**Friedrich Pukelsheim**

# Amendments: Proportional Representation

## Apportionment Methods and Their Applications
## With a Foreword by Andrew Duff MEP
## Second Edition

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| *Page* | *Amendment* |
|---|---|
| 80 | Amended text |
| 155 | Rectified typesetting |
| 214 | Addition of material |
| 253 | Rectified typesetting |
| 262 | Rectified typesetting |
| 271 | Rectified typesetting |

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On page 80, replace the second paragraph by the following:

The assumption that there exists a solution $y \in A(h;x)$ that is distinct from $x$ leads to a contradiction. Indeed, if $y$ has divisor $D(y) < 1 = D(x)$ then $[\![v_j/D(y)]\!] \geq [\![v_j/D(x)]\!]$ implies $y_j \geq x_j$ (Sect. 3.9). Since $y$ and $x$ share the component sum $h$, they would have to be identical which they are not. A similar argument excludes $D(y) > 1$. Hence we have $D(y) = 1 = D(x)$, and $x_j, y_j \in [\![x_j]\!]$ for all $j \leq \ell$. Because of the common component sum $h$ there exist two parties $i \neq k$ with $x_i > y_i$ and $x_k < y_k$. Balancedness implies $y_i = x_i - 1$ and $y_k = x_k + 1$. This yields $x_i, x_i - 1 \in [\![x_i]\!]$, which is possible only when the number to be rounded, $t = x_i$, hits the signpost $s(x_i)$ and this signpost is positive: $t = s(x_i) > 0$. Similarly $x_k, x_k + 1 \in [\![x_k]\!]$ entails $x_k = s(x_k + 1) > 0$. Setting $n := x_i \geq 1$ and $m := x_k + 1 \geq 2$, the joint fulfillment of $s(n) = n$ and $s(m) = m - 1$ violates the left-right disjunction (Sect. 3.8.c). Hence the solution set is a singleton, $A(h;x) = \{x\}$, and the claim is established.

Acknowledgement: I am grateful to Sebastian Goderbauer and Leonie Ermert, Aachen, for pointing out the awkward wording in the printed version.

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**Table 8.1** Majorization paths

*Majorization path of stationary divisor methods,* $\mathrm{DivSta}_r$

| *Rank* | *Votes* | $r = 0$ | | .3 | | .44 | | .47 | | .5 | | .8 | | .97 | | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 42 919 | 41 | +1 | 42 | | 42 | +1 | 43 | | 43 | +1 | 44 | | 44 | +1 | 45 |
| 2 | 13 048 | 13 | | 13 | | 13 | | 13 | | 13 | | 13 | | 13 | | 13 |
| 3 | 10 879 | 11 | | 11 | | 11 | | 11 | | 11 | | 11 | | 11 | | 11 |
| 4 | 10 581 | 10 | | 10 | +1 | 11 | | 11 | | 11 | | 11 | | 11 | | 11 |
| 5 | 9 547 | 10 | −1 | 9 | | 9 | | 9 | +1 | 10 | −1 | 9 | +1 | 10 | | 10 |
| 6 | 5 708 | 6 | | 6 | | 6 | | 6 | | 6 | | 6 | −1 | 5 | | 5 |
| 7 | 2 502 | 3 | | 3 | | 3 | | 3 | −1 | 2 | | 2 | | 2 | | 2 |
| 8 | 1 898 | 2 | | 2 | | 2 | | 2 | | 2 | | 2 | | 2 | −1 | 1 |
| 9 | 1 461 | 2 | | 2 | | 2 | −1 | 1 | | 1 | | 1 | | 1 | | 1 |
| 10 | 1 457 | 2 | | 2 | −1 | 1 | | 1 | | 1 | | 1 | | 1 | | 1 |
| Sum | 100 000 | 100 | | 100 | | 100 | | 100 | | 100 | | 100 | | 100 | | 100 |

