Multilinear Estimation of Skewness and Kurtosis in Linear Models

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Summary: Estimation of the coefficients of skewness and kurtosis in a classical linear model situation is presented as an application of multilinear algebra and standard theory of mean estimation. The resulting estimators have optimality properties among all estimators that are invariant under mean translations, polynomials of degree three (skewness) or four (kurtosis) in the observations, and unbiased.

1. Introduction

Consider a classical linear model

\[ Y \sim (X\beta; \sigma^2 I_n) \]  

in whose definition we include the assumptions that the components of the random \( \mathbb{R}^n \)-vector \( Y \) are independent, and have common coefficients \( \gamma_1 \) of skewness and \( \gamma_2 \) of kurtosis. In other words, \( Y \) has independent components, and \( E Y = X\beta, \text{D}Y = \sigma^2 I_n \), and for all \( i = 1, \ldots, n \),

\[ E \left( \frac{Y_i - EY_i}{\sqrt{\text{Var} Y_i}} \right)^3 = \gamma_1; \quad E \left( \frac{Y_i - EY_i}{\sqrt{\text{Var} Y_i}} \right)^4 = 3 = \gamma_2. \]  

A distinguished case satisfying these requirements, and hence possibly underlying model (1.1), is the normal law, i.e., \( Y \sim N_n (X\beta; \sigma^2 I_n) \).

The purpose of this paper is to present polynomial estimators for skewness (Section 3) and kurtosis (Section 4) which are unbiased under every distribution that fulfills the assumptions of model (1.1), and which are of minimum variance under normality. The concept of unbiased estimation with minimum variance at a distinguished parameter value is of particular interest in linear model theory. For a more general context see the textbook of Schmetterer [1974, p. 273]. It seems to me that there are three major grounds to justify this interest: (i) In a theoretical exposition, investigation of unbiased estimators with local minimum variance properties provides a possible approach to uniform minimum variance unbiased estimation procedures. (ii) In cases

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when uniformly best procedures do not exist unbiased estimation with local minimum variance properties may be an applicable procedure, possibly iterative. (iii) For testing hypotheses, special interest is directed toward the distribution of the test statistic under the null hypothesis, i.e., under one (or more) distinguished parameter values: local minimum variance properties of an estimator provide information in this direction.

When estimating second and higher moments, a natural requirement is that of location invariance, or equivalently, that the estimates depend on the observations \( Y \) only through the residual statistic

\[
Z := MY, \ M := I_n - XX^* \tag{1.3}
\]

where \( M \) projects orthogonally onto the orthogonal complement of the range of \( X \). \( Z \) is the vector of residuals obtained from a simple least squares fit for \( \beta \), and this statistic is also maximal invariant with respect to the translation group \( \{ y \to y + Xb \mid b \in \mathbb{R}^p \} \), where the matrix \( X \) is assumed to be of order \( n \times p \). Estimating the third moment coefficient \( \gamma_1 \), we shall restrict attention to the class

- \( \Gamma_1 \) of all homogeneous polynomials of degree three in the residuals \( Z_1, \ldots, Z_n \), and to its subclass
- \( G_1 \) with pure cubes \( Z_1^3, \ldots, Z_n^3 \) only. As for the fourth moment coefficient \( \gamma_2 \), we analogously investigate the class
- \( \Gamma_2 \) of all constants plus homogeneous polynomials of degree four in \( Z_1, \ldots, Z_n \), and its subclass
- \( G_2 \) of constants plus polynomials with pure fourth powers \( Z_1^4, \ldots, Z_n^4 \) only.

