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Seat biases of apportionment methods for proportional representation

Karsten Schuster [a], Friedrich Pukelsheim [b,*], Mathias Drton [c], Norman R. Draper [d 1]

[a] *Gymnasium Marktoberdorf, D-87616 Marktoberdorf, Germany*
[b] *Universität Augsburg, Institut für Mathematik, Universitätstrasse 14, D-86135 Augsburg, Germany*
[c] *University of Washington, Department of Statistics, 98195-4322 Seattle, WA, USA*
[d] *University of Wisconsin, Department of Statistics, 53706-1685 Madison, WI, USA*

Abstract

In proportional representation systems, an important issue is whether a given apportionment method favors larger parties at the expense of smaller parties. For an arbitrary number of parties, ordered from largest to smallest by their vote counts, we calculate (apparently for the first time) the expected differences between the seat allocation and the ideal share of seats, separately for each party, as a function of district magnitude, with a particular emphasis on three traditional apportionment methods. These are (i) the quota method with residual fit by greatest remainders, associated with the names of Hamilton and Hare, (ii) the divisor method with standard rounding (Webster, Sainte-Laguë), and (iii) the divisor method with rounding down (Jefferson, Hondt). For the first two methods the seat bias of each party turns out to be practically zero, whence on average no party is advantaged or disadvantaged. On the contrary, the third method exhibits noticeable seat biases in favor of larger parties. The theoretical findings are confirmed via empirical data from the German State of Bavaria, the Swiss Canton Solothurn, and the US House of Representatives.

Keywords: Hamilton; Hare; Hondt; Jefferson; Sainte-Laguë; Webster

* Tel.: +49-821-598-2206; fax: +49-821-598-2280.
E-mail address: pukelsheim@math.uni-augsburg.de (F. Pukelsheim).
[1] The authors are listed in reverse alphabetical order. N.R.D. and F.P. gratefully acknowledge support from the Alexander von Humboldt-Stiftung via a Max-Planck-Award for cooperative research.

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1. Introduction

In proportional representation systems, the electoral votes must be translated into specific seat allocations. A central issue is to investigate the consequences of any particular apportionment method. The seat allocations are of course integer numbers, and the votes are almost continuous quantities by comparison. The translation of votes to seats nearly always involves adjusting, in some manner, the fractional seats that would arise if a literal calculation were made. One of the pertinent problems is to measure the effect of the adjustment process. Whereas previous studies have made inferences about the proportionality of electoral formulas from empirical data, this paper derives the information deductively.We concentrate on the three most popular apportionment methods:

- (H) the quota method with residual fit by greatest remainders, also known as the Hamilton method, or Hare method,
- (W) the divisor method with standard rounding (Webster, Sainte-Laguë), and
- (J) the divisor method with rounding down (Jefferson, Hondt).

Assuming repeated applications of each method, we evaluate the seat biases of the various parties. These seat biases are averages, over all possible electoral outcomes, of the differences between the (integer) seats actually apportioned, and the (fractional) ideal share of seats that would have been awarded had fractional seats been possible (Section 2).

For three parties, we provide exact seat bias formulas, as a function of the district magnitude, for each of the apportionment methods (Sections 3–5). These formulas allow one to predict, for a given apportionment method, how much the seat allocation for the largest (middle, smallest) party is expected to deviate from the ideal share of seats this party could claim. Two three-party data sets, from the German State of Bavaria and the Swiss Canton Solothurn, are investigated and seen to be in excellent agreement with our theoretical findings (Section 6).

For four or more parties, we present seat bias formulas only up to remainder terms that vanish as the district magnitude becomes large (Section 7). These formulas are applied to empirical data from the US House of Representatives (Section 8). In this application, seats are apportioned to states rather than parties, but the numerical apportionment problem is identical. Compared to the number of parties typically involved in an electoral context, the number of US States is much larger, from 15 in 1792 to today's 50. Nevertheless, our theoretical formulas fit the empirical data quite well.

Combination of the theoretical and empirical results leads to our main conclusion (Section 9) that the quota method with residual fit by greatest remainders (Hamilton, Hare) has practically unbiased seat allocations for all parties, as has the divisor method with standard rounding (Webster, Sainte-Laguë). By contrast, the divisor method with rounding down (Jefferson, Hondt) is seat biased, favoring larger parties at the expense of smaller parties. This has been observed over many years on the basis of empirical data, but our formulas permit specific calculations about the numerical sizes of the seat biases. For example, the largest party in a system of three can expect five extra seats per twelve elections in excess to their ideal share, that is, a rate of about one excess seat every other election, under the Jefferson method.

In passing from the Jefferson method to other apportionment methods, many authors jump to the opposite extreme, that a method favors smaller parties at the expense of larger parties, as if this were the only alternative. The middle ground of unbiasedness seems to be neglected. To quote but one source from many, Lijphart (1986, pp. 171–172), calling upon six other expert witnesses who voice the same opinion, writes:

> *The consensus is that d'Hondt disproportionally favors the larger parties, and that the largest remainder formula is more proportional and more favorable to the smaller parties.*

On the other hand, Lijphart (1994, p. 157) states that

> *Sainte-Laguë treats all parties in an even-handed manner.*

From the point of view of seat biases, we show that the largest remainder method is as even-handed as the Sainte-Laguë method, rather than being more favorable to the smaller parties.

We do not feel that there is a contradiction between our findings and the sort of statements just quoted. Rather, electoral systems are complex structures and there are plenty of ways of discussing any sort of "disproportionality", or any other deviation from an assumed ideal state. Therefore we would like to emphasize that the notion of "seat bias" is a very specific way of turning the general idea of "disproportionality" into a manageable numerical quantity. There are many alternative measures, some of them aggregations over all parties, see, for example, Gallagher (1991), Oyama and Ichimori (1995), Pennisi (1998), Benoit (2000). The present paper, however, is confined to seat biases on a per-party basis.

Moreover, the electoral literature uses the term "bias" often in a generic, non-quantitative sense indicating some general deviation from the proportional representation ideal. For instance, the index of Nohlen (2000, p. 457) lists under "bias" eight page references, none of which use the term in the distinct statistical sense of the present paper. The glossary (Nohlen, 2000, p. 465) defines "bias" quite generally as favoring one party compared to another.

Taagepera and Shugart (1989, Chapter 10) speak of *deviation from proportional representation* rather than "bias" and discuss a variety of numerical notions of disproportionality used for distinguishing between methods. Monroe (1994) proposes an axiomatic approach for classifying indices of disproportionality and malapportionment. These indices are peculiar to a given method, being summary aggregates over all parties, and serve to distinguish between competing methods. Lijphart (1994, p. 64) advocates various one-number aggregates along these lines. An example with specific design decisions based on expected disproportionality is the 1951 French parliamentary election, as discussed by Riker (1982, pp. 25–28). A study on the general implications of representative disproportionality is put forward by Powell and Vanberg (2000). The sources quoted treat the notion of electoral system bias in its entire complex generality, leaving no more than an impressionistic scope for the numerical analysis.

Cox and Shugart (1991) emphasize the limited information inherent in a single number when it comes to capturing the complex phenomenon of deviating from proportionality. Rather, the political character of disproportionality leads to additional questions (Cox and Shugart, 1991, p. 350),

> *is it the small parties that tend to be over-represented? the middle-sized ones? the big ones? or is there no clear relationship between over-representation and party size?*

The present paper proposes quantitative answers to these questions, focusing on the specific notion of seat biases, for which a formal analysis can be carried out in a surprisingly rigorous fashion.

2. Seat bias formulas

In Sections 2–6 we consider the outcome of an election with three parties, numbered 1, 2, 3, with respective vote counts v_1, v_2, v_3. In a proportional representation system, the number of seats allocated to a party ought to be proportional to the relative weight of their vote counts. Hence, if $V = v_1 + v_2 + v_3$ is the total number of votes cast, there is no loss of generality to convert the vote counts into vote proportions, or weights, $w_1 = v_1/V$, $w_2 = v_2/V$ and $w_3 = v_3/V$. Assuming all possible weights to be equally likely, we calculate the average behavior of the seat allocations. Technically, our assumption is that the three weights w_1, w_2, w_3 follow a uniform distribution over all their possibilities, that is, over the set of any three non-negative numbers summing to one.

The total number of seats to be apportioned is denoted by M, and is called the *district magnitude*. The numbers w_1M, w_2M, w_3M are the *ideal shares of seats of parties* 1, 2, 3. These would be the “fractional numbers of seats” to which, ideally, each party would be entitled if that were possible. In real life, parties 1, 2, 3 are apportioned an integral number of seats m_1, m_2, m_3, using the apportionment method in the applicable electoral law.

