody>
<tr>
<td>Telephoning</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>on-net</td>
<td>0,05 €</td>
<td>0,19 €</td>
<td>–</td>
</tr>
<tr>
<td>landline (Germany)</td>
<td>0,19 €</td>
<td>0,19 €</td>
<td>0,19 €</td>
</tr>
<tr>
<td>off-net</td>
<td>0,19 €</td>
<td>0,19 €</td>
<td>0,19 €</td>
</tr>
</tbody>
</table>

**Minimum term**

<table>
<thead>
<tr>
<th></th>
<th>0 months</th>
<th>24 months</th>
<th>24 months</th>
</tr>
</thead>
</table>

First, the students work with a partner on at least three of the six problems. In a second phase, they choose one of the problems they have been working on and join an accordant panel of experts for discussing their problem and comparing ideas and results. Finally, every group should come to an agreement about a common presentation, which will be presented afterwards at the projector and discussed in the whole class. These presentations can be combined in one data file and handed out to each student.

**4 Summary**

Altogether a spreadsheet program can be an effective didactic tool, even though it has to be noted, that the didactical and educational realisation is a factor that is not to be sneezed at for the software's effectiveness. According to this, the educational realisation should be calculated, because using the computer is not unconditionally advantageous. If so the spreadsheet may have a share in the improvement of school by allowing the pupils a fast variation and minimization of pure calculus. Thereby the students get to focus on the proportionality itself without time-consuming typing amounts of data into the calculator. Using table calculation cannot compensate the school book and a teacher's didactic preparatory work. But it may offer additional practice and even help to understand complex relationships easier.

**Literature**


Interactive Whiteboards – a Pilot Project

Abstract

Under the roof of InnoMathEd we have established a pilot project using Interactive Whiteboards (IWB) in class. By an open request for proposal we offered schools in Augsburg (Germany) to use an IWB within their school for one school year. The project started in July 2009 and the teachers and pupils have worked with the IWBs intensively since then. There are regular meetings in which the teachers get the opportunity to exchange experiences, ideas and material. We as researchers accompany these meetings to give input regarding didactics and methods and we also evaluate the project. This article deals with the advantages of ICT in class and gives an insight to the procedures of the pilot project with the Interactive Whiteboards. There will also be a first look into the evaluation results of the project.

1 The Pilot Project

InnoMathEd aims at the development of pupils’ deep mathematical understanding and for them to acquire key competences that are essential for lifelong learning (OECD, 2007). Pupils should also get more familiar in the use of ICT, especially dynamic mathematics to foster individual and cooperative learning processes. In order to achieve these goals the focus of the project lies on the development of didactic concepts, pedagogical methodologies and innovative learning environments for pupils. These new approaches are furthermore tested, evaluated, disseminated and exploited on European level. The project focuses on pupils’ active, self-responsible and exploratory learning. Every partner of the consortium brings in a different perspective and has various fields of expertise.

Under the roof of the project we have set up a pilot project with Interactive Whiteboards for schools in Augsburg, Germany. The pilot project is still running, but due to the consecutive meetings of all participants and the formative evaluation we can give first results on the short-term effects of the use of the IWB in class. With this pilot project we can reveal strategies and patterns for innovation and integration of ICT in the educational system and in-service training for teachers within a very concrete example.

1.1 Background

In Germany IWBs are not widely-used. Since the federal states are in charge of their school system it is difficult to give all-embracing information. The educational authority of Hamburg, in the northern part of Germany, for example launched a project in 2005 to integrate IWBs in schools. Due to the good evaluation results (Bianco, 2009) the city decided on equipping every school with IWBs (Landesinstitut für Lehrerbildung und Schulentwicklung, 2009) until the end of 2010. In Bavaria, where the pilot project is located, the authorities have been rather reluctant in purchasing Interactive Whiteboards so far. Other countries, like the United States of America, Australia or the United Kingdom have more experiences with using IWBs in their classrooms (SMART Technologies, 2004).

1.2 Project Members

Since InnoMathEd is coordinated in Augsburg, many activities within the EU-project originate here. With the advantage of having another partner of InnoMathEd in the same town, namely Project Education Institute, we are able to roll out projects efficiently. Project Education Institute which works with IWBs at their institution was able to encourage SMART Technologies Germany (SMART) to support the project with their equipment. SMART offered to give IWBs to three schools in Augsburg in order for their classes to work with the technology and to get them more proficient with the use of ICT in daily lessons. Since InnoMathEd is a project focussing on mathematics it was planned to give the offer to teachers of mathematics who are interested in using the computer in class and would embrace the idea of trying out a new approach of teaching.

We sent out an open request to all secondary schools in Augsburg and its surrounding area asking them for application for the project. The aim was to get the schools motivated to take part in this project and give every school an equal chance in receiving an IWB. The call got a large response – many schools asked for further information, prepared letters of motivation and applied for the project via e-mail, mail or in person. After thorough evaluation of the application we settled for three schools. Each chosen school represents certain characteristics that we thought might be interesting for this scientific approach.

(1) The first school is a secondary school with boys and girls in a suburb of Augsburg. In this school, seminars for aspiring teachers are held regularly: Trainee teachers of mathematics

1 URL: http://www.projekt-bildung-gmbh.de/.
2 URL: http://www.smarttech.de/.
come here from all over Bavaria to work together on didactical and pedagogical concepts and try out new approaches for teaching and learning. We figured since this school offers access to (future) teachers of a large geographical area we would give many practitioners the opportunity to work with the IWB and give them hands-on experience in this field.

(2) An all-girls-school was also chosen for the pilot project. Studies show that girls often fall back behind their male counterparts when it comes to performance in mathematics (Faulstich-Wieland, 2004; Bos, Lankes, Prenzel, Schwippert, Walther & Valtin, 2003). This lack of competence does not come automatically: Girls do not start with a weaker performance in school, they fall back over time (OECD, 2009). Meanwhile many initiatives have been started to foster girls’ performance in science and maths, but we agreed on letting this school take part in the project to have a different set of pupils working with the board.

(3) Our third school that received an IWB is also a secondary school which does not exclusively work with boys but has mostly male pupils in their classes. The local education authority heard of the approach and got really interested in the project. It was decided to give three additional schools the opportunity to have an IWB installed: Two of them are general schools1 and the third one is another secondary school. The local education authority decided on two general schools to give those pupils, who might normally not have access to the best educational equipment, the chance to get more experience with the work on computers and IWBs. Due to the great interest in the project it was decided to include not only teachers of mathematics but every other teacher of the participating institutions to take part, and actively work with the boards within their classes. During the summer holidays another school was chosen to be part of the pilot project: This school is the International School in the area and most of its faculty comes from foreign countries where they have already worked with IWBs in class. The International School moved-up late, that is why they were not part of the first meeting held in July (see 1.3).

1 In Germany there is a three-tiered school system: After primary school, which lasts four years, the pupils are (depending on the specific federal state – the pilot project is in Bavaria) split up in three types of schools. A general school (= “Hauptschule”) is the lowest level, followed by two types of secondary schools: “Realschule” is a secondary school which leads up to grade 10 and which is usually followed by an apprenticeship. A “Gymnasium” is a secondary school with a more academical approach, preparing for a continuing education e.g. at a university.

1.3 Project Implementation

We scheduled a first project meeting in July 2009 where we invited all teachers of our schools that were interested in the project. The IWBs were not yet installed, so this first get-together was meant to show them what the boards are capable of and in which way the teachers might use them in class. The meeting was held in the rooms of Project Education Institute, since they have several boards available for use. Also invited were professionals from SMART Technologies to give all members of the project the opportunity to ask rather technical questions that may come up during the meeting. After an introduction and an overview of the timeline of the project, including all milestones, the plenum was split up into smaller groups: Teachers of mathematics (which are the majority in the project) and teachers of languages etc. have different ways of using the boards so we wanted to give all participants a rewarding meeting in matching them with the content suitable for them. All groups were hosted by one representative of SMART and at least one member of InnoMathEd. This approach enabled us to deliver insight into the needs and wishes of the teachers for further planning of the project. It also gave us the chance to give them helpful advice in didactics, pedagogical methods and also technical concerns. For most of the teachers it was the first time to ever work with an IWB. Nearly all of them were eager to try out the various tools and functions. The teachers involved are of all age groups – from young teachers at the beginning of their career to long-time educators who serve as teachers for decades. During our observation we could not find a correlation between age and interest in the project: All teachers showed similar (strong) interest in the technology and asked constructive questions and expressed needs and concerns likewise.

Every school was asked about their specific needs regarding the boards: Some of the problems that had be solved prior to the installment were for example the (1) specific rooms (e.g. internet-access should be available, electricity cables should be in reachable distance) or the (2) board itself e.g. some had to be adjustable in height (e.g. when pupils of different ages work on the board). Each school was asked to clarify the specific needs and give a timely feedback to the coordinating-team. Since the school year in Bavaria starts in September an installation date was scheduled before the beginning of the new term in fall.

The meeting can be regarded as a success: All participants were motivated enough to spend five hours of their spare time to work together to set up the next steps for the project. Since the meeting was scheduled on a regular school day in the evening, it was positively surprising that all participants (approximately 30 persons) stayed until the end and were really satisfied with the meeting.
1.4 First Experiences with the SMART Boards

All SMART Boards were installed in the participating schools in September 2009. There were only smaller problems that had to be solved, e.g. finding the right spot for the board: Some schools did not want to remove the regular blackboard in order to give room for the IWB. Therefore some IWBs had to be hung up at the back of the classroom or next to the blackboard in front. Some schools decided to use a regular classroom for the board, other preferred to use a room which is mainly used for seminars etc. in Germany it is normal that each class has its own classroom. Each teacher changes room and visits the next class. Unlike other countries Germany usually does not apply the teacher’s room concept (except for subjects like chemistry etc. where special equipment is needed). This can of course cause some problems: If the board is installed in one single classroom that is mainly used by one individual class, other pupils that do not get the opportunity to work with the board might experience jealousy or lack of motivation. We therefore encouraged all schools to choose a room that is accessible to many classes in order to get a lot of pupils to actively work with the learning technology.

With the beginning of the term all schools began working with the IWBs. They were given the opportunity to contact the team of InnoMathEd and SMART Technologies whenever they would need support. We did not want to be passive partners of this approach and wait for questions and problems to be verbalized independently. Since the teachers were mostly new in working with the boards, we wanted to give them active support. That is why we arranged another meeting in October 2009 for an exchange of first experiences.

Likewise to the first meeting we had many teachers joining the meeting. Some of them were even accompanied by pupils who were enthusiastic about the use of the board. Their attendance was more than welcome: Teachers are the intermediary between the educational system and the pupil. At the end, it is important not only to train the teacher but to support the acquisition of key competences and the use of ICT for pupils (Bianco & Brandl, 2010).

First, we used some time on answering questions that had arisen in the last weeks from the teachers. Some of them were experiencing similar problems and by sharing the solutions and advice with the plenum we wanted to establish a common dynamic for the group: this way, we wanted to show the persons involved that their questions are openly welcome and that experiences of the others are related to their own. Each school was asked to give a short summary on their first weeks with working with the IWBs. Afterwards we once again split up the group into teachers of mathematics and languages to get together in smaller workshops where they had to develop three goals for the project. We wanted to find out which needs, wishes and hopes the teacher have regarding the use of the Interactive Whiteboard in class. The answers given were very fruitful: The participants wanted to get more proficient with the use of ICT in class and aimed at the development of competences not only for them but their pupils as well. Teachers said that they wanted to motivate the pupils by the use of the IWB. Some said that their goal was to have more time for discussion and active work in class by using the IWB. The subject matters do not have to be transcribed by the pupils but can e.g. be sent by e-mail which can save time for group work etc.

The next step during the workshop was to assemble a set of tools which would be helpful for the teachers to work with the IWB. Everyone could express their wishes and the moderators (again team members of SMART and InnoMathEd) could instantly give precise answers on the possible solutions. If an adequate tool was not on hand immediately, we tried to come up with an answer afterwards. The participants were challenged to think about the use of each tool and really reflect their daily work in class: we wanted to equip them with helpful tools that could actually be used in the classroom, not just gadgets that are not suitable for their needs. Afterwards, we met again in the whole group and exchanged the ideas and thoughts that were shared in the workshops. Material on teaching mathematics (especially dynamic mathematics) was handed out to the teachers to be used with the board. By using software for dynamic mathematics, e.g. Geonext® or Geogebra®, the advantages of the IWB and the software can achieve synergy. The mentioned software work very well on the SMART Board and some of the teachers of mathematics were not experienced with using a SMART board but with dynamic mathematics. There is evidence that pupils can profit from using IWB with mathematics (Becta, 2007). For them it was a lesser hurdle to take when making the transition from the computer to the IWB.

Again, the meeting was met with a lot of enthusiasm and engagement on all sides. We decided to meet again for an exchange of materials in the beginning of 2010.

4 URL: http://geonext.uni-bayreuth.de/.
5 URL: http://www.geogebra.org/cms/.
1.5 Meeting for Exchange of Material

In December 2009 the teachers were invited to take part in seminars held by SMART Technologies in Munich. In these seminars the teachers were offered further support for using the IWBs and got introduced to another set of helpful tools for their teaching.

In March 2010 we had scheduled the next meeting for all teachers of the pilot project. Prior to this gathering we had asked the participants to prepare individual presentations of their work on the IWBs in the last months. We also asked them to collect all learning materials that they may have created alone or with their pupils. Some evaluations of the use of IWBs show, that teachers often hesitate to use the IWB because they avoid the creation of new learning environments (Weißer, 2007). Our aim was to overcome that obstacle by sharing the work within the group: This way all members of the project can profit from the work of the other and we thought the motivation to create a learning unit on one’s own would increase.

All in all the prepared presentations from the schools affirmed the impression of a good integration of the provided IWBs in class. Some teachers were still reluctant of the use of the IWB. The reasons for this hesitation are manifold: One school e.g. chose to place the board within a room were normally no lessons are taught. This means that each time a teachers wants to use the board he has to ask for permission to enter the room with the class and also has to check if the room is not booked by somebody else. This procedure of course objects the idea of an interactive medium which should be readily available for use at any time. In this school, teachers said, the pupils are very eager to work with the board, but due to the insecurity of the teachers with the use of the IWB, it is hard to meet the demands of the pupils. After the meeting we spoke to the teacher of this school again and tried to find out about the reservation towards the IWB. The constructive talk led to a change of perspective with this educator. This school came in very late to the project and felt less informed about the whole procedure. Talking to the teacher face to face was a good way of clarifying the expectations and demands on the project.

Experiences presented by the other schools were really satisfying. The general school which also came in somewhat later to the project has really devoted a lot of time on the project and the teacher works almost exclusively on the IWB now. He showed many learning units that he had created on his own and was highly motivated to continue to work with the board. His experience was very important, since the pupils of his school often have a non-academic background and have little experience with learning technology of this elaborated level. From what the teacher told us, his pupils really appreciate the benefit of the IWB and they enjoy working with the board.

The International School also presented a large variety of learning units that were created during the time of the project. Pupils as well as teachers seem to embrace the idea of the IWB in that institution. Since the teachers from the International School have more experience with working with IWBs the other teachers from the plenum were really intrigued to hear rather long-term experiences firsthand.

Our secondary schools on the pilot project reported that they used the SMART Board for selected content. The use of dynamic mathematics in class worked very well for some mathematics teachers and they said they would continue the work on the boards. In another school there seemed to be a problem with scheduling the classes to the room with the board as well. The person in charge of the schedule said, that it is very difficult to take all wishes of the entire school.

As an interim result we can state that it seems to be an advantage to have the IWB not in a room were no regular lessons are held. We are aware of the fact that we do only have one SMART Board per school, but from what we have heard during our meetings, the use of the board seems to be a lot more consistent, if the IWB is installed in a regular classroom. Other studies have made similar experiences (Miller, Glover & Averis, 2003). This may be due to the fact that the pupils (and the teachers) regard the IWB as a normal medium, like the chalkboard, that is intended for actual usage. Having the IWB installed in a room where normally only teachers have access and the door is even locked, creates a daunting atmosphere that is not conducive for learning.

1.6 Next Steps

The pilot project is ongoing and we have another meeting planned for July 2010 where the final evaluation of the project is scheduled. Also, the meeting is aimed at giving all participants the opportunity to exchange further experiences and materials.

Over the time of the project many interest groups e.g. media representatives, decision-makers from school authorities and parents have reached out to us to get more information about our pilot projects and the SMART Boards.
2 Evaluation

As mentioned above, since this is an ongoing project, the evaluation has not been completed yet. In this section there will therefore only be a short presentation of the results we have collected so far.

The evaluation of the pilot project consists of various methods: (1) There is an online evaluation which is used for InnoMathEd. Three surveys were set up according to the specific target groups: pupils, teachers and students. In the case of the pilot project we only used the surveys for pupils and teachers since there are no students involved in the project. (2) For a more qualitative approach we have collected data during out meetings and informal talks that took place between members of the consortium and participants. (3) For the final meeting in July we are planning to make a final evaluation that will be more specific to the assembly of the pilot project.

2.1 Selected Results

In the following there are some of the results from the online evaluation mentioned above. InnoMathEd is a project focussing on mathematical content; therefore the general evaluation displays a mathematical point of view. The answers that were given by the teachers participating in the pilot project were isolated to give an overview of the results. Sometimes teachers who do not teach mathematics used “not applicable” (n/a) as the appropriate answer. All in all, ten teachers from the pilot project took part in the evaluation. The number of participants varies from meeting to meeting, because we encourage all interested teachers of the schools to take part. This is why we do not have a fixed number of participants. In average we have about 20 to 25 teachers taking part in the meetings.

General Data
The average age of the teachers is 44 years, ranging from 24 to 60 years. Six male and four female teachers took part in the evaluation. Six of the teachers used the SMART Board more than 20 times in class; two stated they used it eleven to twenty times; one teacher used it six to ten times and one said that he never used the IWB in class.

The following questions had to be rated according to the following scale:

a) Strongly agree.
b) Agree.
c) Disagree.
d) Strongly disagree.
e) Not applicable or don’t know.

Previous Knowledge
Regarding the question, whether the teacher is familiar in using computer in mathematics lesson, there seems to be a quite confident self-assessment: Nine out of ten teachers agreed (two of them even strongly) to this statement. For one teacher this statement was not applicable, because he or she is a teacher for languages (which can be found out by taking a look at the open question at the end of the survey). The next question covers the regular use of the computer in mathematics lessons: Seven teachers agree (two strongly), two disagree and once again one answer that is “not applicable”. “I use mathematical software to prepare my lessons.” This statement received the following answers: Nine teachers agreed (three strongly); the remaining answer is also answered as “n/a” by the language teacher. Almost all teachers feel confident in the use of the IWB in class: Nine agree (three strongly), only one disagrees.

Acceptance
All teachers support the idea of using the IWB in class (strongly agree: N=4; agree: N=5; n/a: N=1). Still some teachers see also limits in the use of computer in class; six out of ten agree to the limited benefits. Nonetheless most of them think it is important for their pupils to use a computer in mathematics class (strongly agree: N=3; agree: N=5; disagree: N=1; n/a: N=1).

Learning
During the use of the computer most teachers experienced that they had to give little additional explanation (six out of ten) and they agreed that the pupils liked that they got various ways of solution (strongly agree: N=4; agree: N=6). Pupils seem to know what they have to do next – only two teachers answered that they had pupils that acted differently.

Behavior of the pupils
All teachers agreed that they think their pupils enjoyed the work on the IWB (strongly agree: 6; agree: 4). Regarding the continuation of the use of the IWB (for mathematics class) only one teacher disagreed, the others agreed (N=5) or strongly agreed (N=3). One answer is “n/a”.

Work with the computer in mathematics class
“I had the impression my pupils had difficulties while working with the computer.” The answers to this statement were mixed: Most teachers disagreed (N=6) whereas two agreed. The rest (N=2) were obviously no teachers of mathematics, since they stated that this statement is not applicable to their needs. We also asked, if they think that their pupils would have understood the subject better
with a classical method of teaching: Five teachers disagreed (strongly: 1), two agreed, two answered n/a. One question covers the advantage for the use of ICT for weak learners. Do the teachers think that the use of the IWB supports weak learners? Only one teacher does not think so, the rest believes that weaker pupils can profit from the use of ICT in class. Most teachers also believe that the memorization of the subject is fostered by the technical approach (eight out of ten).

Open question
To give the teachers the chance to state anything else they might have in mind in relation to the project we included an open question. Five out of ten teachers used this opportunity. Motivation seems to be quite high with pupils and teachers as shown in the following statement given by a participant: “I appreciate working with the Smartboard, [...] in general I can strongly confirm that my students like working with the computer and I have the feeling that it helps them to memorize and work in teams etc.” One teacher mentions that he thinks the IWB may cause lack of attention in classes over 30 pupils.

