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Seat biases in proportional representation systems with thresholds

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Abstract In proportional representation systems, apportionment methods are used to convert the number of votes of a party into the number of seats allocated to this party. An interesting characteristic of any such method are the seat biases, that is, the expected differences between the actual seat allocation and the ideal share of seats, separately for each party, when parties are ordered from largest to smallest. For electoral systems with a threshold, that is, with a minimum percentage of votes that parties must reach in order to be eligible to participate in the apportionment process, we show that seat biases decrease from their maximum to zero, as the threshold increases from zero to its maximum, and that all seat biases decrease linearly.

1 Introduction

Proportional representation systems calculate the number of representatives in a political body proportionally to some input data. Important examples are the apportionment of the 435 seats of the US House of Representatives to the 50 states of the Union, proportionally to the decennial population counts. Another example is the apportionment of parliamentary seats to parties, proportionally to the vote counts on the eve of an election day. There are plenty of apportionment methods available to carry out these calculations, see Balinski and Young (2001) for a historical account as well as for the foundations of an apportionment theory, or Taagepera and Shugart (1989) for an exposition from the political science point of view.

An important issue with any such apportionment method is whether it is biased, that is, whether it exhibits indications of disproportionality of some sort or other.

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Following Schuster et al. (2003) we give the general notion of "bias" a specific, operational meaning, and define the *seat bias of the kth largest party* to be

$$B_k(M, 0) = E[m_k - Mw_k | w_1 \geq w_2 \geq \cdots \geq w_\ell \geq 0], \tag{1}$$

that is, the expected difference between the actual number of seats allocated to the kth largest party and the ideal share of seats which the party could claim if fractional seats were available. Here, m_k denotes the number of seats apportioned to party k, M is the district magnitude or house size, that is, the total number of seats to be apportioned, and w_k designates the proportion of votes won by party k. Moreover, we assume that ℓ parties are eligible to participate in the apportionment process, and that they are numbered from largest to smallest, $w_1 \geq w_2 \geq \cdots \geq w_\ell \geq 0$. The letter "$E$" indicates expectation, which is calculated under the assumption that all feasible weight configurations are equally likely. Schuster et al. (2003) present seat bias formulas and empirical data, and also include an extensive review of the literature.

In the present paper, we extend their results to electoral systems where, in order to be eligible to participate in the apportionment process, the proportion of votes which a party wins must exceed a certain threshold t. Many systems impose a 5% threshold, $t = 0.05$.

There exists an extensive literature on thresholds in electoral systems, see Taagepera (1998), Palomares and Ramírez (2002), and the references given there. Those papers do not address the impact of thresholds on seat biases, instead calculating minimum thresholds which a party must pass in order to possibly be allocated a given number of seats, and maximum thresholds beyond which a party is certain to be allocated that many seats.

In contrast, we consider a threshold fixed by the applicable electoral law, whence the smallest party must have a proportion of votes above the threshold t. In such systems, the *seat bias for the kth largest party* depends on the threshold t, and will be denoted by

$$B_k(M, t) = E[m_k - Mw_k | w_1 \geq w_2 \geq \cdots \geq w_\ell \geq t]. \tag{2}$$

In other words, we condition on the event that, while parties are still ordered from largest to smallest, the weight of the last party cannot be arbitrarily close to 0, but must exceed the threshold t, $w_\ell \geq t$.

In Sect. 2 we show that, as the threshold parameter t increases from zero to its maximum, the seat biases decrease from their maxima to zero. This is not at all surprising. After all, the minimum threshold permits a maximum disparity between the largest party and the smallest party. The maximum threshold equals $1/\ell$ and forces all party weights to be identical, $w_1 = w_2 = \cdots = w_\ell = 1/\ell$. What is surprising, though, is that the decrease of the seat biases is practically linear, thus taking the simplest possible form.

