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ROUNDING PROBABILITIES: UNBIASED MULTIPLIERS

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Abstract: When rounding a finite set of probabilities to integral multiples of $1/n$, multiplier methods guarantee that the rounded probabilities again sum to one. For those multiplier methods that are stationary, we discuss the expected discrepancy and calculate unbiased multipliers, under the assumption of uniformly distributed probabilities.

1. Introduction

Usual, *standard rounding* is unfit to round a finite set of *weights* or probabilities to integral multiples of $1/n$, where n is a given *accuracy* or common denominator. Specifically, if weights are rounded to percentages, $n = 100$, then standard rounding yields numbers that often fail to add to 100 percent. Real data abound with examples suffering from this deficiency [1–3, 6–8, 10–14].

For instance, in the 1992 IMS membership survey [9] the authors evidently apply standard rounding to multiples of a tenth of a percent. They report 56 tables of three to eight categories, of which 34 round to 100.0 percent, 12 round to 99.9 percent, and 10 round to 100.1 percent.

Standard rounding yields weights summing to one with a probability that vanishes as the number of categories becomes large [12]. To be precise, standard rounding uses the rounding function $r_{1/2}(x)$ that rounds $x \geq 0$ to the nearest ~~even~~ integer; hence fractional parts are rounded down when they are smaller than $1/2$, and rounded up when they are larger than $1/2$. Assume there are *c categories*, and let $(W_1, \ldots, W_c)$ be a set of random weights that is uniformly distributed in the probability simplex of $\mathbb{R}^c$. DIACONIS–FREEDMAN [6] show that then $\lim_{n\to\infty} \mathrm{P}\left(\sum_{i\leq c} r_{1/2}(nW_i)/n = 1\right) = O(1/\sqrt{c})$. The reason for this deficiency is that there is nothing built into standard rounding to preserve a linear side condition such as summing to one.

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However, there are plenty of rounding methods that do preserve the side condition of adding up to one. They have been proposed and investigated by politicians and political scientists, in the study of apportionment problems for electoral bodies. BALINSKI–YOUNG, in their authoritative monograph [3], prove that among all rounding procedures only quotient methods are free from irritating paradoxes. For the purpose of our investigations we prefer to speak of multiplier methods rather than quotient methods.

A fundamental tool are *rounding rules*, they are reviewed in Section 2. Special emphasis is put on q-stationary rules. Given the *stationarity parameter* $q \in [0,1]$, the q-stationary rule uses the rounding function $r_q(x)$ that rounds to the nearest integer below x when the fractional part of x is smaller than q, and to the nearest integer above x when the fractional part is larger than q. This includes the classical rounding functions of always rounding up $(q = 0)$, always rounding down $(q = 1)$, and rounding up or down in the usual, standard way $(q = 1/2)$.

Every rounding rule induces a corresponding *multiplier method* of rounding, as described in Section 3. A multiplier method may lead to two or more, equally legitimate roundings. Theorem 1 counts how many roundings a multiplier method contains.

In Section 4 we propose a two-step algorithm to calculate the roundings of a multiplier method. The first step is called the *multiplier step* and gets close to a result, but may leave a nonzero discrepancy. The *discrepancy step* then consists of a few iterations, to augment some of the rounded weights if there is a negative discrepancy, or to reduce some of them if the discrepancy is positive. An Emacs Lisp implementation of the algorithm is available [7].

Section 5 investigates the discrepancy moments for uniformly distributed weights. For a finite accuracy n, Theorem 2 secures the existence of a multiplier $\widetilde{\nu}_n$ for which the expected discrepancy vanishes. However, $\widetilde{\nu}_n$ is hard to calculate.

