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The discrepancy distribution of stationary multiplier rules for rounding probabilities

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Abstract. The problem of rounding finitely many (continuous) probabilities to (discrete) proportions N_i/n is considered, for some fixed rounding accuracy n. It is well known that the rounded proportions need not sum to unity, and instead may leave a nonzero discrepancy $D = (\sum N_i) - n$. We determine the distribution of D, assuming that the rounding function used is stationary and that the original probabilities follow a uniform distribution.

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Key words: Discrepancy distribution, Multiplier methods of rounding, Round-off error, Rounding rules, Stationary rounding function, Uniform distribution

1 Introduction

In December 1999, a public German television station conducted a telephone poll to rank five pop music groups. The group that came out top gained 29 percent, and had the good fortune to win by the narrow margin of one percent. However, the five percentages that flashed over the television screen add up to one percent in excess:

$$29 + 28 + 25 + 13 + 6 = 101.$$

According to a subsequent press release, the observed discrepancy of $101 - 100 = 1$ percent was not instrumental in determining the winner (Bild am Sonntag 1999).

It is well known that probabilities, after rounding, need no longer add to unity. We demonstrate the sensitivity of the rounding rule by three one-line examples [1]–[3] of rounding probabilities to percentages leaving, in turn, the discrepancies $-1, 0, 1$:

	Probabilities			Total		Percentages			Total	Discrepancy
[1]	0.534	0.322	0.144	1	$\mapsto$	53	32	14	99	-1
[2]	0.534	0.320	0.146	1	$\mapsto$	53	32	15	100	0
[3]	0.536	0.318	0.146	1	$\mapsto$	54	32	15	101	1

The discrepancy depends on the rounding rule used. Here, we have employed the standard rounding rule which, after multiplying the probabilities by 100, rounds down when a fractional part remains that is less than $1/2$, while it rounds up when the fractional part is greater than $1/2$.

In this paper we show how likely the various possible values for the discrepancy are. The discrepancy distribution is stated in Section 2, together with a discussion of how it relates to the existing literature. The derivation of the distribution is outlined in Section 3.

2 The discrepancy distribution

Our result is not specific to standard rounding, but extends to the wider family of *stationary rounding functions* r_q depending on a parameter $q \in [0, 1]$. The rounding function r_q rounds down when a fractional part remains that is less than q, while it rounds up when the fractional part is greater than q.

Formally, let the nonnegative number $x \geq 0$ be decomposed into its integer part, $\mathrm{IntegerPart}(x) = \lfloor x \rfloor$, and into its fractional part, $\mathrm{FractionalPart}(x) = x - \lfloor x \rfloor$. Then the rounding function r_q is defined as follows:

$$r_q(x) = \begin{cases} \lceil x \rceil = \mathrm{IntegerPart}(x) + 1 & \text{when } \mathrm{FractionalPart}(x) > q; \\ \lfloor x \rfloor = \mathrm{IntegerPart}(x) & \text{when } \mathrm{FractionalPart}(x) < q. \end{cases}$$

A tie occurs when $\mathrm{FractionalPart}(x) = q$, and there the definition may stipulate either $r_q(x) = \lfloor x \rfloor$ or $r_q(x) = \lceil x \rceil$. Our probabilistic assumptions imply that such ties form a nullset, so that the ambiguity arising from ties leaves the distributional results unaffected. By definition the rounded value $r_q(x)$ comes to lie in a unit length interval containing x,

$$x - q \leq r_q(x) \leq x + 1 - q.$$

These *range inequalities* are fundamental, and will be referred to frequently.

The family of stationary rounding functions starts with rounding up ($q = 0$), smoothly passes through standard rounding ($q = 1/2$), and ends with rounding down ($q = 1$). Generally, whatever $q \in [0, 1]$, the points $k + q$ determining the rounding decision in the integer intervals $[k, k+1]$ have the same position relative to the boundaries. Because of this stationarity property our probabilistic analysis extends from the three classical rounding functions r_0, $r_{1/2}$, and r_1, to cover an arbitrary stationary rounding function r_q.