2 Seat biases as a function of the threshold

The threshold t can range from 0 to $1/\ell$, where ℓ is the number of parties that are eligible to participate in the apportionment process. In the beginning, when the threshold is equal or close to 0, the disparity between the largest and the smallest party is most pronounced. On the other hand, when the threshold is equal or close to $1/\ell$, all parties have their proportion of votes close to $1/\ell$ and hence are more or less equal. It is therefore to be expected that, if at all an apportionment method suffers from nonzero seat biases, they will be largest for small thresholds, and wear away as the threshold grows close to $1/\ell$. Indeed, the dependence on t turns out to be practically linear in t,

$$B_k(M, t) = (1 - \ell t) B_k(M, 0). \tag{3}$$

In deriving formula (3) some mild approximations are needed. However, it transpires that these approximations are practically negligible.

Figure 1 exhibits the straight-line decrease, for two-, three-, and four-party systems. Overlaid are dots for thresholds of 5, 10, and 15% that are generated by computer simulations (100 000 realizations). The house size $M = 598$ is appropriate for the German Bundestag. If the approximations in deriving formula (3) were not negligible, the dots would show some deviation from the straight lines, which is not the case. Rather, the dots are perfectly aligned, and re-confirm formula (3).

The apportionment method used in Fig. 1 is the divisor method with rounding down (Jefferson, Hondt) which, among the traditional methods, is the one with the most prominent seat biases, see Fig. 3 in Schuster et al. (2003). In that paper two empirical data sets were investigated. One data set refers to the Swiss Kanton Solothurn, where thresholds were never implemented.

The other data set comes from Bavaria 1966 to 1998, where the threshold was at 5% throughout. With no threshold, theoretical seat biases in a three-party system are a gain of $5/12 = 0.42$ seat fractions for the largest party, and losses of $-1/12 = -0.08$ for the middle party, and $-4/12 = -0.33$ for the smallest party. With a 5% threshold, these seat biases need to be multiplied by the factor $1 - 3/20 = 0.85$. The changes are so small that the concordance with the empirical data set from Bavaria, which after all embraces just 49 apportionments, persists.

Appendix: derivation of formula (3)

The arguments leading to formula (3) extend the geometric approach pioneered by Pólya (1918), and employed by Schuster et al. (2003). The lines of reasoning follow the more detailed approach by Drton and Schwingenschlögl (2004), and Schwingenschlögl and Drton (2004). Those papers also provide a stringent proof of the seat bias formula for multi-party systems which in Schuster et al. (2003, p. 672) was put forward as a conjecture. An alternate proof of formula (3) is given by Heinrich et al. (2005, p. 123), based on weak convergence results.

The approximation step needed to derive the linear relationship (3) consists in a transition from the vote region to the seat region. Theoretically, we restrict attention to those situations where the smallest party has a weight exceeding the threshold t, $w_\ell \geq t$. Practically, we substitute this condition by demanding that the smallest party has a seat proportion exceeding t, $m_\ell / M \geq t$. While the threshold t is a continuous variable, the proportion of seats is discrete. However, for district magnitudes M that are practically relevant the approximation works perfectly well, while the computational simplification appears to be substantial.