In Section 2 the Kronecker and Hadamard third and fourth powers of the residual vector \( Z \) yield derived models in which (i) the unknown coefficient \( \gamma_i \) appears as a mean parameter, and (ii) \( \Gamma_i \) and \( G_i \) turn out to be the classes of all linear estimators; \( i = 1, 2 \). Standard theory of mean estimation, then, is all that is needed for finding unbiased estimators of skewness (Section 3) and kurtosis (Section 4) which, under normality, are of minimum variance. The technical difficulties reduce to appropriately displaying the moments of the residual statistic \( Z \). We shall use two tools to facilitate this task, the minor one of which is the use of the vec operator: this allows to avoid summation signs through expressions like \( \sum_{i=1}^{n} e_i \otimes e_i = \text{vec} I_n \). A major tool is the second one, the symmetrizer \( \pi_s \). For example, if \( U \) is \( N_n (0; I_n) \) - distributed and the sixth moments are to be computed, then \( U_o^{\otimes 2} \) has expectation 3 regardless of the particular permutation \( \sigma \) of the components \( U_1, \ldots, U_n \); and this suggests a two-stage approach: In the first step expectations of \( U_1^6, U_1^4 U_2^2, U_1^2 U_2^4, U_2^6 \) are computed, and in the second step the symmetrizer \( \pi_s \) is employed in order to generate the necessary permutations. It is found that these means greatly simplify the representation of the covariance operators in the derived models.

Our approach is a natural extension of the dispersion mean correspondence that reduces the estimation of variance covariance components to standard mean estimation [Pukelsheim]. When estimating skewness and kurtosis, it would appear natural to also insist on scale invariance. This, however, excludes the use of estimators which are
mere polynomials of the observations, and hence surpasses means and methods of multilinear algebra. Hence we sacrifice scale invariance to theoretical elegance, and neglected scale invariance does, indeed, secure a manageable theory. But it also results in estimators that depend on the unknown value of $\sigma^2$. There are, of course, possibilities of various degrees of arbitrariness of how to replace $\sigma^2$ by an estimate $\hat{\sigma}^2$, and some courses of action will be discussed in the examples. This discussion will show that while the proposed multilinear approach yields procedures of direct applicability only for the genuine third and fourth central moments, it nevertheless motivates a guideline of how estimators for the coefficients of skewness and kurtosis may be obtained as well.

The examples are taken from Anscombe's paper [1961] who used estimates of skewness and kurtosis to study departure from normality.

2. The Derived Models

Extensive use will be made of the Kronecker product $A \otimes B := [[A_{ij}B]]$ for two matrices $A$ and $B$ of arbitrary order, and of the Hadamard product $A \ast B := ((A_{ij}B_{ij}))$ for matrices $A$ and $B$ of the same order, see Rao [1973, pp. 29–30], Styan [1973]. Powers are written as $A^{\otimes 2} := A \otimes A$, $A^{\ast 2} := A \ast A$, etc.; recall the fundamental rule $(A \otimes B)(C \otimes D) = AC \otimes BD$.

The components of the third Kronecker power $Z^{\otimes 3}$ furnish the $n^3$ mixed third powers of $Z_1, \ldots, Z_n$, whence any homogeneous polynomial of degree three may be written as a linear form $w'Z^{\otimes 3}$. The fewer number of components of the third Hadamard power $Z^{\ast 3}$ suffice for investigation whenever the polynomial involves pure cubes $Z_1^3, \ldots, Z_n^3$, only:

$$
\Gamma_1 = \{w'Z^{\otimes 3} \mid w \in \mathbb{R}^{n^3}\}, \quad \mathcal{G}_1 \equiv \{w'Z^{\ast 3} \mid w \in \mathbb{R}^{n}\}. \tag{2.1}
$$

Hence we study the derived models generated by $Z^{\otimes 3}$ and $Z^{\ast 3}$. Computations are facilitated by the fact that $Z^{\ast 3} = H_3 Z^{\otimes 3}$ is a linear transformation of $Z^{\otimes 3}$, where

$$
H_3 := \sum_{i=1}^{n} e_i (e_i^{\otimes 3})', \tag{2.2}
$$

is an $n \times n^3$ matrix; $e_i$ is the $i$-th Euclidean basis vector in $\mathbb{R}^n$, with $i$-th component equal to one and zeroes elsewhere. Analogously, define the $n \times n^4$ matrix $H_4 =

= \sum_{i=1}^{n} e_i (e_i^{\otimes 4})'$. We shall also use $1_n := \Sigma e_i = (1, \ldots, 1)'$. The following lemma shows that both $Z^{\otimes 3}$ and $Z^{\ast 3}$ give rise to models whose expectation is linear in $\sigma^3 \gamma_1$.