For more detailed results, in Section 6, we restrict attention to q-stationary multiplier methods. Theorems 3 and 4 provide asymptotic formulas for the expectation and variance of the discrepancy, when the accuracy n tends to infinity. It follows that the expected discrepancy vanishes asymptotically when the multiplier is taken to be

$$\nu_n = n + c\Big(q - \frac{1}{2}\Big). \tag{1}$$

In particular, the recommended multiplier for the *method of Adams* $(q = 0)$ is $\nu_n = n - c/2$ [13, 14], while for the *method of Jefferson* $(q = 1)$ it is $\nu_n = n + c/2$. The *method of Webster* $(q = 1/2)$ has multiplier $\nu_n = n$, as suggested by the Rule of Three.

Usual, standard rounding is just the same as the multiplier step with $\nu_n = n$ for the method of Webster. The reason for its frequent failure to add to one is that it misses out on the discrepancy step of the algorithm. Our result on the discrepancy moments provide a first explanation for the observed discrepancies in the examples that are mentioned above. The exact, finite discrepancy distribution is given by Happacher [10].

2. Rounding rules

The definition of a rounding rule R is based on a *signpost sequence* $s(k) \in [k, k+1]$, for $k = 0, 1, \ldots$ [3]. The signposts are assumed to be strictly increasing, in order to avoid three-way ties. When $x = s(k)$ coincides with a signpost, there is a two-way tie and $R(x)$ is defined to be the *two-element set* $\{k, k+1\}$. When $x \geq 0$ lies between two signposts, $x \in \big(s(k-1), s(k)\big)$, the set $R(x) = \{k\}$ is a singleton. With starting value $s(-1) = -1$, we define for all $k = 0, 1, \ldots$ and for all $x \geq 0$

$$R(x) = \begin{cases} \{k, k+1\} & \text{for } x = s(k), \\ \{k\} & \text{for } x \in \big(s(k-1), s(k)\big). \end{cases}$$

Alternatively, the signpost sequence and the rounding rule fulfill the *basic relation*

$$k \in R(x) \qquad \Longleftrightarrow \qquad s(k-1) \leq x \leq s(k). \tag{2}$$

We concentrate on *q-stationary rounding rules*, for some fixed value $q \in [0, 1]$. They are defined through the signpost sequence

$$s_q(k) = k + q \qquad \text{for } k = 0, 1, \ldots \tag{3}$$

This rule appears implicitly in [6] with a view towards equivariance, and in [11] with a view towards linearity. Our terminology is inspired by Balinski–Rachev [2].

The *p-mean rounding rules* play a greater historical role [3]. With $p \in (-\infty, \infty)$ fixed, the defining signpost sequence is

$$\widetilde{s}_p(k) = \left(\frac{k^p + (k+1)^p}{2} \right)^{1/p} \qquad \text{for } k = 0, 1, \ldots \tag{4}$$

These are of the order $k + 1/2 + O(1/k)$ as $k \to \infty$. Hence their asymptotic behavior is the same as with $p = 1$, which in turn is the same as (3) with $q = 1/2$. Furthermore, the two extreme p-mean roundings are $\widetilde{s}_{-\infty}(k) = k = s_0(k)$, and $\widetilde{s}_{\infty}(k) = k+1 = s_1(k)$. They coincide with the extreme members from (3).

Hence the stationary rounding rules (3) appear to form a richer family than the p-mean rounding rules (4). Both contain the *classical rounding rules* of rounding up $(q = 0, p = -\infty)$, standard rounding $(q = 1/2, p = 1)$, and rounding down $(q = 1, p = \infty)$.

The fact that a rounding rule R is set-valued is computationally unpleasant. Therefore we also introduce *rounding functions r that are compatible with R*, by demanding $r(x) \in R(x)$ for all $x \geq 0$. Hence r is an increasing, piecewise constant function. It jumps at $s(k)$

where it takes one of the two values, k or $k+1$. Evidently a rounding rule R admits many rounding functions r that are compatible with it.