We consider the task of rounding to integer multiples of $1/n$, where the *rounding accuracy* n is assumed to be given. For example, when rounding to percentages we have $n = 100$, while when rounding to multiples of a tenth of a percent we have $n = 1000$.

The discrepancy problem arises when the rounding rule is applied to a set of probabilities or *weights* $W_1, \ldots, W_c$, for a given number $c \geq 2$ of categories. The Rule of Three suggests first to multiply each weight W_i by n, and then to round the resulting product to obtain $r_q(nW_i) = N_i$, say. Now W_i is rounded to N_i/n, visibly a multiple of $1/n$. It also affords a reasonable approximation of W_i since the range inequalities entail $-(1-q)/n \leq W_i - N_i/n \leq q/n$. And if the total $\sum_{i=1}^{c} N_i$ happens to be equal to n, then the rounded proportions $N_1/n, \ldots, N_c/n$ sum to unity and form a valid set of probabilities.

However, the total may well fail to be equal to n and instead leave a nonzero discrepancy, $(\sum_{i=1}^{c} N_i) - n \neq 0$. The range inequalities imply that the total comes to lie in the interval $[n - qc, n + (1-q)c]$. For example, when $q = 0$ and fractional parts are always rounded up, the target interval is $[n, n+c]$ and the discrepancy is nonnegative. At the other extreme, when $q = 1$ and fractional parts are always rounded down, the target interval is $[n - c, n]$ and the discrepancy is nonpositive.

In order to center the total at the rounding accuracy n we introduce a continuous *multiplier* $v > 0$, to substitute for the constant Rule-of-Three multiplier n. We emphasize the dependence on the multiplier v by denoting the associated *total* by

$$T_{c,q}(v) = \sum_{i=1}^{c} r_q(vW_i).$$

A rounding rule that uses a stationary rounding function r_q and a fixed multiplier v to round weights W_i to proportions N_i/n, with $N_i = r_q(vW_i)$, is called a *stationary rounding rule*. If, for a given rounding accuracy n, the total leads to a vanishing discrepancy, $(\sum_{i=1}^{c} N_i) - n = 0$, then the proportions N_i/n again form a valid set of probabilities.

The discrepancy may be nonzero, however, even when the choice of the multiplier is adapted to the accuracy n. Indeed, the range inequalities imply that the total $T_{c,q}(v)$ comes to lie in the interval $[v - qc, v + (1-q)c]$. If the multiplier is taken to be

$$\mu_{c,q,n} = n + (q - \tfrac{1}{2})c,$$

then $T_{c,q}(\mu_{c,q,n})$ ranges over the interval $[n - c/2, n + c/2]$ centered around the accuracy n, and the discrepancy falls into an interval symmetric around zero,

$$D = T_{c,q}(\mu_{c,q,n}) - n \in \left[-\frac{c}{2}, \frac{c}{2}\right].$$

The multiplier $\mu_{c,q,n}$ thus focuses on the nondefective outcomes, $D = 0$, but also admits the defective events $D = d$ with a nonzero integer d between $-c/2$ and $c/2$.

The following theorem shows how likely the possible discrepancy values d are, assuming that the weight vector $W = (W_1, \ldots, W_c)$ is uniformly distributed on the probability simplex $\mathscr{S} = \{w = (w_1, \ldots, w_c) \in [0, \infty)^c : \sum_{i=1}^{c} w_i = 1\}$.