**Lemma 2.1:** For every classical linear model:

a) $EZ^{\otimes 3} = \Sigma_{i=1}^{n} (Me_i)^{\otimes 3} \cdot \sigma^3 \gamma_1$.

b) $EZ^{\ast 3} = M^{\ast 3} 1_n \sigma^3 \gamma_1$.

c) There exists an unbiased estimator for $\gamma_1$ in $\Gamma_1$ or in $\mathcal{G}_1$ if and only if $1_n'M^{\ast 3} 1_n > 0$. 

Proof: The independent components of \( U := \sigma^{-1}(Y - \bar{E}Y) \) have mean zero, variance one, and coefficient of skewness \( \gamma_1 \). Hence the expected value vanishes for powers like \( U_1 U_2 U_3 \) and \( U_1 U_2^2 \), and equals \( \gamma_1 \) for \( U_3^2 \). a) then follows from \( EZ^{*3} = \sigma^3 M^{*3} EU^{*3} \); and applying \( H_3 \) in a) yields b). As for c), \( \Gamma_1 \)-estimability is ensured whenever \( \Sigma (Me_i)^{*3} \neq 0 \), i.e., \( \| \Sigma (Me_i)^{*3} \| \geq \Sigma e_i^3 M^{*3} e_j \geq 0 \). \( G_1 \)-estimability requires \( M^{*3} 1_n \neq 0 \). The assertion then follows from the nonnegative-definiteness of \( M^{*3} \) [Styan, p. 221]. □

Before proceeding to higher powers we introduce the symmetrizer \( \pi_s \) [Greub, pp. 90f.]. As an endomorphism of a \( p \)-fold Kronecker power \( \otimes^p R^n \), it suffices to have \( \pi_s \) defined on the spanning set of all \( p \)-fold Kronecker powers \( x_1 \otimes \ldots \otimes x_p \) of \( p \) \( R^n \)-vectors \( x_1, \ldots, x_p \) by

\[
\pi_s (x_1 \otimes \ldots \otimes x_p) := \frac{1}{p!} \sum_{\sigma \in S_p} (x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(p)})
\]

where \( S_p \) is the symmetric group (permutation group) of order \( p \). A point in \( \otimes^p R^n \) is called \textit{symmetric} if it lies in the range of \( \pi_s \). A mapping \( L \) defined on \( \otimes^p R^n \) is called \textit{symmetric} if \( L = L \circ \pi_s \).

For example, consider the case \( p = 2 \). Then \( \pi_s (x \otimes y) = \frac{1}{2} (x \otimes y + y \otimes x) \). Using the isomorphism \( \text{vec}: R^{n \times n} \rightarrow R^{n^2} \), \( xy' \rightarrow x \otimes y \), the result of \( \pi_s \) on the space \( R^{n \times n} \) of \( n \times n \) matrices is exactly that it projects a square matrix \( A \) into its symmetric part \( \frac{1}{2} (A + A') \); further properties of the \( \text{vec} \) operator are \( \sum_{i=1}^n e_i \otimes e_i = \text{vec} 1_n \), and \( \text{vec} ABC = A \circ C' \circ \text{vec} B \) [Pukelsheim, p. 326]. Notice that \( z^{*3} \) is symmetric; hence in a linear form \( w'z^{*3} \) we may assume \( w \) to be symmetric, too. Symmetric mappings are those given by \( H_3 \), or \( \Sigma (e_i' M)^{*3} \).

The classes \( \Gamma_2 \) and \( G_2 \) of estimators are now represented by using the fourth Kronecker and Hadamard powers \( Z^{*4} \) and \( Z^{*4} \):

\[
\Gamma_2 = \{ w'Z^{*4} + c, w \in R^{n^4}, c \in R \},
\]

\[
G_2 = \{ w'Z^{*4} + c_G, w \in R^n, c_G \in R \}.
\]

Analogously to Lemma 2.1 we next investigate the expectations of \( Z^{*4} \) and \( Z^{*4} \). To this end, define

\[
m := (M_{11}, \ldots, M_{nn})'.
\]

Lemma 2.2: For every classical linear model:

a) \( EZ^{*4} = \sum_{i=1}^n (Me_i)^{*4} \cdot \sigma^4 \gamma_2 + 3 \sigma^4 \pi_s (\text{vec } M)^{*2} \).