Standard rounding, $q = 1/2$, is usually carried out with the rounding function $r_{1/2}(x)$ that maps $x \geq 0$ to the closest integer when the fractional part of x is distinct from $1/2$ (and the closest integer is thus unique), and to the closest *even* integer otherwise [15, page 175; 4, Section 2.1.1.2]. For rounding down, $q = 1$, a convenient rounding function is the *floor function* or integer part $\lfloor x \rfloor = \max\{k \mid k \leq x\}$. For rounding up, $q = 0$, the counterpart is the *ceiling function* $\lceil x \rceil = \min\{k \mid k \geq x\}$.

3. Multiplier methods

Any rounding rule R induces a multiplier method of rounding. The multiplier methods that come with the classical rounding rules of rounding up, standard rounding, or rounding down are historically associated with the names of Adams, Webster, and Jefferson [3].

Multiplier methods introduce a new continuous variable, the *multiplier* $\nu \geq 0$. This additional degree of freedom is used to fit the side condition that rounded weights sum to one. It is convenient to assemble the weights into a vector $\mathbf{w} = (w_1, \ldots, w_c)$. Without loss of generality we assume $w_i > 0$ for all $i = 1, \ldots, c$. For a given integer $n \geq 1$, the goal is to round w_i to a rational number of the form n_i/n, that is, to find appropriate numerators n_i. The condition $\sum_{i \leq c} n_i/n = 1$ turns into $\sum_{i \leq c} n_i = n$.

Rounding rules do not resolve two-way ties, nor do multiplier methods. Hence a *set* of possible numerators is proposed [3]. Given a rounding rule R, the set of roundings for a weight vector $\mathbf{w}$ and an accuracy n is defined by

$$M_R(\mathbf{w}, n) = \left\{ (n_1, \ldots, n_c) \;\middle|\; \exists \nu \geq 0 \quad \forall i \leq c: \; n_i \in R(\nu w_i) \quad \text{and} \quad \sum\nolimits_{i \leq c} n_i = n \right\}.$$

In the rare, special case when $s(0) = 0$ and $0 \leq n < c$, we define $n_i = 1$ or $n_i = 0$ according as w_i is among the n largest weights or not. In general we adopt the convention $0/w_i < 0/w_j$ for $w_i > w_j$.

There is an alternative characterization in the form of a Max–Min inequality. It uses the signposts $s(k)$ that determine the rounding rule R, and follows from the basic relation (2). A set of integers $(n_1, \ldots, n_c)$ with $\sum_{i \leq c} n_i = n$ belongs to $M_R(\mathbf{w}, n)$ if and only if

$$\max\nolimits_{i \leq c} \frac{s(n_i - 1)}{w_i} \leq \min\nolimits_{i \leq c} \frac{s(n_i)}{w_i}. \tag{5}$$

In particular, the set $M_R(\mathbf{w}, n)$ is always nonempty.

What happens when we start with some member $(n_1, \ldots, n_c)$ in $M_R(\mathbf{w}, n)$ and vary the precision n? It is easy to step up to a member of $M_R(\mathbf{w}, n+1)$, or to step down to a member of $M_R(\mathbf{w}, n-1)$. Let $\mathcal{J}$ and $\mathcal{K}$ be the set of those subscripts that attain the minimum and maximum in (5), respectively,

$$\mathcal{J} = \Big\{ j \le c \;\Big|\; \frac{s(n_j)}{w_j} = \min_{i\le c} \frac{s(n_i)}{w_i} \Big\}, \qquad \mathcal{K} = \Big\{ k \le c \;\Big|\; \frac{s(n_k-1)}{w_k} = \max_{i\le c} \frac{s(n_i-1)}{w_i} \Big\}.$$

Proposition 3.3 in [3] or Theorem 12.5b in [13] say that $\mathcal{J}$ consists of the *augmentation candidates* and $\mathcal{K}$ of the *reduction candidates*:

$$\begin{aligned} j \in \mathcal{J} \quad &\Longleftrightarrow \quad (n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_c) \in M_R(\mathbf{w}, n+1), \\ k \in \mathcal{K} \quad &\Longleftrightarrow \quad (n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_c) \in M_R(\mathbf{w}, n-1). \end{aligned} \tag{6}$$

Moreover, we can enumerate how many roundings appear in the set $M_R(\mathbf{w}, n)$.