Theorem: *Suppose the weights $W_1, \ldots, W_c$ are rounded to integer multiples of $1/n$, where the number c of categories is fix, the stationary multiplier rule used is based on a rounding function r_q for some $q \in [0, 1]$, and the rounding accuracy n is given.*

Then the discrepancy $D = T_{c,q}(\mu_{c,q,n}) - n$ takes on integer values $d \in [-c/2, c/2]$. If the weights are uniformly distributed, the distribution $\mathrm{P}(D = d) = h_{c,q,n}(d)$ is given by

$$h_{c,q,n}(d) = \frac{1}{\mu_{c,q,n}^{c-1}} \sum_{j=0}^{c} (-1)^j \binom{c}{j} \sum_{k=j}^{c} \binom{c-j}{k-j} \binom{n+d+j-1}{c-1+j-k}$$

$$\times \left\{ \frac{c}{2} - d - qj - (1-q)k \right\}_+^{c-1}. \qquad \square$$

The binomial coefficient $\binom{b}{a}$ is taken to be zero unless $0 \le a \le b$. The third binomial coefficient thus contributes an upper limit $k \le c - 1 + j$, active for $j = 0$. Also adjoined are two lower limits, $1 - n - d \le j$ and $c - n - d \le k$. But only large accuracies n are of interest, $n \ge (3/2)c$, in which case $1 - n - d < c - n - d \le (3/2)c - n \le 0$. Hence the two lower limits remain practically inactive. The last factor is defined by $\{y\}_+^{c-1} = y^{c-1}$ when $y > 0$, and by $\{y\}_+^{c-1} = 0$ otherwise. The proof of the Theorem is outlined in Section 3.

The discrepancy distribution $h_{c,q,n}$ conforms well with empirical data, as we illustrate with rounding the fractions of votes in the 1996 presidential elections in the United States, and in the Russian Federation. In both examples we use standard rounding ($q = 1/2$), and round to multiples of a tenth of a percent ($n = 1000$). The examples follow the lead of Balinski and Rachev (1993, page 492), with the present data taken from Happacher and Pukelsheim (1998, page 95; 2000, pages 146–149).

The 1996 US presidential election featured three candidates ($c = 3$). The 50 States, the District of Columbia, and the Candidate's Totals provide a sample of size 52. The observed counts for the possible discrepancy values $-1, 0, 1$ are as follows:

Discrepancy values	-1	0	1	Sum
Observed counts	5	39	8	52
Predicted counts	7	39	6	52

The discrepancy probabilities $h_{3,1/2,1000}(d)$ for $d = -1, 0, 1$, are $1001/8000$, $6000/8000$, $999/8000$, respectively. These are multiplied by 52 to obtain the expected counts $6.5065, 39, 6.4935$. Standard rounding yields the predicted counts $7, 39, 6$, for a sample of size 52.

In the 1996 Russian presidential election, there were ten candidates plus the option to vote against all of them ($c = 11$). The 89 Constitutional Subjects, the Votes Abroad, and the Candidate's Totals provide a sample of size 91. The values ± 5 and ± 4 in the discrepancy support $-5, \ldots, 5$ are not observed. The counts for the other values are:

Discrepancy values	-3	-2	-1	0	1	2	3	Sum
Observed counts	0	9	18	37	20	6	1	91
Predicted counts	0	4	23	38	22	4	0	91

The probabilities $h_{11,1/2,1000}(d)$ that the discrepancy equals $d = \pm 5, \pm 4$ are practically zero. For the middle values $d = -3, \ldots, 3$ we obtain, in turn, $0.002, 0.048, 0.246, 0.411, 0.243, 0.048, 0.002$. Multiplication by 91 gives the expected counts, $0.182, 4.368, 22.386, 37.401, 22.113, 4.368, 0.182$; standard rounding yields the numbers $0, 4, 22, 37, 22, 4, 0$ which, however, total 89 and fall short by two units, $89 - 91 = -2$, for a prediction of 91 cases. This defect is cured by increasing the multiplier to anywhere between 91.5 and 92.5. Multiplication by 92 produces the numbers $0.184, 4.416, 22.632, 37.812, 22.356, 4.416, 0.184$; now standard rounding yields the predicted counts $0, 4, 23, 38, 22, 4, 0$ quoted above.