*Majorization path of power-mean divisor methods,* $\mathrm{DivPwr}_p$

| *Rank* | *Votes* | $p = -\infty$ | | 0 | | .4 | | .6 | | 1 | | 18 | | 50 | | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 42 919 | 41 | +1 | 42 | +1 | 43 | | 43 | | 43 | +1 | 44 | | 44 | +1 | 45 |
| 2 | 13 048 | 13 | | 13 | | 13 | | 13 | | 13 | | 13 | | 13 | | 13 |
| 3 | 10 879 | 11 | | 11 | | 11 | | 11 | | 11 | | 11 | | 11 | | 11 |
| 4 | 10 581 | 10 | | 10 | | 10 | +1 | 11 | | 11 | | 11 | | 11 | | 11 |
| 5 | 9 547 | 10 | −1 | 9 | | 9 | | 9 | +1 | 10 | −1 | 9 | +1 | 10 | | 10 |
| 6 | 5 708 | 6 | | 6 | | 6 | | 6 | | 6 | | 6 | | 6 | −1 | 5 |
| 7 | 2 502 | 3 | | 3 | | 3 | | 3 | −1 | 2 | | 2 | | 2 | | 2 |
| 8 | 1 898 | 2 | | 2 | | 2 | | 2 | | 2 | | 2 | −1 | 1 | | 1 |
| 9 | 1 461 | 2 | | 2 | | 2 | −1 | 1 | | 1 | | 1 | | 1 | | 1 |
| 10 | 1 457 | 2 | | 2 | −1 | 1 | | 1 | | 1 | | 1 | | 1 | | 1 |
| Sum | 100 000 | 100 | | 100 | | 100 | | 100 | | 100 | | 100 | | 100 | | 100 |

Top: Stationary divisor methods. Bottom: Power-mean divisor methods. Both paths start with DivUpw, pass through DivStd, and finish with DivDwn. In contiguous seat columns, a seat is given up from a weaker party (−1) to a stronger party (+1).

On page 214, append the following text:

The formulas in Table 11.1 entail meaningful relations between the divisor method with standard rounding (DivStd) and the Hare-quota method with residual fit by greatest remainders (HaQgrR). For two parties the methods coincide (Sect. 9.10). For three or more parties the methods are distinct. The following relations assume at least as many seats as there are parties: $h \geq \ell \geq 3$.

**Theorem**

a. *The thresholds of representation for* HaQgrR *and for* DivStd*, and the natural thresholds for* DivStd *and for* HaQgrR *are increasing:*

$$a_{\text{HaQgrR}}(1) < a_{\text{DivStd}}(1) < b_{\text{DivStd}}(0) < b_{\text{HaQgrR}}(0).$$

b. *The midpoint of the vote share interval given one seat for* HaQgrR *is* $1/h$*, and the corresponding midpoint for* DivStd *is larger than* $1/h$*.*

*Proof*

a. The first inequality claims $1/(h\ell) < 1/(2h-2+\ell)$. A passage to reciprocals and a rearrangement of terms yield $(\ell-2)(h-1) > 0$, which holds by assumption. The second inequality is obvious. The third inequality claims $1/(2h+2-\ell) < (\ell-1)/h\ell$. Cross multiplication and a rearrangement of terms yield $(\ell-2)(\ell-1) < (\ell-2)h$. Cancellation of $\ell - 2 > 0$ leaves $h \geq \ell$, which holds by assumption.

b. Let $m(1) := \big(a(1) + b(1)\big)/2$ denote the midpoint of the vote share interval given one seat. For HaQgrR, we realize $m_{\text{HaQgrR}}(1) = 1/h$. For DivStd, the inequality $m_{\text{DivStd}}(1) > 1/h$ transforms into $h + \ell - 2 > 0$, which holds by assumption. □

The region that extends from the threshold of representation to the natural threshold begins with the smallest vote share which possibly enables representation, in least competitive constellations, and ends with the smallest vote share which certainly guarantees representation, even in most adverse situations. Part a says that the region for DivStd is a proper subset of the region for HaQgrR:

$$\big[a_{\text{DivStd}}(1); b_{\text{DivStd}}(0)\big] \subset \big[a_{\text{HaQgrR}}(1); b_{\text{HaQgrR}}(0)\big].$$

The transition from hope to certainty of obtaining a first seat starts later and materializes sooner for DivStd than for HaQgrR.