b) \( EZ^{*4} = M^{*4} 1_n \sigma^4 \gamma_2 + 3 \sigma^4 m^{*2} \).

c) There exists an unbiased estimator for \( \gamma_2 \) in \( \Gamma_2 \) or in \( G_2 \) if and only if the rank of \( X \) is smaller than \( n \).
Proof: Again use the standardized $U$ as in the proof of Lemma 2.1. The expected values of $U_1 U_2 U_3 U_4, U_1 U_2 U_3^2, U_1 U_2^2$ vanish, leaving the cases $U_1^2 U_2^2$ and $U_1^4$. From (1.2), the latter has expectation $EU_1^4 = \gamma_2 + 3$. The former has expectation 1 and may arise from $U_1 U_2 U_2 U_2, U_1 U_2 U_2 U_2$, or $U_1 U_2 U_2 U_2$. This gives $EU_1^4 = \Sigma e_i^4 \cdot (\gamma_2 + 3) + + \Sigma_{i,j} (e_i e_j e_j e_j + e_i e_j e_j e_j - e_i e_j e_j e_j - e_i e_j e_j e_j).$ The last term equals $3 \pi_s \Sigma_{i,j} e_i e_i e_j e_j$, and this proves a). b) results when applying the transformation $H_4$ in a). In c) we first get $\Sigma M_{ij}^4 \neq 0$; but this is the same as $M \neq 0$, i.e., range $X \neq R^n$. □

A classical linear model tacitly assumes that the rank of $X$ is less than $n$, hence $\gamma_2$ is "always" unbiasedly estimable. Lemma 2.2 also provides the values for the constants $c_\Gamma$ and $c_G$:

**Corollary 2.3:** For the classes $\Gamma_2$ and $G_2$, unbiasedness of $w'Z_{\gamma}^4 + c_\Gamma (w \in R^n, c_\Gamma \in R)$ for $\gamma_2$ implies

$$c_\Gamma = -3 a^4 w' \pi_s \cdot (\text{vec } M)^{\otimes 2},$$

and unbiasedness of $w'Z_{\gamma}^4 + c_G (w \in R^n, c_G \in R)$ for $\gamma_2$ implies

$$c_G = -3 a^4 w'm^{*2}. \quad \Box$$

This section is concluded with some upper and lower bounds associated with the projector $M$. Let $m$ be as in (2.4); define $p$ to be rank of $X$, and $\nu$ to be the rank of $M$, so that $p + \nu = n$.

**Lemma 2.4:**

a) For every $k = 1, 2, \ldots$, one has $0 \leq m'M_{\gamma}^k m \leq \nu$ and $\nu^k / n^{k-1} \leq \Sigma_{i=1}^n M_{ii}^k$.

b) One has $\Sigma_{i,j}^n M_{ij}^3 \nu^2 = \nu^2 \nu^2 / n^2$, and $\nu^2 - np$ is positive if and only if

$$n > (3 + \sqrt{5}) p / 2.$$

**Proof:** a) Since $M$ is a projector, it follows from Theorem 3.1 and Corollary 3.2 in Styan [1973] that the eigenvalues of $M_{\gamma}^k$ lie in the unit interval $[0, 1]$, whence $0 \leq m'M_{\gamma}^k m \leq m'm$. But

$$M_{ii} = \Sigma_{j=1}^n M_{ij}^2$$

implies $M_{ii} \in [0, 1]$ as well, so that $m'm \leq \Sigma M_{ii} = \nu$. The second part is Jensen's Inequality: $\nu^k / n^k = (\Sigma M_{ii} / n)^k \leq \Sigma M_{ii}^k / n$.

b) Certainly, $\Sigma M_{ij}^3 \geq \Sigma M_{ii}^3 - \Sigma_{i \neq j} | M_{ij} | M_{ij}^2$. But (2.5) implies $M_{ij}^2 \leq | M_{ij} |$, and $\Sigma_{i \neq j} | M_{ij} |^2 = M_{ij}^2 - M_{ii}$. This gives $\Sigma M_{ij}^3 \geq \Sigma M_{ii}^3 + \Sigma M_{ii}^2 - \Sigma M_{ii}$. Now use a) to obtain the first assertion.
For fixed $p$, the zeros of $x^2 - 3xp + p^2$ are $(3 \pm \sqrt{5})p/2$. Noting that $n \geq p$ finishes the proof. \( \square \)

The lower bound $\nu (\nu^2 - np)/n^2$ for $\Sigma M_{ij}^3$ is rather crude since it is not always nonnegative. Yet for the estimability criterion in Lemma 2.1(c), one obtains the sufficient condition $n \geq 3p$. The case of two observations with a homogeneous mean provides an example for $n = 2p$ and $\Sigma M_{ij}^3 = 0$.