A common approach for evaluating the goodness of an apportionment method is to compare, for each party k, their actual seat allocation m_k with their ideal share of seats w_kM. We find it convenient to call the difference m_k-w_kM the *seat excess of party k*. In a single instance it is most unlikely that the ideal share of seats turns out to be an integer, whence a seat excess typically deviates from zero above or below. Thus some parties are favored by enjoying seat excesses that are positive, others are disadvantaged with seat excesses that are negative. The question is whether, apart from the variability that is unavoidable, the seat excess of a party additionally shows some systematic trend. The main results of our paper are formulas for the expected value of the seat excesses, for each party k, under a given apportionment method.

It is well known that the seat allocation for a party is contingent on its weight relative to the weights of the others, see Balinski and Young (2001, p. 46). Hence we condition the averaging process on the event that party 1 is largest, party 2 is middle, and party 3 is smallest. With "largeness" referring to party weights, we thus assume $w_1 \geq w_2 \geq w_3$. Under this condition, we study the expected value of the seat excess m_k–w_kM as a function of the district magnitude M. The resulting quantity is denoted by $B_k(M) = \mathrm{E}[m_k - w_kM | w_1 \geq w_2 \geq w_3]$ and is called the *seat bias of the k-th largest party*. This notion of "bias" is standard statistical terminology, indicating the expected difference between all possible observable values of a quantity and its ideal value. The "E" indicates the statistical average [expectation] over all theoretically possible values of the seat excess m_k–w_kM. Note that we use the term "seat bias" only in the specific, quantitative sense just defined.

The seat allocations m_1, m_2, m_3 always sum to M, the total number of seats, as do the ideal shares of seats w_1M, w_2M, w_3M. Hence the seat biases of all parties together must sum to zero, $B_1(M) + B_2(M) + B_3(M) = 0$.

3. The quota method with residual fit by greatest remainders

Our first set of seat bias formulas is for the quota method with residual fit by greatest remainders (Hamilton, Hare). The method is based on the fixed quota $Q = V/M$, the quotient of the total number of votes and the total number of seats, and operates in two stages. In the principal apportionment stage, party k with v_k votes is allocated a number of seats equal to the integer part in the quotient v_k/Q. In the residual apportionment stage, the remaining seats are allocated, one by one, to the parties for which the fractional part in v_k/Q is largest, second-largest, and so on until all M seats are given away.

We remark in passing that, although common in continental Europe, association of the method with the name of Thomas Hare (1806–1891) is not quite appropriate. Hare did introduce the quota $Q = V/M$, see Hart (1992, p. 27), but the system he proposed was the Single Transferable Vote system.

Let the seat bias $B_k^{\mathrm{H}}(M)$ be the expected Hamilton seat excess of the k-th largest party, as the weights $w_1 \geq w_2 \geq w_3$ attain all their possible values with equal probability. The largest ($k = 1$), middle ($k = 2$), and smallest ($k = 3$) parties face a seat bias of

$$B_1^{\mathrm{H}}(M) = \frac{1}{6M} - 2\frac{\left\lfloor \frac{M+1}{3} \right\rfloor - \frac{M}{3}}{3M^2},$$

$$B_2^{\mathrm{H}}(M) = \frac{1}{6M} + 4\frac{\left\lfloor \frac{M+1}{3} \right\rfloor - \frac{M}{3}}{3M^2},$$

$$B_3^{\mathrm{H}}(M) = -\frac{2}{6M} - 2\frac{\left\lfloor \frac{M+1}{3} \right\rfloor - \frac{M}{3}}{3M^2}.$$

As previously stated, the three seat biases sum to zero. The notation $\left\lfloor \frac{M+1}{3} \right\rfloor$ means that the quotient $\frac{M+1}{3}$ is rounded down to its integer part. Thus, as the district magnitude grows from one multiple of 3 to the next, $M = 3n$, $3n + 1$, $3n + 2$, $3(n + 1)$, the numerator $\left\lfloor \frac{M+1}{3} \right\rfloor - \frac{M}{3}$ cycles through 0, –1/3, 1/3, 0, and, in particular, stays bounded.

In each of the three formulas, the first term is a multiple of $1/M$, while the second term is of the order $1/M^2$. Therefore, each of the three seat biases converges to zero as the district magnitude M becomes large. Even for small district magnitudes the numerical values of the seat biases are quite small. Therefore we feel justified in calling the Hamilton seat allocations "practically unbiased".

Fig. 1 presents a graphical display of the seat biases, as a function of the district magnitude M. The display starts from $M = 6$ (when there are twice as many seats available as there are parties participating), and runs up to a chosen cut-off point $M = 30$. The seat bias curves for the largest party (solid curve) and for the middle party (dashed curve) are overlaying and visually indistinguishable. They stay in the positive, while the (dotted) seat bias curve of the smallest party lies in the negative. Of course, magnifying the vertical scale would allow us to distinguish between the three curves, see Pukelsheim (2000, p. 265). However, the numerical values of the seat biases are too small to warrant an extra figure.

All seat biases stay below a twentieth of a seat, indicating a gain or a loss of at most one seat in twenty elections. With growing district magnitude M all three seat biases become smaller yet, and eventually converge to zero. The graphical picture confirms our assessment that the Hamilton seat allocations are practically zero, that is, that the Hamilton seat allocations are practically unbiased.

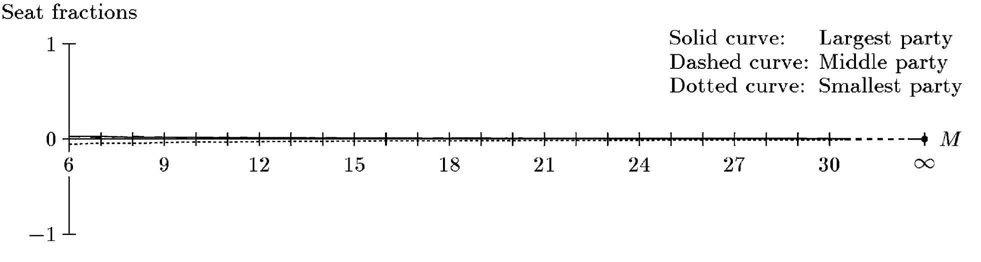

Fig. 1. Theoretical seat bias curves of a three-party system when the district magnitude M increases from 6 to 30, for the quota method with residual fit by greatest remainders (Hamilton, Hare). The seat bias of each party is practically zero, and the dependence on the district magnitude is virtually negligible.

4. The divisor method with standard rounding

The next set of results refers to the divisor method with standard rounding (Webster, Sainte-Laguë). There are many equivalent ways of specifying the method, one of which is as follows. The vote counts v_k are divided by some divisor D, and the resulting quotients v_k/D are rounded in the standard way. That is, if the fractional part of the quotient v_k/D is less than one half then v_k/D is rounded down, if the fractional part is greater than one half then v_k/D is rounded up. The divisor D is adjusted so that the resulting seat allocations m_k^{W} sum to the district magnitude M.

Let $B_k^{\mathrm{W}}(M)$ denote the Webster seat bias of the k-th largest party, that is, the expected difference between the Webster seat allocations and the ideal share of seats. For $k = 1,2,3,$ we obtain

$$B_1^{\mathrm{W}}(M) = \frac{37}{144M} - 8\frac{\left\lfloor\frac{M+1}{3}\right\rfloor - \frac{M}{3}}{3(4M^2-1)} + \frac{37-216\left(\left\lfloor\frac{M+1}{2}\right\rfloor - \frac{M}{2}\right)}{144M(4M^2-1)},$$

$$B_2^{\mathrm{W}}(M) = \frac{7}{144M} + 16\frac{\left\lfloor\frac{M+1}{3}\right\rfloor - \frac{M}{3}}{3(4M^2-1)} + \frac{7+216\left(\left\lfloor\frac{M+1}{2}\right\rfloor - \frac{M}{2}\right)}{144M(4M^2-1)},$$

$$B_3^{\mathrm{W}}(M) = -\frac{44}{144M} - 8\frac{\left\lfloor\frac{M+1}{3}\right\rfloor - \frac{M}{3}}{3(4M^2-1)} - \frac{44}{144M(4M^2-1)}.$$

In each formula the three terms are of the order $1/M$, $1/M^2$, and $1/M^3$, respectively. Again the seat biases disappear as the district magnitude M grows.

Fig. 2 displays the three seat bias curves for district magnitudes M from 6 to 30. Again the seat bias effects are so small as to be of no relevance in reality. The Webster seat allocations for the three parties are justifiably called practically unbiased.

