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ON THE EXISTENCE OF UNBIASED NONNEGATIVE ESTIMATES OF VARIANCE COVARIANCE COMPONENTS[1]

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The existence of unbiased nonnegative definite quadratic estimates for linear combinations of variance covariance components is characterized by means of the natural parameter set in a residual model. In the presence of a quadratic subspace condition the following disjunction for nonnegative estimability is derived: either standard methods suffice, or the concepts of unbiasedness and nonnegative definiteness are incompatible. For the case of a single variance component it is shown that unbiasedness and nonnegative definiteness always entail a reduction to a trivial model in which the variance component under investigation is the sole remaining parameter. Several examples illustrate these results.

1. Introduction and summary. In variance component estimation, common methods may lead to negative estimates for variances which, obviously, are nonnegative parameters. Since many procedures in the analysis of variance construct their estimator as a quadratic form that is unbiased for the parameter under investigation, the question arises when this can be done by simultaneously satisfying the inherent nonnegativity constraint. In the sequel linear combinations of variance-covariance components for which there exists an unbiased nonnegative definite quadratic estimator will be called *nonnegatively estimable* for short.

Various alternatives are at hand as to how a statistician may react to negative variance estimates; see the discussion in Searle (1971), page 406. Also a number of estimators have been proposed which, by sacrificing unbiasedness, evade the negativity defect; see, e.g., Horn and Horn (1975), P.S.R.S. Rao and Chaubey (1978) and Hartung (1981). For the present case in which unbiasedness is maintained, LaMotte (1973) showed that of all individual variance-covariance components, at most the error variance σ_e^2 is nonnegatively estimable. The situation gets more complicated, however, as soon as linear combinations $\sum q_j\tau_j$ of l variance covariance components $\tau_1, \cdots, \tau_l$ are considered, where the coefficients $q_1, \cdots, q_l$ are allowed to be arbitrary numbers.

In Section 2 nonnegative estimability of a form $\sum q_j\tau_j$ is characterized by means of the natural parameter set in a residual model. This leads to the surprising alternative that in the presence of a quadratic subspace condition, as introduced by Seely (1971), either the standard unbiased estimate $\sum q_j\hat{\tau}_j$ does provide an unbiased nonnegative definite quadratic estimate, or no such estimate exists. The familiar multivariate linear model is chosen as an example since it includes variance components as well as covariance components. All proofs for Section 2 are presented in Section 3.

In Section 4 it is shown that considerations of variance-covariance component estimation also cover the case of pure variance component estimation, and details are given for

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two examples from the analysis of variance. In addition it is proved that estimation of a single variance component σ_j^2 always entails a reduction to a model in which σ_j^2 is the only remaining parameter. The model with heteroscedastic variances is used to illustrate these results and the relation to the work of Balestra (1973), Kleffe and Zöllner (1978), and P.S.R.S. Rao and Chaubey (1978).

2. Variance-covariance component estimation. In order to characterize nonnegative estimability in those terms which define the underlying model it seems advantageous to represent a *linear model* by its moments, according to

$$\text{LM:} \quad Y \sim (X\beta; \textstyle\sum_{j=1}^{l} \tau_j V_j).$$

This means that given the real $n \times k$ matrix X and the l real symmetric $n \times n$ matrices V_j the random $\mathbb{R}^n$-vector Y has mean vector $X\beta$ and dispersion matrix $\sum \tau_j V_j$, with unknown values $\beta \in \mathbb{R}^k$ of the mean parameter, and $\tau = (\tau_1, \cdots, \tau_l)' \in \bar{G}$ of the dispersion parameter. The natural parameter set $\bar{G}$ for τ which ensures nonnegative definiteness of the dispersion matrix is

$$\bar{G} = \{t = (t_1, \cdots, t_l)' \in \mathbb{R}^l \mid \textstyle\sum t_j V_j \in \mathrm{NND}(n)\},$$

where $\mathrm{NND}(n)$ denotes the set of all real symmetric nonnegative definite $n \times n$ matrices; a prime indicates transposition. The dispersion parameter τ may comprise variance components as well as covariance components. When $Y \sim \mathcal{N}_n(X\beta; \sum \tau_j V_j)$, i.e., Y additionally follows a n-variate normal distribution, then τ is restricted to the set G of those values $t \in \mathbb{R}^l$ such that $\sum t_j V_j$ is positive definite. The region [Gebiet] G is assumed to be nonempty, and in this case the closure of G coincides with $\bar{G}$.

