

From Experimental Design to Proportional Representation – A Tribute to Bikas Kumar Sinha and Bimal Kumar Sinha on Their 75th Birthday

Friedrich Pukelsheim

Institute for Mathematics, University of Augsburg, Germany

Received: 21 March 2020; Revised: 06 July 2020; Accepted: 10 July 2020

Abstract

The fields of design of statistical experiments and of proportional representation systems share the problem of approximating virtually continuous weights by distinctly discrete proportions. We explain the common aspects of the problem, review three highlights of the underlying theory, and illustrate the usefulness of the results by examples relating to electoral systems, sampling plans, and experimental design.

Key words: Apportionment rules; Divisor methods; Coherence theorem; Seat bias theorem; Goodness-of-fit theorem.

1. Prologue

The common denominator of Professor Bikas Sinha and Professor Bimal Sinha and myself is our joint research in the design and analysis of statistical experiments, dating back to the last millennium. I have fond memories of the discussions with one or the other of the twin professors when visiting them in Delhi 1988, at UMBC 1991, in Kolkata 1994 and, conversely, playing host in Augsburg 1993. Our relations culminated in the joint paper Pukelsheim and Sinha (1995) which merged Bikas' expertise in exact block designs with my interest in optimal approximate designs.

The two fields, optimality analysis of approximate designs and combinatorial construction of block designs, exhibit a complementary character. The first forms part of continuous mathematics, the second, of discrete mathematics. The transition from the continuous domain to the discrete domain was one of the topics dealt with in Pukelsheim and Sinha (1995). Beyond the statistical origin, the transition problem turned out to be quite intriguing by itself. When I stumbled into the problem I did not know nor preview that it would keep me busy to date. In the sequel I shall review three highlights that I found particularly intriguing.

2. Apportionment Rules

Suppose there is a set of experimental conditions, labeled $j = 1, \dots, \ell$, for which a (continuous) experimental design indicates that a share w_j of all observations is to be realized under condition j . That is, the shares w_1, \dots, w_ℓ are nonnegative weights (*i.e.* “continuous” real numbers) summing to unity.

Practically, limited funds restrict the experimenter to a finite sample size h , say. What we seek, then, is an (exact) experimental design x_1, \dots, x_ℓ consisting of frequencies (*i.e.* “exact” natural numbers) summing to h such that the proportion x_j/h of observations becomes as equal as can be to the optimal weight w_j , that is,

$$x_1 \approx hw_1, \quad \dots, \quad x_\ell \approx hw_\ell.$$

If all scaled weights hw_1, \dots, hw_ℓ happen to be natural numbers, the exact solution is $x_1 = hw_1, \dots, x_\ell = hw_\ell$ and the job is done.

However, the quantities hw_j generally fail to be natural numbers and cannot serve as the frequencies sought. Thus, in general, pure proportionality is impossible. The question arises how to split the sample size h into frequencies x_1, \dots, x_ℓ that are reasonably – in some sense or other – proportional to the weights w_1, \dots, w_ℓ .

Evidently the terminology is not geared towards the setting of experimental design. The notation originates from a field that comes with an isomorphic problem, the study of proportional representation systems in parliamentary democracies.

The typical setting is the following. At an election of a parliament of house size h , ℓ political parties finish with vote shares w_1, \dots, w_ℓ . The electoral law stipulates an apportionment rule allotting the h parliamentary seats to the ℓ parties by way of seat contingents x_1, \dots, x_ℓ . The apportionment rule should be such that the seat contingent x_j of party j is close to what the party would claim under pure proportionality, $x_j \approx w_j h$. Alas, since seats are assigned to human beings who are indivisible, the seat contingents x_j must be natural numbers and cannot in general become equal to the fractional shares $w_j h$.