3 Evaluation

Summing up, the pilot project with Interactive Whiteboards can so far be regarded as a success. Most of the teachers are highly motivated to work with the board and they appreciate the opportunity to have a SMART Board in their school. Still, they do critically evaluate the use of the IWB and try to put forward the didactical use for the learning experience for their pupils without overrating the technical aspect of the IWB. Teachers seem to have experience with computers but not necessarily with the IWB. Nonetheless they use the regular meetings to get more familiar with the use of the IWB and try to learn from experiences of others. It is crucial for teachers to harness the benefits of the technology to pass on a positive learning experience to their pupils without totally neglecting the technical aspect of the IWB. Teachers seem to have experience with computers but not necessarily with the IWB. Nonetheless they use the regular meetings to get more familiar with the use of the IWB and try to learn from experiences of others. It is crucial for teachers to harness the benefits of the technology to pass on a positive learning experience to their pupils (Miller, Glover & Averis, 2003). During the meetings it was quite obvious that the teachers do not only come to the meetings, but some of them are really engaged in a meaningful use of the IWB. For the boards to be effectively used it seems to come as an advantage to have the board in a room that is easily accessible by teachers and pupils.

The exchange between teachers of different schools and even school types was very rewarding. It seemed as if everyone was able to profit from the experiences of the others. The teachers in return seem to like the idea of having several contact persons: Regarding didactical issues in general Project Education Institute was contacted most often, for questions concerning mathematics didactics the University of Augsburg was the first contact person. SMART Technologies was also contacted when teachers needed support that were rather technical or more specific to the IWB. This is of course only an overview that refers to short‐time experiences (less than one year) with the IWB. We will therefore keep up the evaluation process to find out what effects can be seen over a longer period of time.

This documentation can be regarded as one strategy or pattern of how ICT can be integrated within an educational context. It is important for all educators to be able to keep up with the demands of today’s society which is dependend on new media and ICT. This paper aimed at giving an example of how it is possible to get these new methods and strategies into the classroom to foster innovation processes even within a relatively small project.

Literature


Learning Environments in the Context of the School’s Needs

Abstract

A relatively short time ago, we could hardly imagine the depths of the changes which today’s information society is experiencing now. These changes are influencing various areas of our lives and they are also very significantly influencing the educational system. For the institutions which are engaged in education, these changes brought, amongst other things, the necessity to define teacher standards and graduate profiles. Future teachers must, during their university education, get new competences, which would reflect the requirements of society which are oriented to the use of information technologies. One of these competences is the skill of creating modern education environments for both elementary school pupils and secondary school students. In this chapter, we will try to describe how we create this skill having long experience with the education of future teachers. We will present some environments which were created and developed by our students, who are future teachers of mathematics.

1 The present state

The use of computers in mathematics lessons has included very complicated developments recently. The era of computer implementation in mathematics lessons, when we were discussing in conferences and seminars, various possibilities of their use and advantages of newly discovered software for presenting pictures in lessons of geometry or solving systems of equations graphically, has definitely ended. However, it does not mean that teachers can use all of the possibilities offered by various information technologies. It is common that these technologies are rather used as a demonstrational instrument, which enables teachers to visualize processes, which could not be written down with chalk on the black board. Nowadays, there are many available programs, also free-source programs, which teachers can use for this kind of purpose. It is more complicated when talking about the use of computers by students. How can we reach the state of computers helping students to understand subject matters; help them to eliminate com-
Learning Environments in the Context of the School’s Needs

character. Concrete material such as blocks for the learning of counting and early arithmetic, or mechanical drawing systems, or audio-visual technologies do not embody the key feature of a computer-based learning environment: it computes formal representations of mathematical objects and relationships. The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner input, and the feedback of the environments is provided in the proper register allowing its reading as a mathematical phenomenon. This cognitive characteristic has bred lofty expectations in mathematics education on the assumption that computers will enable a deeper, more direct mathematical experience. This educational expectation is of a different nature from the expectation rooted in Computer Aided Instruction, because it involves changing the mathematical experience of learners at the epistemological level, rather than facilitating or automating a particular pedagogical style."

Interesting description of learning environments were also introduced by Jonassen and Land (Jonassen, 2000).

2 Preparation of Future Teachers

Let us shortly describe our new experience with preparing future teachers at our institutions. It is important to say that the education of teachers does not end with graduation from university but it is necessary to continue with their systematic education also after coming to schools. From the very beginning of the future teachers’ studies, the teacher trainees are introduced to ways of creating their own educational environments so that computers would serve as effective tools for making a motivating atmosphere, helping to develop new concepts, and making a suitable environment for practice.

At the beginning of their studies, future teachers are assigned with the creation of an educational environment as their semester work. The criteria which have to be met, serving also for the assessment, are as follows:

1. It has to be evident that the environment focuses on the creation of particular concepts.
2. The environment supports motivation and the students’ interest in the new subject matter.
3. The environment offers changes of organizational forms of teaching when necessary.
4. The records of each lesson can be displayed on the Internet, so students who are absent can catch up with their others.
5. It is possible to switch from frontal teaching during each presentation to practice in integrated groups.

plicated and time consuming calculations and constructions, which could divert the students’ attention from the real understanding of presented problems, but without having negative effects on, very often underestimated, craft skills? This kind of problem was discussed at various levels in many scientific papers and literature. We can name for example the works of Kutzler (1998), or Healy and Sutherland (Healy, 1990).

These days, our attention is mainly drawn to more sophisticated ways of using computers. How can we use computers effectively so that they can help students with more independent discoveries of new concepts and help them to support their creative innovation? Could computers give effective help with the factual creation of various knowledge structures? (These questions are discussed for example in (Balacheff, 1996; 2006) and (Binterova, 2010).) So it is important to make all teachers aware of the fact that the technologies themselves cannot solve problems in education, and that we have to always ask how to motivate our students, how to teach them the correct understanding of concepts, how to develop their necessary numerical skills etc. Modern information technologies can only help us to create suitable working environments. There are many possibilities of implementing these technologies in the educational process. The suitable use of the technologies is influenced mainly by the content, the educational levels of students, and also by teachers’ abilities to implement them in their lessons. Only when meeting several conditions, this implementation can have the right effect in school practice; it is not true that modern technologies used in education automatically make better effectiveness in the learning process.

The opinion is supported by a lot of evidence. We can mention, for instance, results of research conducted in Great Britain about the use of interactive whiteboards in various subject lessons (more in (Moss, 2007). It showed that some of the teachers had problems with creating well balanced didactic materials, which would be well understood by students. The implementation of these technologies without a sufficient methodological background, without professionally made materials, and without the prior preparation of the involved teacher did not bring the expected results.

Therefore, it is necessary for future teachers to know these problems they can face, and they should be given solid educational background during their studies to be able to use the technologies in a suitable and effective way. When talking about this, it is also important to determine which learning environment can be considered as good and if teachers are capable of creating such environments themselves. A brief but very concise description of this environment was made in (Balacheff, 1996, p. 469): "A unique feature of effective computer-based learning environments as compared to other types of learning materials is their intrinsically cognitive
6. The immediate feedback is provided both to students and teachers.
7. All students are actively engaged in the learning process.
8. It supports inter-subject elections.
9. It is suitable for making projects.
10. It affects all senses of the students.
11. It makes the learning process easier for students with learning disorders.
12. It provides a manual for teachers.

2.1 Creating a Learning Environment - Example 1.

As a first example of such an environment suitable for fixing new concepts and practicing previously presented subject matter, we present a project called Mathhill. A student made this program which creates such an environment and which is a suitable complement when presenting the topic of Expressions and Polynomials at the lower secondary level. The student came up with the question whether it is possible to improve the understanding of the concept of the polynomial and if it could be possible to use a kind of visualization to support the learning process. The means could be the learning Mathhill environments which she created (see Fig. 1).

Mathhill was created in the Macromedia Flash program. Flash is a program designed for the creation of interactive multimedia animations. It is attractive for a lot of users thanks to its simplicity and various possibilities. The most common products of this program are above all flash banners, online games, whole websites, and other applications, which can work either online or offline. What is interesting about this program is the fact that it connects the graphics and programming and therefore it is suitable for people who create only graphical animations, banners or various effects as well as people who work with Action Script (the programming language of Flash). The menu of the learning environment has been created in a winter landscape containing four buttons which are links to particular games or animations and whose names are situated under their pictures.

![Fig. 1: Environment of Mathhill](image1)

Learning Environments in the Context of the School’s Needs

The first button called Formulae Animations will take us into the environment suitable for utilization in classes of mathematics. Here are animations and visualizations of particular algebraic expressions (Fig. 2) which are taught and used in secondary schools. Another link called Revision can serve pupils in classes of mathematics or at home for home revision.

![Fig. 2: Visualizations of polynomials](image2)

There are exercises on operations with number expressions and the introduction of variables. The third link is called Rational Expressions which is a game based on calculation with rational expressions. The game goes through all the operations with rational expressions starting with addition and subtraction to multiplication and division of expressions. The last link called Calculation with Expressions contains exercises on all operations with polynomials (sums, differences, factorization by factoring out as the behavior of a player 5 as well as by using formulae). The manual for teachers is added to this environment. It explains new possibilities in the didactics of mathematics. The learning environment has been tested in several lower secondary schools and the following passage presents in brief the results of the experiment.

**First Experiment**

18 students were working with the program Mathhill. Half of them finished the program, the fastest pupils managing to go through it in 85 minutes. None of the pupils succeeded in the game called Rational Expressions. The fastest pupils got to the sixth task. All of the pupils managed the game named Revision. The fastest was a boy and a girl, who finished the game in 35 minutes. The others were working about 40 minutes. The pupils were working individually and they were trying to reach the right answer. In our opinion, they were not used to the more difficult tasks presented in the game and they are also not used to using computers in classes of mathematics. In spite of the fact that half of the pupils did not finish the game, they evaluated the game positively.
The program Mathhill was used by 20 pupils and almost all of them finished it. The fastest girl managed to finish the game in 60 minutes. The second educational environment created by one of our students should function as a complete environment for all phases of time. Some of the pupils were browsing Formulae Animations or interesting ways. It can be used with various sources which are connected with video files, audio files or web pages. The sources are prepared before the lesson so they're ready to be used immediately. It saves time and keeps the lesson fluent. Teachers can easily change their notes and show the new interaction. Teachers have to make up a new way of interpretation, new examples. As results from Moss's research (Moss, 2007) show, higher demands on teachers, e.g., learning how to use new procedures and technologies, can be time necessary for preparation of a single lesson. Teachers have to make up a new way of interpretation, new examples, new imperfections but some of them involve relevant mistakes. Our aim was to make material for teachers that would offer students another possible way of learning and solving exercises. As conclusions, the students more often understand a problem faster. What is the role of technologies in the teaching and learning process, how is it possible to use them in the teaching and learning of mathematics? According to Kutzer (2003) it is possible to use computers simplifying experiments, for visualization and concentration. A computer simplifies the complex of such a quality we would like to. These presentations have their own imperfections but some of them involve relevant mistakes. We do not want to say that we get rid of pairs of compasses and rulers. However, the one who taught geometry using a sketchbook equipment presented before. It is a good example of an educational environment with high level technical elaboration in which it is easy to orientate. The environment functions as a kind of trainer, providing feedback and visualization of Multinomial. Learning environments were developed in Czech and English language. They are available on www.pf.jcu.cz.
appreciates the possibility of visualization of such situations which can be hardly demonstrated on the board during a single period. We can demonstrate that it is possible to teach every mathematical matter using a computer and suitable software.

Student created the teaching environments which are made in the SMART Board software for IWBs. The environment themes were ‘functions’ at secondary school. The manual for teachers was added. This manual is called “Guide to individual chapters, including the results of problems and didactical analysis.” The subject matter was elaborated into key competencies. It makes the interactive elements really useful if we are looking at them from the side of the didactics of mathematic. These elements are offered in IWBs and they are also pretty often used by mathematical programs like CAS, Derive6 and GeoGebra for creating and fixing new conceptions. “Use of the program Derive” in the textbook is divided into three levels: Expert Level, Advanced Level and Beginner Level, with a flower icon. After clicking on the flower, the student will enter the educational environment of the program Derive6. If the student is working at expert level, the environment in the program Derive6 is red; if the student is working at advanced level, the environment in the program Derive6 is blue; if the student is working at beginner level, the environment in the program Derive6 is green.

You can see the example in the picture (Fig. 3), where the students are solving problems at the expert level. At this level, we expect that students know the Derive program and that they can work with it. We just assign a problem to them and then it is up to them to choose a procedure leading to the solution.

At advanced level, we expect that students have already worked with the Derive6 program (Fig. 4). They can use online help when solving the problem. The help shows the problem than already solved from that part. At beginner level (Fig. 5), it is the first time that students work with the program Derive6. They can solve the given problems with a model or a manual. The model helps the students to solve different problems. The presented interdisciplinary subject matter is within the scope of Physical Education, Physics, Czech Language ...

Learning Environments in the Context of the School’s Needs
There are some games in this environment, which allow students to check their knowledge about functions. The students need knowledge and experience which they have to have in order to solve it. Thanks to this, students induct new pieces of knowledge. You can also find here the examples to check the right answers, or problems which have to be solved by programs Derive 6 or GeoGebra. Each of these problems is available including the example of the solution which is hidden. If you want to see it, all you have to do is just to click on the icon, which looks like the key, which allows you feedback. You can find tables, graphs or pictures in the text. There are some games in this environment, which allow students to check their knowledge about functions.

The interactive educational environment could be used in all parts of a lesson (the introduction, main part, and conclusion) and in all periods (motivational, exposure, fixation, diagnostic (feedback), and application). This way, we can shortly draw up the basic use of the textbook in single periods of a lesson:

- **Motivational period:** We can better motivate our students to study with this textbook. We can also activate them in a better way and keep their focus.
- **Exposure period:** Thanks to original examples, the teacher can better pass on pieces of knowledge, which students can get on with having fun.
- **Fixation period:** Thanks to the notebook, the teacher can reach for a faster memorization of the subject matter, longer retention in the memory and practicing of the subject matter in an original way.
- **Diagnostic (feedback) period:** During the solution of the problem, the teacher immediately gets feedback from the adoption of a piece of knowledge.
- **Application period:** In single problems, the student applies the adopted piece of knowledge, which he can use later in his practical life.

The most important page in the interactive environment focusing on Functions is the introduction menu (Fig. 6).

![Interactive environment – menu](image)

On this page the user can choose the theme. If it is the first time the user works with the textbook, it is good to start with the link Introduction, where the user can find important information on how to work with the textbook. Then, the user can choose any chapter and should end with the link Conclusion. In the conclusion, students can revise and test their knowledge. This whole chapter is centered on playing games. Students revise the knowledge of functions using various games such as mathematical crosswords or a mathematical memory game (*Find the Pair*).

One part of the User Guide for teachers contains also a list of skills and competences which the students, after going through all the chapters, should acquire. For example, after going through the first chapter, students can represent a point in the right angle reference frame, they can explain the conception of the right angle reference frame, they can control the orientation in the right angle reference frame (see the problem in the picture Fig. 7), knows the conception Cartesian coordinates, can add and write down the point into the right angle reference frame and can read the coordinate of the point, can solve the problems from the right angle reference frame.

![Orientation in the Cartesian coordinate system](image)

**Fig. 7: Orientation in the Cartesian coordinate system**

**First Experiment:**
At first, we tried this education experiment with the created interactive textbook in a Math lesson of 15 year old students at secondary school. The lesson was oriented on the theme of *Right angle reference frame* and the theme *What is and what is not the function?* 19 students were engaged in this experiment. The students already discussed the points on the themes right angle reference frame and *What is and what is not the function?* in Math class. So this education with the interactive environment was more like revising. During
this education, we used frontal teaching, instructive and constructive methods (interpretation, explanation), clearly demonstrative methods, work with the text and dialogical methods (conversation, discussion). The lesson passed without problems. The students interested in interactive textbooks and they were active. There were no students who would not work or bother us in the classroom. The students enjoyed the IWB environment and subs-

tantard problems. The students were really excited about a problem in the football environment, during the solving of this problem in the program Derive6. That was the first time the students worked with this program.

Second Experiment:

That was the second time we worked with the interactive environment during the Math lesson of 14 year old students. The theme was the same: The right angle reference frame. 21 students took part in this experiment. The students had not talk about the theme so far, that means that the education with the interactive textbook was an interpretation, when the students got the new subject matter. During this education, the heuristic method was used. We used frontal teaching, instructive and constructive methods (interpretation, explanation), clearly demonstrational, we also used games and competitions; work with a text and the dialogical methods (conversation, discussion). The lesson showed no problems. The students were interested in the work and all of them worked actively. Most of all they enjoyed the work with the Derive6 program during solving of the problem with the football environment and the game at the end, where the boys competed against the girls.

Conclusions: During the creation of the education environment, I was afraid that students would not understand substandard problems from the field of sport. I was really surprised about the students’ interest during their work with worksheets, their feedback on the created interactive environment, the interest of the teacher about this kind of education, the nice atmosphere in the classroom, the involvement of the students and the fact that they were able to get into the situation and solve it. The students got on the higher level with their knowledge in the subjects of Math, Geography, Physics and Czech etc. They created a nice working atmosphere in the classroom with their concern and active work and this atmosphere motivated them. They cooperated. The teachers really liked this education with the interactive environment, even though their preliminary preparation was different. One of them had already some experience with an Interactive Whiteboard and for the second one this was the first time she met this technology.

Tutor Evaluation of the Project

The learning environment meets the conditions in all three points. The project has 290 pages in total and it is an environment for education in this theme. The interactivity of the environment is exhaustive from the didactics angle and the programs CAS a DGS used were suitable in single parts of the process of creation of conceptions. The creation of conception was alright, so was the motivation for the practice. The learning environments from Czech are available at this website: www.pf.jcu.cz.

3 Conclusions

The good learning environments develop the key competences of students and have the expected outputs according to NCTM Standards in mind. The sheets stress the process of creating concepts, student’s sense perception and their imagination. We think, we can reach a better motivation using these sheets, make students interested in the subject and a particular topic and make them participate in lessons in an active way. Why should the pupils be prevented from using these methods? Pupils learn only for an effect on others, for their parents, school and very rarely for themselves and only some of them can use their knowledge effectively in real life as they do not see the mutual coherence of it. The use of the modern learning environments can save a lot of time and open the door to experimentation and the making of separated models. However, stereotypes still prevail in Czech education and for this reason the transformation will need some time.

At the end, we want to say that it was very positive to watch the lesson with a high quality of education environment, which was created by our students. The education in the environment which will help the natural discussion about mathematical problems in the classroom, when the wrong answer is not a mistake, when real interest is needed not for the marks or success with the teacher, but for real interest. If the environment is right, prepared instead of a standard series of problems, which is usually done for kids, these problems naturally lead to related problems which will check the level of understanding of all and pretty often with use of the software, which can be purposefully used by students, with the possibility of visualization of conceptions and the dynamics of a new educational environment. This kind of education will evaluate all efforts, which are needed during the preparation of an education environment.
Peter Baptist

On Going for a Walk with an Artist and a Famous Mathematician

Abstract

Often people equate mathematics with arithmetic and focus on computational skills. But mathematics involves more than computation. It is a study of patterns and relationships, a way of thinking and a science that is characterized by order and internal consistency, a language that uses carefully defined terms and symbols, a tool that helps to explain the world. The famous British number theorist G.H. Hardy (1877 – 1947) pointed out: "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

This article exemplarily provides ideas how to make students familiar with the above aspects. Such an orientation does not only give an adequate view of the nature of mathematics, it also is a prerequisite for an adequate understanding of mathematical concepts and relationships. We need more opportunities for students’ active engagement. They do not learn mathematics by memorizing formulas and rules or dull computations.

1 Introduction

"Mathematics is no spectator sport", as George Polya (1887 – 1985) clearly said. "To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place, it means to be able to solve problems." These problems should be viewed as a challenge for thinking. Therefore we have to create situations in which students (can) develop interest and try to go their own ways. To the characteristics of an experimental access to mathematics there belong

- open-ended problems that allow an active engagement by students,
- a vivid discourse among students about analyzing, solving, interpreting a problem,
- encouragement of students to generate questions and generalizations of their own,
the insight that mathematics is a stimulating and challenging discipline.

2 Seeing Mathematics through a Painter’s Spectacles

The paintings of the Swiss artist Eugen Jost convince by their diversity. They contain elementary and more complex problems, they attract kids, students and adults with and without mathematical knowledge. They often stimulate to try an experimental access to the underlying mathematics that is more or less hidden. For Eugen Jost mathematics is a beautiful, lavishly landscaped, colourful garden with many paths, partly broad and even, partly narrow and winding, partly fairly steep. Jost strolls through this garden, not as a botanist or a gardener but as a lover of flowers. On his way he goes from one flower to the next, picks a beautiful one from time to time and after a walk he has collected a magnificent bunch of flowers. In a lot of his paintings we can find such bouquets. This article is concerned with a painting that describes the beginning of a new mathematical field, the graph theory.

3 Going back into the 18th Century

This time Eugen Jost does not want to stroll in a garden but in a town and he chooses an outstanding companion. He goes for a walk with Leonhard Euler (1707 – 1783), one of the most prolific and one of the greatest mathematicians of all times. He wrote more than 800 research papers and a lot of books. Born in Basel (Switzerland) Euler made his academic career in Berlin (Prussia) and St. Petersburg (Russia).

In the 18th century Königsberg (now Kaliningrad in Russia) was a well-known Prussian university town that was flown through by the river Pregel. The famous philosopher Immanuel Kant (1724 – 1804) lived and worked there. Let’s have a look at a map of the downtown area.
People in old Königsberg loved to take walks along the river, on the islands and over bridges. In the early 1700s they wondered if it was possible to take a journey across all seven bridges without having to cross any bridge more than once, and return to the starting location. Finding no solution the citizens asked the famous Euler. He proved that such a tour is impossible.