Unbiased nonnegative definite quadratic estimation always entails an initial reduction to the *residual model*

$$\text{RM:} \quad MY \sim (0; \textstyle\sum \tau_j M V_j M), \qquad M = I_n - X(X'X)^- X',$$

in which the natural parameter set for τ is

$$\overline{G_M} = \{t = (t_1, \cdots, t_l)' \in \mathbb{R}^l \mid \textstyle\sum t_j M V_j M \in \mathrm{NND}(n)\}.$$

For when $Y'AY$ is an unbiased nonnegative estimate for $q'\tau$, then $AX = 0$ (Atiqullah, 1962, page 84), and $A = MAM$, whence the estimate $Y'AY = (MY)'A(MY)$ depends on the observation Y only through the residual statistic MY. The parameter set $\overline{G_M}$ of the residual model is now used to give a first characterization of nonnegative estimability.

THEOREM 1. *Suppose $q \in \mathbb{R}^l$. Then the form $q'\tau$ is nonnegatively estimable if and only if all numbers $q't$, $t \in \overline{G_M}$, are nonnegative.*

Since $\bar{G}$ is a subset of $\overline{G_M}$ it follows from this theorem that if $q'\tau$ is nonnegatively estimable then $q't \geqq 0$ for all $t \in \bar{G}$. This relation reflects the fact that only forms $q'\tau$ that are nonnegative on $\bar{G}$ call for nonnegative estimation.

Next a class of linear models is described that allows more explicit answers to the questions of nonnegative estimability. Associated with the l decomposing matrices of the residual model is their Gramian matrix $S_M = ((\operatorname{trace} MV_i MV_j)) \in \mathrm{NND}(l)$, and their span, i.e., the subspace $\mathscr{B}_M = \{\sum \lambda_j M V_j M \mid \lambda_1, \cdots, \lambda_l \in \mathbb{R}\}$ of symmetric $n \times n$ matrices. The subspace $\mathscr{B}_M$ has dimension l if and only if the matrix S_M has rank l. Following Seely (1971), $\mathscr{B}_M$ is called a *quadratic subspace of symmetric matrices* if $B^2 \in \mathscr{B}_M$ for all $B \in \mathscr{B}_M$. When q lies in the range of S_M a distinguished estimate (MINQUE given I_n) for $q'\tau$ is $\widehat{q'\tau} = Y' \sum \lambda_j M V_j M Y$, with $\lambda = (\lambda_1, \cdots, \lambda_l)'$ being an arbitrary solution of $S_M \lambda = q$ (C. R. Rao, 1973, page 303; Kleffe, 1977, page 223).

Now assume that $\mathscr{B}_M$ forms a l-dimensional quadratic subspace. Then $\widehat{q'\tau}$ is the unbiased translation-invariant quadratic estimate for $q'\tau$ which has uniformly minimal variance (UMVU) under normality (Seely, 1971, page 718). The estimate $\widehat{q'\tau}$ also coincides with the maximum likelihood (REML) estimate in a normal residual model when at least $(n - \text{rank } X)$ replicates are available (Pukelsheim and Styan, 1979). Since all these concepts coincide $\widehat{q'\tau}$ may be called *the standard unbiased estimate for* $q'\tau$. In particular, the standard unbiased estimates $\hat{\tau}_j$ for τ_j can be used to form the l-dimensional statistic $\hat{\tau} = (\hat{\tau}_1, \cdots, \hat{\tau}_l)'$. Then for every form $q'\tau$ one has $\widehat{q'\tau} = q'\hat{\tau}$, and $\hat{\tau}$, having similar optimality properties as $q'\hat{\tau}$, is termed *the standard unbiased estimate for* τ. The following theorem shows that (i) for a particular form $q'\tau$ nonnegative estimability reduces to the alternative that either old methods suffice or nothing else can be done, while (ii) no problem arises with the vector statistic $\hat{\tau}$.