Nonzero discrepancies are remedied quite generally in that, rather than using one multiplier for all possible weights, the multiplier is adjusted with a view towards the weights under investigation. This gives rise to *rounding methods*, as opposed to rounding rules. Rounding methods treat the multiplier v as a degree of freedom to accommodate the restriction that the rounded proportions must still sum to unity. A fast rounding algorithm turning a multiplier rule into a multiplier method is proposed in Happacher and Pukelsheim (1996, page 378; 1998, page 102), Dorfleitner and Klein (1999, page 147).

The theory of rounding methods is summarized in the seminal monograph of Balinski and Young (1982). Reviewing the history of the apportionments of seats in the US House of Representatives, the authors show that the apportionment methods proposed by eminent politicians such as John Quincy Adams, Daniel Webster, and Thomas Jefferson are, from a systematic viewpoint, the stationary multiplier methods arising from the classical rounding functions of rounding up, standard rounding, and rounding down.

In statistics rounding methods are useful for the design of experiments (multiplier method with rounding up, Adams), contingency tables (multiplier method with standard rounding, Webster), and sampling theory (multiplier method with rounding down, Jefferson), see Pukelsheim (1993, Chapter 12), Wainer (1998), Dorfleitner and Klein (1999). Balinski and Ramírez (1999) use stationary rounding methods for just-in-time production system optimization.

The Theorem also permits a transparent analysis of the asymptotics for large accuracies, $n \to \infty$. Since in the distribution $h_{c,q,n}$ the double sum has terms bounded of the order $n^{-(k-j)}$, the limit depends only on the term with $k = j$:

$$h_{c,q,n}(d) = \frac{1}{\mu_{c,q,n}^{c-1}} \sum_{j=0}^{c} (-1)^j \binom{c}{j} \binom{n+d+j-1}{c-1} \left\{ \frac{c}{2} - d - j \right\}_+^{c-1} + O\left(\frac{1}{n}\right),$$

$$\lim_{n\to\infty} h_{c,q,n}(d) = \sum_{j=0}^{c} \frac{(-1)^j}{(c-1)!} \binom{c}{j} \left\{ \frac{c}{2} - d - j \right\}_+^{c-1} = h_c(d),$$

say. The limit is the same for all stationarity parameters $q \in [0, 1]$. The distribution h_c is the c-fold convolution of the uniform distribution on the interval

$[-1/2, 1/2]$, see Johnson, Kotz and Balakrishnan (1995, Chapter 26.9). The c degrees of freedom are explained by the accumulation of $c-1$ remainder terms, $\sum_{i=1}^{c-1} U_i$ with $U_1, \dots, U_{c-1}$ stochastically independent and uniformly distributed on $[-1/2, 1/2]$, plus a further degree of freedom for a final rounding of the accumulated remainders, see Diaconis and Freedman (1979, page 361), Happacher and Pukelsheim (2000, page 156).

Seal (1950) makes a point that the study of the convolution of a uniform distribution has a long tradition, and that quite a few earlier results on this and related subjects are neglected by later authors, see Appendix B for an excerpt from his note. Yet, to our knowledge, the finite accuracy discrepancy distribution $h_{c,q,n}$ is new. Mosteller, Youtz and Zahn (1967, page 856) derive the discrepancy distributions for three and four categories and standard rounding, $h_{3,1/2,n}$ and $h_{4,1/2,n}$, and indicate an approximation by a normal distribution with mean zero and variance $c/12$, see also Mitra and Banerjee (1971). Kopfermann (1978, page 43; 1991, page 185) proves that for large accuracies the probability of a vanishing discrepancy tends to $h_c(0)$.