Theorem 1. *Let $(n_1, \ldots, n_c)$ be a member of $M_R(\mathbf{w}, n)$. Then the set $M_R(\mathbf{w}, n)$ is a singleton if and only if strict inequality holds in* (5). *Otherwise equality holds in* (5) *and there are $\binom{a+b}{a}$ roundings in $M_R(\mathbf{w}, n)$, where a is the number of augmentation candidates in $\mathcal{J}$ and b is the number of reduction candidates in $\mathcal{K}$.*

Proof. In the first part we show that $M_R(\mathbf{w}, n)$ contains at least two members if and only if equality holds in (5), compare the proof of Theorem 12.7 in [13]. For the direct part, we choose two distinct members $(n_1, \ldots, n_c) \ne (\widetilde{n}_1, \ldots, \widetilde{n}_c)$ in $M_R(\mathbf{w}, n)$, with respective multipliers ν and $\widetilde{\nu}$. If $\nu < \widetilde{\nu}$ then $n_i \le \widetilde{n}_i$. Now $\sum_{i\le c} n_i = n = \sum_{i\le c} \widetilde{n}_i$ forces $n_i = \widetilde{n}_i$ for all $i = 1, \ldots, c$. This contradicts our choice that the two members are distinct. The same argument applies to $\nu > \widetilde{\nu}$. Hence we get $\nu = \widetilde{\nu}$. Another multiplier $\widetilde{\widetilde{\nu}}$ for $(n_1, \ldots, n_c)$ also satisfies $\widetilde{\widetilde{\nu}} = \widetilde{\nu}$. Therefore $\widetilde{\widetilde{\nu}} = \nu$, and the multiplier for $(n_1, \ldots, n_c)$ is unique. This entails equality in (5).

For the converse part, we assume equality in (5), and fix a member $(n_1, \ldots, n_c)$ in $M_R(\mathbf{w}, n)$. Now all $j \in \mathcal{J}$ and $k \in \mathcal{K}$ satisfy

$$\frac{s(n_i-1)}{w_i} \le \frac{s(n_k-1)}{w_k} = \frac{s(n_j)}{w_j} \le \frac{s(n_i)}{w_i} \qquad \text{for all } i = 1, \ldots, c.$$

We cannot have $j = k$, since $s(n_j - 1) = s(n_j)$ contradicts the strict monotonicity of the signpost sequence. With $j \ne k$, we transfer mass from the kth to the jth entry, by setting $\widetilde{n}_j = n_j + 1, \widetilde{n}_k = n_k - 1, \widetilde{n}_i = n_i$ for all $i \ne j, k$. The new set $(\widetilde{n}_1, \ldots, \widetilde{n}_c)$ satisfies

$$\frac{s(\widetilde{n}_i-1)}{w_i} \le \frac{s(\widetilde{n}_j-1)}{w_j} = \frac{s(\widetilde{n}_k)}{w_k} \le \frac{s(\widetilde{n}_i)}{w_i} \qquad \text{for all } i = 1, \ldots, c.$$

Therefore it is a second member in $M_R(\mathbf{w}, n)$, besides $(n_1, \ldots, n_c)$.

In the second part of the proof we assume equality in (5). We have just seen that then there is a unique multiplier ν. For a given member $(n_1, \ldots, n_c)$ in $M_R(\mathbf{w}, n)$ we have $R(\nu w_j) = \{n_j, n_j + 1\}$ and $R(\nu w_k) = \{n_k - 1, n_k\}$ for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$. Also, $R(\nu w_i)$ is a singleton for $i \notin \mathcal{J} \cup \mathcal{K}$. In other words, for $a + b$ subscripts j and k the rounding rule R produces a two-way tie when applied to νw_j and νw_k. Of these ties, a ties are resolved by the lower option (whence the remaining b ties are resolved by the upper option). There are $\binom{a+b}{a}$ ways to do this. $\square$

4. Rounding algorithm

We can now be more precise about our proposed implementation of a multiplier method. The algorithm is initialized by choosing a rounding function r that is compatible with the rounding rule R. For a given accuracy n it then makes a choice of a multiplier ν that is thought to work reasonably well irrespective of the weight vector $\mathbf{w} = (w_1, \ldots, w_c)$.