Next we compute the sixth and eighth moments under normality, and then turn to minimum variance estimation.

3. Skewness

In the sequel we speak of a \textit{quasinormal} model whenever the assumption of normality is used only to compute the moments involved.

In order to display higher moments we shall use the $(i, j)$-th Euclidean basis matrix $E_{ij} := e_i e_j'$ that carries a one on place $(i, j)$ and zeros elsewhere, and the diagonal matrix $\text{Diag} M$ whose diagonal is copied from $M$.

Lemma 3.1: For every quasinormal classical linear model the dispersion matrices of $Z_{\beta}^3$ and $Z_{\gamma}^3$ are given by $\sigma^6 \pi_s S_\Gamma, \pi_s$ and $\sigma^6 S_G$, respectively, where

\[
S_\Gamma := 6 M_{\beta}^3 + 9 M \otimes (\text{vec} M \cdot \text{vec}' M)
\]

\[
S_G := 6 M_{\beta}^3 + 9 (\text{Diag} M) M (\text{Diag} M).
\]

Proof: Define $S := 6 I_{n,3}^3 + 9 \frac{n}{\sum_{i,j,k=1}^{n} E_{ii} \otimes E_{jk} \otimes E_{jk}}$. It suffices to show that for a $N_n (0; I_n)$-distributed $U$ one has $DU_{\beta}^3 = \pi_s S \pi_s$. Since $M_{\beta}^3$ and $\pi_s$ commute, this implies $DZ_{\beta}^3 = \sigma^6 M_{\beta}^3 (DU_{\beta}^3) M_{\beta}^3 = \sigma^6 \pi_s \cdot M_{\beta}^3 S M_{\beta}^3 \cdot \pi_s$. Using $\text{vec} ABC = A \otimes C' \cdot \text{vec} B$ one obtains

$M \otimes M_{\beta}^2 \{6 I_{n,3}^3 + 9 I_n \otimes (\text{vec} I_n \cdot \text{vec}' I_n)\} M \otimes M_{\beta}^2 = S_\Gamma$. Employing $H_3$, one also gets $DZ_{\beta}^3 = \sigma^6 H_3 S_\Gamma H_3' = \sigma^6 S_G$, as asserted.

Note that $E U_{\beta}^3 = 0$. The equality $E (U_{\beta}^3) \cdot (U_{\beta}^3)' = \pi_s S \pi_s$ is now verified elementwise. Let $\alpha, \beta, \gamma = 1, \ldots, n$ be pairwise different. The only nonvanishing sixth moments arise from $U_6^6$ with expectation 15, and $U_\alpha^2 U_\beta^2$ with expectation 3, and $U_\alpha^2 U_\beta U_\gamma$ with expectation 1. In the first case, $(e_\alpha^3)' \pi_s S \pi_s (e_\alpha^3) = 15$. For the second case use $\pi_s (e_\alpha \otimes e_\beta \otimes e_\beta) = (e_\alpha \otimes e_\beta \otimes e_\beta + e_\beta \otimes e_\alpha \otimes e_\beta + e_\alpha \otimes e_\beta \otimes e_\alpha) / 3$, and obtain $(e_\alpha^3)' \pi_s S \pi_s (e_\alpha \otimes e_\beta \otimes e_\beta) = 3$ and $(e_\alpha \otimes e_\beta \otimes e_\beta)' \pi_s S \pi_s (e_\alpha \otimes e_\beta \otimes e_\alpha) = 3$. Along the same lines it is found that $(e_\alpha \otimes e_\alpha \otimes e_\beta)' \pi_s S \pi_s (e_\beta \otimes e_\gamma \otimes e_\gamma) = 1$, and, finally, $(e_\alpha \otimes e_\beta \otimes e_\beta)' \pi_s S \pi_s (e_\beta \otimes e_\alpha \otimes e_\gamma) = 1$. For vanishing moments of order six the corresponding entry of $\pi_s S \pi_s$ vanishes, too. \( \square \)

Note that $S_G$ is also given by equation (15) in \textit{Anscombe} [1961].