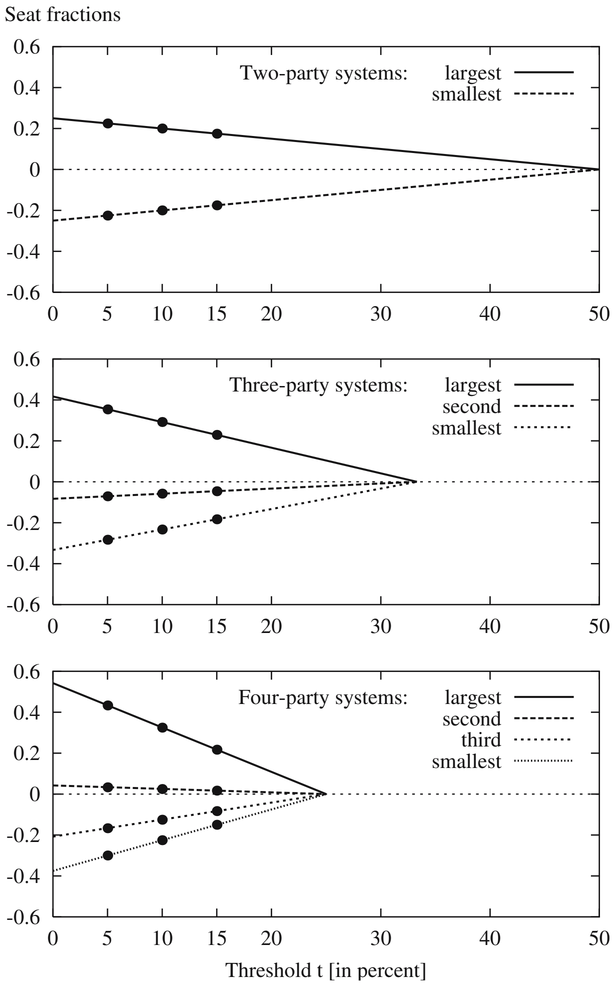

Fig. 1 Linear decrease of seat biases for systems with $\ell = 2, 3, 4$ parties and $M = 598$ seats, for the divisor method with rounding down (Jefferson, Hondt). With threshold t growing from 0 to $1/\ell$, the linear decrease is seen to be in perfect agreement with the simulated seat biases, indicated by bold dots, for thresholds of 5, 10, and 15%

Therefore we condition on $m_\ell/M \geq t$, and approximate the threshold seat bias of Eq. (2) according to

$$B_k(M, t) \approx E[m_k - Mw_k | w_1 \geq w_2 \geq \cdots \geq w_\ell \text{ and } m_\ell/M \geq t]. \tag{4}$$

From Eqs. (1) and (5) in Schwingenschlögl and Drton (2004) it transpires that, except for constants not depending on t, Eq. (4) is the quotient of two sums

$$B_k(M, t) = \sum_m \frac{m_k}{b(m)} \Big/ \sum_m \frac{1}{b(m)}, \tag{5}$$

where the summations extend over all seat allocation vectors $m = (m_1, m_2, \ldots, m_\ell)$ with $\sum_{i=1}^{\ell} m_i = M$ satisfying $m_1 \geq m_2 \geq \cdots \geq m_\ell \geq tM$, while the *boundary factor* $b(m)$ counts the number of permutations leaving the seat allocation m invariant.

Let n be the integer part of $(M-1)/\ell$, and let s be the smallest integer bigger than or equal to tM. Theorem 3 in Schwingenschlögl and Drton (2004) implies that, except for constants not depending on s, the sum in the numerator of Eq. (5) equals

$$\sum_m \frac{m_k}{b(m)} = \sum_{j=s}^{n} \left(j^{\ell-1} + O\left(j^{\ell-2}\right)\right) = n^\ell - (s-1)^\ell + O\left(n^{\ell-1}\right). \quad (6)$$

Similarly, the sum in the denominator of Eq. (5) is seen to equal

$$\sum_m \frac{1}{b(m)} = \sum_{j=s}^{n} \left(j^{\ell-2} + O\left(j^{\ell-3}\right)\right) = n^{\ell-1} - (s-1)^{\ell-1} + O\left(n^{\ell-2}\right). \quad (7)$$

Being the quotient of two polynomials in s, of degree ℓ in the numerator and of degree $\ell - 1$ in the denominator, Eq. (5) is linear in s and hence t, except for lower order remainder terms. By neglecting the remainder terms we obtain

$$B_k(M, t) = a + bt, \quad (8)$$

and it remains to determine the constants a and b. Clearly we have $a = B_k(M, 0)$. The other endpoint $B_k(M, 1/\ell) = 0$ yields $b = -\ell B_k(M, 0)$. Thus, Eq. (8) turns into Eq. (3).

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