- The first, multiplier step rounds the weights w_i to n_i/n with $n_i = r(\nu w_i)$.
- The second, discrepancy step evaluates the *discrepancy*

  $$d = \left(\sum\nolimits_{i \leq c} n_i\right) - n.$$

  While $d \neq 0$ it loops to augment or reduce $n_1, \ldots, n_c$ according to (6), and terminates when $d = 0$.

Upon termination the set $M_R(\mathbf{w}, n)$ may be enumerated using Theorem 1.

For standard rounding with multiplier $\nu = n$, the results of [6, 12] say that the algorithm does *not* terminate with the first step, with probability one as $c \to \infty$. This statement should not be construed as evidence against the first step. Instead it emphasizes the need to continue on into the second step.

5. Random weights

The choice of the multiplier ν depends on the distribution of the weight vectors $\mathbf{w}$ to which the algorithm is applied. In the sequel we assume that the weight vector $(W_1, \ldots, W_c)$ follows the uniform distribution on the probability simplex of $\mathbb{R}^c$. The number of categories, c, remains fixed. Let R be a rounding rule with associated signposts $s(k)$.

The event that for a multiplier $\nu > 0$ a component hits a signpost, $\bigcup_{i \leq c} \bigcup_{k \geq 0} \{\nu W_i = s(k)\}$, has probability zero. Hence, almost surely, $R(\nu W_i)$ is a singleton and any two rounding functions r and $\widetilde{r}$ that are compatible with R satisfy $R(\nu W_i) = \{r(\nu W_i)\} = \{\widetilde{r}(\nu W_i)\}$, for every multiplier $\nu > 0$. Therefore we are free to choose *any* rounding function r that is compatible with R.

Given a multiplier $\nu > 0$ we define the *total*

$$T_c(\nu) = \sum\nolimits_{i \leq c} r(\nu W_i).$$

This is an integer-valued random variable. By choice of ν we would like to bring it close to n, in order to achieve a small discrepancy $T_c(\nu) - n$. Indeed, there is a unique multiplier $\widetilde{\nu}_n$ that makes the *expected total* equal to n.

Theorem 2. *For $\nu > 0$ we introduce $k = \max\{i \geq -1 \mid s(i) \leq \nu\}$. Then we have*

$$\mathrm{E}[T_c(\nu)] = c\sum\nolimits_{j=0}^{k} \left(1 - \frac{s(j)}{\nu}\right)^{c-1}. \tag{7}$$

In particular, for $n \geq c$ there exists a unique multiplier $\widetilde{\nu}_n > 0$ satisfying $\mathrm{E}[T_c(\widetilde{\nu}_n)] = n$.

Proof. Define the integer-valued random variable $N_1 = r(\nu W_1)$. By exchangeability we get $\mathrm{E}[T_c(\nu)] = c\,\mathrm{E}[N_1]$. For $j = 0, 1, \ldots$ we have $\{N_1 > j\} = \{W_1 > \frac{s(j)}{\nu}\}$. This entails $\mathrm{P}(N_1 > j) = (1 - \frac{s(j)}{\nu})^{c-1}$ for $j \leq k$, and $\mathrm{P}(N_1 > j) = 0$ for $j > k$. Now $\mathrm{E}[T_c(\nu)] = c\sum_{j=0}^{\infty} \mathrm{P}(N_1 > j)$ yields (7). The function $f(\nu) = \mathrm{E}[T_c(\nu)]$ is continuous on $(0, \infty)$, and strictly increases to infinity. The right limit for $\nu \downarrow 0$ equals zero or c according as $s(0)$ is positive or zero. Hence for $n \geq c$ the equation $f(\nu) = n$ has a unique solution $\widetilde{\nu}_n > 0$. $\square$