The number of parties contesting an election usually varies from one election to the other. Hence an apportionment rule is suitable for an electoral law only if its formulation does not involve the size of the party system, ℓ . To this end a “vote vector” (v_1, v_2, \dots) is taken to be an infinite sequence of nonnegative numbers that breaks off after a last nonzero term v_ℓ and ends in a tail of zeros. Similarly, a “seat vector” (x_1, x_2, \dots) is taken to be a sequence of natural numbers terminating with zeros. This convention allows an effectual introduction of apportionment rules.

By definition, an “apportionment rule” A maps every house size h and every vote vector $v = (v_1, v_2, \dots)$ into a non-empty “solution set” $A(h; v)$ consisting of seat vectors $x = (x_1, x_2, \dots)$ that have a component sum equal to the house size h and that inherit all zeros of the vote vector v : $v_j = 0 \Rightarrow x_j = 0$ for all j .

The notion of an “allotment method” in Hylland (1978, page 5) is quite similar.

An apportionment rule as defined above is taken to be a set-valued mapping in order that it may accommodate tied situations. A prototype tie arises when three seats are apportioned among two equally strong parties, with 5000 votes each say. Either the first party is allotted one seat and the second two, or the first party two and the second one. As both solutions are equally justified, the solution set comprises both: $A(3; (5000, 5000)) = \{(1, 2), (2, 1)\}$. See Table 3 below for another example.

Note that vote vectors and seat vectors, which a minute ago were agreed to be infinite sequences that terminate with zeros, are jotted down as vectors of finite length simply by omitting the vanishing tails.

Contemplation of which apportionment rules are practically reasonable or not leads to a subclass of procedures called apportionment methods.

3. Apportionment Methods

The abstract notion of apportionment rules embraces procedures obviously unfit for concrete usage. For instance, whatever the vote vector v , all seats could be allocated to the party listed first, $A(h; v) = \{(h, 0, 0, \dots)\}$, a dictatorial rule. The ensemble of all apportionment rules is reduced to a reasonable subset by imposing a set of desirable principles.

There are five basic principles. The first four are anonymity, balancedness, concordance, and decency. They suggest themselves as soon as they are formulated. The fifth principle, exactness, has a more technical flavor.

Anonymity. An apportionment rule A is called “anonymous” when every rearrangement of a vote vector induces the same rearrangement of the accompanying seat vector. Whether a party is listed first or last does not matter, its seat contingent stays the same.

Balancedness. An apportionment rule A is called “balanced” when any two parties that are equally strong differ by at most one seat: $v_i = v_j \Rightarrow |x_i - x_j| \leq 1$. It is unrealistic to insist on equality, but a difference of two or more seats will not be tolerated.

Concordance. An apportionment rule A is called “concordant” when of any two parties the stronger party is allotted at least as many seats as the weaker party: $v_i > v_j \Rightarrow x_i \geq x_j$. A discordant result, giving the stronger party fewer seats than the weaker party, is rejected.

Decency. An apportionment rule A is called “decent” when scalings of the vote vector do not change the solution set: $A(h; \frac{1}{d}v) = A(h; v)$ for all $d > 0$. Hence absolute vote counts v_j and relative vote shares $w_j = v_j/(v_1 + v_2 + \dots)$ yield the same solutions.

Exactness. An apportionment rule A is called “exact” when every sequence of vote vectors $v(n)$, $n \geq 1$, that converges to a seat vector x induces sequences of solution vectors $y(n) \in A(h; v(n))$, $n \geq 1$, that converge to x , too, provided $x_j = 0 \Rightarrow v_j(n) = 0$ for all $n \geq 1$.

Persuasive as they are the five principles suffer from a common weakness. They are insensitive to the house size h and the size of the party system ℓ . They solely deal with variations in the vote vector v_1, \dots, v_ℓ . Anonymity permutes its components, balancedness and concordance compare them by pairs, decency rescales them, and exactness addresses the case when the vote vector coincides with a seat vector or converges to a seat vector. Exactness links the continuum character of the input domain, the space of vote vectors, to the discrete nature of the output range, the grid of seat vectors.