Euler pointed out that the choice of the route inside each land area is irrelevant. The only important feature of a route is the sequence of bridges crossed. Let’s have a look on Euler’s original diagram and the corresponding version by Eugen Jost:

What matters is how everything is connected. This finding allowed Euler to reformulate the problem in more abstract terms. Replacing each land area with a dot (or vertex) and each bridge with a line (or edge) Euler confines himself to the essential. The resulting mathematical structure is called a graph. Our problem is equivalent to asking if the graph on four vertices and seven edges has a so-called Eulerian cycle.

More generally Euler gave a criterion for any network of this kind. He showed that one could transverse such a graph by going through every edge just once only if the graph had fewer than three vertices of odd degree. By the degree of a vertex we understand the number of edges that start or end at the vertex. All the vertices in the above graph are of odd degree, therefore his answer was negative.
The English Canadian number theorist William Thomas Tutte (1917 – 2002) wrote a nice poem on the Königsberg bridges problem and Euler’s solution. Admittedly the following lines are not a masterpiece of 20th century poetry but they contain all the essential facts.

Some citizens of Koenigsberg
Were walking on the strand
Beside the river Pregel
With its seven bridges spanned.

O, Euler come and walk with us
Those burghers did beseech
We’ll walk the seven bridges o’er
And pass but once by each.

“It can’t be done” then Euler cried
"Here comes the Q. E. D.
Your islands are but vertices,
And all of odd degree.”

Since Euler generalized his result to journeys on any network of vertices and edges, the problem of the Königsberg bridges represents the beginning of graph theory. Today this part of mathematics is used in countless fields, from the study of car traffic flow, logistics in transportation to link structures of a website. Euler’s very simple representation of the situation with the bridges connecting the land areas (without regard of the specifics of the street map of Königsberg) also was the forerunner of the mathematical field of topology. We often use this simplification of a real situation for example to get a clear overview of the connections and stops of the public transportation in a city or region.

4 Experimental Mathematics – Stimulating Acts

The Königsberg bridges are an excellent example for a problem that can be solved by experimental methods. The situation can be investigated even with primary school kids. We encourage them to develop their own informal methods to solve the problem. Fig. 6 shows how. Exploring, observing, discovering, assuming are the main activities in this kind of mathematics lesson. After that the kids are asked to try to explain their findings and to express their impressions. To get sustainability the activities together with the results have to be recorded in a study journal. By doing so the students are forced to work carefully and to think thoroughly.

Fig. 6: Children exploring the situation in Königsberg

The above notes show that the pupil is very astonished to be confronted with a problem that has no solution – apparently a new but very important experience.

Fig. 7: Written comments of a child
5 Variation: Königsberg nowadays

Two of the seven original bridges were destroyed by bombs during World War II. Two others were later demolished and replaced by a modern highway. The three other original bridges have been preserved. Google allows us a view of Königsberg in the 21st century. We recognize the three remaining bridges and the two new highway bridges. And in the aerial photo we discover four additional bridges.

![View of Königsberg nowadays](image)

We have a new situation and we have to check the degrees of the vertices. Do we get a round trip this time where we transverse each bridge only once?

![Variation of Jost’s painting](image)

6 Variation: The House of Santa Claus

There is a close connection between the Königsberg bridges and an old children’s game. You have to draw a house, but you may not lift your pencil and you may not repeat a line. While drawing each line segment you have to pronounce a syllable of “This is the house of Santa Claus”.

![The House of Santa Claus](image)

This house is a graph, too, consisting of five vertices and eight edges. (Note: The diagonals in the square have no point of intersection and the artist was not accurate in drawing one diagonal!) Euler’s result helps to find out that there exists at least one solution. Altogether there are 88 solutions.

7 Variation: Knight’s Tours

At the beginning of a game of chess there are 32 pieces of six types (king, queen, rook, bishop, knight, pawn, each with its own style of moving) on the 8x8 chessboard. The knight has an unusual move. It can jump two squares horizontally or vertically, followed by a single square perpendicular to that, and it leaps over intermediate pieces. Mathematicians use the chessboard for a special game, they only need one chess piece. To create a knight’s tour, a knight is required to make a series of moves, visiting each square on the chessboard exactly once. If the start and finish squares are one knight’s move
apart we speak of a closed tour. Euler was the first to write a mathematical paper analyzing knight’s tours. The first algorithm for completing such a tour was described in 1823 by H. C. Warnsdorff.

Fig. 11: Chessboard and the problem of the knight’s tour

It’s not so easy to find a closed tour on a chessboard by trial and error, although there are 13 267 364 410 532 solutions. Here is one of Euler’s examples:

![Chessboard and the problem of the knight’s tour](image)

**Fig. 11: One of Euler’s solutions**

By the way, the exact number of open tours is still open.

Now we reduce the number of rows and columns of the chessboard. Can you find a knight’s tour on a 4x4 board? If not, what is the largest number of squares that the knight can visit?

Can you find an open and/or a closed tour on a 5x5 board?

Comparing the problem of finding a knight’s tour with the Königsberg bridges we notice an essential difference. On the “Königsberg” graph we want to pass each edge exactly once while during a knight’s tour we want to visit each vertex exactly once and no edge more than a single time.

Find a tour along the edges of a dodecahedron such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point.

![Diagram of the dodecahedron](image)
8 An Excursion in Greek Mythology

From Greek mythology we know the tale of Theseus and Ariadne. The location of this story is a labyrinth at Knossos on the island of Crete that was built for King Minos by the constructor Daidalus. He had made the labyrinth so cunningly that he himself could barely find the way out after building it.

In this labyrinth there lived the minotaur, a creature that was half man and half bull. It was fed with the bodies of young men and women who had yearly to be sent to Minos by the Athenians as a tribute. Theseus was among those who were sent from Athens as the third tribute to the minotaur. When he arrived, Ariadne, one of King Minos’ daughters, fell in love with him and offered him help if he agreed to marry her and take her with him to Athens. She gave Theseus a ball of thread, which he fastened to the door when he went in, so that, after killing the minotaur, he could make his way out by winding up the thread.

The design in the above detail of Eugen Jost’s painting became associated with the minotaur labyrinth. Having a closer look at it we definitely recognize: If this is really the design of minotaur’s housing, then Theseus had no need of Ariadne’s thread. There is no chance to get lost. Therefore we can be sure that the minotaur was trapped in a complex branching labyrinth, a so-called maze. Often the notion labyrinth is synonymously used with maze, but there is a subtle distinction between the two. Maze refers to a complex branching puzzle with choices of path and direction. Such a maze can be described by a graph – and here we have the connection to Euler. A labyrinth by contrast has only a single, non-branching path, which leads to the centre. It has a clear route to the centre and back and is not designed to be difficult to navigate.

To construct the Knossos labyrinth we start by drawing a cross and four dots. The diagram shows what to do. Now it’s your turn.

9 Final Remark

The walk with Leonhard Euler and Eugen Jost shows: Mathematics is much more than mere computing, mathematics is a part of our culture. Therefore historical aspects should be integrated in our teaching. Abe Shenitzer from York University (Toronto) underlines this aspect when saying: “One can invent mathematics without knowing much of its history. One can use mathematics without knowing much, if any, of its history. But one cannot have a mature appreciation of mathematics without a substantial knowledge of its history”.

Literature

Abstract

JSXGraph is a library for displaying dynamic mathematics, e.g. dynamic geometry, function plotting, turtle graphics, in a web browser. It is written in JavaScript and runs on a broad variety of devices from desktop computers down to smart-phones and tablet PCs. JSXGraph is able to import various file formats like GEONEXT, GeoGebra, Intergeo, and – at least partially – Cinderella. At the moment, this seems to be the only possibility to display content from these sources on upcoming small computing devices, which makes them usable in the classroom.

1 Introduction

In the late 1990s the availability of graphical web browsers that enabled easy access to the World Wide Web brought many fresh ideas to the classroom and to mathematics education. The programming language Java became the dominant tool to raise interactivity in dynamic mathematics to a new level. Countless new Java applets came to existence to visualize many aspects of mathematics from kindergarten level to university level. Also, powerful software systems were developed that combined geometry and calculus under one graphical user interface. The most prominent examples are Cinderella¹, GEONEXT² and GeoGebra³ to name a few of them.

But now a new hardware generation is on the horizon which appears to be better suited for the classroom than the old clumsy desktop PC. The revolution started with the success of small and cheap netbooks and the appearance of powerful smart-phones. Now, these two complementary worlds seem to melt together into tablet PCs. The success of the iPad made by Apple confirms this. Probably, very soon many other hardware manufacturer will follow and produce cheaper tablet PCs having more features than Apple’s iPad.

For use in the classroom the advantages of these devices over the desktop PC are the long battery life and their small size and weight. Also, they offer much more possibilities than the still popular graphical, programmable pocket calculators. These features weigh out the difficulties in using these devices – especially typing – which is still easier on the desktop PC with a keyboard.

Now, mathematics education faces the challenge that most of the existing web-based software for dynamic mathematics is implemented in Java and embedded in web pages as so called Java applets. But there will be no Java plug-in available on most of these new machines.

Without good software the new hardware is useless for learning mathematics in the classroom.

With the project JSXGraph⁴ at the University of Bayreuth we try to take up this challenge and offer first class dynamic mathematics software that runs on every device including smart-phones, netbooks, tablet PCs and desktop PCs. Moreover, the goal is to provide compatibility for existing resources for mathematics education.

2 What is JSXGraph?

JSXGraph is a software library implemented in the programming language JavaScript for dynamic mathematics. JSXGraph can be easily embedded into web pages, the download size for the library, when used on-line, is a mere 80 kByte. The software is open-source, released under the Lesser GNU General Public License (LGPL). The source code is hosted by Sourceforge⁵.

For graphical output, JSXGraph uses the vector graphics format SVG (scalable vector graphics) on all web browsers supporting that format. This covers the popular web browsers firefox, chrome, safari and opera. The widely used web browser “internet explorer” does not support SVG, but instead uses the vector graphics format VML (vector markup language) – at least up to version number 8. The internet explorer version 9 supports SVG. Since JSXGraph is usable with SVG as well as VML, this means that JSXGraph still runs on older desktop PCs. In many cases, these outdated machines are restricted – for various reasons – to the use of Internet Explorer 6. With JSXGraph it is possible to access modern mathematical content even with these old machines.

¹ http://cinderella.de
² http://geonext.de
³ http://geogebra.org
⁴ http://jsxgraph.org
⁵ http://sourceforge.net
Many smartphones come with the operating system Android, also many already announced tablet PCs are expected to be Android based. The default web browser on Android does support neither SVG nor VML, but it allows to draw bitmap graphics with the new HTML element canvas. Starting with release 0.82, JSXGraph supports the canvas element, too. Even on more powerful computers JSXGraph has the advantage over Java based software that the downloading time and the initialization time are much shorter than for comparable Java applets. In summary, JSXGraph is usable on a huge amount of devices and should be able to take up the challenge and support dynamic mathematics on the upcoming hardware generation. At the time of writing, there is no other software for dynamic mathematics that can be used on such a wide range of devices.

3 How to use JSXGraph

There are three possible scenarios:

3.1 Display Existing Content

JSXGraph is able to read the following file formats:
- GEONE\x{T
- Intergeo
- GeoGebra
- Cinderella

The support of the GEONE\x{T file format by JSXGraph is close to 100%. Only very few GEONE\x{T resources are misinterpreted by JSXGraph. In Fig. 1 the construction to the right is the GEONE\x{T Java applet, to the left is the same file displayed by JSXGraph.

![Fig. 1: Importing GEONE\x{T](image)

3.2 Write Custom-made Applets

JSXGraph provides an API (application programming interface) to build dynamic mathematics applications for the web browser. The differential equation plotter on the JSXGraph home page is one example for using JSXGraph in mathematics education on the university level. Other applications are function plotters, turtle graphics, and support for various possibilities to create charts. This is especially interesting for publishers of e-books or providers of e-learning content. In this way, JSXGraph meanwhile is used in situations that are different from mathematics education, like medical information systems or landslide prediction.

The JSXGraph wiki contains more than 150 examples of dynamic mathematics, covering many areas like charts, function plotting, calculus, geometry, and turtle graphics, to name a few.

3.3 Geometric Construction Language

JSXGraph comes with a simple geometric construction language called JessieScript, which is closely related to the syntax students use in school to describe their constructions by compass and ruler. An example is shown in Fig. 2, the online version is available at http://jsxgraph.uni-bayreuth.de/jessie. The whole web page consists of three elements: the form for the text input of the construction, the display of the construction and a log window.

The most important commands are:

---

6 http://www.android.com/
7 http://i2.geo.net
8 See http://jsxgraph.uni-bayreuth.de/wiki/index.php/DifferentialEquations
9 http://swiftcareolutions.com/index.html
10 http://www.rhok.org/2010/06/rhok-1-0-washington-d-c-winning-hack-chasm/
11 http://jsxgraph.uni-bayreuth.de/wiki
### Challenge of a New Hardware Generation to Mathematics Education

#### 4 Conclusion

JSXGraph enables the usability of existing mathematical resources on a broad variety of new, small computing devices. These devices seem to be very well suited for use in classroom, but up to now there is a lack of good mathematical software, since Java applets are not longer supported. The goal of JSXGraph is to change this situation.

### Literature


<table>
<thead>
<tr>
<th>A(1,1)</th>
<th>Point with name 'A' at position (1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZY(0.5,1)</td>
<td>Point with name 'ZY' at position (0.5,1)</td>
</tr>
<tr>
<td>[AB]</td>
<td>Straight line through points A and B</td>
</tr>
<tr>
<td>[AB]</td>
<td>Ray through points A and B, stopping at A</td>
</tr>
<tr>
<td>[AB]</td>
<td>Ray through points A and B, stopping at B</td>
</tr>
<tr>
<td>g=[AB]</td>
<td>Segment through points A and B</td>
</tr>
<tr>
<td>k(A,1)</td>
<td>Circle with midpoint A and radius 1</td>
</tr>
<tr>
<td>k(A,B)</td>
<td>Circle with midpoint A through point B on the circle line</td>
</tr>
<tr>
<td>k(A,[BC])</td>
<td>Circle with midpoint A and radius defined by the length of the (not necessarily existing) segment [BC]</td>
</tr>
<tr>
<td>k_1=k(A,1)</td>
<td>Circle with midpoint A and radius 1, named by 'k_1'</td>
</tr>
</tbody>
</table>

The JSXGraph homepage contains the full description of the syntax.

![Fig. 2: JessieScript](image_url)
Abstract

The paper deals with didactical scenarios on the notion of geometric congruence based on previous successful experience with using a specific model which reflects the relations among structures, activities and intelligences. We present a series of problems tuned to various interests, intelligences and age of the learners. The learning process is in harmony with the fundamental ideas of the constructivism and the inquiry-based learning. It involves computer modelling of artefacts created in the best tradition of the Bulgarian folk art – wood-carved ceilings painted ceramics, embroidery, crocheting, etc., to be presented and shared within a larger audience. Our experience with students and teachers alike suggests that harnessing the strong intelligences of the learners enhances the understanding and appreciation component of the learning process.

1 Introduction

The symmetry and the asymmetry co-exist in unity both in nature and art. A creative process involves specific decisions about the symmetry-asymmetry ratio of the object being created, and this very ratio is a crucial aspect when evaluating its esthetics. The design and the creation of objects sharing the features of fantasy and nature is often based on geometric transformations such as congruences (translations, rotations, reflections and compositions of those). Thus it seems natural to encourage learning this geometry topic by integrating it with the design and reconstruction of artistic artefacts. With this idea in mind we have developed didactic scenarios on geometric congruences in the frames of the InnoMathEd European project (Kenderov, 2010). The design and the implementation of these scenarios is just an element of a more ambitious goal – we expect our students to look for manifestations of geometric congruences, discover them and use them in various activities, and thus – to be able to find patterns and relationships deepening their knowledge and understanding of the surrounding world.

2 The Congruences

The notion of congruence can be introduced by various methods and activities involving different intelligences. The students:

- observe objects from nature, architecture, popular customs, science and discover (recognize) congruencies
- create congruent objects by means of various ideas and tools (e.g. by folding a sheet of paper, turning a slide or a piece of glass with a drawing on it, by means of an appropriate software, etc.)
- transform figures in order to get a specific one
- reach the level of free use of geometric transformation as a means for achieving the goal of a more complex project

The first tasks for the students are to discover and describe the harmony based on congruences in different sports and folk dances (Fig. 1). Then they are expected to perform symmetrical movements with arms and legs. And finally – to present (just with their hands) geometric figures such as triangle, trapezoid, etc.

Fig. 1: Recognizing congruences in sports and dances

Next the students can use various computer environments for exploring congruences\(^1\). They can draw by "free hand", work with specific tools in graphic editors, grids in 2D and 3D, dynamic constructions, and of course - programming in Logo (Fig. 2).

\(^1\)Sources are e.g.: GEONeXt [http://geonext.de]; GeoGebra [http://geogebra.org]; Elica [http://www.elica.net]; National Library of Virtual Manipulatives [http://nlvm.usu.edu/edu].
In order to make the right choice for a specific project it is important for the students to realize that each environment has its advantages and disadvantages. For those who are fond of music, the symmetry could be demonstrated in the context of famous music fragments: (Fig. 3). Here is how a 16-year old student with a high motivation for science reacted to this musical illustration of symmetry during a lecture on music and mathematics:

When Prof. Noam Elkies played the traditional Paganini theme in reversed it turned into variation 18 of Rachmaninoff’s Rapsody on the Theme of Paganini. For anyone with at least an ounce of knowledge in classical music, this had to have been an epiphany moment...

Thus the symmetry could not only be seen but could also heard.

Fig. 3: Part of 24th capriccio theme by Paganini and the 18th variation theme by Rachmaninoff

A good project for the musically inclined is to find the musical interpretation of the geometric congruences (Fig. 4) (Hart, 2009).

Congruences could be found in various aspects of a music piece when considered as a sequence of clear-cut units. The repetition (translation) and the symmetry in a melodic structure are easily recognized in its formal description (by letters or digits) (e.g. AABBCDDE, ABA, 11122332, 11122111) where the same letter (digit) would mean that the corresponding units have the same melodic structure possibly modified by transposition (keeping the intervals). The same idea is used for describing the harmonic structure, e.g. the string TTSDDSTT reflects the harmony of 8 bars being based on the tonic (T), subdominant (S) and dominant (D) chords. Analyzing the structure of a set of music pieces in terms of congruences can be very helpful within a project for writing algorithmic music (Sendova, 2001). Similarly, it is a good challenge for the students to recognize congruences in the natural language and the literature works (e.g. to search and check for palindromes, to analyze the structure of a drama, of poetic forms such as sonnets, haiku, etc.). Then they could describe formally these structures, and finally – to get acquainted with the standard description, e.g. in the case of drama – with Freytag’s pyramid (Fig. 5) (Freytag, 1863).
Especially interesting in mathematics context are problems with the following operations: *adding, elimination, displacement, replacement* and *exchange* (Chehlarova, 2009; Grozdev & Chehlarova, 2007; Grozdev & Chehlarova, 2009; Chehlarova & Sendova, 2009; Grozdev & Chehlarova, 2008). What follows are fragments of our experience with design and reconstruction scenarios involving congruences. We present a series of problems and the response of learners of different age.

## 3 Congruences and Design

### Problem 1
Study the tessellation below and create your own tessellation design.

![Fig. 6: Painted ceramics from Veliki Preslav and a modern tessellation](image)

After observing the above samples of tiles the students are expected to realize that since the tile has the form of a square and decoration with 4 axes of symmetry the options for different tessellations based on these tiles are very limited (if the tiles have to have common vertices).

![Fig. 7: Tessellations with a ready-made and with learners’ own tile-model.](image)

### Problem 2
Make a model in the style of “Troyan drop”.

![Fig. 8: Samples of handmade ceramic plates and computer models in the same style](image)

When painting in a *free hand* style the students would start with the axes. All of them used an even number of axes and then continued the painting by coupling the motives. The zoom mode was used to improve the precision of the work.

### Problem 3
Help the archeologists to reconstruct the artefacts by creating computer models.

![Fig. 9: Models of ancient artefacts to be reconstructed](image)

One of the shortcomings of using Paint is that only the rotation at angle multiple of 90° is possible. If the students choose dynamic software, e.g. GeoGebra, they can insert twice the picture and then rotate the first copy around a center at an appropriate angle. The first figure is made transparent and then the two figures are rotated until they fit (Fig. 10).

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The idea could be extended in the case of rotational artefacts.

**Problem 4**
Create computer models of the artefacts in Fig. 11.

Some of the students decided to use a pattern to be copied and translated appropriately (Fig. 12). The difficulties when passing from 2D to 3D were discussed with the teacher.

**Problem 5**
Make a model of a wood-carved ceiling as the ones in Fig. 13.

In the case of the last model in Fig. 13 the students started with a procedure drawing a Y-shaped element, then added parameters for the size and the angle, made a branch with a varying number of Y-shaped elements and rotated it around the starting point giving rise to a whole class of ceiling models.

The richness of the results we gained in various setting (in-service and pre-service teachers alike) inspired us to introduce the notion of rotational symmetry in the context of the ceiling dynamic geometry models when developing a scenario in the frames of the InnoMathEd project.