THEOREM 2. *Assume $\mathscr{B}_M$ to be a l-dimensional quadratic subspace of symmetric matrices, and suppose $q \in \mathbb{R}^l$.* (i) *Then either the standard unbiased estimate for $q'\tau$ is nonnegative, or $q'\tau$ is not nonnegatively estimable.* (ii) *The matrix estimate $\sum \hat{\tau}_j MV_jM$ for the dispersion matrix in the residual model, obtained from the standard unbiased estimate $\hat{\tau}$ for τ, always is nonnegative definite.*

Part (ii) means that $\hat{\tau}$ maps the sample space $\mathbb{R}^n$ into the parameter set $\overline{G_M}$ of the residual model. An example will show that $\hat{\tau}$ does not, in general, map into the parameter set $\bar{G}$ of the original model (LM). Horn and Horn (1975), page 876, draw attention to the model $(Y_1, Y_2, Y_3)' \sim ((\mu, \mu, \mu)'; \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2))$, with standard unbiased estimates $\hat{\sigma}_j^2 = \prod_{i \neq j} (Y_j - Y_i)$. Since their product is $-(Y_1 - Y_2)^2(Y_1 - Y_3)^2(Y_2 - Y_3)^2$, the vector statistic $\hat{\sigma}^2$ is not in $\bar{G}$, almost surely with respect to every continuous distribution. But $\mathscr{B}_M$ does form a 3-dimensional quadratic subspace of symmetric 3×3 matrices, whence $\hat{\sigma}^2$ must map into $\overline{G_M}$, by Theorem 2(ii). In fact, one obtains $\sum \hat{\sigma}_j^2\ Y'MV_jMY = MYY'M$.

In many analysis of variance models the matrices MV_jM also commute, besides spanning a quadratic subspace (R. D. Anderson, et al., 1979). In this case $\mathscr{B}_M$ has a basis of pairwise orthogonal projectors $R_1, \cdots, R_l$ (Seely, 1971, page 714), let T be the $l \times l$ matrix with entries t_{ij} defined by $MV_iM = \sum_j t_{ij}R_j$. Here the standard unbiased estimate $q'\hat{\tau}$ for $q'\tau$ also coincides with the analysis of variance estimate $\sum \lambda_j Y'R_jY/\text{rank } R_j$, which combines the l sums of squares $Y'R_jY$ in order to obtain unbiasedness for $q'\tau$, i.e., $(\lambda_1, \cdots, \lambda_l)' = T^{-1}q$. For $q'\tau$ to be nonnegatively estimable it is then obviously sufficient that all λ_j be nonnegative. That this is also necessary is shown by the following corollary, thus strengthening Theorems 1 and 2(i).

COROLLARY 3. *Assume $\mathscr{B}_M$ to be a commutative l-dimensional quadratic subspace of symmetric matrices, and suppose $q \in \mathbb{R}^l$. Then the form $q'\tau$ is nonnegatively estimable if and only if all components of $T^{-1}q$ are nonnegative.*

For these models the alternative on nonnegative estimability attains its simplest form: either the analysis of variance method does provide an unbiased nonnegative estimate, or the concepts of unbiasedness and nonnegative definiteness are incompatible.

An example that also involves covariances is furnished by the multivariate linear model (T. W. Anderson, 1958, page 178; C. R. Rao, 1973, page 544; Searle, 1978, page 183). Taking N independent p-variate observations $Y_\alpha \sim (B'x_\alpha; \Sigma)$ to be the subvectors of the grand $\mathbb{R}^{Np}$-vector Y, one obtains the model $Y \sim ((X \otimes I_p)\beta; I_N \otimes \Sigma)$. Here the $N \times k$ matrix X has rows x_α, as usual, and the vector parameter β is the lexicographic reordering of the $k \times p$ matrix parameter B. The residual model is $(M \otimes I_p)Y \sim (0; M \otimes \Sigma)$, with $M = I_N - X(X'X)^-X'$. Since $\mathscr{B}_{M\otimes I_p}$, consisting of all products $M \otimes T$ with symmetric $p \times p$ matrices T, is a quadratic subspace, the matrix estimate $\hat{\Sigma}$, obtained from the standard unbiased estimates of its components, is nonnegative definite.