An apportionment rule that is anonymous, balanced, concordant, descent and exact is called an “apportionment method”. Almost all procedures that can be found in electoral laws qualify as apportionment methods.

4. Fairness or Coherence Principle

There is a sixth principle, fairness, also known as coherence. Fairness properly incorporates the two parameters missed out so far, the house size h and the size of the party system ℓ . Essentially, given a large house size H and a large party system $1, \dots, L$ with a solution x_1, \dots, x_L , every subsystem $1, \dots, \ell$ with its induced seat total $h = x_1 + \dots + x_\ell$ admits the subvector x_1, \dots, x_ℓ as a solution.

Fairness implements the idea that the whole and its parts must fit together in a coherent way. Balinski and Young (2001, page 141) put it this way: “An inherent principle of any fair division is that *every part of a fair division should be fair.*”

Fairness. An apportionment method A is called “fair”, or “coherent”, when it satisfies (a) coherence of subproblems and (b) coherence of substituted solutions.

(a) “Coherence of subproblems” means that, given a grand seat vector $(x_1, \dots, x_L) \in A(H; v_1, \dots, v_L)$ for a system of L parties, the subvector (x_1, \dots, x_ℓ) is a member of the ℓ -subsystem solution set $A(h; v_1, \dots, v_\ell)$, where $h = x_1 + \dots + x_\ell$ and $\ell < L$.

(b) “Coherence of substituted solutions” means that, given a grand seat vector $x = (x_1, \dots, x_L) \in A(H; v_1, \dots, v_L)$ and an ℓ -subsystem seat vector $y = (y_1, \dots, y_\ell) \in A(h; v_1, \dots, v_\ell)$, substitution of y into x yields a grand solution $(y_1, \dots, y_\ell, x_{\ell+1}, \dots, x_L) \in A(H; v_1, \dots, v_L)$, where $h = x_1 + \dots + x_\ell$ and $\ell < L$.

Coherence of subproblems (a) is a top-down concept. It demands that every subvector that is extracted from a grand solution is a valid solution for the associated subproblem. Coherence of substituted solutions (b) is a bottom-up idea. Tied solutions for subproblems, when substituted into the grand solution, yield tied grand solutions.

The above formalization of fairness makes sense only in the presence of anonymity. With anonymity, the order in which parties are listed is negligible. The system may be rearranged so that the ℓ -subsystem parties are assembled in the initial section $1, \dots, \ell$. For this reason the notion of fairness asks for apportionment methods, not just for apportionment rules.

5. Divisor Methods

My first highlight of apportionment theory is the Coherence Theorem. It states that the six principles characterize an important class of apportionment rules, divisor methods. Divisor methods scale the votes v_j into interim quotients v_j/d and then round the quotients to a neighboring integer to obtain the seat numbers x_j . The methods differ by which rounding rule they apply. In turn, the applicable rounding rule determines which divisors d are feasible to exhaust the preordained house size, $x_1 + \cdots + x_\ell = h$.

Generally, a rounding rule maps an interim quotient v_j/d that lies in the integer interval $[n - 1; n]$ to one of the endpoints. To this end the interval is equipped with a “signpost” $s(n)$. Below $s(n)$, the quotient is rounded downwards to the singleton $\{n - 1\}$, above, upwards to the singleton $\{n\}$. If the quotient is equal to $s(n)$, it is rounded ambiguously to the two-element set $\{n - 1, n\}$. The ambiguous rounding at the signpost proper turns a “rounding rule” R into a set-valued mapping:

$$R\left(\frac{v_j}{d}\right) = \begin{cases} \{n\} & \text{in case } \frac{v_j}{d} \in (s(n); s(n+1)), \\ \{n-1, n\} & \text{in case } \frac{v_j}{d} = s(n) > 0, \\ \{0\} & \text{in case } \frac{v_j}{d} = 0. \end{cases}$$

