



Illustration: Medienstern Zürich

# Zurich's New Apportionment

*How mathematics has played an essential role in giving an ancient democracy a new electoral system*

By Friedrich Pukelsheim

On Sunday, 28 September 2008 the people of the Swiss canton of Schaffhausen were called to the ballot box to vote in the cantonal council elections. This was their first encounter with a new electoral system that had just recently been adopted in a popular referendum and won the confidence of

the people. This system uses a new method to calculate the allocation of votes to parliamentary seats that had made its way directly from the ivory tower of science to the world of politics.

The first trial of the new electoral system took place in the canton of Zurich in 2004. Not only has it been used on numerous occasions elsewhere in Switzerland since then,

but it has also found widespread acceptance. Dubbed "Zurich's new apportionment procedure" – also referred to mathematically as the "biproportional divisor method with standard rounding" – it is especially good at conforming to the Swiss tradition of popular democracy and proportional representation. This is particularly well demonstrated by the cantonal elections in the canton of Zurich on 15 April 2007, which are taken as an example here.

As is the case in all Swiss cantons, the canton of Zurich is divided into several electoral districts for the purpose of these elections, a system with a long tradition. In the middle of each legislative period the 180 seats on the cantonal council are allocated in proportion to the size of the population to each of the 18 districts. At present, the smallest district (Andelfingen) has four and the largest (Bülach) 17 seats.

Left: A city council meeting in Zurich's town hall. The new apportionment procedure saw its debut in the city council elections on 12 February 2006. Below: A snapshot of events on election night.

In past elections, serious difficulties arose in the small districts, because if there are nine or more parties running for only four seats, for instance, then the voters for more than half of the parties are, unavoidably, left unrepresented. It is pure common sense to surmise that the objectives of proportional representation can only be achieved if – to put it bluntly – the term proportionality is taken very loosely in such cases. However, the electoral system is not governed by common sense; what matters is what the constitutional courts say.

In 2002 the Swiss Federal Court ruled that the constitutional right of all citizens to a system of proportional representation is indeed violated if the constituencies are so small that there are too few seats to allow proportionality. The reasoning behind this judgement was that it is not acceptable for the guaranteed right to equality, which the voters are entitled to under a system of proportional representation, to take 18 different forms in different districts of a common electoral region, such as a canton. No voter should be put at a disadvantage simply because they live in a small district. The right to equality applies equally to all voters, wherever they live in the region where the election is taking place, in this case the canton of Zurich. The electoral system thus needs to deliver this equality, as far as is practically possible. In the wake of this groundbreaking judgement, the parliament of the canton of Zurich felt obliged to revise its electoral system. Basically, either the small districts had to go, or another solution was called for.

This is where maths came in, as a better electoral system, capable of overcoming the weaknesses that had been identified in the old system, had in fact already been developed by mathematicians about a decade previously. The mathematician Balinski from the École Polytechnique in Paris had described a

doubly proportional method of seat apportionment and presented it using empirical data from an election in Mexico as an example. When Christian Schuhmacher, head of the legislative service for the canton of Zurich, consulted the author (a mathematician at the University of Augsburg) for advice it proved relatively easy to adapt Balinski's method to create "Zurich's new apportionment procedure" (Neues Zürcher Zuteilungsverfahren, NZZ).

The new system is a two-stage seat apportionment procedure. In the first stage, a general distribution of seats called "super-apportionment" is performed for the entire electoral region, in which all of the 180 seats are apportioned to the parties in proportion to the total number of votes they received across the canton. This is done by dividing each party's canton vote total by a common divisor (which is specific to the election) and rounding the result to the nearest number of seats. In the 2007 elections, the divisor used was a "canton divisor" of 1531. The rounding to the nearest number of seats is necessary because dividing

vote totals by the divisor does not result in an integer. The canton divisor is chosen by the returning officer administering the election so that all 180 seats are filled.