Diaconis and Freedman (1979, page 361) show that, under standard rounding, the asymptotic distribution h_c holds for any weight distribution that is absolutely continuous, and this result is extended to all stationary rounding functions by Balinski and Rachev (1993, page 479). Both results apply as the accuracy tends to infinity. Whether, for finite accuracy, the discrepancy distribution can be determined for a weight distribution other than uniform remains a challenge. The challenge may be more theoretical than practical, though, since among the many empirical examples that we have studied none grossly invalidates the distribution given above. Moments of the discrepancy distribution $h_{c,q,n}$, and optimality properties of the multiplier $\mu_{c,q,n}$ are discussed by Happacher and Pukelsheim (1996, 1998, 2000). The present theorem originates with the dissertation Happacher (1996), which includes additional literature, more examples, and further asymptotic ramifications.

3 Derivation of the distribution

We derive the distribution of the total $T_{c,q}(v)$, with $c \geq 2$ and $q \in [0, 1]$ fixed. For a given multiplier $v > 0$ and an integer $t \in [v - qc, v + (1-q)c]$, the probability of the event $\{T_{c,q}(v) = t\}$ is claimed to be

$$g(v, t) = \frac{1}{v^{c-1}} \sum_{j=0}^{c} (-1)^j \binom{c}{j} \sum_{k=j}^{c} \binom{c-j}{k-j} \binom{t+j-1}{c+j-k-1} f(j, k),$$

where $f(j, k) = \{v + (1-q)c - t - qj - (1-q)k\}_+^{c-1}$. The identity $g(\mu_{c,q,n}, n+d) = h_{c,q,n}(d)$ then establishes the Theorem. We concentrate on the case $v \geq qc$, whence $t \geq 1$. Smaller multipliers create boundary effects at $t = 0$ and need extra care, see Happacher (1996).

Scaling with the multiplier v, $(w_1 \dots, w_c) \mapsto (vw_1, \dots, vw_c)$, maps the uniform distribution on the probability simplex $\mathscr{S}$ into the uniform distribution on the scaled simplex $\mathscr{S}(v)$. The surface volume of $\mathscr{S}(v)$ is well known to be $v^{c-1}\sqrt{c}/(c-1)!$.

The event $\{T_{c,q}(v) = t\}$ is decomposed by classifiying the rounding results according to how they relate to the "simplicial walls" $N_i = 0$. For $j = 0, \ldots, c$ we define the classes

$$\mathscr{A}(j) = \left\{ N = (N_1, \ldots, N_c) \in \mathbb{N}^c : \#\{i : N_i = 0\} = j \text{ and } \sum_{i=1}^{c} N_i = t \right\}.$$

Because of $t \geq 1$ the class $\mathscr{A}(c)$ is empty. Otherwise, the cardinality of $\mathscr{A}(j)$ is determined by the number of distinguishable distributions of t balls into $c - j$ cells in which no cell remains empty. The binomial formula in Feller (1968, (ii) on page 38) yields

$$\#\mathscr{A}(j) = \binom{c}{j}\binom{t-1}{c-j-1}.$$

For $N \in \mathscr{A}(j)$ the range inequalities, in the format $r_q(x) - 1 + q \leq x \leq r_q(x) + q$, motivate the introduction of the following *relevance set anchored at N*,

$$R(N) = \left\{ (x_1, \ldots, x_c) \in [0, \infty)^c : x_i \in [N_i - 1 + q, N_i + q] \forall i \text{ and } \sum_{i=1}^{c} x_i = v \right\},$$

It is not hard to show that $\{T_{c,q}(v) = t\} = \bigcup_{j=0}^{c-1} \bigcup_{N \in \mathscr{A}(j)} R(N)$. The decomposition is almost surely disjoint, since the intersection of two relevance sets is empty or consists exclusively of ties. The following Lemma states that each anchor class $A(j)$ leads to relevance sets with a constant surface volume.