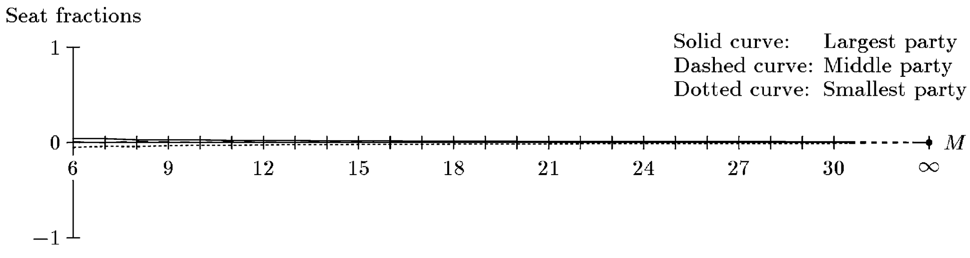

Fig. 2. Theoretical seat bias curves of a three-party system when the district magnitude M increases from 6 to 30, for the divisor method with standard rounding (Webster, Sainte-Laguë). The seat bias of each party is practically zero, and the dependence on the district magnitude is virtually negligible.

5. The divisor method with rounding down

Under the divisor method with rounding down (Jefferson, Hondt), the vote counts v_k are divided by some common divisor D but, in order to obtain the seat allocations m_k^{J}, the quotients v_k/D are now rounded down to the integer part contained in them. The divisor D is adjusted to make the resulting seat allocations sum to the district magnitude M.

For $k = 1,2,3$, let $B_k^{\mathrm{J}}(M)$ be the Jefferson seat bias of the k-th largest party. All three bias formulas start out with a term not depending on the district magnitude M, while the subsequent three terms are of order $1/M$, $1/M^2$, $1/M^3$ as with the previous method:

$$B_1^{\mathrm{J}}(M) = \frac{5}{12} - \frac{11}{36M} + \frac{1-8\left(\left\lfloor\frac{M+1}{3}\right\rfloor - \frac{M}{3}\right) + 18\left(\left\lfloor\frac{M+1}{2}\right\rfloor - \frac{M}{2}\right)}{12(M+1)(M+2)}$$
$$+ \frac{11}{18M(M+1)(M+2)},$$

$$B_2^{\mathrm{J}}(M) = -\frac{1}{12} - \frac{5}{36M} + \frac{7+16\left(\left\lfloor\frac{M+1}{3}\right\rfloor - \frac{M}{3}\right) - 18\left(\left\lfloor\frac{M+1}{2}\right\rfloor - \frac{M}{2}\right)}{12(M+1)(M+2)}$$
$$+ \frac{5}{18M(M+1)(M+2)},$$

$$B_3^{\mathrm{J}}(M) = -\frac{4}{12} + \frac{16}{36M} - \frac{8+8\left(\left\lfloor\frac{M+1}{3}\right\rfloor - \frac{M}{3}\right)}{12(M+1)(M+2)} - \frac{16}{18M(M+1)(M+2)}.$$

The first, constant term determines the behavior when the district magnitude is large, as it is in practice.

Fig. 3 illustrates that the seat biases of the three parties are almost constant, that is, the variation due to M is virtually negligible. However, the levels attained are

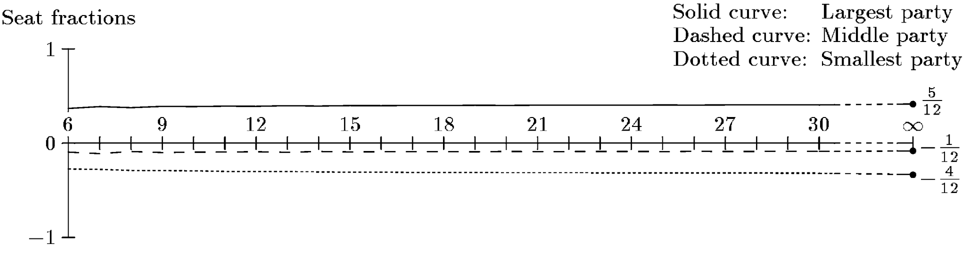

Fig. 3. Theoretical seat bias curves of a three-party system when the district magnitude M increases from 6 to 30, for the divisor method with rounding down (Jefferson, Hondt). The seat biases of all parties are visibly distinct from zero, the dependence on the district magnitude still being virtually negligible.

visibly distinct from zero. On average per twelve elections, the largest party enjoys an advantage of five excess seats, paid for by one seat from the middle party, and by four seats from the smallest party.

6. Empirical three-party examples: Bavaria and Solothurn

Two data sets for three-party allocations are analyzed, the state legislature elections [Land tagswahlen] in the German State of Bavaria from the period after World War II, and the renewal elections [Erneuerungswahlen] in the Swiss Canton Solothurn from the last century. In both instances the applicable electoral laws were subject only to minor changes during the observational period, whence the voting attitude of the electorate can be considered stable during that time. We compiled the data from official Bavarian and Solothurn publications, and deposited them on the Internet at http://www.uni-augsburg.de/bazi/.

The Bavarian data arise from fourteen elections in 1948–1998. Whether a party takes part in the seat apportionment process is decided on the state level, leaving seven elections with seat apportionment between three parties. There are seven electoral districts, in Table 1 numbered I–VII, resulting in 49 apportionments for evaluation. We do not distinguish by district magnitude, since M ranges from 19 to 65 where the seat bias curves in Figs. 1–3 stay fairly constant.

The observed Hamilton seat excesses of the largest party are displayed in the top part of Table 1. The average of the 49 values is –0.04 fractions of a seat (with a standard error of 0.04); this is a viable estimator for the seat bias of the largest party. A single average cannot, however, mirror the variation that is visible in the data. We prefer to describe the empirical seat excess distribution in a more informative manner.

This is achieved by using five numbers. For the largest party, these are the minimum value –0.58, the first quartile (delimiting the lower quarter of the data) –0.26, the median (middle value) –0.08, the third quartile (delimiting the upper quarter) 0.17, and the maximum value 0.59. The quoted numbers appear in the five-number seat excess statistics of the largest party in the first line of the middle part of Table 1. It is followed by the corresponding five-number statistics of the middle party, and of the smallest party.

These numbers give rise to the box plots on the left side of Fig. 4. In each box plot, the vertical lines extend from the minimum to the first quartile, and from the third quartile to the maximum. The central portion of the data is represented by the rectangular box, and the dashed line inside the box is the median. We add a dot at zero, as an indication that the theoretical seat biases are practically zero.

The Swiss Canton Solothurn introduced proportional representation in 1896, and since then has held 27 elections. There are ten electoral districts, providing a total of 270 apportionment instances. Of these, 143 involve exactly three parties, with district magnitude M varying between 7 and 29. The distribution summary statistics for the seat excesses that are calculated from these date are listed in the bottom part of Table 1. In Fig. 4, the box plots showing the Hamilton seat excesses for Solothurn are placed on the right side, for direct comparison with those from the Bavarian data.

Table 1
Observed seat excesses in three-party systems, for the quota method with residual fit by greatest remainders (Hamilton, Hare). Top: Largest party raw data from 49 post-WWII Bavarian apportionments. Middle: Five-number statistics of the largest, middle, and smallest parties for the Bavarian data. Bottom: Five-number statistics from 143 Twentieth Century's Solothurn apportionments

Observed seat-excesses of the largest party, from 49 apportionments in Bavarian elections							
	I	II	III	IV	V	VI	VII
1966	–0.33	–0.08	0.33	0.13	0.02	–0.12	0.32
1970	–0.08	–0.19	0.01	0.22	–0.36	–0.24	0.41
1974	0.05	0.20	–0.33	–0.14	–0.01	–0.16	0.28
1978	–0.23	–0.21	–0.40	0.05	0.23	0.51	–0.06
1986	–0.03	–0.35	–0.17	0.59	–0.36	0.12	0.59
1994	–0.27	–0.14	–0.49	0.02	–0.39	0.19	–0.35
1998	0.03	–0.17	–0.58	0.16	–0.53	–0.12	0.36
Five-number statistics of observed seat excesses (N = 49), Bavarian data (19 ≤ M ≤ 65)							
	Minimum	First quartile	Median	Third quartile	Maximum		
Largest party	–0.58	–0.26	–0.08	0.17	0.59		
Middle party	–0.48	–0.20	–0.03	0.29	0.63		
Smallest party	–0.51	–0.25	0.02	0.19	0.49		
Five-number statistics of observed seat excesses (N = 143), Solothurn data (7 ≤ M ≤ 29)							
	Minimum	First quartile	Median	Third quartile	Maximum		
Largest party	–0.56	–0.25	–0.04	0.22	0.59		
Middle party	–0.61	–0.19	0.10	0.31	0.52		
Smallest party	–0.63	–0.25	–0.05	0.26	0.51		

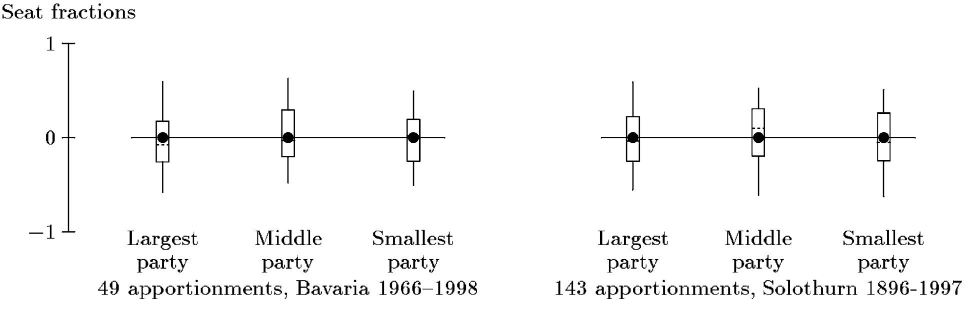

Fig. 4. Observed seat excess distributions in three-party systems, for the quota method with residual fit by greatest remainders (Hamilton, Hare). The box plots of the data are located around zero, varying about half a seat above and below. Dots (●) indicate theoretical seat biases, which are practically zero.