3. Proofs for Section 2. Recall that for a convex cone K in $\mathbb{R}^l$ its dual cone K^{dual} is defined to consist of all those $\mathbb{R}^l$-vectors q such that $q't \geqq 0$ for all $t \in K$. Let $\mathcal{Q}$ be the set of all $\mathbb{R}^l$ vectors q such that the form $q'\tau$ is nonnegatively estimable. Clearly $\mathcal{Q}$ is a convex cone, as is $\overline{G_M}$. Theorem 1 thus states that $\mathcal{Q} = \overline{G_M}^{\,dual}$.

PROOF OF THEOREM 1. Since the interior G of $\bar{G}$ is assumed to be nonempty, any two forms which coincide on $\bar{G}$ must have the same coefficients. Applying this to $q'\tau$ and $E(Y'AY)$, one finds that $\mathcal{Q} = \{(\text{trace } AMV_1M, \cdots, \text{trace } AMV_lM)' \in \mathbb{R}^l | A \in \text{NND}(n)\}$. Now $t \in \mathcal{Q}^{dual}$ if and only if trace $A \sum t_jMV_jM \geqq 0$ for all $A \in \text{NND}(n)$, or, equivalently, $\sum t_jMV_jM \in \text{NND}(n)$. Hence $\mathcal{Q}^{dual} = \overline{G_M}$, and $\overline{G_M}^{\,dual} = (\mathcal{Q}^{dual})^{dual}$. The latter is the closure of $\mathcal{Q}$, so that the assertion follows provided $\mathcal{Q}$ is closed.

To this end choose a full rank decomposition $M = QQ'$ where the $n \times \nu$ matrix Q satisfies $Q'Q = I_\nu$, with $\nu = n - \text{rank } X$. By setting $H_j = Q'V_jQ$ one has $\mathcal{Q} = \{(\text{trace } H_1Z, \cdots, \text{trace } H_lZ)' \in \mathbb{R}^l | Z \in \text{NND}(\nu)\}$. Since there exists some $c \in G$ which makes $\sum c_jH_j$ positive definite, the lemma of Bellman and Fan (1963), page 2, yields closedness of $\mathcal{Q}$. □

Whenever q lies in the image $S_M[\overline{G_M}]$ a choice $q = S_m\lambda$ with $\lambda \in \overline{G_M}$ ascertains that $Y' \sum \lambda_jMV_jMY$ is an unbiased nonnegative estimate for $q'\tau$. Hence the convex cone $S_M[\overline{G_M}]$ is always contained in $\mathcal{Q}$, their precise relation being as follows.

LEMMA 1. *The image of $\overline{G_M}$ under S_M is a closed convex subcone of $\mathcal{Q}$. If S_M has a rank l, then $S_M[\overline{G_M}]$ and $\mathcal{Q}$ coincide if and only if the standard unbiased estimate $\hat{\tau}$ for τ maps into the natural parameter set $\overline{G_M}$.*