**Scenario – Harnessing the rotational symmetry dynamically in a design context**

What is the common feature of a wood-carved ceiling and a ceramics plate in the style of a Trojan drop? – A mathematical phenomenon known as rotational ($k$-fold) symmetry. To explore this concept you will use a Dynamic Software (GeoGebra in our case).

**Hint**
- Identify the smallest geometric element to undergo rotation so as to get a virtual model of original you have chosen to model.
- Choose an angle of rotation and rotate the figure as many times as necessary to close the figure (rosette).
- Carry out an exploration for different angles of rotation and fill in the Table 1.
- Write your conjecture and check it.

### Table 1

<table>
<thead>
<tr>
<th>Angle $\alpha$</th>
<th>Number of rotations $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Warming Up

#### Problem 5.1

What are the center and the angle of rotation in Fig. 14?

![Fig. 14](image-url)

#### Problem 5.2

Which of the figures below have rotational symmetry? Verify your conjecture.

![Fig. 15](image-url)

#### Problem 5.3

Which of the figures below have rotational symmetry?

![Fig. 16](image-url)

#### Problem 5.4

Generate a regular $n$-gon by means of a $n$-fold rotational symmetry for:

- a) $n = 4$
- b) $n = 6$
- c) $n = 8$
- d) $n = 12$
- e) $n = 36$

Try to find different solutions.

![Fig. 17](image-url)

![Fig. 18](image-url)
Create your own button (icon)

- with a fixed angle of rotation

![Image](image1)

Fig. 19

- for rotational symmetry of a circle

![Image](image2)

Fig. 20: Visit our library of virtual models of ceilings:

Problem 6
Create a computer motive for Bulgarian embroidery (Fig. 21).

![Image](image3)

Fig. 21: Bulgarian folk dancers and computer models inspired by the embroidery of the costumes

Here the students are expected to assess the specifics of the software and to choose the most convenient one for creating the basic motive and then to decide what geometric transformation to apply so as to get the desired composition.

The project for creating crochet models involves precision and option to play with various parameters, thus the choice of Logo is natural:

Problem 7
Create a design of a lace napkin for serving tea or coffee (Fig. 22).

![Image](image4)

Fig. 22: Masterpieces of older and younger Bulgarian women

It is interesting to note that this project is usually offered at the beginning of a Logo course for graduates in different fields who want to qualify for teachers in IT. As a rule they get excited of the “no threshold, no ceiling” motto of Logo and enjoy playing with parameters to produce various rosette models, and further – to create an interactive program. Of course, the ceiling could be only your own imagination. What if the crochet model contains a unique element which happens to be symmetric (Fig. 23)... this gives rise to the next project.
Problem 8
Create a stylized model of a butterfly:

Fig. 23: Crochet and Logo models of a butterfly

Coming back to our main object of study, the symmetry, let us recall what the great mathematician Hermann Weyl thinks of it: Symmetry is one of the ideas by which man through the ages has tried to comprehend and create order, beauty, and perfection. Is there an innate link between symmetry and beauty? Could symmetrical composition be an intricate part of defining beauty? Plastic surgery advertisements often read: The symmetry is beauty. At the same time a recent study by Harrison (2007) shows that although we commonly perceive faces as symmetric, in fact there are asymmetries in them. Maybe this interplay between symmetry and asymmetry in search of beauty was what inspired an in-service teacher (working in a school for fashion design) to create a pattern which was originally symmetric and then became asymmetric after she added a decoration (Fig. 24):

Fig. 24: Creating balance of symmetry and asymmetry

4 Conclusions
Our experience with learners of different age (pupils, pre-service and in-service teachers alike) suggests that harnessing the strong intelligences of the learners enhances the understanding component of the learning process. Furthermore, assisting the learners in their choice of a computer environment appropriate for a specific project goal is a crucial factor for the students to experience satisfaction, joy, self-confidence and motivation for further activities, and even to reach thoughts impossible for them before that.

In conclusion, we felt very encouraged to hear what one of the teachers we have recently been working with wrote after a Logo course: It is not possible to “feel” informatics, music, art, poetry, even dancing, not to forget the dreams, if they are in isolation.

Literature
GEONExT, URL: http://geonext.uni-bayreuth.de/.
GeoGebra, URL: http://www.geogebra.org/cms/.
Elica, URL: http://www.elica.net/site/index.html.
20 Years Later – Inquiry-based Learning Again

Abstract

In 1988 a team of Bulgarian specialists in mathematics and informatics education published a series of mathematics textbooks for 8th to 11th grades. Specific scenarios were developed for explorations within Geomland – a Logo-based computer environment for Euclidean Geometry explorations. Within the framework of the project InnoMathEd we have developed some lessons about geometric transformations, based on the methodology in the previously mentioned textbooks. In the present paper we are demonstrating schematically the methodology behind the learning environments on Geometric transformations with Dilation as an example.

1 Introduction

More than 20 years ago (in 1988), when the terms Inquiry-based learning and Dynamic mathematics were still not widely spread, a team of Bulgarian specialists in mathematics and informatics education published a series of mathematics textbooks for 8th to 11th grades (Sendov et al., 1988-1990) which could be described as revolutionary.

The themes on exploring functions, geometric transformations, 2D and 3D topics were presented in harmony with the learning by doing and exploring approach and the principle of integrating mathematics, informatics and languages (Dicheva, Nikolov, & Sendova, 1989). The lessons contained the so-called "turtle corners" where the topics were complemented with a Logo version of the concepts presented. In addition, specific scenarios were developed for explorations within Geomland – a Logo-based computer environment for Euclidean Geometry explorations (Sendov & Sendova, 1995). Regrettably, these ideas could not achieve their full potential. The teachers were not prepared on the necessary level to meet the challenge of teaching in exploratory style. Most of the schools were lacking the necessary technical requirements (Nikolov & Sendova, 1989). Even in those cases when the teachers would choose to use
these very textbooks (to be given a choice was a new phenomenon at the time) many of them would just skip the parts designed for dynamic explorations.

Today we witness well equipped schools but the general opinion is that the information technologies (IT) should be used mainly for the IT classes. The teachers in mathematics and other (non-IT) subjects are still not self-confident enough (both technically and psychologically) to use the full potential of IT in their classes (Stefanova, Nikolova, Kovatcheva, Boytchev & Sendova, 2009).

With all this in mind it was with a great satisfaction that we, the lecturers within the InnoMathEd project, realized how eager the teachers we educated were to learn new things in terms of both technical and soft teaching skills. In all of the workshops we organized in a number of Bulgarian towns, the participants showed a great enthusiasm and persistence in acquiring the skills necessary to work with a Dynamic Geometry software, as well as to modify their teaching style in harmony with the inquiry-based learning. Not only were they ready to test and apply the scenarios developed by the InnoMathEd experts, but also to develop their own teaching/learning materials in the same spirit. For a relatively short time, they gained self-confidence and showed their readiness to work with their students as partners in a research team.

2 A New Construction behind an Old Scaffolding

We have developed an InnoMathEd inspired series of scenarios on geometric transformations, based on the methodology in the previously mentioned textbooks. The scaffolding metaphor, although different from the term coined by Wygotski to mean a specific teaching strategy for providing individualized support of the learner, could be considered as closely related since it provides supports to facilitate the learner’s development (Van Der Stuyf, 2002). The aim is to present the material in a “classical” way, but using the possibilities which the new information technologies give us – visualization, interactivity and dynamic explorations (Dimkova, 2010a, 2010b, 2010c). So far we have developed the learning environments Reflection in line, Dilation and Inversion in English and Bulgarian. The reason of our choice is that the teachers will have the possibility to practice the lessons in different grades. The Reflection in line is taught in the 8th grade, and the Dilation - in the 9th grade of the Bulgarian schools. The Inversion is introduced in some extracurricular activities, in specialized math schools and at the universities. The learners are expected to pass several times during five basic phases.

- **Exploration** – the students make experiments with moving objects in a dynamic construction. They answer questions and make hypothesis about the properties of the objects.
- **Sketch (Dynamic construction)** – the students make new dynamic constructions.
- **Investigation** – the students investigate the construction which they have made.
- **Conjectures** – the students have to write all the conjectures which they have reached in their note books or in text files.
- **Theory** – the teacher gives the definitions, theorems, proofs, and consequences in a classical way.

There are problems including questions, dynamic constructions, investigation or proofs for all students and more difficult problems as challenges at the end of the modules.

3 Dilation

Below we are demonstrating schematically the methodology behind the learning environments on Geometric transformations with Dilation as an example. The module starts with some definitions and reminds some known concepts and properties.

*We know that two polygons are similar when their corresponding angles are equal and their corresponding sides are proportional. In real life it is often necessary to construct a figure similar to a given figure. This may happen, for example, when making maps, blueprints or photos in different scales. In such cases appropriate geometrical transformations are used to construct similar figures (Fig. 1).*

![Fig. 2: Similar figures](image)

The next phases include explorations and investigations, during which the learner can study the behavior of the image of a point A when the point is moved along a specific figure (Fig. 2). After their
investigations the learners are expected to describe their conclusions, conjectures, reasoning, and proofs.

**Exploration**

Let O, A, X be collinear points and let the point O be fixed. If point A is moving around the contour of a geometric figure (a circle, a polygon, etc.) in a way that the ratio \(OX/OA\) stays constant, what figure is described by the movement of point X?

**Sketch**

You can move the point A around the contour of the given figures. Dragging the point A on the slider you can change the ratio \(OX/OA\); you can change the radius (\(r\)) and the angle (\(\alpha\)).

**Investigation**

Watch what figure will be drawn by point X. Compare the shape of this figure with the shape of the given one. Do you think that these two figures are similar?

Clear the trace and make new investigations changing the ratio \(OX/OA\), the radius and the angle.

**Conjectures**

Write your conjectures, please!

The next step is to give to the learners a problem illustrated on Fig. 3.

**Problem**

Given a pentagon \(ABCDE\) and a point \(O\), the points \(A', B', C', D', E'\) are such that \(OA'=2.OA\), \(OB'=2.OB\), \(OC'=2.OC\), \(OD'=2.OD\) and \(OE'=2.OE\). Prove that the pentagons \(ABCDE\) and \(A'B'C'D'E'\) are similar.

**Instruction**

In order to prove that the pentagons are similar we have to show that their sides are proportional and their corresponding angles are equal. The pentagon \(A'B'C'D'E'\) is obtained from the pentagon \(ABCDE\) under geometrical transformation which maps the point \(X\) to the point \(X'\) so that \(OX'=2OX\).

Then we introduce the formal definition of dilation.

**Definition**

Let \(O\) be a point and \(k \neq 0\) be a real number. A geometrical transformation which maps each point \(X\) into the point \(X'\), such that \(OX' = kOX\), is called a dilation with the centre \(O\) and the coefficient \(k\).

We denote by \(\chi(O,k)\) the dilation with center \(O\) and coefficient \(k\). If the point \(X'\) is the image of the point \(X\) under dilation \(\chi\), we write: \(X \rightarrow X'\) or \(\chi(X) = X'\).

After the definition, the students have the possibility to explore the dilation with positive and negative coefficient (Fig. 4) to make the following conclusion: given a dilation with center \(O\) and the coefficient \(k\), than the image of every point \(X\) is single-valued by the equation \(OX'=kOX\). Furthermore, any dilation is single-valued and well-defined by its center and coefficient.
Fig. 4: Dilation with positive and negative coefficient

We make again exploration, sketch, investigation (Fig. 5) and conclusions, which will prepare the student for the first theorem.

**Exploration**

Let \( O \) be a point, \( AB \) be a segment and \( X \) be an arbitrary point on it. Let \( X' \) be the image of \( X \) under dilation with center \( O \) and coefficient \( k \). If \( X \) is moving along \( AB \), what will be the locus of \( X' \)? In other words, **what will be the image of a segment under dilation?**

- Move the point \( X \) along the segment \( AB \). What will be the trace of \( X' \)?
- Explore the trace of \( X' \) after changing the position of the center of dilation and the coefficient of dilation \( k \) (dragging point \( C \)).

**Sketch**

- Chose option Dilation from the Menu and select the points \( A \) and \( O \) (exactly in this order). In the Dilation menu type \( k \) (the coefficient of dilation). In this way you will receive the point \( A' \), the dilation image of point \( A \).
- Make the same with the point \( B \). You will receive point \( B' \).
- Construct the segment \( A'B' \).
- Measure the lengths of \( AB \) and \( A'B' \).
- Calculate the ratio: \( \frac{A'B'}{AB} \).

**Investigation**

- Move \( X \) again and compare the trace of \( X' \) with the segment \( A'B' \).
- Compare the ratio \( \frac{A'B'}{AB} \) with the coefficient of dilation.
- Do this investigation for different positions of the center \( O \) and different values of the coefficient of dilation.

**Proof:** Let \( \chi \) be a dilation with center \( O \) and coefficient \( k \neq 0 \). It is sufficient to find out that if \( X \) and \( Y \) are two arbitrary points and \( X' = \chi(X) \), \( Y' = \chi(Y) \), then \( XY' = v \cdot XY \), where \( v = |k| \).

Thus we obtain the important properties of dilation:

**Corollary 1**

The image under dilation of a line through the center is the same line.

**Corollary 2**

The image under dilation of a line, not passing through the center, is a line parallel to the given line.

**Corollary 3**

Since the image of an angle under similarity transformation is an angle equal to the first one, then dilation preserve angles. Moreover – each angle and its image are angles with parallel sides. Two figures which correspond under dilation are called homothetic.

Here we give a definition of homothetic figures. The students can make investigations with three given homothetic figures on three
dynamic constructions (Fig. 6). They can change the coefficients, the position of the center of dilation and the figures.

**Exploration**

Let $O$ be a point, $F$ - geometric figure, $F'$ - the image of $F$ under dilation with center $O$ and coefficient $k$. How does the position of $O$ and the value of $k$ influence on the positions of $F$ and $F'$?

**Sketch**

- You can change the position of the center of dilation by dragging the point $O$.
- You can change the coefficient of dilation by dragging the point $C$.
- You can change the figures by dragging any point on the original figures.

**Investigation**

- Investigate the behavior of the corresponding polygons when varying the values of the coefficient of dilation and the position of its center.

**Conjectures**

Is it possible for two polygons to be similar but not homothetic?

If $k = -1$ where is the center of the dilation?

Write your conjectures, please!

Following is Theorem 2 with its classical proof (Fig. 7).

**Theorem 2**

If $O$, $A$, $A'$ are three points laying on the same line, then there is only one dilation with center $O$, which carries the point $A$ in the point $A'$.

**Fig. 6: Homothetic figures**

**Fig. 7: Dynamic construction for the Theorem 2**

The theoretical part ends with finding the coordinates of the homothetic image of a point with center of dilation the origin of the coordinate system (Fig. 8).

**Fig. 8: Coordinates of homothetic points**

Vectors $OA$ and $OA'$ have coordinates $(x, y)$ and $(x', y')$. Since $A$ and $A'$ correspond to each other under dilation, then $OA=k$, $OA'$.

Thus $x'=kx$, $y'=ky$ and $A'(kx,ky)$, i.e. $\chi(o,k):(x,y)\rightarrow(kx,ky)$.

At the end, problems for self-preparation are given. They include questions, constructions and demonstrations. The students have the possibility to make constructions in the applets in the site or (by double-click on the applet) to pass in the GeoGebra environment. For the advanced students challenging problems are prepared. Here we give only part of these problems.

1. Given dilation $\chi(o,k)$ and an arbitrary point $C\neq O$.
   Construct the image of $C$ under the dilation $\chi$ if:
   - $k=2$, $k=-2$, $k=2/3$, $k=-2/3$, $k=5/2$, $k=-5/2$.
2. Construct an arbitrary triangle $\triangle ABC$.
   Construct the image of $\triangle ABC$ under dilation $\chi(S,k)$ if:
   - $k=1.5$ and $S$ does not coincide with neither $A$, nor $B$, nor $C$;
   - $k=1/2$ and $S=A$;
   - $k=-1$ and $S$ is the midpoint of $AB$.
3. Construct a trapezoid $ABCD$ ($AB||CD$). Find its image under dilation $\chi$:
   - using vertex $A$ as dilation center and $k=4/3$; $k=-4/3$;
   - using the intersection of the diagonals as dilation center and $k=1/2$.
4. Construct the dilation image of a square and a rhombus.
5. Where is the image of the midpoint of a segment?
6. Find the coefficient of the dilation $\chi(O)$ which maps $M_i$ into $M'_i$ (Fig 9).

![Image 1]

Fig. 9: Dynamic construction for the Problem 6

8. Let $A$, $B$, $M$ be three collinear points and let $A'$, $B'$, $M'$ be their images under dilation $\chi(O,k)$. We know that the points $A'$, $B'$, $M'$ are collinear, too. What will be the relation between the ratio $AM/MB$ and $A'M'/M'B'$? Change the position of point $O$ and the coefficient of dilation and then compare the ratios again.

9. Let the point $M$ be on the segment $AB$ and let $AM:MB = m$. Points $A$, $B$, $M$ map under the dilation $\chi(O,k)$ into points $A'$, $B'$, $M'$. What can you say about the ratio $A'M'/M'B'$?

10. Using the properties of dilation divide a given segment into three equal parts.

11. Given a trapezoid $ABCD$ $(AB\parallel CD)$, find the center of the dilation which maps $A\rightarrow B$ and $B\rightarrow C$; $A\rightarrow C$ and $B\rightarrow D$ (Fig 10).

![Image 2]

Fig. 10: Dynamic construction for Problem 10

4 Discussion

The environments are developed by means of the HTML code, Java and GeoGebra as dynamic geometry software. We recommend using Google Chrome browser in view of its larger screen. There are some reasons for our choice to use HTML.

If the teacher has the possibility to teach in a computer room, all the students can explore, draw, and investigate the new concepts with the teacher simultaneously.

If not, the teacher can only present the lesson and demonstrate the constructions with multimedia. After that, when the students have the possibility to use a computer they can review the lesson on their own.

The lessons are published on the Internet and the students can learn on their own when necessary.

During our courses with teachers we guide them through most of the options of GeoGebra. Thus they can make their own applets and can implement their own modifications of the InnoMathEd scenarios into practice.

As for the students, they do not need preliminary knowledge on the Dynamic software used – with the help of the teacher they can immediately immerse in the learning process.

Literature


Roman Hašek & Vladimíra Petrášková

A Way to Improve Financial Literacy of Future Teachers

Abstract

The article deals with the activity of the Faculty of Education of the University of South Bohemia in the field of financial education. The aim of its effort is to provide teachers, in both pre-service and in-service teacher training courses, with a practical insight into the operation of the financial world. It starts from the premise that a teacher prepared in such a way plays an important role in the process of the successful improvement of financial literacy. An effective way of using the computer algebra system Maple for the introduction of students of mathematics teaching into the nature of basic financial products is shown.

1 Introduction

For most people it is not easy to have a good grasp of the basic terms in the field of finance and of their interrelations, in particular due to the fact that the offer for financial products keeps changing and its statement is frequently unclear, insincere and confusing. But, notwithstanding these facts, the financial products such as current account, building saving, private pension scheme, instalment selling, consumer or mortgage credit, form a natural part of our lives. This means that each citizen should have a fund of knowledge and skills necessary for his/her orientation on the issue of money and prices including administration of financial assets and liabilities with regard to changing life situations. This fund of knowledge and skills can be designated as financial literacy (MF CR, 2007; Vitt et al., 2000; Atkinson et al., 2006). The importance of taking care of the improvement of people's financial literacy can be illustrated with the next personal story, which is fictional but based on real stories known from T.V. news, newspapers and dialogs of the authors with students and representatives of banks.

Personal story: Peter started work with the starting net salary of 16,000 CZK after graduation from school. He continued to live with his parents and he contributed 2,000 CZK a month to the household. After one year he decided to buy a small flat at a cost of...
Roman Hašek, Vladimíra Petrášková

1.000.000 CZK. His parents gave him 300.000 CZK. To cover the remaining amount of money Peter decided to use a mortgage credit. A bank fixed the amount of the monthly installments at 5.011 CZK with a repayment period of 20 years. Now, he should have calculated his personal budget. If he had done this he would have learnt that the surplus of the budget was only 29 CZK per month (see Tab. 1).

The above mentioned situation is not in any case exceptional. Many people acquire things without considering their actual financial situation. Then they are not able to pay back all the incurred debts. They frequently move without restraint to a warrant of execution or personal bankruptcy. In connection with this phenomenon of irresponsible handled personal money, a low level of financial literacy of the affected debtors is mentioned as the main factor of their behavior.

Without a doubt the financial literacy should be included in the curricula of grammar and secondary schools. Financial tasks along with corresponding concepts have more and more been included in the textbooks. It is evident that the teachers should also be properly prepared for such situations. A teacher with practical insight into the operation of the financial world plays an important role in the process of the successful improvement of financial literacy. His/her deep understanding of related issues enables him/her to develop pupils’ knowledge in an effective, constructivist way. We will show a way of using the computer algebra system Maple for the introduction of students of mathematics teaching into the nature of basic financial products.