Super-apportionment guarantees is that each voter carries equal weight, irrespective of whether they live in a small district or a large one, thus complying with the constitutional principle of equality. Moreover, this procedure is clear and easy to understand. Since every vote total is divided by the same divisor, relative proportions remain unaffected, meaning that the concept of proportionality is implemented in practice. However, because only whole seats can be shared out between the parties, the result has to be rounded off at the end.

The second stage, "sub-apportionment", where the seats are apportioned to the party lists in the electoral districts, now is subject to more stringent requirements. This is because on the one hand the pre-specified districts sizes have to be adhered to, while, on the other hand, the total number of seats per party for the entire canton, calculated in the super-apportionment, must be met. Surprisingly, the sys-



Illustration: Zachariasen

	SVP	SP	FDP	Grüne	CVP	glp	EVP	EDU	AL	Kantonsdivisor	
<b>Oberzuteilung im gesamten Wahlgebiet</b>											
[ Wählerzahl / Kantonsdivisor -> Parteisitze ]											
Kanton Zürich	180	85056-56	54363-36	44622-29	29155-19	20235-13	16071-10	14608-10	7865-5	3532-2	1531
<b>Untorzuteilung an die Wahlkreise</b>											
[ Stimmzahl / (Wahlkreisdivisor x Parteidivisor) -> Wahlkreissitze ]											
Stadt Zürich, Kreise 1&2	5	7896-1	10749-2	8460-1	5241-1	2649-0	2215-0	679-0	413-0	850-0	6400
Stadt Zürich, Kreise 3&9	12	47555-3	54764-3	15438-1	24609-2	16226-1	10255-1	8042-0	1829-0	6322-1	16100
Stadt Zürich, Kreise 4&5	5	4183-1	9220-2	1890-0	5827-1	1231-0	2124-0	312-0	102-0	2606-1	5000
Stadt Zürich, Kreise 6&10	9	27483-2	41117-3	20345-1	21607-1	9041-1	9159-1	5017-0	1194-0	3872-0	15000
Stadt Zürich, Kreise 7&8	7	13793-1	19537-2	21011-2	15182-1	5741-1	6405-0	2995-0	496-0	1735-0	12000
Stadt Zürich, Kreise 11&12	12	47929-4	42863-3	16143-1	16368-1	13339-1	7208-1	6638-1	2572-0	2126-0	12000
Bezirk Dietikon	11	57231-4	26684-2	25153-2	10047-1	16580-1	4333-0	6269-1	1906-0	981-0	12500
Bezirk Affoltern	6	19914-2	11816-1	12410-1	4651-1	3499-0	3927-0	4923-1	2163-0	316-0	9100
Bezirk Horgen	15	106836-4	62794-2	78517-4	37310-1	32885-2	18088-1	20059-1	7089-0	2190-0	24600
Bezirk Meilen	13	105029-4	52763-2	88948-3	20940-1	22387-1	20755-1	11895-1	507-1	1016-0	28000
Bezirk Hinwil	11	67980-4	27705-1	24812-1	20718-1	17620-1	10231-1	14428-1	15379-1	1138-0	18400
Bezirk Uster	16	120851-5	64071-2	55030-3	24819-1	24893-1	45108-2	15894-1	12036-1	2250-0	24000
Bezirk Pfäffikon	7	32800-3	12988-1	11569-1	12146-1	4968-0	3512-0	8561-1	4178-0	317-0	12700
Winterthur-Stadt	13	70175-3	64288-3	39605-2	37676-2	25834-1	14670-1	20193-1	7976-0	6637-0	24000
Winterthur-Land	7	34299-3	12341-1	12112-1	7777-1	5371-0	3305-0	8761-1	3322-0	316-0	13000
Bezirk Andelfingen	4	12845-2	4019-1	5122-1	3143-0	897-0	954-0	1534-0	1298-0	444-0	7000
Bezirk Bülach	17	140090-6	67833-3	55185-3	33619-1	25010-1	20238-1	23301-1	13584-1	1587-0	23000
Bezirk Dielsdorf	10	57674-4	21520-2	15743-1	12294-1	8485-1	6820-0	4713-0	4943-1	439-0	13000
Parteidivisor		1.02	1.07	0.91	1.013	0.89	1.1	1	0.67	0.7	