Hence a rounding rule R is specified by its signposts. A general “signpost sequence” $s(0), s(1), s(2), \dots$ needs to fulfill three properties. (a) It starts with $s(0) = 0$. (b) For $n \geq 1$ the term $s(n)$ is localized in the integer interval $[n - 1; n]$. (c) If there is a signpost hitting the left limit of its localization interval, $s(m + 1) = m$, then all signposts stay below their right limits, $s(n) < n$, and if there is a signpost hitting the right limit, $s(m + 1) = m + 1$, then all signposts stay above their left limits, $s(n) > n - 1$. The “left-right disjunction” (c) becomes instrumental when verifying exactness of the accompanying divisor method.

By definition, the “divisor method D with rounding rule R ” maps a house size h and a vote vector v_1, v_2, \dots into the set of seat vectors $x = (x_1, x_2, \dots)$ given by

$$D(h; v) = \left\{ x \mid x_1 \in R\left(\frac{v_1}{d}\right), x_2 \in R\left(\frac{v_2}{d}\right), \dots \text{ for some } d > 0, \text{ and } x_1 + x_2 + \cdots = h \right\}.$$

That is, the seat contingent x_j of party j is obtained by scaling its vote count v_j by a divisor d and rounding the interim quotient v_j/d to an adjacent natural number x_j .

The role of the divisor d is to ensure that all h seats are meted out. If d is too small then the interim quotients v_j/d are too large for their roundings to sum to h . If d is too large then the quotients are too small. Thus the divisor acts as a “sliding controller” which is adjusted until the desired total is met, $x_1 + x_2 + \cdots = h$.

Set-valued mappings again emerge due to the handling of ties. Suppose there are two parties whose interim quotients hit the $(m + 1)$ -st and n -th signposts, $v_1/d = s(m + 1)$ and $v_2/d = s(n)$, whence $R(v_1/d) = \{m, m + 1\}$ and $R(v_2/d) = \{n - 1, n\}$. If the parties' fair apportionment is $m + n$ seats, the first quotient may be rounded downwards and the second upwards, or vice versa. It is not up to mathematics to select one of the two options. The decision is left open by offering two solutions, $D(m + n; v_1, v_2) = \{(m, n), (m + 1, n - 1)\}$.

Coherence Theorem. *An apportionment rule A is anonymous, balanced, concordant, decent, exact and fair if and only if A is “compatible” with a divisor method D , in the sense that the inclusion $A(h; v) \subseteq D(h; v)$ holds true for all house sizes h and for all vote vectors v .*

The significance of the Coherence Theorem is that it provides helpful practical guidance. If we agree that the six principles are conditions *sine qua non*, there is no need to look outside the class of divisor methods.

Compatibility of a method A with a method D implies that they agree whenever the solution set $D(h; v)$ is a singleton. For, if $D(h; v) = \{x\}$ then $\emptyset \neq A(h; v) \subseteq D(h; v)$ forces $A(h; v) = \{x\}$.

Yet, in the presence of ties, A may differ from D . Then the solution set of D contains two or more seat vectors. In fact, a divisor method D is “complete” in the sense that it enumerates all tied solutions possible. However, a fair apportionment method A may abstain from completeness by implementing a tie resolution strategy.

For example, the electoral law for the Spanish Congreso de los Diputados resolves ties by following the motto “Stronger Parties First”. If there are two parties whose interim quotients hit signposts, then the party with more votes is rounded upwards and the party with fewer votes is rounded downwards. Completeness is lost, yet the six principles persist.

The direct part of the proof of the Coherence Theorem is challenging. Starting from an apportionment rule A that satisfies the five basic principles and fairness, a signpost sequence needs to be constructed so that the induced divisor method D is such that A is compatible with D . Conversely, it is easy to verify that every divisor method satisfies the five basic principles and fairness. For details Balinski and Young (2001, page 141) or Pukelsheim (2017, page 168).