2 Financial Education in the Czech Republic

2.1 The Issue of Financial Literacy

The issue of financial literacy is also discussed within the international organization for Economic Co-operation and Development (OECD). In 2003, the OECD initiated an intergovernmental project ‘Financial Education Project’, (OECD, 2003), which focused on the introduction of an integrated system of financial education. Its main objective will be to increase the level of financial literacy. The results of the whole project were summarized in the publication Improving Financial Literacy, (OECD, 2005). This book represents the first major study dealing with financial education at an international level and it is an important contribution to the development of the financial literacy of consumers.

In the Czech Republic, attention to the activities in the field of financial education is paid by the Ministry of Finance of the Czech Republic, which is concerned with consumer protection in the financial market. The Ministry of Finance followed the recommendations stated in the publication "Improving Financial Literacy" (the Czech Republic has been a member of the OECD since 1995) and issued the document Strategie finančního vzdělávání (Strategy of financial education) (MF CR, 2007). The objective of this strategy is to create an integrated system of financial education that will con-

### Table 1: Personal budget

<table>
<thead>
<tr>
<th>Earnings / CZK</th>
<th>Spending / CZK</th>
</tr>
</thead>
<tbody>
<tr>
<td>net salary 16.000</td>
<td>monthly installment 5.011</td>
</tr>
<tr>
<td>spending on living 3.900</td>
<td></td>
</tr>
<tr>
<td>phone, internet 1.000</td>
<td></td>
</tr>
<tr>
<td>entertainment and culture 1.500</td>
<td></td>
</tr>
<tr>
<td>transport to job 260</td>
<td></td>
</tr>
<tr>
<td>food 4.000</td>
<td></td>
</tr>
<tr>
<td>drugstore 300</td>
<td></td>
</tr>
<tr>
<td><strong>Total 16.000</strong></td>
<td><strong>Total 15.971</strong></td>
</tr>
</tbody>
</table>

Peter had never been interested in the functioning of the household and therefore he had the impression that after the deduction of the mandatory expenses of the installment (5.011 CZK), culture (1.500 CZK), phone, the Internet (1.000 CZK) and transport (260 CZK) from his earnings he would have enough spare money (ca. 8.000 CZK). To get this amount of money he could use a consumer credit from a bank, an installment plan arranged with some credit company or a credit card. Peter decided to use an offer of a credit company because he gained the goods at place immediately after drawing up the corresponding contracts. First he obtained a computer at a cost of 25.000 CZK, laying down a deposit of 2.500 CZK with a duty to pay 2.500 CZK a month for the next ten months. The credit company immediately offered him a credit card with an initial credit of 35.000 CZK. Peter used it without hesitation to cover the purchase of a television set (15.000 CZK), a hifi system (10.000 CZK) and a DVD recorder (10.000 CZK). The corresponding monthly installment amounted to 5% of the debt, i.e. 1.750 CZK. Now, according to the table of Peter’s personal budget, we can see that his budget is deficit to the extent of an imbalance of about 4.250 CZK. No matter how he tried to reduce his less important expenses on culture (1.500 CZK) and phone (1.000 CZK) his budget would have still been deficit with an imbalance of 1.750 CZK. It is clear that he would sooner or later face the executors.
tribute to an increase in the level of financial literacy in the Czech Republic. The document Strategie finančního vzdělávání (Strategy of financial education) is followed by the document Systém budování finanční gramotnosti na základních a středních školách (The system of establishment of financial literacy at primary and secondary schools) (MEYS ČR, 2007b), that was approved by the government in December 2007. This is a common document of the Ministry of Finance, Ministry of Education, Youth and Sports and the Ministry of Industry and Commerce of the Czech Republic. This document includes particular standards determining the target status of financial education for primary and secondary education. In 2008, the specified financial literacy standards were fully implemented into the 'General educational programmes' for education at grammar schools as well as at trade and technical schools in the Czech Republic (MEYS ČR, 2007a). The incorporation of the financial literacy standards into the 'General educational programme for primary education' has not been executed yet. Until this, primary schools and lower classes of long-term grammar schools can implement the financial issue into their curriculum only optionally. In compliance with this document (MEYS ČR, 2007b), the faculties of education were invited to include the standards of financial literacy in the content of the relevant university programmes of education. The Faculty of Education of the University of South Bohemia (USB) in České Budějovice reacted to this invitation in such a way that it included the subject 'An Introduction to Finance' in the fields of study concerning the teaching of mathematics.

2.2 Financial Education at Grammar and Secondary schools

Three standards of financial literacy were defined in the document Systém budování finanční gramotnosti na základních a středních školách (The system of establishment of financial literacy at primary and secondary schools) (MEYS ČR, 2007b), with respect to the target groups of affected pupils:

- Financial literacy standard of a first-grade pupil (ages 6 – 10 years).
- Financial literacy standard of a second-grade pupil (ages 11 – 15 years).
- Financial literacy standard of a secondary school pupil (ages 16 – 19 years) – the same as the financial literacy standard of an adult.

All three standards include the following areas: money, household management and financial products. The secondary school pupil standard contains an extra topic of consumer rights. Let us look at the last two standards in detail.

Financial literacy standard of a second-grade pupil (ages 11 – 15 years)

i. Money
Content: the handling of money, generation of a price, inflation.
Gained skills: appropriate use of various tools of payment - both cash and direct debit, ability to generate a price as a sum of costs, profit and VAT (value added tax), to be able to explain an influence of supply and demand on the value of money.

ii. Household management
Content: household budget, types of budgets, their differences.
Gained skills: a pupil is able to draw up a simple household budget, to give main incomes and expenses of a household, to differentiate regular incomes and expenses from the one-off ones, to consider the necessity of particular expenses, to explain principles of balance, deficit and surplus budgets, a pupil knows the consumer rights and is able to seek his/her redress if necessary.

iii. Financial products
Content: banking services, asset and liability accounts, financial market products to invest and gain money, interest running.
Gained skills: a pupil knows in which situation to use debit cards and credit cards, is able to explain their limits; he/she gives the most usual ways of surplus money handling (consumption, saving, investment); gives and compares usual methods of deficit coverage (loans, installment plan, leasing); explains the meaning of paid and received interest; gives the most usual kinds of insurance and suggests when to use them.

Financial literacy standard of a secondary school pupil (ages 16 – 19 years)

i. Money
Content: payment (in both domestic and foreign currency), price generation, inflation.
Gained skills: a student is able to use the most frequent payment tools, to change money according to the actual exchange rate, to generate a price as a sum of costs, profit and VAT (value added tax), to explain the price differences with respect to consumer, place, period etc.; a student recognizes the usual tricks hidden in prices (a price without VAT etc.) and various false advertisements; he/she explains the nature of inflation and its fallout on incomes, deposits, loans and long-range financial planning; gives examples of defense against inflation.
ii. Household management
Content: household budget.
Gained skills: a student differentiates regular incomes and expenses from the one-off ones and draws up a simple household budget with respect to them; he/she can offer a solution of the deficit household budget and knows what to do with the surplus household budget.

iii. Financial products
Content: surplus money, money shortage, insurance.
Gained skills: a student offers a way of handling surplus money (saving, offers which include a state contribution, bills of exchange, immovables etc.); he/she chooses the best product in which to invest the surplus money and gives an explanation; a student finds the kind of credit that suits him/her best; a student considers the way of ensuring the credit and explains a way of keeping from insolvency; he/she explains the ways of interest rate definition and the difference between the interest rate and the annual percentage rate (APR); he/she chooses the best insurable product with respect to his/her needs.

iv. Consumer rights
Content: consumer protection rules; patterns of arrangements.
Gained skills: through an example a student explains the possibilities of claiming consumer rights (purchase of goods and services, including financial products); he/she shows the possible effects of ignorance of an arrangement including its general conditions.

It examines students’ skills to solve the particular example but in real life we face a broad spectrum of offers and we need to select the best one for us. More effective is the method of education through the solving of pupils projects. They give students a chance to be in touch with real-world problems. Here we can see an example of such an educational project that was realized at the gymnasium in České Budějovice in cooperation with its mathematics teacher and a couple of students of mathematics teaching at the Faculty of Education of the USB.

The above standards were implemented into the so called ‘General educational programs for gymnasiums and secondary schools’ (MEYS ČR, 2007a), which play the role of curriculum texts in the Czech Republic, into two educational areas: ‘A man and the world of labour’ and ‘Mathematics and its application’. Either area gives pupils the necessary knowledge that they need to apply during the solving of educational projects. Education through the solving of pupils projects, which enables them to keep in touch with the real-world problems, has appeared as the most effective method of improving the financial literacy of young people.

2.3 The Educational Project as an Effective Method of Financial Education
To have some kind of loan is a normal situation in these days. No matter what your income, it can help you to solve some of the obvious demands: living, a new car, a holiday abroad, furniture for a flat, tuition fees etc. We face a lot of advertisements in the communication media, banks and other financial institutions. To select a loan that suits our demands and financial capacity could be very crucial for our future life.

The role of the teacher is to prepare his/her student with the ability to consider all the pros and cons of offered financial products. Such a goal can be fulfilled only with an educational method which reflects an actual situation of financial products and that enables students to touch on all possible and necessary decisions he/she should make. In the days of dynamic movements on the financial market it is not sufficient to follow textbooks. They provide students with a good theoretical base but, with respect to the static character of typed text, they cannot introduce them to the actual problems of the real world. Let us look at a typical text book problem.

Example: A Nováčkovi couple plan the reconstruction of their flat costing 300,000 CZK but they don’t have such an amount at their disposal. A bank offered them a loan over two years with an annual percentage rate of 8.2 %. They will repay it in monthly installments, the first one due one month after the approval of the loan. The bank compounds the interest monthly. Calculate the annuity and the whole interest rounded off to the nearest hundred crowns.

It examines students’ skills to solve the particular example but in real life we face a broad spectrum of offers and we need to select the best one for us. More effective is the method of education through the solving of pupils projects. They give students a chance to be in touch with real-world problems. Here we can see an example of such an educational project that was realized at the gymnasium in České Budějovice in cooperation with its mathematics teacher and a couple of students of mathematics teaching at the Faculty of Education of the USB.

The educational project assignment
Problem: A young couple with a baby has found out that the purchase of a washing drier would simplify the functionality of their household. To realize the purchase they need an amount of 18,000 CZK. Find out what possibilities they have to gain the necessary capital.
Solution: Students had found two possible ways of solving the problem of the lack of capital. The first one was to ask a bank for a consumer credit. The second possibility was to use installment payments. The real period of repayment corresponding to the amount of borrowed money was the maximum of one year.
In the case of a consumer credit students had learnt that there were two kinds of such credit: cash and cashless. In the case of cash credit you can use the gained money for any purpose, whereas the cashless credit is connected to a particular purpose and the bank
A Way to Improve Financial Literacy of Future Teachers

Roman Hašek, Vladimíra Petrášková

Itself pays out the corresponding invoices, without giving you any cash. At first the students liked the cash consumer credit because of the possibility of freely disposing of the borrowed cash, but they learnt that this possibility is paid with an increase of the interest rate by about 1 – 2%. Considering the fact that the purpose of the loan, the buying of a washing drier, was given, they finally chose the cashless credit. The annual interest rate of the credit was 12%. Then the bank required a charge of 400 CZK for the evaluation of the loan application, paid on approval of the application, and 250 CZK p.a. for administration of the credit account, to be paid at the end of the year. The borrowed money would be paid back in just one year in monthly installments, always at the end of the month, with monthly interest calculated. Then the amount of a monthly installment would be 1.599 CZK. Therefore the couple would pay for the washing drier the amount of 12.1.599 + 400 + 250 = 19.838 CZK. It means that the extra-payment would be 1.838 CZK.

The installment payments, as the second possibility, were mostly charged with interests higher than consumer credits had. Instead of paying cash for the whole cost of goods, a client lays down a specified percentage of this cost, so called 'advance payment'. The rest of the loan is then paid off by regular installments. A repayment timetable, mostly monthly, depends on the actual offer of an installment company. Students appreciated the simplicity and speed of the arrangement of such payment – only two IDs and information about monthly incomes were sufficient to conclude a contract within 20 minutes. Sometimes installment companies provide some promotional offers, mostly in the form of loans without charges. You would not pay any extra money in such a case.

Unfortunately, at the time when our students searched for possibilities to gain money for buying a washing drier, no installment company offered such promotional action but they found that an installment company offered so called credit "1/10". This credit consists of the fact that we will pay 10% of the price of the goods in cash, plus subsequent ten installments amounting to 10% of the price of the goods. Therefore in the case of the washing drier at a cost of 18.000 CZK we would pay an extra amount of 1.800 CZK.

Conclusions: The students concluded that there are no significant differences between a consumer credit and an installment plan. The cost of the washing drier is in both cases approximately the same. They did not see any reason that could change this equality on behalf of one of the possibilities, just as the majority of other people. But such reason existed and the role of the student's teacher was to notice it. The possible selector between the given offers is the time factor. Its meaning is simply described by the words: "The immediate value of money is higher than its future value". It means that it could be better to postpone your installments and use the saved money to earn additional value instead. What does it bring to the solution of our problem? First, in the case of the consumer credit the amount of the installment is lower by approximately 200 CZK than the monthly payment of the installment plan. Second, in the case of installment payment you must immediately lay down CZK 1.800 at the beginning of the contract, contrary to the consumer credit the first installment of which is paid after one month of the contract duration.

With respect to these new facts the students finally decided to use the consumer credit. Justice of this choice is strengthened by the values of the annual percentage of rate (APR) – the time factor indicator. The APR of the consumer credit is to the value 22,19% and the credit "1/10" has the APR to the value 26,27%. To give the APR value for each financial offer is the statutory duty of banks and installment companies.

2.4 Financial Education at the Faculty of Education of the University of South Bohemia

The aim of the faculty effort is to provide teachers, in both pre-service and in-service teacher training courses, with a practical insight into the operation of the financial world. We started from the premise that a well educated teacher is the basic necessity for the effective education of pupils and therefore plays an important role in the process of the successful improvement of financial literacy, i.e. to teach people to handle their money responsibly.

For a number of years, the Faculty of Education of the University of South Bohemia has been teaching a course 'Financial Mathematics', which content is partly identical to the above mentioned course 'An Introduction to Finance'. This subject was in particular designed for students of a specialized bachelor's field of the study of Financial Mathematics preparing its students for a career in the financial world. Although these students gained education in both mathematics and economy, long-term experience (approx. 10 years) had shown that they had difficulties in understanding the functioning of particular financial products (credits, securities etc.). Due to this fact questions arise: "How will the terms in the world of finance be understood by the students of teaching of mathematics who have not completed the courses focused on economy?", "Can a specially designed teaching environment be helpful for a better understanding of respective schoolwork?"

Therefore, the authors of the text decided to realize a research focused on the identification of an effective methodological support of the subject ‘An Introduction to Finance’ before its implementation. A brief introduction of this research together with its results and conclusions is mentioned at the end of this article. Detailed information about this research, accompanied with tables and
graphs can be found in Hašek and Petrášková (2010). Information resulting from the research led the authors to the decision to support the teaching of the subject ‘An Introduction to Finance’ by means of an interactive hypertext tool in form of a web page accessible on the Internet. A description of the implementation and functioning of this web page can be found further along in this text. It provides an explanation of the theory and handling of tasks in finance and banking with the utilisation of the computer algebra system Maple. The hypertext solution of this tool ensures the possibility of continuous updating according to the current condition of the market with financial products.

The content of the subject ‘An Introduction to Finance’ is in compliance with financial standards designed for primary and secondary education in the Czech Republic. The main topics that are taught in the subject are the payment of interest, current account and its management, handling of surplus money (saving, building saving, saving account, valuables – bills of exchange, shares, bonds), solving of a money shortage (bank loans – consumer credit, mortgage loan, credit cards, leasing). The subject ‘An Introduction to Finance’ belongs to the sphere of financial mathematics and uses its methods, notions and theoretical base. Financial mathematics deals simply with the utilization of mathematics in the financial area. Some parts of this discipline manage with secondary school mathematics (for instance the notion of function – linear, indirect proportion, exponential and logarithmic function and the notions of consequences and series), whilst other parts demand a knowledge of some principles of college mathematics (for example Taylor expansion, correlation coefficient, least squares method and so on).

3 The Web Interactive Tool

For the purposes above the authors of the paper decided to create an integrated set of teaching materials focused on the creation of students’ capabilities to settle model problems in terms of the handling of finance. An essential teaching material is an interactive tool in form of a web page (see Fig. 1).

![Fig. 1: “An Introduction to Finance” web page](image)

It provides a user with an explanation of the theory, links to external sources and handling of selected tasks in finance and banking with the utilisation of Maple. Next we will see that this software provides a user with an invaluable combination of symbolical, numerical and graphical possibilities for the modelling of financial relations. In addition, versions 11 and higher provide the option to interconnect with the program Microsoft Excel. The problems are thoroughly solved in a way that explains all aspects of a problem and uncovers possible risks which are mostly kept back from the eyes of an uninformed user. An open format of the web tool ensures the possibility of continuous updating according to the current condition of the market with financial products.

The design of the page is shown in Fig. 1. On the left side of the window we can see the content of the page. Its items comply with the basic topics of the curriculum of the course ‘An Introduction to Finance’. After clicking onto some of them, e.g. ‘Spotřebitelské úvěry’ (‘Consumer credits’), a special page devoted to the topic with a detailed teaching text and selected real-life tasks is opened. For each task, the user can select from several Maple applications (smart documents so-called Tutorials, maplets and maple code, see Fig. 2) related to the task, the number of which depends on the topic of a task.

In the first place we take advantage of the features of interactive smart documents. They enable us to combine text, symbolic and numerical computation and graphs in one worksheet all focused on

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1 URL: http://www.pf.jcu.cz/stru/katedry/m/uf/. Information about this tool was also published in SBML (2009).
solving, explaining and modeling some particular phenomenon of a financial issue. In addition to smart documents we use assets of such other Maple features as maplets and classical worksheets. And, moreover, each topic is followed with links to relevant external links, mostly equipped with financial calculators.

3.1 Smart Document

Handling of the tasks on the web page (http://www.pf.jcu.cz/stru/katedry/m/uf/) is executed in program Maple 11. The environment of the so-called smart document was selected as the main tool for the presentation of a pattern solution and modelling of the functioning of the relations hidden behind the computations of the parameters of respective financial products. On the page, these interactive documents are titled Tutorial (Fig. 2).

Fig. 2: Consumer credit - assignment of Task 1

They are available both in MW format that can be run in Maple version 11 and higher and in PDF format that can be used as an adequate teaching text detailing the respective topic. Particular tasks are also completed with the solution code in MWS format generating the environment classic worksheet Maple 11 and, if it is beneficial, selected tasks are followed by specially programmed applications called maplets. The smart document (see the screenshots in Fig. 5 and 6), also called an interactive document, which represents a unique environment enabling the user to combine the text, formulae for symbolic and numerical computations entered in the Maple code, graphs and tools for the control of input parameters such as e.g. input and output fields, scroll bars or radio buttons. At first sight, a smart document seems to be a solution to a specific task with a respective specification. Mathematical formulae used in the document are a part of the text and do not differ in colour. However, its advantage lies in the possibility of a change in input values and subsequent conversion through the Maple function 'Execute the entire worksheet'. Thus the user gets a powerful tool for the experimentation and examination of the dependence of a solution upon input values. Smart documents, also called interactive documents, are created in the environment Document mode of the program Maple 11. For this activity we can utilise both the detailed help of the program (after setting command "?Document mode") and materials available on the Internet (Maplesoft, 2010). For illustration we state the solution of Task 1 from the chapter 'Spotřebitelské úvěry' ('Consumer credits') in Fig. 5 and 6. Although the task is specified with particular values, a respective smart document (Tutorial) enables the user to change these values arbitrarily and subsequently convert the task in compliance with the specification. Thus by means of this single document the user can change the parameters of consumer credit for arbitrary values of input data with a direct comparison of various alternatives of the annual percentage rate of a bank. The next Fig. 3 displays a detailed cut from the smart document devoted to the solution of a task from the topic 'Mortgage credit' ('Hypotéční úvěr').

Fig. 3: Mortgage credit – embedded classical code

It illustrates how the environment of a smart document (Document mode) can be combined with a conventional working environment Maple (Worksheet mode) in which we can execute e.g. supplementary symbolic or numerical computations. In this case, Worksheet mode was used for the derivation of the relation between the amount of debt and the amount of each installment. Thus a student, by means of Maple, derives the formulae that are or should be used in computations of the parameters of respective financial products. In the same way we also utilise the numerical tools of the program as it is illustrated in the following sample of Tutorial (Fig. 5, 6) to Task 1 from the topic Consumer credit.
3.2 Maplets

Program Maple enables a user to create special interactive applications so-called maplets. This application runs in a separate window that utilizes computation possibilities of the core of program Maple. The window of a maplet can be equipped with various controls, input and output text and graphic fields, help, commentaries etc. Thus a user will get a tool by means of which he/she can utilize all functions of program Maple without controlling this program. It is not necessary to have started Maple during utilization of a maplet, however unfortunately it must be installed on a respective computer. We can see a maplet devoted to the Mortgage credit in Fig. 4.

![Fig. 4: Mortgage credit – maplet](image)

It provides a user with the appropriate size of the annuity (per year) together with the corresponding curtail schedule. At first glance such a maplet could appear as a standard financial calculator but there is a difference between these two applications. The evident advantage of maplets over web based financial calculators can be illustrated for instance with the above curtail schedule. The calculators provide only the resulting figures (Měšec.cz, 2010) whereas the properly programmed maplets can reveal some hidden but important information. An interested reader acquires detailed information on the creation of maplets by setting the Maple command “?roadmap”.