PROOF. To prove closedness of $S_M[\overline{G_M}]$ let $(t^\iota)_{\iota\in\mathbb{N}}$ be a sequence in $\overline{G_M}$ such that S_Mt^ι converges to $q \in \mathbb{R}^l$, say. Define $Z_\iota = \sum_j (t^\iota)_jMV_jM$, then Z_ι lies in both $\mathscr{B}_M$ and $\text{NND}(n)$. Choose $c \in G$ so that $H_0 = \sum c_jV_j$ is positive definite. Since trace $H_0Z_\iota = c'S_Mt^\iota$ converges to $c'q$, by assumption, it follows that a subsequence $(Z_{\nu_\iota})_{\iota\in\mathbb{N}}$ converges to $Z \in \mathscr{B}_M \cap \text{NND}(n)$, say; see Bellman and Fan (1963), page 2. Hence $Z = \sum t_jMV_jM$ for some $t \in \overline{G_M}$, and because of $S_Mt^\iota = (\text{trace } V_1Z_\iota, \cdots, \text{trace } V_lZ_\iota)'$ the two limits q and $(\text{trace } V_1Z, \cdots, \text{trace } V_lZ)' = S_Mt$ must coincide. Thus $q \in S_M[\overline{G_M}]$, as asserted.

Next assume $S_M[\overline{G_M}] = \mathcal{Q}$. Then $q'\hat{\tau} \geqq 0$ for all $q \in \mathcal{Q}$, and $\hat{\tau} \in \mathcal{Q}^{dual} = \overline{G_M}$, by Theorem 1. Conversely, if $\hat{\tau} \in \overline{G_M} = \mathcal{Q}^{dual}$, then $q'\hat{\tau} \geqq 0$ for all $q \in \mathcal{Q}$. But $q'\hat{\tau}$ is the quadratic form $Y'\sum \lambda_jMV_jMY$, $S_M\lambda = q$, whence $\mathcal{Q} \subset S_M[\overline{G_M}]$. Since the other inclusion holds in general, the proof is complete. □

The quadratic subspace property enters into Theorem 2 through the following lemma. Recall that the *positive part* B_+ of a symmetric matrix B is obtained from the spectral decomposition of B by deleting all negative eigenvalues and their associated projectors. The *negative part* B_- then is $B_+ - B$, and satisfies trace $B_+B_- = 0$.

LEMMA 2. *Assume $\mathscr{B}$ to be a quadratic subspace of symmetric matrices, and let P be the projector onto $\mathscr{B}$ orthogonal with respect to the Euclidean matrix inner product $\langle A, B\rangle = \text{trace } A'B$. Then if A is nonnegative definite so is $P(A)$.*

PROOF. Fix $A \in \text{NND}(n)$, and assume $B = P(A) \notin \text{NND}(n)$, i.e., $B_- \neq 0$. The fact that both A and B_- are nonnegative definite implies $\langle A, B_-\rangle \geqq 0$, and $\|A - B\|^2 = \|A - B_+\|^2 + 2\langle A - B_+, B_-\rangle + \|B_-\|^2 > \|A - B_+\|^2$. But B_+, also being a member of $\mathscr{B}$ (Seely 1971, page 711), cannot be closer to A than the projection B. Hence $B_- = 0$, so that $B = B_+$ is nonnegative definite. □

PROOF OF THEOREM 2. It suffices to show that $S_M[\overline{G_M}]$ and $\mathcal{Q}$ coincide, since then $q'\hat{\tau} \geqq 0$ unless $q \notin \mathcal{Q}$, and $\hat{\tau} \in \overline{G_M}$ by Lemma 1. With $\mathcal{Q}$ represented as in the proof of Theorem 1, the inclusion $\mathcal{Q} \subset S_M[\overline{G_M}]$ means that for all matrices $A \in \text{NND}(n)$ there exists some $t \in \overline{G_M}$ such that for all $i = 1, \cdots, l$ one has $\langle MV_iM, A\rangle = \langle MV_iM, \sum t_jMV_j$

$M\rangle$. If $B = \sum t_j MV_jM$ is the projection of A onto $\mathscr{B}_M$, then certainly $\langle MV_iM, A\rangle = \langle MV_iM, B\rangle$. From Lemma 2 one concludes $B \in \mathrm{NND}(n)$ and $t \in \overline{G_M}$, so that this t has the required properties. □

Corollary 3 is an immediate consequence of Theorem 1: $q'\tau$ is nonnegatively estimable if and only if $0 \leqq q's = (T^{-1}q)'T's$ for all $\mathbb{R}^l$-vectors s for which $T's$ has nonnegative components. Here $\mathscr{Q}$ is polyhedral, being the image $T[\mathbb{R}^l_+]$ of the nonnegative orthant under T.