6. Stationary Divisor Methods

The multitude of divisor methods still is huge. There are as many divisor methods as there are rounding rules, and there are as many rounding rules as there are signpost sequences. Within this universe there is a one-parameter family, stationary divisor methods, lining up the three apportionment methods that in many respects serve as reference procedures: the divisor method with upward rounding, the divisor method with standard rounding, and the divisor method with downward rounding.

Stationary divisor methods are indexed by a “split” parameter $0 \leq r \leq 1$. The stationary divisor method with split r has signposts $s_r(n) = n - 1 + r$. As a consequence the interval $[n - 1; n]$ is split into the section $[n - 1; n - 1 + r]$ where numbers are rounded downwards to $n - 1$, and the section $[n - 1 + r; n]$ where the rounding is upwards to n . The proper split point $n - 1 + r$ may be rounded either way. The methods are termed “stationary” because of the stationary position of the signposts in their localization intervals. Whatever the interval, the distance from the signpost to the left endpoint is r , to the right endpoint, $1 - r$.

Three members of the family of stationary divisor methods stand out to be of particular importance.

The “divisor method with upward rounding” comes with split $r = 0$. If an interim quotient v_j/d has a nonzero fractional part, it is rounded upwards. If the quotient happens to be a whole number, it may stay as is or it may be rounded to the whole number above.

The “divisor method with standard rounding” belongs to split $r = 1/2$. An interim quotient v_j/d is rounded downwards or upwards according as its fractional part is less than one half or greater than one half. If the quotient happens to have a fractional part equal to one half, it may be rounded either way, downwards or upwards.

The “divisor method with downward rounding” has split $r = 1$. If a quotient v_j/d has a nonzero fractional part, it is truncated to its integer part. If the quotient happens to be a whole number, it stays as is or is rounded to the whole number below.

My second highlight of apportionment theory is the Seat Bias Theorem. Parliaments typically are hesitant to amend an apportionment method once it has found its way into the electoral law. When a method is used repeatedly at several elections, the question arises whether it predictably benefits some participants and disadvantages others. More pointedly, does a method on average favor stronger parties at the expense of weaker parties?

To answer this question we rearrange parties by decreasing vote shares. Some electoral laws stipulate a threshold t lest a party should be dropped from consideration. For example Germany requires at least five percent of all valid votes for a party to participate in the seat apportionment process. Thus parties are taken to be ordered according to $w_1 \geq \dots \geq w_\ell \geq t$. The key figure for the k -th strongest party is $x_k - hw_k$, the deviation of the actual seat contingent x_k from the proportional seat fraction hw_k . Assuming the vote shares to be uniformly distributed over the probability simplex $\Omega_\ell = \{ (w_1, \dots, w_\ell) \in [0; 1]^\ell \mid w_1 + \dots + w_\ell = 1 \}$, the expected value of $x_k - hw_k$ for large house sizes and conditional on decreasing vote shares designates the “seat bias” of the k -th strongest party. This seat bias acquires a telling format.

Seat Bias Theorem. *If seats are apportioned using the stationary divisor method with split r and if the threshold is set at t then the seat bias of the k -th strongest party is*

$$\lim_{h \rightarrow \infty} \mathbb{E} \left(x_k - hw_k \mid w_1 \geq \dots \geq w_\ell \geq t \right) = \left(r - \frac{1}{2} \right) \left(H_k^\ell - 1 \right) (1 - \ell t),$$

where $H_k^\ell = \sum_{n=k}^\ell (1/n)$ is a partial sum of the harmonic series.

The seat biases of all parties must sum to zero since $x_1 + \dots + x_\ell - h(w_1 + \dots + w_\ell) = h - h = 0$. That is, if some parties are advantaged, others are disadvantaged. Conversely, if some parties are disadvantaged then others are advantaged. One man's meat is another man's poison.

The three factors of the bias formula mirror three distinct aspects of the problem.

The “method factor” ($r - 1/2$) reflects the influence of the stationary divisor method under investigation. The factor is positive, zero, or negative according as the split r is larger than one half, equal to one half, or smaller than one half.