3.3 Maple Code

The selection "Maple code" (Fig. 2) offers the task solution code entered in the conventional line mode of program Maple. It allows the user, who is familiar with the syntax of the program Maple a prompt insight into the task solution and it enables him/her to detect hidden links between particular task parameters by means of experimentation with a specification.

3.4 Application of the Interactive Tool

Next, we will illustrate the impact of the interactive tool on the deeper understanding of the functionality of financial products via the specific example. Among the most frequently used financial products we can include the mortgage loan and consumer credit. Consider the next example from the interactive tool - Task 1 from the chapter Consumer credit.

On the following screenshots (Fig. 5 and 6) we can see that the interactive aid enables us to change all given parameters of the consumer credit (the amount of loan, annual interest rate, number of installments per year, duration of the service in years and additional charges) and in accordance with these changes, it computes the appropriate size of the annual percentage rate related to the credit.

A student can observe various relationships. For example the change of the annual percentage rate corresponding to the change of additional charges, number of installments or the amount of the installment. Further, a student can easily learn about the functionality of the relationship between the price of a consumer credit and the values of the task parameters.

Thanks to detailed analysis of various kinds of consumer credit he/she can learn more about the role of the time factor that is reflected by the annual percentage rate. This analysis is done in the smart document as a solution of an additional example. Thus we can see that smart documents enable us to simply compare the advantages and disadvantages of various solutions to a particular case of a money shortage. Reasonable parameters of a loan are charges. The charges change according to the actual situation within the financial market.
The interactive aid enables a student to change this parameter and to explore the impact of such change on the price of the credit (Fig. 6). The smart document also illustrates the connection between financial mathematics and pure mathematics. A student must have knowledge of functions, sequences (particularly the geometric se-
quences) and series. From the above it is clear that the interactive tool enables a student, based on his/her own experiments, to explore various regularities and relationships in the area of financial products. In this way it helps to improve his/her financial capability.

4 Pedagogical Research on the Impact of the Presented Method

The presented web interactive tool was tested with students of the Faculty of Education of the University of South Bohemia. The aim of this test was to discover a possible impact of the utilization of the tool on the improvement of the financial literacy, or perhaps the financial ability (Vitt et al., 2000) of the students.

The results of this comparative test, which are minutely presented in Hašek & Petrášková (2010), revealed almost no impact on the improvement of the financial literacy, but they showed the considerable effect of the tool usage on the increase of the students’ financial ability. Students submitted to the test were divided into two groups – the first group regularly used the presented web tool while studying the subject ‘An Introduction to Finance’, the second group studied without using the tool. Students from the first group, using the tool, were twice as successful in the solving of financial problems compared to the students from the second group, using only standard learning materials. This result is very favorable and corresponds to the obvious fact that pure theoretical knowledge is useless without the ability to use it in practice.

The results of the test research showed still other interesting facts: For example, almost nobody had problems with the tasks dealing with the bills of exchange business. However, hardly any of them will see a bill of exchange in his/her life. On the contrary, the most difficult tasks appeared to be those dealing with consumer credit, mortgage credit and inflation, situations that without doubt everybody will face in life. Students often did not even try to solve these actual real-life problems. The test also pointed out the fact that students often cannot resist a simple marketing strategy of banks’ commercials based on the fact that a monthly interest rate looks visually better than an annual interest rate. As we mentioned in the text, the results of the research helped to finalize the environment of the web interactive tool to support the subject ‘An Introduction into Finance’.

5 Conclusion

The authors of the article presented a new teaching method that can be used for the teaching of the subjects related to the field of financial mathematics. This method utilizes an interactive webpage to support such teaching. An advantage of this educational aid is the possibility of continuous updating according to the current condition of the market with financial products. It reacts both to the creation of new products and to changes in the conditions (e.g., changes in interest rates, the creation of new products) in banking and financial spheres. Its efficiency was verified by means of the research executed with the students of the Faculty of Education of the University of South Bohemia in České Budějovice, whose results and conclusions are the main message of the article of Hašek and Petrášková (2010).

Literature


Motzer Renate

Pictures with Rational Functions

Abstract

To work with rational functions in an active-discovering way, students of the 11th grade should create pictures with Geogebra on their own. They can decide by themselves, how sophisticated their picture should be, and they can combine calculating and planning with playful trying on the computer. Individualization is therefore possible concerning mathematical as well as aesthetic aspects. Computer-aided exploration helps the students to extend their knowledge from their personal point of view.

1 What Should Students Learn in these Lessons and Why?

The purpose of this teaching unit, which was done in the Year of Mathematics in 2008, was to get the students independently into contact with whole-rational functions and their graphs. Students of a grade 11 of the design master of a Bavarian high school should get acquainted with whole-rational functions as tools to generate pictures with the help of Geogebra. Due to the fact that these were students of a design class, I hoped that they would enjoy experimenting with functional graphs and with forming a picture. Besides, they should also be allowed to experience the aesthetic aspect of mathematics by themselves. From the design lessons the students were already accustomed with putting together something on their own and to deliver it within a given deadline, even if they had no experience with this from mathematics lessons so far. For this unit they had a time limit of three school weeks (with four teaching hours per week): First they had to work out how one finds the equations of functions. Finding and arguing with equations of functions were also part of the next test. Geogebra was unfamiliar to the class until then. I chose it because the training time was fairly short compared to what was necessary for the task. Furthermore, this software is accessible to all students also at home free of charge. Due to the fact that the students can check with a click whether the given functions have the desired graph and can mend them by shifting the graphs and are able to see some mistakes directly at the PC, they get a broader access to the subject “function” (above all, to the inquiry or control of the qualities).
The aim of this was to get the students acquainted with parametric functions as modeling tools. Here they model self-elected motives. What they have learnt during the exercise they can also apply later on, whenever data is given (e.g., physical measuring data or other statistical data) for which “suitable” equations of functions are supposed to be found.

2 Which Previous Knowledge Did the Students Have?

A teaching unit, in which the students work out the investigation of whole-rational functions, especially of 3rd and 4th degree with the help of an expert’s puzzle, was given ahead. Besides, the equations of functions are given and the characteristics of the graphs are searched. The youngsters learnt calculating zeros with substitution or with the aid of the division of polynomials, in addition there is the investigation of the symmetric behavior and the behavior for $x \to \infty$ and $x \to -\infty$.1

The teaching form “Expert puzzle”:

- The teaching material is split up (e.g., in four subunits).
- The students work on one of these subjects in groups. Afterwards they are experts on this subject.
- The groups are mixed again (in this case in groups of 4 persons), so that each student always teaches three others what he has learnt and is now knowing as an expert.

3 Which Knowledge, which Skills Should be Acquired in this Unit “Putting up Equations of Functions”?

Now it is a matter of turning around the perception: Properties of the graphs are given and the accompanying equation of functions should be determined. Above all, however, the graph from which one maybe has a vague image in mind can be exactly drawn. According to the circumstances it may happen that the calculated graph looks different to the one intended, at least to a certain extent. Consequently, the interplay of equations of functions and the graphs can be explored. How important this interplay is for the development of functional thinking (cf. Vollrath, 1989), is stressed over and over again (e.g. Leuders & Prediger, 2005; Vogel, 2006).

Besides, the “principle of the surgical exercise is realized also, i.e. practice tasks of the same kind should be generated for the purposes of the surgical principle as a systematic variation of the data to recognize thereby legitimacies and to achieve therefore knowledge profie” (Winter, 1984).

Important questions concerning the functional relationships which the students are supposed to deal with in this context are:

- Which meaning do the coefficients have which appear in the equations of functions?
- How does it affect the graph if one is changed?
- How does it affect it, if the power of $x$ becomes raised or degraded?

In order to learn how one finds equations of functions, the students received information from me2. There, examples are given and the calculations are founded. Furthermore, the students were able to find some practice-tasks which had to be solved by the students. Students could work by themselves or in small groups. The material offers several ways of solutions to the same purpose and also covers special cases which do not have to be worked through by all students. It is shown how systems with up to four equations for four variables can be solved.

Most students started with the practice tasks. They started with a (clearly defined) aim and then looked in the text what one should do with the respective task. Some students worked through the text from the beginning to the end.

If no Geogebra knowledge is available, a short introduction should be given at the beginning. The class did not know Geogebra until then. To get familiar with the software, I first gave some handling tips in the classroom (on the laptop with beamer). Then we went to the computer-room for one and a half hour where the students were able to work with the program in small groups.

The important tools of Geogebra which play a role in the production of the pictures:

- Write the equation of a function in the field “Input”
- Restrict a functional graphs on a given interval $[a, b]$ by “function [equation or name of the function, a, b]”
- Show and hide graphs and/or her names
- Show and hide the co-ordinate system
- Show the construction protocol

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1 Material can be found on my website. URL: http://www.math.uni-augsburg.de/prof/dida/team/motzer/downloads/expertenpuzzle/

4 What do the students have to do concretely?

After this preliminary work the students could come along to the major tasks which were finished in the essentials at home:

Pictures from whole-rational functions

Provide with Geogebra (www.geogebra.org) a picture which consists at least of five whole-rational functions. Co-operate with others and combine your individual parts to common bigger picture. Next to the picture you should also present the “history of your work”:

- pre-considerations,
- an installation of the used functions with the accompanying intervals,
- understandable calculations of the equations of the functions,
- what you have learnt in the course about the treatment of the task for yourselves.

You should also be able to explain your calculations orally and to explain other properties of the functions used.

Example:

![Graph of eight functions](image)

This picture is composed of the graph of eight functions:

5 What does the Teacher Introduce?

In order to give an idea of the task, it is advisable to explain the task to the students with the help of an example. I have chosen a car as an example. On the board I explained the students how I have found the equations of the functions for the car. In this way they could get an insight into how one can work (but not has to work). By the production of the “wheels” I told them that sometimes one has to reject ones ideas or how one can change them. At first I wanted the students to use only whole-rational functions and not to use circles or semicircles. Hence, one should not draw the wheel as a semicircle. My next idea was to take a parabola. For the left wheel it has to be on the floor in \((-2; -1)\). So we start with \(n(x) = a(x + 2)^2 - 1\). Since the function must have a zero at \(-1\), we get \(a = 1\). If one draws the parabola in the interval \([-3; -1]\), one sees that it corresponds not enough to a wheel. The next idea could be a function of 4th degree. If one draws \(y = (x + 2)^4 - 1\), you see the wheel is too flat: So one should try to take the 3rd degree instead. \(y = (x + 2)^3 - 1\) fits for \(x\) from \([-2; -1]\). The analogous calculations can be done for the second wheel. One can see at this example how changing the power affects the “flatness” of the graph of \(y = x^n\) in the origin, and how one can shift the graph in the coordinate system. For the function \(k\) I planned, that the graph should start with the turning point \((-4,5; 0)\). Hence, one can try the form: \(k(x) = a(x + 4,5)^3 + b(x + 4,5)\). The parameters \(a\) and \(b\) are chosen in a way that the function has an inflection point at \(x = -2\) and has an adequate height. Because the class hasn’t learnt any differential calculus, they...
cannot determine $a$ and $b$ by two equations, but only by trying. With $x = -2$ a function of $3^{rd}$ degree is connected: $y = a(x + 2)^3 + b$, so that with $x = 0$ the height 2,5 is reached. For this, I put up with the students both equations for $a$ and $b$ and solved them. Because the roof of the car should become very flat, a function of $4^{th}$ degree $y = a(x - 1,5)^4 + b$ was used by me. Here an equation can be given for $a$ and $b$, so that the composition is continuous at $x = 0$. Then $a$ and $b$ were varied by me, until the picture corresponded to my images. The smaller one chooses $a$, the flatter the roof becomes. I wanted to show to the students that one can combine calculation and systematic trying, and hoped to be a model to them in this way.

6 What did the Students do?

Of course, I cannot check how far they got help from a third person. I found the results were, in any case, overwhelming. This concerns in particular the creative part, but some students handled also the systematically calculating part very well. Even one of the weakest female students, who also had no special knowledge of using the PC, still had a vague image of the relationship between a functional equation and the corresponding graph of the function: She has made a picture on her own and could learn quite a lot of about functions for herself just by "trying".

Translate

My picture, a "Heartagramm" has results from five straight lines and two semicircles. I have created the picture mainly by trying. First, I wanted to produce the semicircles with a function, after a long time of trying I also found one which went through all three necessary points but which was too flat. So I decided to substitute them with semicircles.

The function was $f(x) = -0,04x^4 + 5$ and I found that I could meet exactly the middle point with +5. Furthermore, I discovered that with $0,001x^4 + 5$ the parabola became broader, even in general with $0,01x^4$ or $0,1x^4$. At the beginning it was difficult to find the right gradient for the straight lines. I have tried first with $f(x) = 1x + 4$, until I found $f(x) = 1,25x + 5$, what corresponded to the picture then.

The protocols and calculations provided show that many students have worked by systematical trying. Nevertheless, some of them wrote down observations which show that they have also extended their algebraic understanding of functions, at least a little bit (see the example above). Several students limited themselves only to straight line and parabolas (for example the umbrella). Nevertheless, some students consciously put a function of $3^{rd}$ or $4^{th}$ degree in
7 The Example “Sail Boat”

The graph on the screen not always looked like the one intended. For example, one female student liked to generate a wave and tried to find the equation using the given zeros (the sine function is not a topic in the design class). However, the generated wave did not look like the desired picture, one part of the “wave” went too widely down, another part was too flat. By playing with the parameter $a$ in $f(x) = a(x - x_1)^2(x - x_2)^2(x - x_3)^2(x - x_4)^2(x - x_5)^2$ the wave looked somehow in a way that the student was satisfied. By the movement of the ship the wave may not go equally deep everywhere (see “sailing-boat”). In addition, in the calculation enclosed one sees that the student made a mistake in dissolving the equation (she calculated the reciprocal value of $a$). While trying the counted value she noticed that the result could not be right. When she tested the reciprocal value, the picture fitted.

8 The Use of Other Types of Functions

Another student liked to insert a wave and he remembered the sine function which he was taught in secondary school. I helped him somewhat to refresh his knowledge to enable him generating the desired wave with the help of the sine function. When the students experimented with Geogebra themselves and discovered fast how one can draw circles there, I allowed that they may also insert circles or parts of a circle. Indeed, now the introduced construction of the “wheels” had become superfluous (wheels have been inserted by the students, however, not in their pictures), but what was shown with the help of the wheels about the fitting of equations of functions could be used in other scenarios by many others. Instead of simply drawing straight lines with the DGS the accompanying equations always should be calculated. In the end, however, only one part of the class did it this way. The students also discovered that they could shift functional graphs in the coordinate system and that the equations were adapted automatically. What they have learnt about the parabola could be transferred. They were able to experience it in some larger generalization.
Sketch a picture with Geogebra from rational functions (there should at least 5 equations of functions be involved). In order to find the equations you can solve the equation system with the help of computers if necessary, e.g.,

- http://mitglied.lycos.de/nhable/scripts/gleisyst.htm
- http://mathestuff.de/mathematik/lgs_online).

Examine in each case whether the compound function is continuous and differentiable.

Calculate the area included. (Hint: With Geogebra one can calculate it with the tool “integral” surfaces).

Note down what you have learnt during the production of the picture and the calculation of the surface.

11 Pictures Designed in the 11th Grade

Literature

Abstract

On Proving and Discovering Theorems by Computer in Teacher Training

Computer in Teacher Training

Introduction

In the last thirty years new technologies in computer algebra were developed. These technologies influence teaching processes as well. Both, dynamic geometry systems (DGS) like Cabri, Geonext, GeoGebra, Sketchpad, etc., and computer algebra systems (CAS) like Derive, Mathematica, Maple, etc., appeared at schools of all levels in teaching mathematics all over the world. In this article we will aim at the use of computers in proving and discovering mathematical theorems in initial teacher training.

Nowadays DGS and CAS are used at schools to solve various tasks in mathematics. DGS is mostly used by teachers and students to demonstrate a geometric situation of a problem and then influence the process interactively. For instance, when demonstrating the orthocenter of a triangle, students realize by moving a vertex of a triangle that all three heights intersect at one point in infinitely many cases.

This demonstration and interactive method is suitable for students of all school categories, but as a "proving" method we recommend it especially for students at basic schools (age up to 15 years). It is known that DGS technology is based on numerical computations of every individual geometric situation and does not enable symbolic computation. Therefore, verification is not a proof!

We think that now it is time to introduce new technologies which make possible to prove and discover theorems by computer. First we give a basic theory of automated theorem proving.

Automated Theorem Proving

In this part we outline basic concepts of the theory of automated theorem proving (Cox, Little & O'Shea, 1997; Pech, 2007). This theory is based on results in commutative algebra in the last thirty years. In this period powerful computers were constructed and very fast, these made essential changes very effective. At the same time efficient mathematical software was developed. In this paper we outline basic concepts of the theory of automated theorem proving in elementary geometry. We are able to prove and discover geometric theorems. The first step of proving such a statement is its translation into algebra. In this stage we describe assumptions (hypotheses) of the theorem and the conclusion. Then we transform them to a set-theoretical language, where $H$ is the set-of-hypotheses and $C$ the conclusion. The first step of proving such a statement is its translation into algebra. In this stage we describe assumptions (hypotheses) of the theorem and the conclusion. Then we transform them to a set-theoretical language, where $H$ is the set-of-hypotheses and $C$ the conclusion.
Proving and Discovering Theorems by Computer in Teacher Training

Let us demonstrate the above theory by two examples. In the first example we show the method of proving of a statement. The second example deals with discovering a new statement. We will proceed similarly to students in their seminar works. Besides computer proofs we also give classical proofs, since we think that both methods of proving – classical and computer – should be combined. We also add the first stage of “proving” – verification – which could be used especially for pupils at elementary schools. Whereas verification uses the tools of DGS, computer proofs require CAS.

Example 1

Prove that altitudes of a triangle are concurrent.

Verification: Let the point $O$ be the intersection of the heights $h_1$ and $h_2$. We want to show that the point $O$ lies also on the height $h_3$, see Fig. 1.
In Cabri we will use the icon Member? We ask whether the point $O$ lies on $h_b$ with the answer “This point lies on the object.” Moving the vertices $A, B, C$ arbitrarily, the answer is still the same. We verified the validity in many situations. But despite of it this verification is not a proof!!

**Classical proof:** Through the vertices $A, B, C$ of a triangle $ABC$ lead parallels with opposite sides $BC$, $AC$, $AB$. We get a new triangle $A'B'C'$ whose side bisectors are the heights $h_a, h_b, h_c$ of the original triangle $ABC$, Fig. 2. Now it suffices to show that the side bisectors $h_a, h_b, h_c$ of $A'B'C'$ are concurrent, which is much easier. We can do it in a similar way as in verification and computer proof. Suppose that $O \in h_a \cap h_b$. We are to show that $O \in h_a$. If $O \in h_b \cap h_c$, then $O \in h_b \land O \in h_c$ which is equivalent to $|OC|=|OB|$, which is equivalent to the conclusion $O \in h_a$.

**Computer proof:** First, we will choose an appropriate coordinate system so that the relations, by means of which a geometric situation will be described, are as simple as possible, see Fig. 3.

Denote by $A=[0,0]$, $B=[a,0]$, $C=[b,c]$ the vertices of a triangle $ABC$. Now we express equations of the altitudes $h_a, h_b, h_c$ of $ABC$. 

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**Fig. 1: Verification**

**Fig. 2: Classical proof**

**Fig. 3: Computer proof**
\[ h_a : (b-a)x + cy = 0, \ h_b : bx + cy - ab = 0, \ h_c : x - b = 0. \]

Suppose that the altitudes \( h_b \) and \( h_c \) are concurrent at a point \( O = [p,q] \), then it holds:
\[
O \in h_b \Leftrightarrow h_1 : bp + cq - ab = 0, \\
O \in h_c \Leftrightarrow h_2 : p - b = 0.
\]

We want to show that the conclusion \( z \): “the altitude \( h_a \) contains the point \( O \)” is consequently fulfilled, i.e.,
\[
O \in h_a \Leftrightarrow z : (b-a)p + cq = 0.
\]

Hence we are to prove the following statement:

\[
\forall p,q \ [(bp + cq - ab = 0 \land p - b = 0) \Rightarrow ((b-a)p + cq = 0)]. \tag{5}
\]

In this very simple case we are able to show that the statement (5) is true even by hand. Indeed, realize the equality
\[
(b-a)p + cq = 1 \cdot (bp + cq - ab) - a \cdot (p-b). \tag{6}
\]

We expressed the conclusion polynomial \((b-a)p + cq\) as a linear combination of hypotheses polynomials \(bp + cq - ab\) and \(p-b\).

From the validity of the equations \(bp + cq - ab = 0\), \(p - b = 0\) and from (6) the validity of the equation \((b-a)p + cq = 0\) follows. We showed that the polynomial \((b-a)p + cq\) belongs to the ideal \(I = (bp + cq - ab, p - b)\).