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**Table 13.4** Step 2 of the house size adjustment calculations, election to the 18th Bundestag 2013

| 18BT2013-Step 2 | | | CDU | | | |
|---|---|---|---|---|---|---|
| | State divisor | Dir. | Second Votes | Quotient | DivStd | Target |
| SH | 61 000 | 9 | 638 756 | 10.47 | 10 | 10 |
| MV | 60 000 | 6 | 369 048 | 6.2 | 6 | 6 |
| HH | 60 000 | 1 | 285 927 | 4.8 | 5 | 5 |
| NI | 66 000 | 17 | 1 825 592 | 27.7 | 28 | 28 |
| HB | 65 000 | 0 | 96 459 | 1.48 | 1 | 1 |
| BB | 60 000 | 9 | 482 601 | 8.0 | 8 | 9 |
| ST | 60 000 | 9 | 485 781 | 8.1 | 8 | 9 |
| BE | 62 000 | 5 | 508 643 | 8.2 | 8 | 8 |
| NW | 63 600 | 37 | 3 776 563 | 59.4 | 59 | 59 |
| SN | 61 000 | 16 | 994 601 | 16.3 | 16 | 16 |
| HE | 61 000 | 17 | 1 232 994 | 20.2 | 20 | 20 |
| TH | 60 000 | 9 | 477 283 | 8.0 | 8 | 9 |
| RP | 63 000 | 14 | 958 655 | 15.2 | 15 | 15 |
| BY | 58 300 | — | — | — | — | — |
| BW | 60 600 | 38 | 2 576 606 | 42.52 | 43 | 43 |
| SL | 67 000 | 4 | 212 368 | 3.2 | 3 | 4 |
| Target seat total | | | | | | 242 |

| | SPD | | | | | LINKE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Dir. | Sec. Votes | Quotient | DivStd | Target | Dir. | Sec. Votes | Quotient | DivStd | Target |
| SH | 2 | 513 725 | 8.4 | 8 | 8 | 0 | 84 177 | 1.4 | 1 | 1 |
| MV | 0 | 154 431 | 2.6 | 3 | 3 | 0 | 186 871 | 3.1 | 3 | 3 |
| HH | 5 | 288 902 | 4.8 | 5 | 5 | 0 | 78 296 | 1.3 | 1 | 1 |
| NI | 13 | 1 470 005 | 22.3 | 22 | 22 | 0 | 223 935 | 3.4 | 3 | 3 |
| HB | 2 | 117 204 | 1.8 | 2 | 2 | 0 | 33 284 | 0.51 | 1 | 1 |
| BB | 1 | 321 174 | 5.4 | 5 | 5 | 0 | 311 312 | 5.2 | 5 | 5 |
| ST | 0 | 214 731 | 3.6 | 4 | 4 | 0 | 282 319 | 4.7 | 5 | 5 |
| BE | 2 | 439 387 | 7.1 | 7 | 7 | 4 | 330 507 | 5.3 | 5 | 5 |
| NW | 27 | 3 028 282 | 47.6 | 48 | 48 | 0 | 582 925 | 9.2 | 9 | 9 |
| SN | 0 | 340 819 | 5.6 | 6 | 6 | 0 | 467 045 | 7.7 | 8 | 8 |
| HE | 5 | 906 906 | 14.9 | 15 | 15 | 0 | 188 654 | 3.1 | 3 | 3 |
| TH | 0 | 198 714 | 3.3 | 3 | 3 | 0 | 288 615 | 4.8 | 5 | 5 |
| RP | 1 | 608 910 | 9.7 | 10 | 10 | 0 | 120 338 | 1.9 | 2 | 2 |
| BY | 0 | 1 314 009 | 22.54 | 23 | 23 | 0 | 248 920 | 4.3 | 4 | 4 |
| BW | 0 | 1 160 424 | 19.1 | 19 | 19 | 0 | 272 456 | 4.496 | 4 | 4 |
| SL | 0 | 174 592 | 2.6 | 3 | 3 | 0 | 56 045 | 0.8 | 1 | 1 |
| Target seat total | | | | | 183 | | | | | 60 |