The “party factor” ($H_k^\ell - 1$) captures the impact of the party's rank-order k in a system of ℓ parties. In view of the approximation $H_k^\ell \approx \log \ell - \log k$ the factor changes sign when k passes $\ell/e \approx \ell/3$. Hence the party factor is positive for the top third of stronger parties, and negative for the bottom two thirds of weaker parties.

The “threshold factor” ($1 - \ell t$) describes the impact of the threshold t when ℓ parties are contesting the election. It affects the size of the bias, but not its sign. For seven parties and a five percent threshold, as in Table 1 below, the factor amounts to $0.65 \approx 2/3$.

All in all a method with split r larger than one half favors stronger parties at the expense of weaker parties. In particular the divisor method with downward rounding (which has $r = 1$) is the procedure most widespread in actual electoral laws. It is also known under the names of D'Hondt, Hagenbach-Bischoff, Jefferson.

A method with split smaller than one half favors weaker participants at the expense of stronger participants. An example is the divisor method with upward rounding (which has $r = 0$). Occasionally the method is used to allocate seats between districts by population figures.

The divisor method with standard rounding ($r = 1/2$) has method factor zero. All seat biases are zero, every party may expect its proportional due. On average no party is advantaged, nor is any party disadvantaged. The divisor method with standard rounding is the unique stationary divisor method that is “unbiased”.

The clue to the proof of the Seat Bias Theorem is the identity $x_k - hw_k = (r - 1/2)(\ell w_k - 1) + (x_k - y_k) + u_k(h)$, where $y_k \in R_r(h_r w_k)$ is an auxiliary seat contingent derived from the deterministic multiplier $h_r = h + \ell(r - 1/2)$, and where $u_k(h) = y_k - (h_r w_k - r + 1/2)$ is a rounding residual. It is easy to see that the first term yields the limit formula. The hard part is to show that the other two terms eventually average out to zero. For details see Pukelsheim (2017, page 139).

7. A Closer Look at The Assumptions

The assumptions underlying the Seat Bias Theorem raise suspicion as to its practical usefulness. Nobody would care for parliaments with “large” house sizes h near infinity. Fortunately, when the convergence behavior is scrutinized, the bias formula is seen to fit empirical data perfectly well for all practical purposes provided there are at least twice as many seats as there are parties participating, $h \geq 2\ell$.

Nor would we claim that uniformly distributed vote shares are a realistic model. Strong parties know that they are strong, and weak parties know they are weak. A distribution with pronounced peaks following opinion polls would be more meaningful. Luckily, essential parts of the proof of the Seat Bias Theorem target the rounding residuals $u_k(h)$ for which there is an invariance principle. Rounding residuals transpire to be uniformly distributed assuming no more than that the vote share distribution on the probability simplex Ω_ℓ is absolutely continuous.

In any case, confrontation of the theoretical seat bias formula with practical seat bias data confirms the formula to be an excellent and valid predictor.

Unbiasedness of the divisor method with standard rounding offers a cogent rationale that this is the superior method for use in electoral laws.

Fields other than the political sciences may aim at other features. The many facets of science afford a welcome opportunity for us to return to statistical topics such as sampling allocations and experimental designs.

8. Efficient Rounding of Sampling Allocations and Experimental Designs

The Goodness-of-Fit Theorem is my third highlight. Motivated by statistics and operations research it views the apportionment problem as an approximation task. Let f denote a goodness-of-fit criterion that assesses the quality of an approximation. Given a distribution with virtually continuous weights w_1, \dots, w_ℓ summing to unity, the task is to find a distribution with distinctly discrete weights $x_1/h, \dots, x_\ell/h$ that provides an f -optimal approximation. The optimization takes place over the set $\mathbb{N}^\ell(h)$ of integer vectors with ℓ components summing to h . Not surprisingly, the answer depends on the goodness-of-fit criterion f selected.

Goodness-of-Fit Theorem.