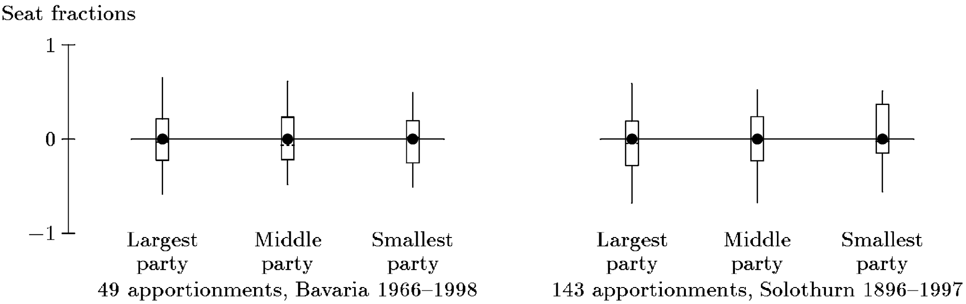

Fig. 5. Observed seat excess distributions in three-party systems, for the divisor method with standard rounding (Webster, Sainte-Laguë). The box plots of the empirical data are located around zero, varying about half a seat above and below. Dots (●) indicate theoretical seat biases, which are practically zero.

Fig. 5 exhibits a similar comparison for the Webster seat allocations, again with the observed Bavarian seat excesses on the left, and the observed Solothurn seat excesses on the right. Figs. 4 and 5 look very similar, as predicted by our theoretical considerations. The remaining differences are small and are accounted for by the variation that is unavoidable in two distinct empirical data sets.

Figs. 4 and 5 send the same message. All box plots show roughly the same amount of seat excess variability, from about half a seat above the median of the observed distribution to about half a seat below the median. Thus we focus on the locations of the boxes. Under the quota method with residual fit by greatest remainders (Hamilton, Hare; Fig. 4), as well as under the divisor method with standard rounding (Webster, Sainte-Laguë; Fig. 5) the observed seat excess distributions are centered around zero for each of the three parties. Hence the empirical evidence confirms the theoretical findings of Sections 3 and 4, that the seat allocations of each of the three parties are practically unbiased for the Hamilton and Webster methods.

Fig. 6 contains the box plots of the empirical seat excesses under the divisor method with rounding down (Jefferson, Hondt), and looks startlingly different. The observed seat excess distributions are seen to differ in location. The largest party

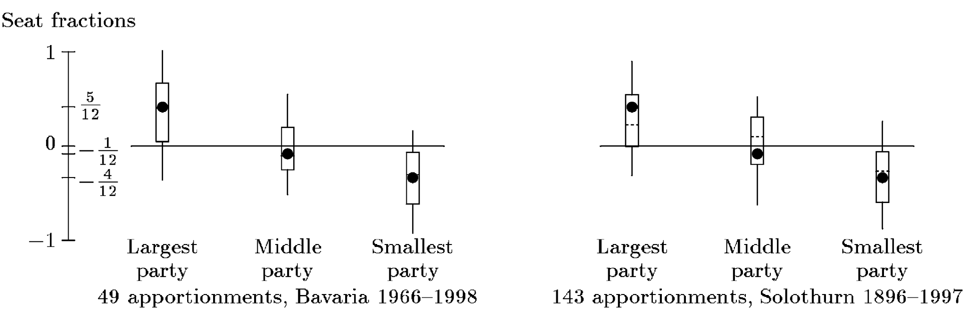

Fig. 6. Observed seat excess distributions in three-party systems, for the divisor method with rounding down (Jefferson, Hondt). The empirical box plots are evidently not centered at zero. For the largest party it is located above the zero line, for the smallest party, below. Dots (●) indicate theoretical seat biases.

has a distribution concentrated above the zero line, the smallest party, below the zero line. The middle party distribution is centrally placed. The Jefferson seat allocations are clearly biased, favoring the largest party at the expense of the smallest party.

7. Seat bias formulas for four or more parties

We now turn to the general case, with an arbitrary number $l \geq 4$ of (lists of) parties participating in the apportionment process. Unfortunately we do not possess proofs for our seat bias expressions as stringent as those for the three-party results in Sections 3–5. Instead, we carried out extensive computer calculations to obtain the following formulas that include small, unspecified remainder terms. These terms appear to be small enough to be irrelevant from a practical point of view.

For the quota method with fit by largest remainders (Hamilton, Hare), the seat biases $B_k^{\mathrm{H}}(M)$ turn out to be identical and slightly positive, for parties $k = 1, \ldots, l-1$ from the largest down to the second-smallest:

$$B_k^{\mathrm{H}}(M) = \frac{l+1}{24M} + O\left(\frac{1}{M^2}\right),$$

$$B_l^{\mathrm{H}}(M) = -(l-1)\frac{l+1}{24M} + O\left(\frac{1}{M^2}\right).$$

The l-th, smallest party carries the deficit that balances the positive accumulation. However, all seat biases turn out to be rather small numerically.

The notation $O(1/\mathrm{M}^2)$ indicates that the remainder term is bounded of the order $1/M^2$. In other words, the remainder term is in absolute value less than a constant divided by M^2. When a large number l of parties participates in the apportionment process, then an even greater number M of seats is needed for allocation, whence the remainder term becomes generally negligible. We can therefore safely neglect the remainder term $O(1/M^2)$ when interpreting the formulas.

Thus all parties, except the smallest, expect seat biases that are slightly positive. The smallest party is the only party disadvantaged, and its negative seat bias must make up for the accumulated advantages of all larger parties. We explain this exceptional behavior as follows. The worst case that can happen to the smallest party is to win no seat at all, $m_l = 0$. Of course, under this side condition the seat bias becomes negative. It transpires that the worst case dominates the calculations and leads to an overall negative seat bias of the smallest party.

Even though the special role of the smallest party may appear disconcerting, its seat bias remains so small numerically as to be invisible in practice. These results confirm our previous assessment that the quota method with fit by largest remainders (Hamilton, Hare) is practically unbiased for any number of parties.

For the divisor method with standard rounding (Webster, Sainte-Laguë), the seat biases of the largest $l-1$ parties $k = 1, \ldots, l-1$ are captured by the first formula below, while the seat bias of the l-th, smallest party is given by the second:

$$B_k^{\mathrm{W}}(M) = \frac{l + \frac{2}{l}}{24M} + \frac{l+2}{24M}\left\{\left(\sum_{j=k}^{l-1} \frac{1}{j}\right) - 1\right\} + O\left(\frac{1}{M^2}\right),$$

$$B_l^{\mathrm{W}}(M) = -(l-1)\frac{l + \frac{2}{l}}{24M} + O\left(\frac{1}{M^2}\right).$$

When the terms in braces are summed over $k = 1, \ldots, l–1$, we obtain zero. Hence a certain amount of balancing goes on between the $l–1$ largest parties alone. The accumulated contribution of the terms $(l + 2/l)/(24M)$ is evened out by the negative seat bias of the smallest party.

Two points are worth mentioning. First, the Webster seat bias of the smallest party extends almost as far into the negative as does the Hamilton seat bias of the smallest party. Second, from six parties onwards the second-smallest party faces a negative seat bias, too. However, all these theoretical effects are so small numerically that we do not consider them practically relevant. That is, the Webster seat allocations are practically unbiased no matter how many parties participate in the apportionment process.

For the divisor method with rounding down (Jefferson, Hondt) the situation changes dramatically. A leading term appears, not depending on the district magnitude M and thus dominating the formulas. The seat bias of the k-th largest party is found to be

$$B_k^{\mathrm{J}}(M) = \frac{1}{2}\left\{\left(\sum_{j=k}^{l} \frac{1}{j}\right) - 1\right\} + O\left(\frac{1}{M}\right).$$

The remainder term, bounded of order $1/M$, appears to be practically irrelevant.