6. Discrepancy moments for stationary methods

From now on we restrict attention to a q-stationary rounding function r_q, with signpost sequence (3). The basic relation (2) almost surely yields $\nu W_i - q < r_q(\nu W_i) < \nu W_i + 1 - q$, and $\nu - cq < T_c(\nu) < \nu + c(1 - q)$. With $\nu_n = n + c(q - 1/2)$ from (1), we almost surely obtain the symmetric support bounds

$$n - \frac{c}{2} < T_c(\nu_n) < n + \frac{c}{2}.$$

For $c = 2$ categories and an accuracy $n \geq 2$, the multiplier $n + 2q - 1$ is positive and the integer-valued random variable $T_2(n + 2q - 1)$ lies strictly between $n - 1$ and $n + 1$. Hence it degenerates to a constant, $T_2(n + 2q - 1) = n$ almost surely. In particular, we have $\widetilde{\nu}_n = n + 2q - 1$ in Theorem 2. In other words, the discrepancy vanishes almost surely when a q-stationary rounding rule with multiplier $n + 2q - 1$ is applied to two categories. For standard rounding this is already pointed out in [12].

For three or more categories, more can be said about the expected total in Theorem 2. With $q \in [0, 1]$ and $\nu > 0$, we introduce $\epsilon = \nu - q - \lfloor \nu - q \rfloor \in [0, 1]$. Reversing the order of summation in (7), we obtain

$$\mathrm{E}[T_c(\nu)] = \frac{c}{\nu^{c-1}} \sum\nolimits_{j=0}^{\lfloor \nu - q \rfloor} (j + \epsilon)^{c-1}. \tag{8}$$

Elementary calculus indicates how to expand the *summa potentatis*:

$$c\sum\nolimits_{j=0}^{k} (j + \epsilon)^{c-1} \approx c\int_{-1/2}^{k+1/2} (x + \epsilon)^{c-1}\, dx \approx \left(k + \frac{1}{2} + \epsilon\right)^{c} = \left(\nu - q + \frac{1}{2}\right)^{c}.$$

Geometrically, the addition of $1/2$ serves as a continuity correction. Numerically, a polynomial in $k + 1/2 + \epsilon$ approximates the sum much better than a polynomial in $k + \epsilon$, in that the exponents drop off in steps of two [5]. This enables us to evaluate the asymptotic behavior of (8).

Theorem 3. *For $q \in [0,1]$ and $\nu > q$, set $\epsilon = \nu - q - \lfloor \nu - q \rfloor$. Then we have*

$$\begin{aligned}
\mathrm{E}[T_c(\nu)] &= \frac{(\nu-q+1/2)^c}{\nu^{c-1}} \left\{ 1 - \frac{1}{12}\binom{c}{2}\frac{1}{(\nu-q+1/2)^2} + \frac{7}{240}\binom{c}{4}\frac{1}{(\nu-q+1/2)^4} \right. \\
&\quad \left. - \frac{31}{1344}\binom{c}{6}\frac{1}{(\nu-q+1/2)^6} + \frac{127}{3840}\binom{c}{8}\frac{1}{(\nu-q+1/2)^8} \mp \dots \right\} + \frac{\pi_c(\epsilon)}{\nu^{c-1}} \\
&= \nu - c(q-1/2) + \rho_c(\nu),
\end{aligned}$$

with a polynomial π_c in ϵ of degree c in the first representation, and a remainder term $\rho_c(\nu) = O(1/\nu)$ as $\nu \to \infty$ in the second representation. If c is even, the sum in the first representation stops at the term with binomial coefficient $\binom{c}{c-2}$.