(a) *The divisor method with standard rounding yields solutions $(x_1, \dots, x_\ell) \in \text{DivStd}(h; w_1, \dots, w_\ell)$ that minimize the squared statistical distance criterion*

$$f_a(x_1, \dots, x_\ell) = \frac{(x_1 - hw_1)^2}{hw_1} + \dots + \frac{(x_\ell - hw_\ell)^2}{hw_\ell}.$$

(b) *The divisor method with downward rounding yields solutions $(x_1, \dots, x_\ell) \in \text{DivDwn}(h; w_1, \dots, w_\ell)$ that minimize the worst-overrepresentation criterion*

$$f_b(x_1, \dots, x_\ell) = \max \left\{ \frac{x_1}{hw_1}, \dots, \frac{x_\ell}{hw_\ell} \right\}.$$

(c) *The divisor method with upward rounding yields solutions $(x_1, \dots, x_\ell) \in \text{DivUpw}(h; w_1, \dots, w_\ell)$ that maximize the worst-underrepresentation criterion*

$$f_c(x_1, \dots, x_\ell) = \min \left\{ \frac{x_1}{hw_1}, \dots, \frac{x_\ell}{hw_\ell} \right\}.$$

The proof of the theorem is straightforward, see Pukelsheim (2017, page 185). We add a few comments for each of the three parts. The examples in Tables 1–3 are evaluated with the free Java program Bazi (www.th-rosenheim.de/bazi).

In part (a) the criterion f_a resembles the familiar χ^2 -statistic. However, the limiting distribution of f_a is a Lévy-stable distribution, not a χ^2 -distribution, see Heinrich et al. (2004). Nevertheless, the criterion is in excellent harmony with the constitutional imperative that all voters should contribute equally to the electoral outcome, see Pukelsheim (2017, page 186). Hence the divisor method with standard rounding is the authoritative and unbiased procedure for the apportionment of seats among parties by vote counts. It meets the ideal of “One Person, One Vote” in a superb manner.

Since 2008 the divisor method with standard rounding has been included in the election law for the German Bundestag. See Table 1 for an illustration.

Table 1: Divisor method with standard rounding. Apportionment of 709 seats, election to the 19th German Bundestag, 24 September 2017.

Political Party	Votes	Interim Quotient	Seats [DivStd]
“CDU”	12 447 656	199.8	200
“SPD”	9 539 381	153.1	153
“AfD”	5 878 115	94.4	94
“FDP”	4 999 449	80.2	80
“LINKE”	4 297 270	69.0	69
“GRÜNE”	4 158 400	66.7	67
“CSU”	2 869 688	46.1	46
Sum (Divisor)	44 189 959	(62 300)	709

In part (b) the criterion f_b pops up when allocating observations in stratified sampling schemes, as discussed by Pukelsheim (1997). The reciprocal of the criterion provides a lower bound for the variance efficiency. Maximizing the lower bound is equivalent to minimizing the criterion f_b . Hence given a target sample size n and strata weights w_1, \dots, w_ℓ , the number of observations per stratum is determined most efficiently by using the divisor method with downward rounding, $(n_1, \dots, n_\ell) \in \text{DivDwn}(n; w_1, \dots, w_\ell)$. See Table 2.

Table 2: Divisor method with downward rounding. Efficient proportional sampling plan for 30 observations, see Example 9.5 in Hedayat and Sinha (1991, page 272).

Stratum	Size	Interim Quotient	No. of Obs. [DivDwn]
“Stratum 1”	60	9.5	9
“Stratum 2”	90	14.3	14
“Stratum 3”	50	7.9	7
Sum (Divisor)	200	(6.3)	30

The divisor method with downward rounding is biased in favor of large weights at the expense of small weights, as noticed above. In sampling schemes, the weight w_j relates to the standard deviation in stratum j . The bias behavior means that overly many observations are allocated to strata where the variance is large and uncertainty is high. This rule of conduct appears to be purposive for the allocation of observations.