In case $\sigma^4$ is known, we are now in a position to exhibit various estimators of the coefficient of skewness $\gamma_1$, and discuss their relationship.
**Theorem 1:** Assume a classical linear model $\mathbf{Y} \sim (\mathbf{X}\beta; \sigma^2 \mathbf{I}_n)$ such that $\sigma^2 > 0$.

a) The Aitken estimate for $\gamma_1$ in the derived model

$$Z^{\ast} \sim (\sum_{i=1}^{n} (M_{i})^{\ast} \cdot \sigma^3 \gamma_1; \sigma^6 \pi_3 S_1, \pi_3)$$

is given by

$$\gamma_1 := (\sigma^3 \sum_{i,j=1}^{n} e_i^{\ast} (\pi_3 S_1, \pi_3)^{+} e_j^{\ast})^{-1} \sum_{i=1}^{n} e_i^{\ast} (\pi_3 S_1, \pi_3)^{+} Z^{\ast}.$$ 

$\gamma_1$ is a homogeneous polynomial of degree three in $Z_1, \ldots, Z_n$, unbiased for $\gamma_1$, and, under normality, of minimal variance among all these estimators.

b) The Aitken estimate for $\gamma_3$ in the derived model

$$Z^{\ast} \sim (M^{\ast} \pi_3 \gamma_1; \pi_6 S_G)$$

is given by

$$\gamma_3 := (\sigma^3 \sum_{i=1}^{n} M^{\ast} S_G M^{\ast} \pi_3 l_n)^{-1} l_n M^{\ast} S_G Z^{\ast}.$$ 

$\gamma_3$ is a homogeneous polynomial of degree three involving pure cubes $Z_1^3, \ldots, Z_n^3$ only, unbiased for $\gamma_1$, and, under normality, of minimal variance among all these estimators.

c) The Least Squares Estimator in the model generated by $Z^{\ast}$ is

$$\bar{\gamma}_1 := (\sigma^3 1_n M^{\ast} l_n)^{-1} l_n Z^{\ast}.$$ 

$\bar{\gamma}_1$ is a homogeneous polynomial of degree three involving pure cubes $Z_1^3, \ldots, Z_n^3$ only, unbiased for $\gamma_1$, and of minimal norm among all unbiased homogeneous polynomials of degree three in $Z_1, \ldots, Z_n$.

Under normality, these estimates are related as stated in d) - g):

d) $\text{Var} \gamma_1 \leq \text{Var} \bar{\gamma}_1 \leq \text{Var} \gamma_3 \leq \frac{1}{\nu} [6 (\nu^2/n^2 - p/n)^{-1} + 9 (\nu^2/n^2 - p/n)^{-2}]$, where $\nu = n - p = \text{rank} M$.

e) $\bar{\gamma}_1$ is of minimal variance in $\Gamma_1$ if and only if

$$\pi_3 S_1, H_3^H S_G^H M^{\ast} l_n = \rho \sum_{i=1}^{n} (M_{i})^{\ast}, \rho \in \mathbb{R}.$$ 