| | GRÜNE | | | | |
|---|---|---|---|---|---|
| | Dir. | Sec. Votes | Quotient | DivStd | Target |
| SH | 0 | 153 137 | 2.51 | 3 | 3 |
| MV | 0 | 37 716 | 0.6 | 1 | 1 |
| HH | 0 | 112 826 | 1.9 | 2 | 2 |
| NI | 0 | 391 901 | 5.9 | 6 | 6 |
| HB | 0 | 40 014 | 0.6 | 1 | 1 |
| BB | 0 | 65 182 | 1.1 | 1 | 1 |
| ST | 0 | 46 858 | 0.8 | 1 | 1 |
| BE | 1 | 220 737 | 3.6 | 4 | 4 |
| NW | 0 | 760 642 | 12.0 | 12 | 12 |
| SN | 0 | 113 916 | 1.9 | 2 | 2 |
| HE | 0 | 313 135 | 5.1 | 5 | 5 |
| TH | 0 | 60 511 | 1.0 | 1 | 1 |
| RP | 0 | 169 372 | 2.7 | 3 | 3 |
| BY | 0 | 552 818 | 9.48 | 9 | 9 |
| BW | 0 | 623 294 | 10.3 | 10 | 10 |
| SL | 0 | 31 998 | 0.48 | 0 | 0 |
| Target seat total | | | | | 61 |

The evaluations operate per state; that is, rowwise. The state divisor that is shown in the top box is applied to this state throughout the other three boxes. The maximum of the direct seats (Dir.) and the proportionality seats (DivStd) yields the target seat numbers.

**Table 14.3** Sub-apportionment, Schaffhausen 2016

| Sh2016Sub-app. | | SVP | SP | FDP | AL | GLP | ÖBS | EDU | CVP |
|---|---|---|---|---|---|---|---|---|---|
| | | 17 | 13 | 9 | 4 | 4 | 2 | 2 | 2 |
| Schaffhausen | 27 | 61 962-5 | 69 104-6 | 43 557-5 | 32 252-3 | 24 078-2 | 13 234-1 | 8 490-1 | 8 548-1 |
| Klettgau | 13 | 25 032-4 | 13 900-2 | 12 144-2 | 2 976-1 | 3 405-1 | 1 415-0 | 5 389-1 | 1 121-0 |
| Neuhausen | 8 | 4 990-3 | 4 708-2 | 2 070-1 | 874-0 | 476-0 | 1 122-1 | 535-0 | 1 742-1 |
| Reiat | 7 | 7 480-2 | 4 232-1 | 3 346-1 | 662-0 | 1 378-1 | 1 063-0 | 889-0 | 392-0 |
| Stein | 4 | 2 656-2 | 1 805-2 | 469-0 | 428-0 | | 358-0 | 155-0 | 955-0 |
| Buchberg-Rüdlingen | 1 | 456-1 | 98-0 | 62-0 | 26-0 | | | | |
| Party Divisor | | 1.1157 | 1.054 | 0.9 | 1 | 0.97 | 1 | 1 | 1.68 |

| (*continued*) | EVP | SVP/A | JSVP | SVP/K | JFSH | JUSO | SVP/S | District Divisor |
|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| Schaffhausen | 6 133-1 | 1 332-0 | 2 277-0 | 2 660-0 | 2 462-0 | 5 241-1 | 2 001-1 | 10 100 |
| Klettgau | 3 144-0 | 2 465-1 | 2 687-1 | 1 169-0 | 772-0 | 955-0 | 927-0 | 5 400 |
| Neuhausen | 177-0 | 257-0 | 164-0 | 196-0 | 210-0 | 261-0 | 178-0 | 1 788 |
| Reiat | 573-0 | 1 250-0 | 734-0 | 1 280-1 | 958-1 | 178-0 | 442-0 | 2 800 |
| Stein | 105-0 | 277-0 | 328-0 | 236-0 | 79-0 | | 62-0 | 1 140 |
| Buchberg-Rüdlingen | | | | | 80-0 | | | 500 |
| Party Divisor | 1.2 | 0.904 | 0.8 | 0.7 | 0.6 | 1 | 0.37 | |

The Schaffhausen SVP party votes (61 962) are divided by the district divisor for Schaffhausen (10 100) and by the party divisor for SVP(1.1157). The resulting quotient 5.4987 justifies 5 seats. Input and output are linked by a hyphen: 61 962-5. The other seat numbers are obtained similarly. The district divisors and the party divisors jointly ensure that each district meets its district magnitude and that each party exhausts its overall party-seats.