Lemma: *For $j = 0, \ldots, c-1$, every relevance set $R(N)$ with anchor $N \in \mathscr{A}(j)$ has surface volume*

$$V(j) = \sum_{k=0}^{j} \sum_{\ell=0}^{c-j} (-1)^{k+\ell} \frac{\sqrt{c}}{(c-1)!} \binom{j}{k}\binom{c-j}{\ell} f(k+\ell, j+\ell). \qquad \square$$

The proof of the Lemma is deferred to Appendix A. We now obtain

$$g(v,t) = \frac{1}{v^{c-1}} \sum_{j=0}^{c-1} \binom{c}{j}\binom{t-1}{c-j-1} \sum_{k=0}^{j} \sum_{\ell=0}^{c-j} (-1)^{k+\ell} \binom{j}{k}\binom{c-j}{\ell} f(k+\ell, j+\ell).$$

After changing the summation from $k + \ell$ to j and from $j + \ell$ to k, then using the identity $\binom{b}{a}\binom{c}{b} = \binom{c}{a}\binom{c-a}{c-b}$ to shift ℓ from two of the binomial coefficents into the third, and finally evaluating the resulting sum over ℓ

through Feller (1968, (12.9) on page 64), we arrive at the formula for $g(v,t)$ claimed above. The proof is complete.

If the integer t lies outside the interval $[v-qc, v+(1-q)c]$ then we find $g(v,t)=0$. Hence $g(v,t)$ automatically takes care of which integers t belong to the support or not. The distribution simplifies if the stationarity parameter q is an integer,

$$q=0,1 \Rightarrow g(v,t)=\frac{1}{v^{c-1}}\binom{t+qc-1}{c-1}\sum_{j=0}^{c}(-1)^j\binom{c}{j}\{v-t-j\}_+^{c-1}.$$

When the rounding rule is based on rounding up ($q=0$) or rounding down ($q=1$), the discrepancy distributions $h_{c,0,n}$ and $h_{c,1,n}$ simplify accordingly.

Appendix A: Proof of the lemma

For $y=(y_1,\dots,y_c)\in\mathbb{R}^c$, let $F_y^\infty(v)$ denote the surface volume of the simplicial polytope $\mathscr{S}_y^\infty(v)=\{(x_1,\dots,x_c)\in[y_1,\infty)\times\cdots\times[y_c,\infty):\sum_{i=1}^c x_i=v\}$. The translation $x'=x-y$ leaves volume invariant, and changes the constraint into $\sum_{i=1}^c x_i'=v-\sum_{i=1}^c y_i$. It follows that $F_y^\infty(v)=F_0^\infty(v-\sum_{i=1}^c y_i)=\{v-\sum_{i=1}^c y_i\}_+^{c-1}F_0^\infty(1)$, where $F_0^\infty(1)=\sqrt{c}/(c-1)!$ is the surface volume of the probability simplex $\mathscr{S}_0^\infty(1)=\mathscr{S}$. More generally, for $a,b\in\mathbb{R}^c$ we consider the set

$$\mathscr{S}_a^b(v)=\left\{(x_1,\dots,x_c)\in[a_1,b_1)\times\cdots\times[a_c,b_c):\sum_{i=1}^c x_i=v\right\}.$$

Its surface volume $F_a^b(v)$ is obtained by using the inclusion-exclusion principle,

$$F_a^b(v)=\sum_{m=0}^c(-1)^m\sum_{y\in Y(m)}F_y^\infty(v)=\sum_{m=0}^c(-1)^m\frac{\sqrt{c}}{(c-1)!}\sum_{y\in Y(m)}\left\{v-\sum_{i=1}^c y_i\right\}_+^{c-1},$$

where the set $Y(m)$ comprises all vectors $y=(y_1,\dots,y_c)$ sharing m components with b and the remaining $c-m$ components with a; compare the discussion of multivariate cumulative distribution functions in Billingsley (1986, page 177).