SVP Schweizerische Volkspartei      Grüne Grüne      EVP Evangelische Volkspartei  
 SP Sozialdemokratische Partei      CVP Christlichdemokratische Volkspartei      EDU Eidgenössische-Demokratische Union  
 FDP Freisinnig-Demokratische Partei      glp Grünliberale Partei      AL Alternative Liste / PdA

Kantonsratswahlen Zürich am 15. April 2007. Jeder Wähler hat so viele Stimmen, wie im Wahlkreis Sitze zu vergeben sind (linke Spalte: 5, 12, 5, usw.). Um die Wählerzahl zu erhalten, die im Wahlkreis hinter einer Partei steht, wird die Stimmzahl durch die Wahlkreisgröße geteilt und standardgerundet ( $7896/5 = 1579.2 \rightarrow 1579$ ,  $47555/12 = 4095.42 \rightarrow 4095$ , usw.). Gesamtkantonal ergeben sich die Wählerzahlen 85056 für die SVP, 54363 für die SP, usw. (erste Zeile). Bei 180 Gesamtsitzen entfällt somit in der Oberzuteilung auf je 1531 Wähler rund ein Sitz ( $85056/1531 = 55.55 \rightarrow 56$ ,  $54363/1531 = 35.51 \rightarrow 36$ , usw.). In der Untorzuteilung sind die Wahlkreisdivisoren (rechte Spalte) und die Parteidivisoren (letzte Zeile) so berechnet, dass sowohl die Wahlkreisgrößen als auch die kantonsweiten Parteisitze genau ausgeschöpft werden. Dazu werden die Stimmzahlen durch beide Divisoren geteilt und dann standardgerundet:  $7896/(6400 \times 1.02) = 1.2 \rightarrow 1$ ,  $47555/(16100 \times 1.02) = 2.9 \rightarrow 3$ , usw.

tem only requires a few minor – and very plausible – modifications in order to fulfil these more stringent requirements.

**D**istrict divisors”, which formed part of the old system and hence predate the introduction of the new system, are used to ensure that, within a given district, all parties are represented proportionally. The new system introduces additional “party divisors” to secure proportionality between the 18 district lists for any given party. Otherwise, the calculation is performed in exactly the same way as before. The number of votes is divided by the corresponding “district divisor” and “party divisor” and then rounded to the nearest number of seats.

The mathematical contribution is evident at two levels. On the one hand, it is necessary to ensure that the divisors (canton divisor, district divisors and party divisors) are easy to calculate. Given modern knowledge of algorithms and computing, this is not a problem. On the other hand, mathematics also provides structural insight that helps us to

understand why, in the light of actual electoral practices, the new biproportional system is so good.

One of the most convincing structural characteristics is what in technical terms is called “coherence”, the relationship between the overall problem and the partial problems contained in the overall structure. This is of practical relevance because, if there is disagreement over a seat, the disagreement does not usually affect all of the parties or all of the districts, but only a few of them. It turns out that, if the candidates wish to redistribute the seats they are entitled to and also do so using the biproportional method, the end result is exactly the same number of seats the method granted them right in the beginning.

To be more precise, the term “coherence” means that partial problems that can be embedded in the overall problem result in the same number of seats as that given by the solution to the overall problem. Whichever other party a dissatisfied party compares itself with – even if it is its closest rival – everything is alright, from a purely mathemati-

cal point of view. The new system reduces the potential for a parliamentary or legal argument whether seats are allocated “correctly”. The theoretical notion of coherence thus contributes a very practical “conflict-reducing” strategy – a very special feature of the new Zurich apportionment procedure.

Mind you, structural elegance and mathematical clarity are no ends in themselves, but need to be integrated into an electoral system that has grown organically. For this to be successful, the historical roots are crucial, as are the constitutional principles and the relevant socio-political goals. Switzerland is a prime example of the fact that such modernisation of the electoral system can even be implemented rapidly in a decidedly traditional environment, while also enjoying broad political and public acceptance.

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