The multivariate linear model, discussed at the end of the previous section, may serve to illustrate the projection argument of Lemma 2. The projector onto $\mathscr{B}_{M\otimes I_p}$ is $P(A) = \sum_{i,j=1}^{p} \langle M \otimes (e_ie_j'), A\rangle M \otimes (e_ie_j')/\mathrm{trace}\, M$, where e_i is the ith column of I_p; the estimated dispersion matrix in the residual model is $M \otimes \hat{\Sigma} = P((M \otimes I_p)YY'(M \otimes I_p))$. Since Y is the lexicographic ordering of the random $N \times p$ matrix U whose rows are Y_α one gets $Y'(M \otimes (e_ie_j'))Y = \langle U'MU, e_ie_j'\rangle$. Hence $(n - \mathrm{rank}\, X)\hat{\Sigma} = U'MU = \sum U'Me_\alpha e_\alpha' MU = \sum_{\alpha=1}^{N} (Y_\alpha - \hat{B}'x_\alpha)(Y_\alpha - \hat{B}'x_\alpha)'$, with $\hat{B} = (X'X)^- X'U$.

4. Variance component estimation. Analysis of variance models have the form of a *variance component model*

$$\text{VCM:}\quad Y \sim (X\beta; \textstyle\sum \sigma_j^2 V_j), \qquad \text{all}\quad V_j \in \mathrm{NND}(n),$$

with parameter sets $\mathbb{R}^k$ for β as before, and the nonnegative orthant $\mathbb{R}^l_+$ for the vector $\sigma^2 = (\sigma_1^2, \cdots, \sigma_l^2)'$ of variance components. For a reparametrization with τ as in LM, the natural parameter set $\bar{G}$ always contains $\mathbb{R}^l_+$, whence $q'\sigma^2$ is nonnegatively estimable in the model VCM if and only if $q'\tau$ is nonnegatively estimable in the same model reparametrized with τ. In other words, there is no need to distinguish whether $q'\sigma^2$, or $q'\tau$, is nonnegatively estimable, and the results of Section 2 apply.

The following two examples are extensively discussed by Corbeil & Searle (1976), page 784. Let $1_n = (1, \cdots, 1)'$ denote the equiangular line in $\mathbb{R}^n$, and set $J_n = 1_n1_n'$. The 2-way crossed classification, mixed model, no interaction corresponds to

$$Y \sim \left([1_{abn}\!:\! I_a \otimes 1_{bn}] \begin{bmatrix} \mu \\ \alpha \end{bmatrix}; \sigma_\beta^2 J_a \otimes I_b \otimes J_n + \sigma_e^2 I_{abn}\right),$$

with $a, b, n > 1$. Here $q_\beta\sigma_\beta^2 + q_e\sigma_e^2$ is nonnegatively estimable if and only if $q_e \geqq q_\beta/(an) \geqq 0$. The 2-way crossed classification, mixed model, with interaction is

$$Y \sim \left([1_{abn}\!:\! I_a \otimes 1_{bn}] \begin{bmatrix} \mu \\ \alpha \end{bmatrix}; \sigma_\beta^2 J_a \otimes I_b \otimes J_n + \sigma_{\alpha\beta}^2 I_a \otimes I_b \otimes J_n + \sigma_e^2 I_{abn}\right),$$

with $a, b, n > 1$. Then the form $q_\beta\sigma_\beta^2 + q_{\alpha\beta}\sigma_{\alpha\beta}^2 + q_e\sigma_e^2$ is nonnegatively estimable if and only if $q_e \geqq q_{\alpha\beta}/n \geqq q_\beta/(an) \geqq 0$. Typical details for the last example are as follows; cf. R. D. Anderson et al. (1979). Let $\overline{J_n} = n^{-1}J_n$ be the projector onto the equiangular line in $\mathbb{R}^n$, and define $M_n = I_n - \overline{J_n}$. Using $R_1 = \overline{J_a} \otimes M_b \otimes \overline{J_n}$, $R_2 = M_a \otimes M_b \otimes \overline{J_n}$, and $R_3 = I_a \otimes I_b \otimes M_n$, one obtains $M = I_a \otimes M_{bn} = MV_3M = R_1 + R_2 + R_3$, $MV_2M = nR_1 + nR_2$, and $MV_1M = anR_1$. From this the triangular matrix T is constructed, and $T^{-1}q \in \mathbb{R}^3_+$ leads to the desired conclusion. For further examples see LaMotte (1973), page 729, and Pukelsheim (1979), page 80.