The largest party enjoys a positive seat bias and thus can expect seats in excess of their ideal share. The seat biases become successively smaller as we pass from the largest party ($k = 1$) to the smallest party ($k = l$). About two thirds of the parties face a negative seat bias, but in each individual case the prospective deficit remains bounded by half a seat on average. Indeed, since a small party does not win many seats anyway, any loss they endure must remain bounded.

As the number of parties grows, the deficits of the smaller parties accumulate and generate a growing surplus of seats for the larger parties. More precisely, the seat bias of the k-th largest party ($k = 1, 2, \ldots$ fixed) steadily increases and, in fact, grows beyond limits. This growth is sublinear and hence rather slow, as outlined in Appendix A3.

In order to supplement these qualitative dependencies by some quantitative comparisons, we group the parties into a top third, a median third, and a bottom third, following the lead of Balinski and Young (2001, p. 75). When the number of states is not divisible by three, the authors enlarge the median group. This unbalanced procedure would generate kinks in the graphical display, in our case. We prefer a smoother way of aggregation. If the number of parties is a multiple of three, $l = 3n$, then clearly the n largest parties form the top third, and so the sum of their seat biases is calculated. If $l = 3n + 1$, one third of the seat bias of the $(n + 1)$-st party is added. If $l = 3n + 2$, two thirds are added. If $l = 3n + 3$ is again divisible by three, three thirds are added, that is, the top group comprises the $n + 1$ largest parties as it should. The other two groups are handled similarly.

Fig. 7 shows the cumulative Jefferson seat biases thus obtained, plotted against the number of parties l. The cumulative seat biases are seen to depend almost linearly on the number of parties. Roughly per six parties that join the race, the top third of parties gains one extra seat. One quarter of the deficit is shouldered by the median third of parties, three quarters by the bottom third. For instance, when there are 24 parties competing, then the top eight parties together benefit by four seats beyond their cumulative ideal shares, while the median eight parties lose one seat, and the bottom eight parties fall short by three seats.

In summary, we can draw three conclusions on how seat biases vary with district magnitude M, and with the number of parties l, under a specific apportionment method:

1. The seat bias formulas in this section, and those for the larger family of stationary apportionment methods in Appendix A, all confirm that the variation due to the district magnitude M becomes irrelevant as soon as $M \geq 2l$. In particular, the Hamilton and Webster methods generate practically unbiased seat allocations, for all parties.
2. The district magnitude M may — possibly — generate a noticeable seat bias effect only when $M < 2l$, that is, when M is too small to be able to proportionally mimic the differences of party weights. Therefore district magnitudes of so small a size should not be used for illustrating seat bias behavior. They may display pathological artifacts that are untypical for practical purposes.

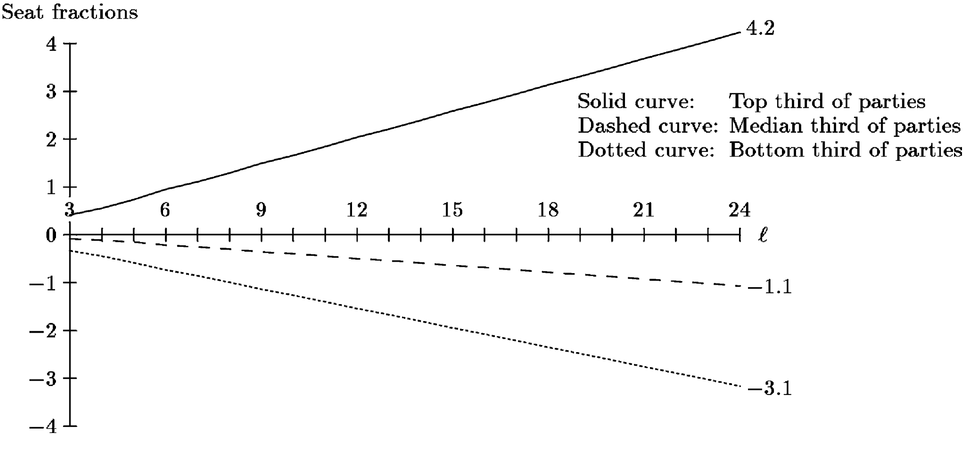

Fig. 7. Theoretical cumulative seat biases of the top, median, and bottom thirds of parties, for number of parties l from 3 to 24, for the divisor method with rounding down (Jefferson, Hondt). The top third's surplus causes a minor seat deficit of the median third, and a major seat deficit of the bottom third.

3. When seat biases do occur, they persist independently of M. But they do respond to the size of a party relative to the other parties. Under the divisor method with rounding down (Jefferson, Hondt), large parties are favored at the expense of small parties, and the more parties participate, the greater becomes the surplus that large parties gain in excess to their ideal share. There is a trend not unfamiliar from other fields in life: a few large enjoy an unbounded surplus, at the expense of the many small who have to make do with less than their fair share.

8. Empirical large-system example: US House of Representatives

An empirical large-system example is provided by the apportionment of the 435 seats in the US House of Representatives among the 50 States. This specific apportionment problem is thoroughly dealt with by Balinski and Young (2001). While the authors address the bias problem from various angles, they do not investigate the notion of seat biases as proposed in the present paper. It is therefore of particular interest to see how our theoretical formulas fare with the empirical evidence from US history. Moreover, bias issues formed a core argument in the court challenges of the current apportionment method, as reported by Ernst (1994).

Allocating House seats to States on the basis of their apportionment populations calls for operational procedures identical to those for allocating parliamentary seats to parties on the basis of their electoral vote returns, so our theoretical formulas should apply. The US Constitution requires that each State receive at least one seat. Our formulas are derived without such a requirement, and hence we disregard it in our subsequent calculations. The currently used “Method of Equal Proportions” is the divisor method with geometric rounding (Hill, Huntington). We defer a discussion of the method to Appendix A4.

Prior to 1960, the number of US States was 48 or lower, and a comparison would require an adjustment of some sort or other which we wanted to evade. We therefore chose the five censuses 1960–2000 for evaluation, when in each case $l = 50$ states competed for $M = 435$ seats. The apportionment populations are listed in Balinski and Young (2001, pp. 174–180) where, for each census, states are ordered from largest to smallest by their apportionment populations. The data can also be retrieved from the Internet at http://www.uni-augsburg.de/bazi/. For the difficulties of how to define the apportionment population see Poston et al. (1999).

For the quota method with residual fit by greatest remainders (Hamilton, Hare), we first determine the apportionments for each of the five censuses. Just as in the three-party examples of Section 6, we then calculate the five-number seat excess statistics of the largest state, of the second-largest state, and so on up to that of the fiftieth-largest — that is, smallest — state. The box plots that go along with these 50 five-number statistics are exhibited in Fig. 8. Viewed across all states, the box plots are centered around zero, deviating up to about half a seat above or below. The US data confirm our theoretical finding that the Hamilton seat allocations are practically unbiased.

For the divisor method with standard rounding (Webster, Sainte-Laguë), a similar

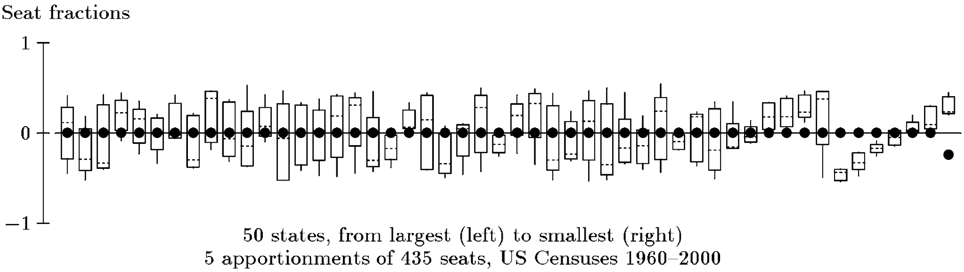

Fig. 8. Observed seat excess distributions in a 50-state system, for the quota method with residual fit by greatest remainders (Hamilton, Hare). The box plots of the empirical data arrange themselves around the zero line, deviating up to half a seat above or below. Dots (●) indicate theoretical seat biases.

analysis leads to Fig. 9. The Webster seat allocations appear to be practically unbiased, as theory predicts. Theoretically, the smallest state suffers a negative seat bias of

$$B_{50}^{\mathrm{W}}(435) = -\frac{49 \times \left(50 + \frac{2}{50}\right)}{24 \times 435} = -0.23$$

seat fractions (for the Hamilton method: –0.24). No such effect is visible in Fig. 9 (nor in Fig. 8). As we have argued in Section 7, the effect originates with the worst case when the smallest state is so small as to win no seat at all. In the five censuses under consideration, all states receive at least one seat under the Webster method (and under the Hamilton method), and the worst case never materializes.