In order to do it by computer in the program CoCoA\(^1\) we enter
\[
\text{Use } \mathbb{Q}[abcpq]; \quad I := \text{Ideal}(bp + cq - ab, p - b); \quad \text{NF}(I); \\
\text{and get the answer } 0 \text{ which means that the normal form of the polynomial } (b-a)p + cq \text{ w. r. t. the ideal } I \text{ equals } 0 \text{ and the statement (5) is true. A computer proof is finished.}
\]

\(^1\)Software CoCoA is freely distributed. See URL: http://cocoa.dima.unige.it.
and CoCoA returns 0. The statement (7) is true.

Remark: How is it possible that by changing heights we first achieved a direct proof whereas in the other case we had to add non-degeneracy conditions? The reason is that in the latter case the expression of a conclusion polynomial by hypotheses polynomials is as follows
\[ p - b = (-1/a) \cdot ((b - a)p + cq) + (1/a) \cdot (bp + cq - ab). \]
Variable \( a \) is in a denominator, that is why we ruled out the case \( a = 0 \).

In the next example we will demonstrate the method of discovering new statements. By automated discovery of theorems (Recio & Vélez, 1998), we mean the process of dealing automatically with arbitrary geometric statements (i.e. statements that could be, in general, false) and aiming to find complementary hypotheses such that the statements are true.

Example 2

Find the locus of points \( P \) such that the feet of perpendiculars \( K, L, M \) which are dropped from \( P \) to the sides \( BC, CA, AB \) of a triangle \( ABC \) are collinear.

Computer proof: Denote by \( A = [a,0] \), \( B = [b,c] \), \( C = [0,0] \) the vertices of a triangle \( ABC \) and let \( K = [k_1,k_2] \), \( L = [l_1,l_2] \), \( M = [m_1,m_2] \), \( P = [p,q] \), see Fig. 4.

We describe the geometric situation as follows:

\[ PM \perp AB \Leftrightarrow h_1 : (p - m_1)(b-a) + (q - m_2)c = 0, \]
\[ M \in AB \Leftrightarrow h_2 : cm_1 + (a-b)m_2 - ac = 0, \]
\[ PK \perp BC \Leftrightarrow h_3 : (p - k_1)b + (q - k_2)c = 0, \]
\[ K \in BC \Leftrightarrow h_4 : bk_2 - ck_1 = 0, \]
\[ PL \perp AC \Leftrightarrow h_5 : (p - l_1)a = 0, \]
\[ L \in AC \Leftrightarrow h_6 : al_2 = 0. \]

Conclusion \( z \) has a form: \( K, L, M \) are collinear
\[ \Leftrightarrow z : k_1l_2 + l_1m_2 + m_1k_2 - m_1l_2 - m_2k_1 - l_1k_2 = 0. \]
First, we try to show that \( K, L, M \) are collinear for an arbitrary point \( P \). We enter

\[ \text{Use } R := \mathbb{Q}[abcpqk_1..2l_1..2m_1..2t]; \]
\[ I := \text{Ideal}( (p-m_1)(b-a) + (q-m_2)c, cm_1 + (a-b)m_2 - ac, (p-k_1)b + (q-k_2)c, bk_2 - ck_1, (p-l_1)a, al_2, k_1l_2 + l_1m_2 + m_1k_2 - m_1l_2 - m_2k_1 - l_1k_2 ); \]
\[ \text{NF}(I); \]

The answer 1 means that the statement is not true. Searching for non-degeneracy conditions does not give any result and we omit this part. Now we will search for additional conditions which have to be added to the assumptions so that the statement becomes true. To do it, we add conclusion polynomial \( z \) to the ideal \((h_1, h_2, ..., h_6)\) and eliminate dependent variables \( k_1, l_1, ..., m_1, m_2 \) in this new ideal \( J \).

\[ \text{Use } R := \mathbb{Q}[abcpqk_1..2l_1..2m_1..2t]; \]
\[ J := \text{Ideal}( (p-m_1)(b-a) + (q-m_2)c, cm_1 + (a-b)m_2 - ac, (p-k_1)b + (q-k_2)c, bk_2 - ck_1, (p-l_1)a, al_2, k_1l_2 + l_1m_2 + m_1k_2 - m_1l_2 - m_2k_1 - l_1k_2 ); \]
\[ \text{Elim}(k_1..m_2, J); \]
We get the condition
\[ h_7 : a c^2 (-a c p + c p^2 + a b q - b^2 q - c^2 q + c q^2) = 0. \]

What is the geometric meaning of this equation? If \( a = 0 \) or \( c = 0 \) then the vertices \( A, C \) coincide or \( A, B, C \) are collinear respectively. These cases can be ruled out. The remaining condition
\[ -a c p + c p^2 + a b q - b^2 q - c^2 q + c q^2 = 0 \]
means that the point \( P \) lies on the circumcircle of \( ABC \) which is easy to prove. Now consider the ideal \( K = I \cup \{ h_7 \} \) and compute the normal form of 1 w. r. t. the ideal \( K \). We enter

Use \( R := \mathbb{Q}[a b c p q k][1..2]m[1..2]t] \);
\( \text{NF}(1,K); \)

The answer is 1, therefore we have to search for non-degeneracy conditions. Eliminating all variables except \( a, b, c \) in the ideal \( K \)

Use \( R := \mathbb{Q}[a b c p q k][1..2]l[1..2]m[1..2]t] \);
\( \text{NF}(1,L); \)

we get the condition
\[ (b^2 + c^2)((a-b)^2 + c^2) = 0. \]  

The geometric meaning of (8) is as follows. As the coordinates of points \( a, b, c \) are real numbers, then \( b^2 + c^2 = 0 \) is fulfilled only for \( b = 0 \land c = 0 \), hence \( B = C \). Similarly \( (a-b)^2 + c^2 = 0 \) holds if

**Theorem [Simson-Wallace]:** Let \( P \) be an arbitrary point of the circumcircle of a triangle \( ABC \). Then the feet of perpendiculars \( K, L, M \) which are dropped from \( P \) to the sides \( BC, CA, AB \) of a triangle \( ABC \) are collinear.
4 Students works

At the end of the course students are obliged to write a seminar work which consists of the following steps:

- Introduction of a problem
- Description in DGS
- Verification in DGS
- Classical proof
- Computer (automated) proof
- Conclusion

Problems that students solve in their seminar works are discussed with a teacher in advance. As a database we mainly use the source http://www.cut-the-knot.org/geometry.shtml, where about 1000 problems are given. Students have to use both DGS and CAS to solve the problem. They also choose a language, either Czech or English. All parts of the prescribed steps of seminar works given above are usually solved by students satisfactorily with one exception – computer proofs. It turns out that computer proofs form the most difficult part of seminar works. There are several reasons of this state. Most common mistakes that students make in computer proofs are as follows (Pech, 2005):

- Inexact translation of a geometric problem into algebra
- Inappropriate choice of a coordinate system
- Searching for non-degeneracy conditions
- Translation from algebra to geometry
- Insufficiently deep knowledge both from geometry and algebra.

The major problem that students encounter when adding non-degeneracy conditions is a translation of an algebraic content given by equations and inequations into geometry. This problem has not been solved satisfactorily yet. The remedy is a deeper knowledge both from geometry and algebra.

5 Conclusion

In the contribution two methods – proving and discovering – using new technologies and new mathematical software were introduced. Proving and discovering theorems by computer is a very powerful tool which should be taught at schools or at least in teacher training. DGS is a very suitable software for verification statements and stating conjectures. At elementary and basic school level, verification by DGS could replace a rigorous mathematical proving. At a higher education, at secondary schools and universities, we strongly recommend proving and discovering by computer. Computer proofs should be combined with classical methods so that one could compare both advantages and disadvantages of both of them.

Literature


An Experience with Wireless Technology and Outstanding Students of Mathematics

Abstract

This study was carried out with outstanding students of mathematics and with use of TIC. The objective was to find a technology-based approach for working with outstanding students of mathematics or, expressed differently, to use technology as a teaching tool. Latest-generation calculators were used and discovery tasks designed in order to achieve this objective. The effect of using these means in the learning process of a student is considered from the pedagogical “I-You-We” (Gallin, Ruf 1998) perspective. During the learning process, the knowledge acquired by the students was subjected to personalised assessments.

1 Introduction

Most young persons of today will use several presently existing – as well as some yet to be introduced – new technologies (OECD, 2006) when they are adults. Due to their current educational potential, these technologies can now be harnessed to create powerful tools that assist the process of learning mathematics. Through the use of these technologies in the classroom, their educational benefit can be maximized. If these tools are used appropriately, the students are able to concentrate on solving the problem and acquainting themselves with the underlying mathematical concepts without necessarily having to execute an algorithmic process (Ursini, 2006). On the other hand, the teacher’s role is different to that in a traditional class: he/she must strive to create a working environment in which reflection is stimulated and students become active participants. Some tools, for instance the TI Navigator, will trigger an immediate feedback in the students about their experience in the classroom, allowing them to discover their own errors and correct them immediately, whether by themselves or with the help of these media.

When they are used in classes with outstanding and fully motivated students of mathematics, these tools make the classroom an ideal place for the development of ideas (Callejo, 1994). These students are generally interested in asking questions and formulating these problems. They neither expect nor want to be given the answers – they use the technology as a resource or supporting means as they try to find the answer by themselves. As a result, the use of these technologies in working with these students creates a special investigative situation where motivational difficulties are minimised (Leu, 1993).

In this context, it is necessary to study some aspects that are specific to these outstanding students in this particular technological environment, namely the development of technological skills and the characteristics of the outstanding students. For this purpose, the most relevant technological skills were chosen, and the development of these skills was assessed individually. Furthermore, special tasks were designed that could be solved with the help of these learning and learning support tools.

A two-week course, consisting of activities that involved the use of technologies to consolidate and/or introduce new mathematical knowledge, was developed within the framework of the BETA (Good students with Academic Talents) Programme of the Pontifical Catholic University of Valparaíso in Chile. This work focused on promoting certain technological skills that would equip outstanding students to better solve mathematical problems – for instance the ability to acquire information or to search and process data. On the other hand, mathematical knowledge was also considered from the very specific angle of the application of these technologies. In our experience, the use of technological tools strengthens the didactic environment and has a positive impact on the learning outcomes of the students, letting them move between different registers of representation that exist in the treatment of mathematics (Duval, 2005).

2 Development of Technological Skills for Outstanding Students of Mathematics

The context in which the investigation was evaluated features two components: outstanding students in mathematics and technological skills. For the students we considered some characteristics of the model, proposed by Reyes-Santander (2009), that is to say: S1: Mastery of areas of mathematical knowledge (Geometric, algebraic, numerical).
S2: Persistence and permanence with mathematical activities that motivate the student, and with metacognitive generation activities.  
S3: Capacity to generate creative, advance and abstract ideas in mathematics.  
S4: Knowledge, performance and involvement in mathematical subjects.  
S5: Independence and depth of the cognitive process, in particular regarding the inference and linking of concepts.  
S6: Capacity to understand and manipulate mathematical information.

The other component is defined by the following didactic technological skills:

C1: Knowledge and manipulation of the most relevant aspects concerning the use of two types of technologies: Latest-generation calculators (TI Navigator 84, Texas Instrument\(^1\)) and the shareware GeoGebra which allows work to be performed both in the area of Euclidean geometry, analytical geometry and in the subject of functions.

C2: Development in the student of skills related to the use of technology to obtain a better understanding of mathematical knowledge, in particular, the data analysis given by the technology, the visualization and understanding of a mathematical object in different registries from representation (Duval, 2005).

C3: Use of technological means as a tool for learning, as a means to support and assist in the resolution of mathematical problems.

C4: Development of social consciousness, within the meaning of promoting collaborative work, in such way that “by knowing the equipment”, individual knowledge is consolidated and new knowledge is generated, there by fostering intellectual growth, creativity and critical thinking (Landesakademie für Fortbildung und Personalentwicklung an Schule, 2002).

These two components describe the framework that we used in this study and allow us to find an appropriate approach for working with outstanding students in mathematics involving the use of new technologies.

\(^{1}\) For more information about this calculator, reference is made to the website: http://education.ti.com/educationportal/sites/US/homePage/index.html.

3 Methodology

The methodology of each session consisted of three phases, based on the pedagogical principle of “I-You-We” (Ulm, 2004). These phases were repeated in the same session, this repetition depending on the development of the class.

I: Individual work consisting of the analysis of the problem or proposed challenge. Each student had the access to a calculator.

You: Phase of sharing and interchanging ideas in group work, each group consisting generally of a maximum of three students.

We: Common conclusion, where the mathematics involved in the activity is formalized. This part was used the “fast scan”\(^2\) to compare answers and perform the institutionalization of the study object.

The following technologies were used:

- Latest-generation calculators, which are used by the students of Engineering of the Pontifical Catholic University of Valparaíso of Chile. This was the instrument of choice to work with and develop the class (Reich, 1979). The technological means are considered in this case as a means to achieve the cognitive objective, respectively the desired skills or learning outcome of the class.

- Network connection to TI Navigator, a software-hardware that permits wireless connection to the calculators, so that the students send and receive information synchronically. This means by which group work (We) the exchange of ideas (You) can be promoted.

- The software GeoGebra, a shareware that permits work to be carried out both in the area of Euclidian geometry and analytical geometry. It works in accordance of indirect learning (Reich, 1979).

- During two weeks, the students were observed to ascertain the development they made in the area of technological skills. In this period, different tasks were carried out. Two of these were chosen for the purpose of evaluating the mathematical skill of the students.

\(^{2}\) The “fast scan” is an evaluation system in line that is realized through the wireless connection of the calculators with TI 84 and Navigator. To the students the question is sent to them by line and they also respond in line, throwing a summary of their answers in a common screen.
4 Task Design

The tasks to have been considered were: “Colour picture” – a translated and modified version of the task “Amplifying algebraic terms” (Sinus, 2007, S. 26) and “Locate”. Both tasks were designed in the three phases “I-You-We”.

Task 1
In the first task, the students were presented with a picture of Richard Paul Lohse (1983) entitled “6 lines of complementary colours”. To these, the letters a, b, c were assigned to indicate the dimensions of the sides. This task was divided into four parts:

- Recognizing that with letters represent and describe the area of rectangles by means of algebraic expressions. (S1).
- Performing four “cut-outs” of the picture. The students were asked to identify the cut-out and to express the surface area obtained by means of an algebraic expression. (S2, S3).
- The students were presented with different algebraic expressions, and they had to cut-out the corresponding section of the picture. They were also asked to come forth with a “reduced” algebraic expression of the area. (S4).
- A cut-out of the picture was presented to the students, with numerical expressions that indicated the dimension of the sides of the colour rectangles. Then the students were asked to find the total area of the cut-out. (S1, S2, S4, S6).

At the end of each part of this task, the students were asked to send the answers they had obtained to the navigator.

Task 2
The “Locate” task consisted of five parts:

- Finding North, South, West and East cardinal points by means of the expressions “advance n number of steps” in different directions (S1).
- Inventing figures and to describing them to the classmate, by means of the cardinal points. (S3).
- In the graphical environment of the calculator, the students had to be located themselves in the quadrants indicated to them; this was seen simultaneously by all the class, and each student was represented by a symbol on the Cartesian plane. In this part, the students were asked to offer personal observations to describe what happening in group form (S1).
- Working in a graphical environment of their calculator, the students had to locate themselves on a specified line. In this way, some conditions were introduced by very simple algebraic representation so that the points (the students) belonged to straight lines. (S4).
- The students had to be located through graphical atmosphere of their calculator on some determined straight line, introducing of this form some conditions so that the points (students) belonged to straight lines with very simple algebraic representation. (S4).
- The fifth part consisted of individual challenges that the students had to solve.

At the end of each section, the students were asked to send their answer or conclusions to the navigator. This was called “fast scan” (S5).

5 Procedure and Data Analysis

The tasks were carried out by a total of twelve students at the age of 13 and 14 years. They were given approximately two hours to solve these tasks. The working teams that attended the sessions consisted of two educators and one student of teacher studies in mathematics.

The data was acquired from the following sources:

- In connection with the performance of the tasks with TI Navigator, they came from special "scan" system.
- By means of questions that had to be answered either individually or as a group on the worksheet or in the work notebook.
- Recording of the class on video.
- Observation and personal appraisal of the students by the working team.

For the analysis of the characteristics of an outstanding student of mathematics, we considered the answers given by the students through "fast scan", the answers the students wrote in their work notebook and the recording of the class reviewing the recording of the class for the analysis of some parts of the tasks.

5.1 Analysis of the Task “Colour Picture”

In the design of the tasks, section 3.1 added the initials of features we expected to be observed in these students.

S1: Mastery of the knowledge “area of a four-sided figure”. The twelve students gave the right answer, using four different forms of algebraic representation that the use of the calculator permitted.
S2; S3: Persistence and perseverance in the activity. The ability to generate abstract, advanced, and creative ideas through algebraic expressions. In order to observe this, the students had to determine and to send the algebraic expressions corresponding to the area of two cut-outs of the picture. The answers obtained by the students were subdivided into two parts, denominated S2S3A and S2S3B. Out of the answers obtained in S2S3A and S2S3B, nine of the students respond correctly, one does not respond at all, and other two do not find an adequate form to represent the area by means of symbols offered by the used technological tools. These answers were not completely wrong, but it was not possible to differentiate between sum and product. Both students, who did not give the right answer in response to the two tasks S2S3A and S2S3B, nevertheless do insist that they understand what is asked of them, but they simply do not know how to do it. Therefore, their failure to solve the tasks is probably due to the non-development of some technological skills or the insufficient development of the skill.

Fig. 1 shows the different algebraic expressions that the students used to determine the area of cut-out A by means of algebraic expressions.

<table>
<thead>
<tr>
<th>Student</th>
<th>Answer</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Catalina</td>
<td>A(H+D)C</td>
<td>0.00</td>
</tr>
<tr>
<td>Cristhian</td>
<td>A(H+B)+A-C</td>
<td>0.00</td>
</tr>
<tr>
<td>Diego</td>
<td>A+H+BC</td>
<td>0.00</td>
</tr>
<tr>
<td>Estefany</td>
<td>A+H+B+C</td>
<td>0.00</td>
</tr>
<tr>
<td>Fabiola</td>
<td>A+AB+BE</td>
<td>0.00</td>
</tr>
<tr>
<td>Jose</td>
<td>A+H+B+C</td>
<td>0.00</td>
</tr>
<tr>
<td>Mauy</td>
<td>A+H+DA+C-A</td>
<td>0.00</td>
</tr>
<tr>
<td>Miguel</td>
<td>A+H+B</td>
<td>0.00</td>
</tr>
<tr>
<td>Stiven</td>
<td>A=B+C+Ad</td>
<td>0.00</td>
</tr>
<tr>
<td>Tavo</td>
<td>A+H+B+K</td>
<td>0.00</td>
</tr>
<tr>
<td>Sofía</td>
<td>A+AB+BE</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Fig. 1: Answers given by the students to part S2S3A.

S4: Knowledge, performance and insertion in the subject of algebraic expressions. Continuing with algebraic expressions, the students have to find the meaning of “reduced”. For this purpose, we give them a cut-out of the picture and ask them for “a reduced” algebraic expression of the area.

We hoped that the product or result of this activity was going to be original in the sense of innovating personal productivity (Reyes-Santander, 2009), that they could find the concept of the “reduced”. From the obtained answers (see Fig. 2), one sees that there are four students, three of them different to those from the previous part, who confuse the symbols of the calculator or do not use them adequately, recognizing that they themselves mistyped. This must be attributable to the fact that in the “fast scan”, the students have a predefined time to respond and in the screen common appears that is first. The other answers are correct, in the sense that an algebraic expression is used to represent the area of the cut-out of the figure, but of these answers only four are reduced expressions.

From the comments of a student who carries out a division and from his/her comments on the matter, it is inferred that doing a division is considered as a concept of reduction of expressions, this is from the point of view of the student, involving that a division is considered as concept of reduction of an algebraic expression, and as the expression continues being equivalent, it must to be considered as a reduced expression.

Fig. 2 shows the answers given by the students, five of them obtain the correct answer, although none of them factorizes by 2. Since in previous parts they worked with simpler cut-outs, they were able to reduce similar terms and to observe properties such as associativity, commutativity of terms and the distributive property.
Fig. 2: Answer given by students for S4.

S6: Capacity to understand and manipulate numerical information. The students were given a cut-out of the picture, this time with real numbers that indicated the dimension of the sides of the color rectangles. The students were asked to find the total area of the cut-out; this cut-out having the same area as the cut-out in the previous question, but a different shape. In this way it was possible to understand the students’ ability to take in the general information that they had been working on until that moment, and their ability to manipulate this information in a concrete way, using the technology that they disposed of.

This task was solved successfully by five of the students, four of them achieving a better response than they had with regard to the previous question, but a different shape. In this way it was possible to understand the students’ ability to take in the general information that they had been working on until that moment, and their ability to manipulate this information in a concrete way, using the technology that they disposed of.

The other three students obtained the triple of the real area. When they were asked how they had obtained this answer, they answered that they had calculated a square and superposed it over the rest of the figure, which was correct when they explained it, but at the time of calculating they counted the same area one time too many, the square they had calculated could only be superposed twice over the total figure and the not three times. When these students gave their explanation, the remaining students told them where they had committed the error. The response of one student out of the twelve was totally wrong, in this case the student calculated the area without applying or using what he had learned previously, but adds and multiplies instead. As the cut-out is not a rectangle, it is probable that he went wrong somewhere, but when it came to explaining, it was very difficult for him to find out where exactly he had made the mistake.