f) $\bar{\gamma}_1$ is of minimal variance in $G_1$ if and only if

$$(\text{Diag} M) M_m = \rho M^{\ast} l_n, \rho = m'M_m/1_n M^{\ast} l_n.$$ 

g) $\bar{\gamma}_1$ is of minimal variance in $\Gamma_1$ if and only if $\pi_3 \cdot (M_m) \otimes (\text{vec} M) = \rho \sum_{i=1}^{n} (M_{i})^{\ast}, \rho = m'M_m/1_n M^{\ast} l_n.$
Proof: In a general linear model \( Y \sim (X\beta; \sigma^2 V) \), the Aitken estimate \( (X'V^*X)^+ X'V^*Y \) for \( \beta \) is of minimum variance iff \( \text{range } X \subset \text{range } V \) [Zyskind, p. 658]. Since for non-negative-definite matrices the range of a sum equals the sum of the ranges, this condition is easily verified for a) and b). c) uses the representation \( X^*Y \) of the Least Squares Estimator. The first two inequalities in d) are immediate, the last one uses Lemma 2.4. e) -- g) follow since, in general, \( \hat{c}'Y \) is Blue of its expectation iff \( V\hat{c} \) lies in the range of \( X \) [Zyskind, p. 653]. For case g), e.g., this gives \( \pi_s S_1 \Sigma e_i^\otimes = \rho \Sigma (M_1)^\otimes \) for some real number \( \rho \). Its essential term is \( \pi_s \cdot \Sigma (M_1) \otimes \text{vec } M \cdot M_{ii} = \pi_s \cdot (Mm) \otimes \text{vec } M \). In each of the assertions e) -- g), the particular value of \( \rho \) can be computed by suitable premultiplication. □

Examples: Anscombe [1961, pp. 14f.] contents himself with estimating the third moment \( \mu_3 = \sigma^3 \gamma_1 \) for which problem Theorem 1 holds true using the estimates

\[ \sigma^3 \gamma_1, \sigma^3 g_1, \text{ and } \sigma^3 \overline{g_1}, \] respectively; the classes \( \Gamma_1 \) and \( G_1 \) remain the same. Anscombe's \( \sigma^3 g_1 \) coincides with our least squares version \( \overline{\sigma^3 g_1} \).

Firstly, for \( Y \sim ((0, 1, 2)'\mu; \sigma^2 I_3) \), Theorem 1 yields \( \overline{\sigma^3 g_1} = \frac{125}{174} (Z_1^3 + Z_2^3 + Z_3^3) \);

this is of minimum variance in \( G_1 \), but not in \( \Gamma_1 \) [Anscombe, p. 14].

Secondly, for \( Y \sim ((0, 1, 2, 3)'\mu; \sigma^2 I_4) \), the variance of \( \overline{\sigma^3 g_1} \) is not minimal in \( G_1 \).

For \( \sigma^3 g_1 \), one obtains \( \overline{\sigma^3 g_1} = 0.365 Z_1^3 + 0.573 Z_2^3 + 1.015 Z_3^3 + 2.404 Z_4^3 \), and

\[ \overline{\text{Var } g_1} = 5.48 < 6.59 = \text{Var } \overline{g_1} \] [Anscombe, p. 15].

Thirdly, consider Anscombe's design-condition I [1961, p. 3] of a common over-all mean component: \( 1_n \in \text{range } X \), and design-condition II of equal variances of the residuals \( Z_1, \ldots, Z_n; M_{ii} = \cdots = M_{nn} \). Under these conditions \( Mm = 0 \), and Theorem 1 g) ensures that \( \sigma^3 \overline{g_1} \) is of minimal variance in the big class \( \Gamma_1 \), and not only in its subclass \( G_1 \), as found by Anscombe [1961, p. 15]. □

4. Kurtosis

The following discussion parallels that of Section 3. First the dispersion matrix of \( Z^{*4} \) is computed.

Lemma 4.1: For every quasinormal classical linear model the dispersion matrices of \( Z^{*4} \) and \( Z^{*4} \) are given by \( \sigma^8 \pi_s A_1, \pi_s \) and \( \sigma^8 A_G \), respectively, where

\[ A_1 := 24 M^{*4} + 72 M \otimes M \otimes (\text{vec } M \cdot \text{vec}' M), \]

\[ A_G := 24 M^{*4} + 72 (\text{Diag } M) M^{\ast 2} (\text{Diag } M). \]

Proof: Define \( A := 24 f_n^{*4} + 72 f_n^{\ast 2} \otimes (\text{vec } I_n \cdot \text{vec}' I_n) + 9 (\text{vec } I_n \cdot \text{vec}' I_n)^{\ast 2} \). As in Lemma 3.1 first prove \( EU^{*4} (U^{*4})' = \pi_s A \pi_s \); using \( EU^{*4} = 3 (\text{vec } I_n)^{\ast 2} \) one then finds \( DU^{*4}, DZ^{*4}, \) and \( DZ^{*4} \) as asserted. □

Note that one may now obtain equation (33) in Anscombe [1961].
Theorem 2 is our main result on estimating the coefficient of kurtosis $\gamma_2$. As in Theorem 1, it is assumed that $\sigma^2$ is known.