In part (c) the criterion f_c arises in the optimality theory of experimental designs, see Pukelsheim (2006, page 311). Here, w_1, \dots, w_ℓ are the weights of an optimal design with ℓ support points. If for a given sample size n the weights are discretized into frequencies n_1, \dots, n_ℓ , then the smallest term of the likelihood ratios $(n_j/n)/w_j$ turns out to be a universal efficiency bound. Universality means that this lower bound is meaningful for all optimality criteria that are of interest in this context (*i.e.* for all information functions). The best lower bound is the one that is largest. According to part (c) it is obtained using the divisor method with upward rounding.

Hence the divisor method with upward rounding is the recommended procedure to convert an optimum design into an efficient exact design for sample size n , $(n_1, \dots, n_\ell) \in \text{DivUpw}(n; w_1, \dots, w_\ell)$. See Table 3.

Table 3: Divisor method with upward rounding. Two equally justified (*i.e.* “tied”) efficient exact designs #1 and #2 for 9 observations that belong to the A -optimal design for cubic regression on $[-1; 1]$, see Pukelsheim (2006, page 224).

A-Optimal Support Point	A-Optimal Weight	Interim Quotient	No. of Obs. [DivUpw]	
			#1	#2
“-1”	0.151	1	1	2
“-0.464”	0.349	2.3	3	3
“0.464”	0.349	2.3	3	3
“1”	0.151	1	2	1
Sum (Divisor)	1	(0.151)	9	9

According to the previous section the divisor method with upward rounding is biased in favor of small weights at the expense of large weights. The consequence is that small weights are likely to be allocated more observations than pure proportionality would demand. In particular, even the tiniest weight is rounded upwards to at least one observation. Hence the discretization process preserves all support points of the optimal design. Thus the divisor method with upward rounding appears to be a purposive rule of conduct for the discretization of optimal designs.

9. Epilogue

In conclusion we realize that although the five apportionment principles and the notion of fairness were introduced with narratives from the proportional representation world, these concepts make perfectly good sense also in the contexts of sampling schemes and experimental designs.

This proves once again that problems that seemingly are far apart actually share common theoretical underpinnings, just like scientists who live far apart — like Bikas in Asia, Bimal in North America, and myself in Europe — stay united by standards common to all fields of science.

Acknowledgements

The author gratefully acknowledges valuable remarks of the referee and the Editor.

References

- Balinski, M. L. and Young, H. P. (2001). *Fair Representation – Meeting the Ideal of One Man, One Vote. Second Edition*. Washington DC: Brookings Institution.
- Hedayat, A. S. and Sinha, B. K. (1991). *Design and Inference in Finite Population Sampling*. New York: Wiley.
- Heinrich, L., Pukelsheim, F. and Schwingenschlögl, U. (2004). Sainte-Laguë’s chi-square divergence for the rounding of probabilities and its convergence to a stable law. *Statistics & Decisions*, **22**, 43–59.
- Hylland, A. (1978). *Allotment Methods: Procedures for Proportional Distribution of Indivisible Entities*. Mimeographed Report, John F. Kennedy School of Government, Harvard University. [Reprinted as Working Paper 1990/11, Research Centre, Norwegian School of Management. Sandvika, 1990.]
- Pukelsheim, F. (1997). Efficient rounding of sampling allocations. *Statistics & Probability Letters*, **35**, 141–143.
- Pukelsheim, F. (2006). *Optimal Design of Experiments. SIAM Classics in Applied Mathematics Volume 50*. Philadelphia PA: SIAM.
- Pukelsheim, F. (2017). *Proportional Representation. Apportionment Methods and Their Applications. With a Foreword by Andrew Duff MEP. Second Edition*. Cham CH: Springer.
- Pukelsheim, F. and Sinha, B. K. (1995). Optimal block designs revisited – An approximate theory detour. *Statistics and Probability Letters*, **24**, 375–380.