A reduction which is useful when the coefficient vector q of a form $q'\sigma^2$ has many zeroes is implied by the following.

LEMMA 3. *Assume a variance component model* VCM *in which a nonnegative form $q'\sigma^2$, $q \in \mathbb{R}^l_+$, is to be estimated. Define V_0 to be the sum of those matrices V_j such that q_j is zero, and let Q be the projector $M - MV_0M(MV_0M)^+$. Then every unbiased nonnegative definite quadratic estimate $Y'AY$ for $q'\sigma^2$ satisfies $Y'AY = (QY)'A(QY)$.*

PROOF. Write $MV_0M = CC'$. Unbiasedness implies $X'AX = 0$, and trace $C'AC = 0$, and nonnegative definiteness yields $AX = 0$, and $AC = 0$. Since $Q = I_n - X(X'X)^-X' - C(C'C)^-C'$, one obtains $A = QAQ$. □

It is easily verified that if $q_j = 0$ then $QMV_jMQ = 0$, and that $QX = 0$. Hence one may restrict attention to the *Q-reduced model* $QY \sim (0; \sum_{\{j \mid q_j>0\}} \sigma_j^2 QV_jQ)$, rather than work in the 'larger' residual model $MY \sim (0; \sum \sigma_j^2 MV_jM)$. In fact, the idea of a transition from the original model VCM to a suitably reduced model extends to arbitrary nonnegatively estimable forms $q'\sigma^2$, emphasizing that one has to pay with a certain number of degrees of freedom when one imposes the non-negativity constraint for unbiased estimation of $q'\sigma^2$.

In case of a single variance component σ_i^2 the reduction suggested by Lemma 3 leads to a necessary and sufficient condition for nonnegative estimability. Here V_0 is $\sum_{j\neq i} V_j$, and the Q-reduced model $QY \sim (0; \sigma_i^2 QV_iQ)$ only involves the parameter σ_i^2. Thus σ_i^2 is nonnegatively estimable if and only if $QV_iQ \neq 0$. This case is also discussed by LaMotte (1973), Kleffe (1977), page 219, Pukelsheim (1977), page 330.

As a final example, consider the model with a common mean and heteroscedastic variances, i.e., $Y \sim (1_n\mu; \sum \sigma_j^2 V_j)$, where $V_1 = \text{Blockdiag}[I_{n_1}:0:\cdots:0]$, $\cdots$, $V_l = \text{Blockdiag}[0:\cdots:I_{n_l}]$, and $n = \sum n_j$. The estimate $\hat{\sigma}_i^2$ (MINQUE given I_n) exists if $n \geqq 3$ and $n_i \geqq 1$, but may attain negative values; see J. N. K. Rao and Subrahmaniam (1971), page 973. In fact, σ_i^2 is nonnegatively estimable if and only if $n_i > 1$, and the natural estimator in the Q-reduced model turns out to be the sample variance of the ith group of observations. This is also observed by P.S.R.S. Rao and Chaubey (1978), page 773, and Kleffe and Zöllner (1978), page 29. These authors admit an arbitrary design matrix X, but make use—as does Balestra (1973), page 26—of the condition $\sum V_j = I_n$ which holds in the present model. Their arguments parallel our Lemma 3: first reduce to $PY \sim (PX\beta; \sigma_i^2 PV_iP)$, with $P = I_n - V_0$, and then reduce to $NPY \sim (0; \sigma_i^2 NPV_iPN)$, with $N = P - PX(PX)^+$.

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