Proof. Section 2 of [5] carries over to the shifted summands $j + \epsilon$ that appear in (8) when the summation starts at $j = 0$ rather than $j = 1$. Now formula (2.11) in [5] provides the first representation. The second follows from the binomial expansion of $(\nu - q + 1/2)^c$. □

In the second representation, the remainder terms are:

$$\begin{aligned}
\rho_2(\nu) &= \frac{q(q-1) - \epsilon(\epsilon-1)}{\nu}, \\
\rho_3(\nu) &= 3\frac{1/6 + q(q-1)}{\nu} - \frac{q(q-\frac{1}{2})(q-1) + \epsilon(\epsilon-\frac{1}{2})(\epsilon-1)}{\nu^2}, \\
\rho_4(\nu) &= 6\frac{1/6 + q(q-1)}{\nu} - \frac{4q(q-\frac{1}{2})(q-1)}{\nu^2} + \frac{q^2(q-1)^2 - \epsilon^2(\epsilon-1)^2}{\nu^3}, \\
\rho_c(\nu) &= \binom{c}{2}\frac{1/6 + q(q-1)}{\nu} + O\left(\frac{1}{\nu^2}\right) \qquad \text{for all } c \geq 3.
\end{aligned}$$

For two categories, the multiplier $\widetilde{\nu}_n = n + 2q - 1$ yields $\rho_2(n + 2q - 1) = 0$, as we know from the remarks following Theorem 2. For three or more categories, Theorem 3 has a companion result for the variance. In general, the variance equals $c/12$ plus a remainder term bounded of order $1/\nu$. For the classical methods of Adams, Webster, and Jefferson, the order surprisingly improves to $1/\nu^2$.

Theorem 4. *For $c \geq 3$ categories, $q \in [0,1]$ and $\nu > 2q$ we have*

$$\mathrm{V}[T_c(\nu)] = \frac{c}{12} + \frac{2}{3}\binom{c}{2}\frac{q(q-\frac{1}{2})(q-1)}{\nu} + O\left(\frac{1}{\nu^2}\right) \qquad \text{as } \nu \to \infty.$$

Proof. Straightforward, though lengthy calculations establish the result [10]. □

Finally we return to the discrepancy $T_c(\nu) - n$. The expectation is $\mathrm{E}[T_c(\nu) - n] = \nu - (n + c(q-1/2)) + O(1/\nu)$, by Theorem 3. Hence $\nu_n = n + c(q - 1/2)$ from (1) generates a discrepancy with an expectation that vanishes asymptotically, $\mathrm{E}[T_c(\nu_n) - n] = O(1/n)$. For three and four categories, the remainder terms for the classical methods are as follows:

$q = 0, p = -\infty$	$q = 1/2, p = 1$	$q = 1, p = \infty$
Adams	**Webster**	**Jefferson**
$\mathrm{E}[T_3(n - \frac{3}{2}) - n] = \frac{1}{2n-3}$	$\mathrm{E}[T_3(n) - n] = -\frac{1}{4n}$	$\mathrm{E}[T_3(n + \frac{3}{2}) - n] = \frac{1}{2n+3}$
$\mathrm{E}[T_4(n - 2) - n] = \frac{1}{n-2}$	$\mathrm{E}[T_4(n) - n] = -\frac{1}{2n}$	$\mathrm{E}[T_4(n + 2) - n] = \frac{1}{n+2}$

These results conform with the empirical examples that we have looked at. For instance, the average discrepancy of the IMS survey data [9] is $-2/56$, with c ranging between three and eight. This is in line with the slightly negative expected discrepancy that comes with the Webster method. The discrepancy step of the algorithm of Section 4 passes through an expected number of loops that is given by $\mathrm{E}[|T_c(\nu_n) - n|]$. Since the integrand is integer-valued, the expectation is bounded from above by $\mathrm{E}[(T_c(\nu_n) - n)^2]$. The latter approximately equals $c/12$, by Theorem 4. This conforms with the empirical number of loops in the IMS example, $22/56 = 0.4$.

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