At the end of this part of the task, we worked on the common screen. There the answers were displayed, but they differed beginning with the first decimal number. This surprised the students; they said that the calculations all had to be equal because they had carried them out with the calculator. The reason was that the students did not consider all the decimal numbers or applied rounding so that they did not have to work with many decimals. Some of them found a way of calculating directly with roots, without making the calculator give them an approximation first. Due to this and the discussion on the matter, the students concluded that the most exact approach was the expression for the root of 2, which the calculator gave with ten decimal digits.

At the end of the complete task, the students mentioned that for them it was easier to work with letters than with decimal numbers. They also said that they knew that working with letters and working with numbers is the same. Another observation they made concerned the strategy that they thought easiest. They said that they found it easiest if, before calculating an area, they took a close look at the figure and tried to form rectangles, square or half-squares. They talked among themselves of the possibility of making more complicated figures with curvilinear cut-outs.

5.2 Analysis of the Task “Locate”

S1: Mastery of areas of mathematical knowledge. In this part all students understood what the task was about and for what this type of instructions could be used. This was reflected in the observations that they wrote in their notebooks and their answers in the “fast sounding”.

S3: Ability to generate abstract, advanced and creative ideas from instructions. Only two of the drawings the students made in their notebooks were “different” from the rest regarding the work form.
know the point where the straight line intersects with main vertical straight line (known as the X-axis), and that the first number that appears in the pair of numbers is always zero. The student worked with another geometric figure, they were not able to find an explanation/instruction referring to an opponent. The second number in the pair of numbers is always zero.

5.3 Analysis of the Technological Skills

The skills to be analyzed are C1, C2, C3, and C4, specifically in section 2. To be analyzed is the results obtained by means of a combination of observation and consideration of each student A1 to A12 during a part of the work sessions.

From Table 1 and the before mentioned categories of credits, it is possible to conclude that no student achieved outstanding (LE) 85% to 100%; pass (L) 75% to 84%; conditional pass (PL) 60% to 74%; fail (NL) 0% to 59%.

Two students of the total achieved a full understanding of mathematical knowledge, like the data analysis given by the technology and the change of a registry to another registry (C2). In this case the computer program GeoGebra is also included. For the semiotic registry (Duval, 2005) the language is also included. From the students' comments we can see that they were able to manipulate with excellence the aspects of the calculator and software GeoGebra (C1).

For ten students of the twelve students, it was difficult to explain in words what they had done with the calculator. This means that they did get the correct answers, but do not know how to express them in natural language. The majority of these ten students needed support to change between registries.

At the end of this part of the task, the students comment that for the horizontal straight lines there is only one condition, namely to
Table 1: Technological skills achieved by students.

<p>| | | | | |</p>
<table>
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<tbody>
<tr>
<td>A1*</td>
<td>LE</td>
<td>LE</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A2</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A3</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
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<tr>
<td>A4</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A5*</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A6</td>
<td>PL</td>
<td>L</td>
<td>PL</td>
<td>PL</td>
</tr>
<tr>
<td>A7</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A8</td>
<td>LE</td>
<td>L</td>
<td>L</td>
<td>LE</td>
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<tr>
<td>A9</td>
<td>LE</td>
<td>L</td>
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<tr>
<td>A10</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A11*</td>
<td>LE</td>
<td>LE</td>
<td>LE</td>
<td>LE</td>
</tr>
<tr>
<td>A12</td>
<td>LE</td>
<td>L</td>
<td>LE</td>
<td>LE</td>
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</tbody>
</table>

Ten students achieved an excellent standard in the use of technological means as a support tool and aid to solve problems (C3), this means that at the time of solving a problem they did not go back to pencil and paper first, but left most of the calculations to the programs including the calculator TI84.

Most of the students did not only achieve an excellent development, but there was also a good working atmosphere and respect. This development shows that social awareness was there, and the "We" was both prominent in and provided focus to the work on the common screen. This was seen clearly in the "Locate" task where each student appears like a point on the projected plane.

6 Conclusion

Regarding the objective of using technological tools as learning aids in our work with outstanding students of mathematics, we can state:

- that the mathematical knowledge of outstanding students of mathematics can be enhanced through the use of TIC;
- that it was possible to identify certain characteristics of outstanding students, and that the development of these characteristics could be evaluated;
- that it was possible to identify when this knowledge manifested itself and was consolidated;
- like in Guzmán’s experience (quoted in Callejo, 2002), the fact that “some” students were able to access relatively profound and indeed the most creative levels of current mathematics makes us believe that it may be possible to design a project with selected tasks and adequate technological tools in order to confirm a clear vocation for engineering subjects or mathematical investigation in these students.

It would be interesting to evaluate the type of tasks proposed in this study using these tools in a normal classroom setting, in order to see whether the results thus obtained (i.e. in the classroom) also provide evidence that this technology acts as a facilitating medium and a stimulator of better learning outcomes. In this sense, it needs to be considered that competence C4, namely the social component, may be limited because work in this case was performed with only 12 students. If the class is bigger, it will take longer to achieve this.

Regarding the use of TI Navigator, we can offer the following conclusions:

- A common working environment was achieved that stimulated both group and personal reflection. TI Navigator also inspired the students to be active, independent and collaborative participants.
- With this software and the “fast scan”, an immediate feedback is achieved, allowing the students to discover, analyse and correct their errors by themselves or with the help of the entire group, and in some cases the technological tools were again used as a resource.
- It also gave the working group a better control of the learning progress of these students while providing a global and individual view of the class.

In this sense, a relevant aspect of the “Colour picture” task was observed: The students of this class had not treated algebraic expressions at school, but they were familiar with the area of geometrical figures. In the first scan, different expressions for the area are offered, and observing that these terms must be equal, the students then apply a rule that is known although not commonly applied in a classroom setting, namely the distribution property.

We believe that this was one of the relevant findings regarding the application of knowledge. The knowledge of the area of a rectangle...
serves as the starting point. It is then observed that the expressions are equal. It is concluded that therefore there simply has to be a rule to obtain the one from the other. With respect to the application of knowledge, the same occurred in the "Locate" task when the students are "situated, stuck" on a straight line in virtual form and have to find the conditions for “belonging” to this straight line. Now the students must perform a change of register, which is generally treated by going from the equation of the straight line to the Cartesian diagram of the straight line. In this case, the students had to proceed the other way round, namely through the change of register from the Cartesian diagram to the equation.

The amount of semiotic registers (Duval, 2005) that the technology is able to offer, should be highlighted. We know that it is possible, by observing the management of these registers by the students, to observe whether they have acquired a mathematical object. It must be mentioned that this study is the starting point, as it were a first step, for the investigation of the development, skills and potentials of outstanding students of mathematics with special consideration of technological tools as a facilitating medium for classroom learning.

Finally, in the light of the obtained results, we wish to point out that when these two components were combined – one of them being a group of students with interest in mathematics and the other being the use of TIC – we were able to observe the following in the students: better concentration on problem solving, familiarisation with mathematical concepts, and a higher degree of socialisation of knowledge. Therefore they developed transversal skills in the learning process – skills such as teamwork and the efficient use of modern technologies – that are indispensable for success in the society of the future.

**Literature**


With the curriculum of 2006, Norway has started an ambitious program to include ICT tools in mathematics education. In this contribution I will present the historical background as well as elements of the recent development. This article will focus on the means for curriculum change used by the school authorities.

Norway has a national curriculum for all grades 1 through 13. Upper secondary education consists of grades 11 to 13. The first 10 years are compulsory. The school starting age is 6. Mathematics education in upper secondary education in Norway serves several purposes. In upper secondary education there are vocational streams as well as specific mathematics programs. There have been mathematics programs for the different vocational streams, as well as programs for diverse focuses of mathematics, e.g. mathematics for the natural sciences and mathematics for the social sciences. In the recent curriculum from 2006, the various streams and programs have merged into two main directions in the beginning of the upper secondary level: theoretical mathematics and practical mathematics. Each of these directions is further subdivided in the last part of the upper secondary level. The theoretical mathematics program has two parts: The mathematics and science courses, prefixed by “R”, and the social science courses, prefixed by “S”. The system has the purpose of being flexible and taking into account different group of students, hence the structure of upper secondary mathematics is complicated.

The information technology as we know it has been developed since the Second World War. Commercial use started early, and use in education gained momentum during the 1960s. In a much quoted article from 1966 in Scientific American, the mathematician Patrick Suppes presented viewpoints on the use of computers in mathematics education. Although focusing mainly on computer aided instruction (CAI), “the article offered a compelling vision for the role of technology in education” (Suppes, 1966).

The 1960s and 70s saw a rapid development of the introduction of computers and technology in education. More extensive views were
presented. We will mention the different roles of the computers in education as formulated by Taylor: The computer as a tool, as a tutor and as a tutee (Taylor, 1980).

Even if computers have had all these roles in upper secondary education in Norway, the use has been mostly focused on using the computer as a tool. ICT – information and communication technology (formerly IT – information technology) is not only about computers, but denotes also use of other tools, such as calculators, video equipment etc. For mathematics education, calculators as tools have held a strong position since the 1970s. In the presentation that follows, we will focus mainly on mathematics education in upper secondary, for mathematics and natural science.

The curriculum of 1976\(^2\) was a curriculum for development, giving possibilities for a range of different approaches with technology. In the following years, several experimental curricula were constructed and accompanying exams. The situation was quite open, in the sense that there were options for teachers regarding the kind of technology they would use in mathematics education. Around 1990 some students had their exams using computers with CAS (Computer Algebra Systems), whereas others used advanced or graphic calculators. Some teachers might have used computers in other roles than tools, e.g. as a tutor, but looking at the computer (or calculator) as a tool was the dominant conception/role in upper secondary school mathematics.

The curriculum of 1994 changed the situation concerning technology quite substantially. New areas of mathematics such as modeling and problems solving were introduced. These areas focused on applying digital tools. The use of technology became in a sense much more "regulated":

"Means for IT" comprises both calculators and computers. The students should learn to master the calculator, and the schools should provide the best possible conditions for use of computers in mathematics education. Students in the 5-hour-course (remark: the most extensive program) must have a graphic calculator. (RUF, 1993, p. 171, translated from Norwegian, G.G.).

To enforce the use of technology, solving some problems for the final exam required the use of graphic calculators. After some initial resistance, the graphic calculator gained acceptance as a tool by most upper secondary math teachers.

\(^2\) This curriculum appeared in several versions over the following 17 years. Each version was introducing what we might call "minor adjustments" not affecting the basic principles in the curriculum.

Concerning content, the curriculum of 2006 did not change much from the 1994 curriculum. A certain restructuring of content was introduced, and the structure of upper secondary mathematics became more complicated with the introduction of more alternative courses. Most notably there was one curriculum from grade 1 to 13. The structure using aims/objectives was strengthened with detailed objectives for grades 2, 4, 7, and 10 as well as for all levels in upper secondary education. The technological aspect was also detailed further, introducing requirements for using technological tools.

What has been done by the central school authorities to implement the present curriculum reform using technology in mathematics education? Let us first briefly consider one important factor that will influence the implementation – the final exam.

2.1 Implementing Curriculum Reform: The Relationship between Exams and Teaching

As mentioned above, the curriculum reforms in the 1990s introduced the system: management by objectives (MBO). In this model assessment (tests, exams) plays an important role. The model below is an adaptation of a model for business, and taken from Gjone (1996).

![Fig. 1: Model for role of assessment](image-url)
To implement the use of technology in mathematics education in Norway in the mid 1990s, it was communicated to teachers that the national exams would require graphic calculators, as also stated in the curriculum (quoted above)\(^3\).

That assessment has an influence on teaching as illustrated in the figure, has also been explored by several researchers, e.g. “High school exit exams are influencing curriculum and instruction, a study of two school districts suggests”, from *Education Week* (2007). Also formulated in Reflections from the Park City Mathematics Institute (2001): “National exams were seen as having a strong influence on teaching”.

This development has been further strengthened by implementing the present curriculum. The situation is somewhat different since the introduction of PCs changes the situation: It has been decided that equipment for education should be free for the students, also in upper secondary education. This means an added cost for the regional school authorities. They are responsible for upper secondary education.

### 2.2 The Present Development: Implementing the Curriculum of 2006

For some time there has been a discussion about the role of ICT in upper secondary mathematics education in Norway. There have been special developments where different models for the exam have been tried out. After a meeting with a group of selected teachers and university mathematicians it was decided to introduce two part exams.

The tradition with written final exams in mathematics in Norway is to have exams lasting five hours. Starting in 2008, and further extended in 2009, written exams in mathematics in grades 10 and 11 to 13 (upper secondary) consists of two parts: The first part (two hours) is with paper and pencil only, and the second part is three hours is with “everything” (all possible tools which do not allow communication). The organization of the exam is such that the two parts are given out at the beginning of the exam. When answers to the first part are handed in, the students get access to whatever tools they have brought.

There was a small difference between the exam in 10th grade (lower secondary) and the exams in upper secondary. The tradition with exams in 10th grade was that the first part of the exam was answered on the exam paper which was handed out with the problems. This has been continued such that in the 10th grade the exam papers where in two separate parts. Also in the 10th grade the students could hand in the first part at any time – and hence get access to tools, whereas in upper secondary the students had to wait for two hours to hand in the first part. The reason stated by the authorities for this difference, was that the ministry was afraid that the students in 10th grade might disturb the others if they were sitting for a period without anything to do.

With the second part with “all tools”, it was clearly an incentive for students to bring and use computers. However, it was stated that all problems could be solved with a graphic calculator. For the exam in the 2nd year upper secondary in 2009 there were two alternative problems in geometry in the second part of the set. For one of these problems (alternative 2) it was stated that a computer with suitable software was recommended. I have argued that also for the first alternative it was definitely an advantage to have a computer with CAS (Gjone, 2010). Probably the new organization with a part without tools had a marked influence on the teaching. It was reported that training and several tests during the year without tools were common among teachers.

The new form of exams in mathematics was introduced without much discussion among Norwegian teachers in the spring of 2008. For the exam in 2009, Utdanningsdirektoratet\(^4\) decided to have an evaluation. An evaluation of the exam was to have two parts – a quantitative evaluation performed by a private firm – Rambøll management – and a qualitative evaluation, carried out by the Department of Teacher Education and School Development at the University of Oslo.\(^5\) Both evaluations were presented together in one book, and also on the internet (Utdanningsdirektoratet, 2009). The two separate parts of the report could be clearly identified. What kinds of data were collected in these two evaluations?

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\(^3\) As a member of the committee writing the curriculum of 1994 I was part of this process.

\(^4\) Utdanningsdirektoratet: Norwegian Directorate for Education and Training. The Norwegian Directorate for Education and Training is responsible for the development of primary and secondary education. The Directorate is the executive agency for the Ministry of Education and Research.

\(^5\) I was one of two persons from the university department.
3 The Evaluations of the Two-part Exams in the Spring of 2009

The survey by Rambøll management did not evaluate the actual exam problems other than ask the students and teachers about difficulty. Mainly they were concerned about how the organization of the exam functioned, and presenting views from graders, teachers, and students. In the qualitative evaluation we interviewed graders, teachers and school administrators, and also looked at the exam problems to some degree. We – from the University of Oslo – were both present at a couple of exam sites (which were at local schools) to observe the actual organization of the exam. In addition, I was grading some 200 exam papers together with other graders.

Since the data in the report are presented in Norwegian language, I will present excerpts with a mixture of graphics and text. The texts in this presentation are all translated by the author.

There are, in my opinion, several problems with the survey conducted by Rambøll management. The words ‘exam’/‘form of the exam’/etc. used in the survey relates to several different elements, which in my view are not clearly separated or distinguished:

The exam consists of two parts. There was an initial discussion how large each part should be, this is not reflected in the survey. Should it be two hours without any tools, and three hours with ‘everything’ or vice versa? If one favors a two-part exam, is the main reason:

- there is a part where no tools (only paper and pencil) can be used?
- there is a part where ‘everything’ can be used?

These distinctions are important since a two part exam was supported by a majority of the teachers.

3.1 Findings from the Survey

The diagrams presented in this section are all taken from the report (Utdanningsdirektoratet, 2009). We will first return to the question presented above, the influence of the exam (content and structure) on teaching:

How do teachers evaluate the organization of the exam?
Number of teacher answers: 81

More than 40% of the teachers hence report that they have changed how they teach mathematics. We do not know how they have changed, but there are clearly several elements that we can find in this change. They use more technology in their teaching, they use more time on solving mathematical problems with paper and pencil only. Another diagram might give us some input on this question:

How were the students prepared for the exam?
Number of teacher answers: 81, students 603

We see that the preparations used most were trainings in solving problems, and having tests with the same structure. It should be noted that there was the same type of exam the year before, also
that there were officially published sample problems, as well as problem-collections published in connection with textbooks. A similar question asked, was the training using ICT. This of course is dependent on the equipment at the schools. The table below does not detail the type of equipment (software, hardware) but gives an indication of some interesting observations.

Which tools have been trained?
Number of teacher answers: 47, students 218

![Image of bar chart]

Fig. 4: Tools having been trained

It is interesting to note that a substantial number of teachers (about 55%) and students (about 44%) express that there has been education with a graphic calculator without CAS.

In the qualitative study it was stated by some teachers and school administrators that the school did not support use of PCs in mathematics education.6 We also see that the tool most used in preparation was dynamical software/Geogebra. Geogebra is quite dominant in upper secondary mathematics education in Norway. Even if there exist other dynamical software, and we do not have information about the market share, only a few interested teachers use other mathematical software.7 The most interesting question, however, is which ICT tools were actually used by students during the exam. Before we consider ICT tools, let us briefly look at more general tools: 97% of the students wrote that they used tools in Part 2 of the set. Textbooks were the tool used by most students (93%). The students' own notes were also used quite extensively. The situation with ICT-tools is summarized in the following diagram:

The students’ use of tools
Number of student answers = 584

![Image of bar chart]

Fig. 5: Students’ use of tools

From this diagram we see that different types of calculators are the tools that are definitely most used. We can summarize the diagram as follows:

- Graphic calculator without CAS, more than 30% report much use
- Calculator with CAS was also used quite a lot, a little less than 30% report much use
- There was little use of PCs during the exam, even if training with PCs were quite dominant

There is a reason to be somewhat skeptical of the data, especially the alternative “calculator with CAS”. We note that there is some uncertainty about the meaning of CAS. Summing up the first three categories – about 20% of the students did not know if they operated a CAS calculator.8

6 The students in these schools could bring their own, or use the PCs supplied by the school, but the teachers would not engage in the training and use of this technology in mathematics teaching. The main argument for this position was all problems on the exam could be solved with a graphic calculator, and the time in class could be more efficiently used for traditional mathematics training.

7 This statement is my personal impression, but based on interviews with students and their supervisors in schools, in the teacher training program at the University of Oslo.

8 CAS is not the term used in the survey. The phrase used is “calculator with symbol handling tools.”
An important factor for the school authorities was that the form of the exam should motivate the students, and increase their interest for mathematics. The diagram below presents the answers to this question in the survey. Again there is some uncertainty how to interpret the answers, as formulated above – what is the content of the phrase “form of the exam”?

To what extent did you experience that the form of the exam with two parts increased the students’ interest for mathematics?

Number of teacher answers: teachers 81, students 600

![Fig. 6: Increased students' interest](image)

About 40% of the students and about 20% of the teachers were giving a positive answer to this question.

Returning to the statement formulated in the introduction – the means for a curriculum reform – the present implementation illustrates several aspects.

The curriculum consists of objectives/aims for mathematics education. Using technology is formulated as an objective. Also, (lack of) calculating skills of students were getting attention. Norway’s result in the PISA-project9 also contributed to the search for new models for exams. As argued in Fig. 1, presented above, assessment plays an important role in the MBO philosophy in education. To realize these objectives, a new form of the final exam in mathematics was introduced. The Ministry also financed evaluations that should give information on the effects of the form of the exam.

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9 PISA: Programme for Student Assessment. International project comparing outcomes of education as well as other factors.

3.2 Some Consequences of the Evaluations

From the data we can infer that the (form of the) exam influences the teaching in mathematics. Teachers report that they changed their teaching.

An important element in both evaluations was to consider how the content and the organization of the exam were suited to assess the objectives/aims of the curriculum. The data from these evaluations show that the participants in the process (teachers, students and school administrators) were positive to the form of the exam. We might ask what, if any, were the consequences of the evaluation of the exam in the spring of 2009?

One major concern in the qualitative evaluation was that students experienced unequal conditions during the exam. There was a variation in how the exam was organized; there were also differences in the tools used by students. Therefore, in the qualitative evaluation, we recommended that the new form of the exam should be postponed until more equal conditions were established. We also recommended that the part with no tools should be expanded to three hours out of five. The deadlines for the evaluations were September of 2009, hence too late to be taken into account for the exams in the spring of 2010. The decisions by the ministry should have been established well before the school started in August.

4 Reflections

In several countries around the world, there is attention to introducing ICT in mathematics education. In most of these countries there are a variety of arguments either for or against introducing more ICT tools. For many countries it is a question of cost, both to the school system and the individuals.

In this paper I have tried to show the model used by Norwegian school authorities to introduce ICT in mathematics education in upper secondary school.

Literature