For the divisor method with rounding down (Jefferson, Hondt) the picture changes; see Fig. 10. The seat excesses decrease from the largest state's box plot that is positioned above 2 to the smallest state's box plot that is located near –1/2. The theoretical seat biases from Section 8 are indicated by dots. The empirical variabilities are of the same magnitude as with the previous two methods, of at most about half a seat above or below theoretical expectations.

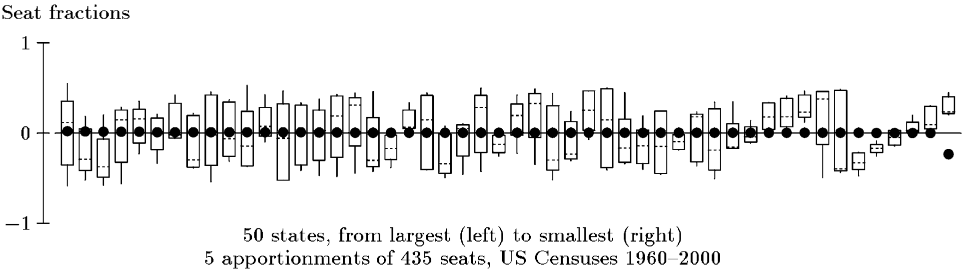

Fig. 9. Observed seat excess distributions in a 50-state system, for the divisor method with standard rounding (Webster, Sainte-Laguë). The box plots of the empirical data arrange themselves around the zero line, deviating up to about half a seat above or below. Dots (●) indicate theoretical seat biases.

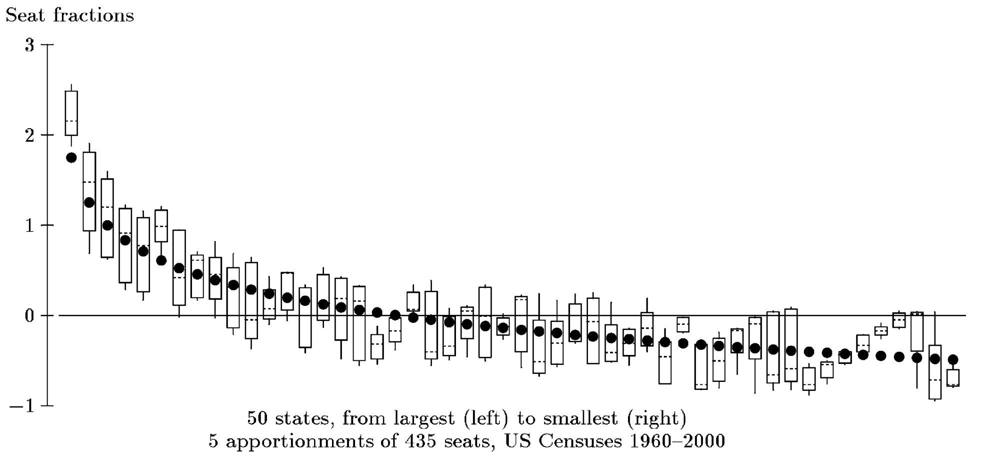

Fig. 10. Observed seat excess distributions in a 50-state system, for the divisor method with rounding down (Jefferson, Hondt). The empirical box plots display a decreasing trend, from above two seats for the largest state, to around half a seat for the smaller states. Dots (●) indicate theoretical seat biases.

9. Discussion and conclusions

The notion of seat bias as used in the present paper captures an intrinsic property within the specific apportionment method being investigated. It is a way of studying how balanced (or unbalanced) the seat allocations are, on average, when the specified method is applied repeatedly. Replications may be generated in three ways, (i) repeated applications over time in just one electoral district, (ii) use of the method in multiple electoral districts at a common point in time, or (iii) varying time and locality simultaneously.

The present approach should be distinguished from comparing two or more methods on the ground of extrinsic numerical indicators. This alternative access to the problem is chosen by Balinski and Young (2001, p. 118) when defining that “one method *favors small states relative to* a second method”, or by Marshall et al. (2002) when saying that “one method *is majorized by* a second method”. Intrinsic and extrinsic analyses coexist, as an amplification of the fact that the question of deviating from proportionality has so many facets.

Averages over repeated applications of a method indicate “typical” properties of a method, and hence are informative from a general viewpoint of electoral system theory. When a specific seat allocation is challenged in court, however, it is a single case that is contested, and the report of Ernst (1994) testifies to the difficulties of persuading a court to view the case contested as a typical manifestation of other, conceivable cases not contested.

In order to calculate theoretical averages one has to make some kind of distributional assumption. In our case we chose the proportions of votes of each party, that is, the weights $w_1,\ldots,w_l$, to be equally likely over all their possibilities. Any distributional assumption is open to debate. Ours was challenged early on by Nybølle

(1922) as a response to Pólya (1919a) who used the assumption for the first time. Nybølle (1922, p. 158) substantiates his objections with depicting a set of 1917 Danish electoral data. His display shows an extremely high degree of symmetry, which we find much more suspicious as far as conformity with reality is concerned than our distributional assumption.

The ultimate test of theoretical calculations is not whether some initial assumptions are pleasing or not, but whether the final results sufficiently conform with empirical data. As far as apportionment methods are concerned, all practical problems arise from the fact that whatever ideal calculations are carried out, they lead to seat fractions which then must be rounded to a nearby integer. Thus the problem dictates a natural range of variability of up to one seat. Indeed, the empirical data shown in Figs. 4–6 and 8–10 uniformly vary over a range of up to roughly one seat.

Therefore, we find that conformity of seat bias calculations with empirical data is sufficiently validated when the theory predicts the central locations of the empirical box plots in such a way that the empirical data vary around them up to about half a seat above or below. Our theoretical formulas pass this test to an astounding extent. For example, in Fig. 10 our theoretical seat biases pick up the apparent trend almost perfectly. Even for the largest and smallest states, where the theoretical seat biases come to lie outside the empirical box plots, the data stay within roughly half a seat of the predicted centers.

Two important variables of an electoral system are the district magnitude M, and the number l of participating parties (or states). Our theoretical results and their empirical confirmations ascertain that the dependence on the district magnitude M becomes negligible whenever there are at least twice as many seats available for allocation as there are parties. In the initial section, when $M < 2l$, the seat bias curves occasionally show some wild oscillations. We have not taken the space to discuss this erratic initial behavior because we think that it is practically of little interest. With only “a few” units to allocate, nobody can manufacture a representation that is proportional. We find it surprising that “a few”, in our context, excludes district magnitudes below $2l$ only. Beyond this threshold, the seat bias curves appear stable enough to validate their predictive power.

As an aside we would add that we find it insufficient to illustrate an apportionment method with numbers where the district magnitude M is smaller than the number of parties l. For instance, Riker (1982, p. 260) allocates ten seats among twelve parties. Such numbers are inconclusive for demonstrating the intricacies of proportional representation methods.

On the other hand, the dependence on the number of parties l is not negligible. If an apportionment method has non-zero seat biases, the distance between them tends to become more pronounced the more parties participate. The divisor method with rounding down (Jefferson, Hondt) is a prototype of an apportionment method that has non-zero seat biases favoring larger parties at the expense of smaller parties.

If all seat biases are zero, then the balanced behavior appears to persist without regard to how many parties are present. Our theoretical calculations did not end up with seat biases that were literally equal to zero. But the final results were numerically small enough to be considered approximately equal to zero, for all practical purposes. In this sense, the quota method with residual fit by greatest remainders (Hamilton, Hare), as well as the divisor method with standard rounding (Webster, Sainte-Laguë) give rise to seat allocations that are practically unbiased.

Acknowledgements

We thank two referees for their valuable suggestions and for their demand to extend beyond three-party systems. We felt, at the time of our original submission, that systems with four or more parties were not readily calculable. Without the referees' insistence that we tackle multi-party systems we would not have tried, and we would not have succeeded. We are grateful to Martin Rasmussen for initial assistance with our computer program.

Appendix A. Derivation of seat bias formulas

The first to aim at numerical seat bias results was the eminent mathematician George Pólya; for a narrative of his life see Alexanderson (2000). Pólya (1918, p. 374) considered his seat bias calculations *das wichtigste Resultat dieser Abhandlung [the most important result of this treatise]*. His approach is based on geometric arguments, as detailed in Pólya (1919a, b). While Pólya did not provide closed-form seat bias formulas of the type given in the present paper, we found his geometric approach sufficiently powerful so that, with some amendments, it led to the results reported here.

The two divisor methods from Sections 4 and 5 emerge from a unified argument by embedding these methods in the wider family of *stationary divisor methods*. Such a method depends on an extra parameter q, which lies in the interval between zero and one, $0 \leq q \leq 1$. In essence, q is the dividing point used to decide if a fractional remainder is small or large. The vote count v_k is divided by a common divisor D, and then the resulting quotient v_k/D is rounded down if its fractional part is less than q, and rounded up if greater than q. The divisor D is adjusted so that the resulting seat allocations sum to M. The divisor method with standard rounding (Webster, Saint-Laguë) has $q = 1/2$, while the divisor method with rounding down (Jefferson, Hondt) has $q = 1$.