**Table 14.5** Hypothetical sub-apportionment for unionwide "parties"

| EP2014 | S & D | EPP | ALDE | GUE/NGL | ECR | GREENS | EFDD | NI | State |
|---|---|---|---|---|---|---|---|---|---|
| Sub-app. | 19 | 19 | 6 | 6 | 6 | 6 | 5 | 5 | Divisor |
| DE 8 | 8003628-2 | 10380101-2 | 1415641-0 | 2535053-1 | 2272817-1 | 3749562-2 | | 485848-0 | 3600000 |
| FR 8 | 2650357-1 | 3943819-1 | 1884565-1 | 1252730-1 | | 1696442-1 | | 4712461-3 | 1880000 |
| UK 8 | 4102240-2 | | 1087632-0 | 159813-0 | 3875987-2 | 1756940-1 | 4376635-3 | 131163-0 | 2000000 |
| IT 8 | 10986853-4 | 6171129-1 | | 1108457-0 | | | 5807362-2 | 1688197-1 | 3000000 |
| ES 4 | 3614232-1 | 4382329-1 | 2087359-0 | 2893058-1 | | 1194889-1 | | | 3600000 |
| PL 4 | 667319-1 | 2752061-1 | | | 2246870-2 | | | 505586-0 | 1300000 |
| RO 4 | 2093234-2 | 2213046-2 | 379582-0 | | | | | | 900000 |
| NL 4 | 446763-0 | 721766-0 | 1307001-1 | 658333-1 | 364843-0 | 331594-1 | | 633114-1 | 1000000 |
| BE 2 | 1269993-1 | 1129739-0 | 1520431-1 | | 1123355-0 | 739016-0 | | 284856-0 | 2400000 |
| EL 2 | 836136-1 | 1298948-0 | | 1518376-1 | 197837-0 | | | 886255-0 | 1776000 |
| CZ 2 | 214800-1 | 392539-1 | 244501-0 | 166478-0 | 116389-0 | | 79540-0 | | 430000 |
| PT 2 | 1034249-1 | 910647-1 | 234788-0 | 566689-0 | | | | | 1100000 |
| SE 1 | 1103079-1 | 728062-0 | 609615-0 | 234272-0 | | 572591-0 | 359248-0 | | 2000000 |
| HU 1 | 478837-0 | 1193991-1 | | | | 284980-0 | | 340287-0 | 1300000 |
| AT 1 | 680180-0 | 761896-1 | 229781-0 | | | 410089-0 | | 556835-0 | 2000000 |
| BG 1 | 424037-0 | 825370-1 | 386725-0 | | 238629-0 | | | | 1000000 |
| DK 1 | 435245-0 | 208262-0 | 528789-0 | 183724-0 | 605889-1 | 249305-0 | | | 1000000 |
| FI 1 | 212781-0 | 390376-0 | 456642-1 | 161074-0 | 222457-0 | 161263-0 | | | 600000 |
| SK 1 | 135089-0 | 186912-1 | 37376-0 | | 80145-0 | | | | 300000 |
| IE 1 | 124168-0 | 369120-0 | 68986-0 | 477316-1 | 369545-0 | 81458-0 | | | 800000 |
| HR 1 | 137952-0 | 318203-1 | 137952-0 | | 63641-0 | 86806-0 | | | 400000 |
| LT 1 | 197477-0 | 199393-0 | 335980-1 | | 92108-0 | 75643-0 | 163049-0 | | 500000 |
| SI 1 | 32484-0 | 166403-1 | 32662-0 | | | 41525-0 | | | 200000 |
| LV 1 | 57863-0 | 204979-1 | | | 63229-0 | 28303-0 | 36637-0 | | 200000 |
| EE 1 | 44550-0 | 45765-0 | 153268-1 | | | 43369-0 | | | 200000 |
| CY 1 | 47938-0 | 97732-1 | | 69852-0 | | | | | 130000 |
| LU 1 | 23895-0 | 76736-1 | 30108-0 | | | 30597-0 | | | 100000 |
| MT 1 | 134462-1 | 100785-0 | | | | | | | 200000 |
| "Party" Div. | 0.94 | 1.5 | 1.2 | 1.1 | 1 | 0.65 | 0.8 | 1 | |

The double-proportional variant of the divisor method with standard rounding is used. It returns to each Member State the number of seats contributed, and it guarantees each "party" its unionwide party-seats. Sample calculation: The German S & D votes (8 003 628) are divided by the divisor for Germany (3 600 000), and by the S & D divisor (0.94). The resulting quotient is 2.4 (not shown) and justifies two seats.