Any anchor $N\in\mathscr{A}(j)$ has exactly j components equal to zero. We do not change volume by assembling them in the initial section, $N_1=\cdots=N_j=0$. This gives $V(j)=F_a^b(v)$, where $a=(0,\dots,0,N_{j+1}-1+q,\dots,N_c-1+q)$ and $b=(q,\dots,q,N_{j+1}+q,\dots,N_c+q)$. Now let the vector y be composed by choosing $m=k+\ell$ components from b, in that k are from the initial section and ℓ are from the final section of b, and filling in the remaining entries with the components of a, so that $j-k$ are from the initial section and $c-j-\ell$ are from the final section of a. Then we obtain $\{v-\sum_{i=1}^c y_i\}_+^{c-1}=f(k+\ell,j+\ell)$, and

$$V(j) = \sum_{k=0}^{j} \sum_{\ell=0}^{c-j} (-1)^{k+\ell} \frac{\sqrt{c}}{(c-1)!} \binom{j}{k} \binom{c-j}{\ell} f(k+\ell, j+\ell).$$

The proof of the Lemma is complete.

Appendix B: Excerpt from Seal (1950)

12 *December* 1950
The Joint Editors, *The Journal of the Institute of Actuaries Students' Society*

Spot the prior reference

Sirs,

A gem which has fast become a favourite relaxation of the more priggish type of mathematician is one which might be called: Spot the prior reference. The equipment is elementary – a good memory or an extensive system of card records with appropriate cross-references. The object of the game is simple – the infliction of a blow to the self-esteem of a colleague while retaining an appearance of scientific detachment.

The first move is made by an author who inadvertently omits that thorough search through the numerous volumes of *Mathematical Review* and the *Zentralblatt für Mathematik* which nowadays occupies as much of a mathematician's time as the preparation of a supposedly original article. The second move falls to the editor whose referees fail to notice that the work submitted has already appeared in print in a substantially similar form ten, twenty or even a hundred years earlier – and the game is on. The reviewer now appears on the scene and scores one or more points according to the number of years he can span and the amount of scorn he can convey in a politely worded account of the author's limitations. The game continues as a third and fourth writer show that even the reviewer himself has not found the site of original publication of the material presented. Final honours go to the player who has revealed the greatest number of missing references in the previous writers' articles.

An amusing example of the game in progress is to be found in the 1944, 1945 and 1947 volumes of the *Philosophical Magazine* and concerns a subject which has recently been discussed in your pages, namely, the probability distribution of the sum of n continuous or discrete rectangular variates.

[Follows a paragraph mentioning five notes, including one "provided by Mr Packer in this *Journal*".]

Extraordinarily, neither Simpson's (1757), Lagrange's (1773) nor Laplace's (1776, 1781, 1810) names are mentioned in any of these five notes, though a reviewer referred to the latter. In fact, the references uncovered by these writers and by Mr Packer only represent a small part of the numerous independent derivations of the distributions, continuous and discrete, under consideration. As a demonstration of the efficiency of my own card index and to provide your readers with a quiver of weapons with which to participate in the game in future years, I append a list of the post-classical derivations I have encountered, most of which have not, to my knowledge, been collected together previously. In each case I indicate whether the derivation relates to the continuous or the discrete case (c and/or d), whether or not the author refers to any earlier solutions (e or $\bar{e}$), and whether or not the method used was sufficiently different to be considered (by me) original at the time it was written (o or $\bar{o}$).

[Follows a list of thirteen papers from Lobatschewsky (1842) through Auerbach (1933), together with their Seal indicators $c, d, e, \bar{e}, o, \bar{o}$.]

Two comments may be made on the above list. It will be noticed that I disagree with Mr Packer that Rietz's proof was any simpler than Laplace's. In fact it was Laplace's own derivation (for which, in modern notation, see the Appendix to my paper in the 1949 Swiss *Bulletin*) with only formal differences. Secondly, it may be mentioned that a 3-decimal table similar to Mr Packer's Table 3 is provided by Auerbach in the paper cited.

I hasten to assure you that the provision of this list, which contains four original proofs between Laplace and Rietz, is not intended as a criticism of Mr Packer's excellent note. The modest title of your *Journal* would, in any case, forbid the scoring of points on the part of your averagely priggish correspondent who signs himself.

Yours faithfully, H. L. Seal
295 Madison Avenue, New York 17

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