A1. Two-party systems

In two-party systems and under the stationary divisor method with parameter q, the seat bias of the larger of the two parties turns out to be

$$B_1^{(q)}(M) = \frac{1}{2}\left(q-\frac{1}{2}\right) + \frac{\left\lfloor \frac{M+1}{2} \right\rfloor - \frac{M}{2} - 2\left(q-\frac{1}{2}\right)^2}{2(M+2q-1)}.$$

The seat bias of the smaller party is $B_2^{(q)}(M) = -B_1^{(q)}(M)$, since the sum must be zero.

In a system with just two parties, the quota method with residual fit by greatest remainder (Hamilton, Hare) coincides with the divisor method with standard rounding (Webster, Sainte-Laguë; $q = 1/2$), whence

$$B_1^{\mathrm{H}}(M) = B_1^{\mathrm{W}}(M) = \frac{\left\lfloor \frac{M+1}{2} \right\rfloor - \frac{M}{2}}{2M} = \begin{cases} \frac{1}{4M} & \text{if } M \text{ is odd,} \\ 0 & \text{if } M \text{ is even.} \end{cases}$$

With these two methods the seat bias is literally zero when the district magnitude is even, and decreases with $1/(4M)$ when it is odd.

The divisor method with rounding down (Jefferson, Hondt; $q = 1$) has

$$B_1^{\mathrm{J}}(M) = \frac{1}{4} - \frac{\left\lfloor \frac{M+1}{2} \right\rfloor - \left\lfloor \frac{M+1}{2} \right\rfloor}{2(M+1)} = \begin{cases} \frac{1}{4} & \text{if } M \text{ is odd,} \\ \frac{1}{4} - \frac{1}{4(\mathrm{M}+1)} & \text{if } M \text{ is even.} \end{cases}$$

Thus the larger party can expect a surplus of one extra seat in four elections, under the Jefferson method. The derivation of the results is a specialization of the following three-party situation.

A2. Three-party systems

We do not base our derivations on one of the algorithms that may be used to implement an apportionment method. Instead, we appeal to the geometric approach pioneered by Pólya (1919b), and extended by Carter (1982), Balinski and Young (2001, pp. 118-128), Kopfermann (1991, pp. 183-187), Bradberry (1992). Viewed together, all ordered triples of party weights, $w_1 \geq w_2 \geq w_3$, make up a rectangular *weight triangle T*. The set of weights leading to a specific seat allocation m_1, m_2, m_3 is called the *domain of attraction* of m_1, m_2, m_3. Since the weight triples w_1, w_2, w_3 in T are equally likely, any specific seat allocation m_1, m_2, m_3 occurs with a probability proportional to the area of its domain of attraction. In general, shape and area of such a domain depend on the seat allocation m_1, m_2, m_3 where it is anchored and, of course, on the selected apportionment method.

Under the stationary divisor method with parameter q, we decompose the problem into seven subproblems, according to whether the seat proportions m_1/M, m_2/M, m_3/M lie in the interior of the weight triangle T, or on one of its three edges, or on one of its three vertices. Within each of these classes, all domains of attraction are found to share the same area (though not the same shape). Counting the number of seat allocations that occur in each of the seven classes, we may then determine the expected seat allocation $\mathrm{E}[m_k | w_1 \geq w_2 \geq w_3]$. Since the seat bias is defined as the expected difference between m_k and w_kM, it remains to calculate the expected ideal share $\mathrm{E}[w_kM|w_1 \geq w_2 \geq w_3]$. The latter is the product of the district magnitude M and the k-th component of the barycenter 11/18, 5/18, 2/18 of the weight triangle T.

The resulting seat biases take the following forms, for the largest (B_1), middle (B_2), and smallest (B_3) parties, depending on the stationarity parameter q, and expressed as a function of the district magnitude M:

$$B_1^{(q)}(M) = \frac{5}{6}\left(q-\frac{1}{2}\right)$$
$$-\frac{(27q^2-9q-8)M^2 + 2(3q-1)(21q^2-17q + 1)M + 5(3q-1)(3q-2)(2q-1)^2}{12(M + 3q-1)(M + 3q-2)(M + 2q-1)}$$
$$+ \frac{3(2q-1)M + 3(3q-1)(3q-2)-3(2q-1)^2}{2(M + 3q-1)(M + 3q-2)(M + 2q-1)}\left\lfloor\frac{M + 1}{2}\right\rfloor$$
$$-\frac{2}{3(M + 3q-1)(M + 3q-2)}\left\lfloor\frac{M + 1}{3}\right\rfloor,$$
$$B_2^{(q)}(M) = -\frac{1}{6}\left(q-\frac{1}{2}\right)$$
$$-\frac{(9q^2-27q + 16)M^2 + 2(3q-1)(3q^2-11q + 7)M-(3q-1)(3q-2)(2q-1)^2}{12(M + 3q-1)(M + 3q-2)(M + 2q-1)}$$
$$-\frac{3(2q-1)M + 3(3q-1)(3q-2)-3(2q-1)^2}{2(M + 3q-1)(M + 3q-2)(M + 2q-1)}\left\lfloor\frac{M + 1}{2}\right\rfloor$$
$$+ \frac{4}{3(M + 3q-1)(M + 3q-2)}\left\lfloor\frac{M + 1}{3}\right\rfloor,$$
$$B_3^{(q)}(M) = -\frac{4}{6}\left(q-\frac{1}{2}\right)$$
$$+ \frac{4(3q-1)(3q-2)M^2 + 8(3q-1)(3q-2)(2q-1)M + 4(3q-1)(3q-2)(2q-1)^2}{12(M + 3q-1)(M + 3q-2)(M + 2q-1)}$$
$$-\frac{2}{3(M + 3q-1)(M + 3q-2)}\left\lfloor\frac{M + 1}{3}\right\rfloor.$$

When $q = 1/2$, the first term vanishes and we retrieve the seat bias formulas for the divisor method with standard rounding (Webster, Sainte-Laguë) of Section 4. When $q \neq 1/2$, the first term creates non-zero seat biases, while the other terms are of order $1/M$ or lower, and so tend to zero as M increases. When $q = 1$, we obtain the divisor method with rounding down (Jefferson, Hondt), and the seat biases of Section 5 appear. For the quota method of Section 3 the arguments follow similar lines, differing only in detail.

A3. Multi-party systems

Under the stationary divisor method with parameter q, the seat bias of the k-th largest party among $l \geq 4$ parties amounts to

$$B_k^{(q)}(M) = \left(q - \frac{1}{2}\right)\left\{\left(\sum_{j=k}^{l} \frac{1}{j}\right) - 1\right\} + O\left(\frac{1}{M}\right).$$

For the divisor method with rounding down (Jefferson, Hondt; $q = 1$) this specializes to the formula for $B_k^{J}(M)$ given in Section 7.

For the divisor method with standard rounding (Webster, Sainte-Laguë; $q = 1/2$) the first term vanishes, leaving seat biases $B_k^{W}(M)$ that are bounded of order 1/M. We found that this is true also for the quota method with residual fit by greatest remainders (Hamilton, Hare). In order to elucidate the fine structure of the two methods we carried our approach further and isolated the terms of order $1/M$ from those of order $1/M^2$. This gave rise to the other formulas of Section 7.

The seat bias $B_k^{(q)}(M)$ depends on the number of parties l and on the party's rank k through the factor that is enclosed in braces. The sum $\Sigma_{j=k}^{l} \frac{1}{j}$ is a portion of the harmonic series. When k is fixed and l increases, the series—and hence the factor—grows beyond limits. Therefore the largest party ($k = 1$) may look forward to an unlimited positive seat bias, for stationary divisor methods with $q > 1/2$, and must fear an unlimited negative seat bias when $q < 1/2$.

In order to study how quickly the sum grows with increasing l, we approximate it by the integral of $1/x$ over the range from $k - 1/2$ to $l + 1/2$, leading to the approximation

$$\sum_{j=k}^{l} \frac{1}{j} \approx \log\left(l + \frac{1}{2}\right) - \log\left(k - \frac{1}{2}\right) = \log \frac{l + \frac{1}{2}}{k - \frac{1}{2}}.$$

This shows that for fixed k the increase in l is logarithmically slow (in other words, sublinear), and that the same is true of the decrease in k for fixed l. The direction of the seat bias is reversed when the sign of the factor in braces switches from positive to negative. This happens when $k(l) = \frac{l}{e} + \frac{e+1}{2e}$, where $e = 2.718$ is the base of the natural logarithm. The factor is positive for the k-th largest party if and only if $k < k(l)$. As a fraction of all parties when the number of parties l is large, this amounts to $1/e = 36.8$ percent, that is, a bit more than a third.

For the divisor method with rounding down (Jefferson, Hondt; $q = 1$), it is then roughly the top third of parties for which the theoretical seat biases are positive, while for the median and bottom thirds the seat biases turn negative. Splitting the set of all parties into thirds is thus close to maximizing the seat bias effects that are peculiar to the method. Hence Fig. 7 lets the effect stand out as prominently as is possible.

Preferences are reversed for the divisor method with rounding up (Adams; $q = 0$), since the factor $1/2$ switches to $-1/2$. Note that the trisection persists. The top third of parties are now threatened with an unlimited negative seat bias, whereas the two thirds of smaller parties may look forward to an excess of up to half a seat, on average.

We conclude this section with a description of how we obtained our results. The approach from Appendix A2, of investigating the different types of domains of attraction, their volumes and their numbers of occurrences, carries over to the general case. While we were unable to compactify the resulting expressions into formulas worthy of communication, our results were good enough to set up a computer program calculating seat biases for an arbitrary stationarity parameter q between zero and one, for district magnitudes M into the thousands, but for no more than nine parties. Beyond these limits, the task of running through all seat allocations and carrying out the necessary calculations becomes too complex and time consuming.

A combination of theoretical deductions and practical calculations then led to the results reported here. As a prototype, we describe the derivation of the formula for $B_k^{(q)}(M)$. We first plotted our computer results against q, for large M fixed. A zero at $q = 1/2$ emerged, and a linear dependence on q. This identified the first factor, $(q-1/2)$.

On combinatorial grounds we then conjectured that the second factor is of the form $z(l, k)/l!$, with a whole number $z(l, k)$ in the numerator. We used our program to numerically calculate the quantities $l!B_k^{(q)}(M)/(q-1/2)$ which, for large M, should reveal $z(l, k)$. The values in Table 2 were obtained where, for the purpose of orientation, we have adjoined the factorial $l!$ in the second column.

Brief contemplation reveals the recursive relation $z(l, k) - z(l, k + 1) = l!/k$. Hence the numbers $z(l, k)$ for $k > 1$ can all be expressed in terms of the very first $z(l, 1)$, which in turn is determined from the fact that each row sums to zero. One gets $z(l,k) = l!\left\{\left(\Sigma_{j=k}^{l} \frac{1}{j}\right)-1\right\}$, and the end result emerges.The other formulas in Section

Table 2
Partial results of computer calculations on the way to determine the formula for $B_k^{(q)}(M)$. Brief contemplation reveals the recursive relation $z(l, k)$ - $z(l, k + 1) = l!/k$, which is easily solved for $z(l, k)$

l	$l!$	$z(l,1)$	$z(l,2)$	$z(l,3)$	$z(l,4)$	$z(l,5)$	$z(l,6)$	$z(l,7)$	$z(l,8)$	$z(l,9)$
2	2	1	–1							
3	6	5	–1	–4						
4	24	26	2	–10	–18					
5	120	154	34	–26	–66	–96				
6	720	1044	324	–36	–276	–456	–600			
7	5040	8028	2988	468	–1212	–2472	–3480	–4320		
8	40320	69264	28944	8784	–4656	–14736	–22800	–29520	–35280	
9	362880	663696	300816	119376	–1584	–92304	–164880	–225360	–2277200	–322560

7 were obtained in a similar way, though the initial computer experiments were necessarily more elaborate.

We are confident that the formulas given in Section 7 are fairly reliable. The numerical evidence is overwhelming, even though a stringent proof in the sense of a deductive derivation is still missing.

A4. The divisor method with geometric rounding

The apportionment of the US House of Representatives seats to the fifty states is carried out using the divisor method with geometric rounding (Hill, Huntington; Equal Proportions). That is, the apportionment populations v_k are divided by a common divisor D. When a quotient comes to lie in the interval between two neighboring integers n and $n + 1$ and it is less than their geometric mean $\sqrt{n(n + 1)}$, then v_k/D is rounded down to n. Otherwise it is rounded up to $n + 1$. The divisor D is adjusted so as to make the resulting seat allocations m_k^{EP} sum to the district magnitude M.

In the interval from zero to one, the decision point is $\sqrt{0 \times 1} = 0$, whence quotients that fall below one are always rounded up to one. Thereafter the decision point shifts into the interior of the interval. For instance, between one and two it is $\sqrt{1 \times 2} = 1.414$. Thus the chances of being rounded up to two are slightly larger than of being rounded down to one. For large integers n the decision points slowly stabilize to become of the form $n + 1/2$, see Happacher (1996, pp. 12–14). These are the same decision points that are pertinent to the divisor method with standard rounding (Webster, Sainte-Laguë). It follows that the Equal Proportions seat allocations are asymptotically unbiased for large district magnitudes M.

Caution is advised for not-so-large district magnitudes. Using our computer program we calculated the theoretical Equal Proportions seat biases in three-party systems for district magnitudes from six to thirty. The seat biases, shown in Fig. 11, are clearly not negligible. The smallest party is advantaged, the middle party is slightly disadvantaged, and the loser is the largest party. These calculations suggest that all three seat biases are bounded of the order (log M)/M, which accounts for the slow convergence to zero.

The US apportionment example has $M = 435$. The question is whether 435 is

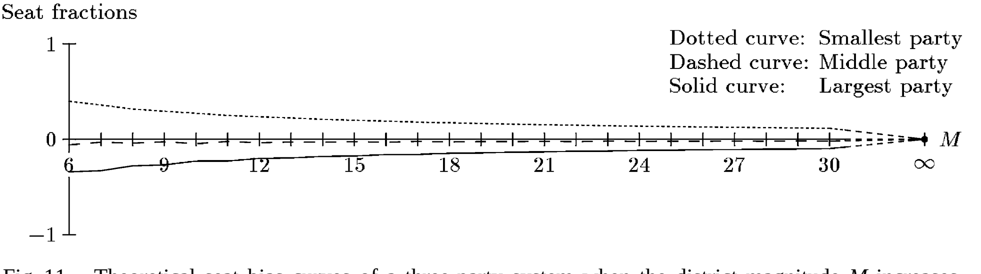

Fig. 11. Theoretical seat bias curves of a three-party system when the district magnitude M increases from 6 to 30, for the divisor method with geometric rounding (Hill, Huntington; Equal Proportions). The seat bias of each party tends to zero rather slowly, the initial dependence on M being clearly noticeable.

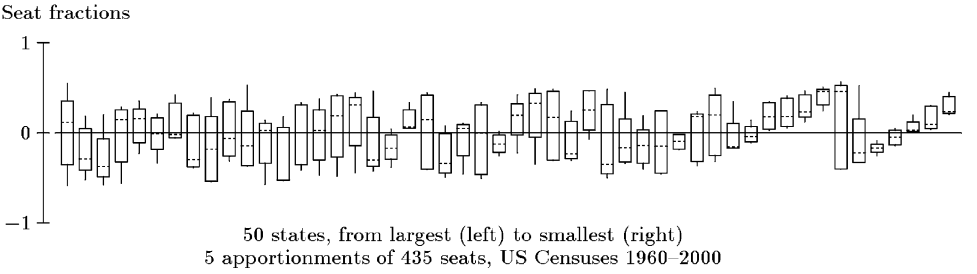

Fig. 12. Observed seat excess distributions in a 50-state system, for the divisor method with geometric rounding (Hill, Huntington; Equal Proportions). The box plots are located around the zero line and promise unbiased seat allocations, similar to Figs. 8 and 9. Theoretical seat biases are not available.

Table 3
Computer calculations of the Equal Proportions seat biases B_k^{EP} when M = 435, for number of parties l from 2 to 5. For the current House size, the theoretical EP seat biases are practically zero

l	B_1^{EP}	B_2^{EP}	B_3^{EP}	B_4^{EP}	B_5^{EP}
2	–0.004	0.004			
3	–0.010	–0.002	0.012		
4	–0.016	–0.006	–0.001	0.023	
5	–0.022	–0.011	–0.004	0.002	0.035

“large” or “not-so-large”, for the seat biases to be practically zero or not. Fig. 12 shows the empirical box plots of the Equal Proportions apportionments for the five censuses 1960–2000. The appearance is similar to Figs. 8 and 9, and we cannot recognize any marked bias effect. Thus the graphical evidence suggests that the seat allocations for $M = 435$ and $l = 50$ are practically unbiased.

In addition we ran our computer program to calculate the precise seat biases for up to five parties. (Since for the Equal Proportions method the domains of attraction have different volumes for different seat allocations, our program failed to handle more parties.) The numbers, listed in Table 3, are rather close to zero. However, extrapolation from five to fifty offers only a weak degree of persuasion.

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