**Theorem 2:** Assume a classical linear model $Y \sim (X\beta; \sigma^2 I_n)$.

a) The Aitken estimate for $\gamma_2$ in the derived model

$$Z^{\cdot 4} \sim 3 \sigma^4 \pi_s \cdot (\text{vec } M)^\circ 2 \sim (\sum_{i=1}^{n} (Me_i)^\circ 4 \cdot \sigma^4 \gamma_2 ; \sigma^8 \pi_s A \pi_s)$$

is given by

$$\gamma_2 := \sigma^{-4} \sum_{i=1}^{n} e_i^{\cdot 4} (\pi_s A \pi_s)^{+} Z^{\cdot 4} - 3 \cdot \sum_{i=1}^{n} e_i^{\cdot 4} (\pi_s A \pi_s)^{+} (\text{vec } M)^\circ 2$$

$\gamma_2$ is a constant plus a homogeneous polynomial of degree four in $Z_1, \ldots, Z_n$, unbiased for $\gamma_2$, and, under normality, of minimal variance among all these estimators.

b) The Aitken estimate for $\gamma_2$ in the derived model

$$Z^{* 4} \sim 3 \sigma^4 m^* 2 \sim (M^{* 4} 1_n \sigma^4 \gamma_2 ; \sigma^8 A G)$$

is given by

$$\overline{\gamma}_2 := \sigma^{-4} 1_n M^{* 4} A_G^{+} Z^{* 4} - 3 \cdot 1_n M^{* 4} A_G^{+} m^* 2$$

$\overline{\gamma}_2$ is a constant plus a homogeneous polynomial of degree four involving pure fourth powers $Z_1^4, \ldots, Z_n^4$ only, unbiased for $\gamma_2$, and, under normality, of minimal variance among all these estimators.

c) The Least Squares Estimator in the model as in a) is

$$\overline{\gamma}_{ls} := (1_n^M \sigma^4 1_n)^{-1} (\sigma^{-4} 1_n Z^{* 4} - 3 \cdot m'm).$$

$\overline{\gamma}_{ls}$ is a constant plus a homogeneous polynomial of degree four involving pure fourth powers $Z_1^4, \ldots, Z_n^4$ only, unbiased for $\gamma_2$, and of minimal norm among all unbiased constants plus homogeneous polynomials of degree four in $Z_1, \ldots, Z_n$.

Under normality, these estimates are related as stated in d) -- g):

d) $\Var \overline{\gamma}_2 \leq \Var \overline{\gamma}_2 \leq \Var \overline{\gamma}_{ls} \leq \frac{1}{\nu} [24 (n^3/\nu^3) + 72 (n^4/\nu^6)]$, where $\nu = \text{rank } M$.

e) $\overline{\gamma}_2$ is of minimal variance in $\Gamma_2$ if and only if

$$\pi_s A \pi_s H'_4 A_G^{* } M^{* 4} 1_n = \rho \sum_{i=1}^{n} (Me_i)^\circ 4, \rho \in \mathbb{R}.$$
f) $\overline{g_2}$ is of minimal variance in $G_2$ if and only if

$$ (\text{Diag} M) M^{*2} m = \rho M^{*4} 1_n, \quad \rho = m^t M^{*2} m / 1_n^t M^{*4} 1_n. $$

g) $\overline{g_2}$ is of minimal variance in $\Gamma_2$ if and only if

$$ \pi_s \cdot (\text{vec} M) \otimes (\text{vec} M (\text{Diag} M) M) = \rho \sum_{i=1}^n (M_i^4)^{\otimes 4}, \quad \rho \text{ as in (f)}. \qquad \square $$

Examples: For the first example of Section 3, $\overline{g_2} = \sigma^{*4} 625 / 914 (Z_4^1 + Z_2^4 + Z_3^4) - 3150 / 914$.

This is of minimal variance in $G_2$, but not in $\Gamma_2$.

In the second case, $\overline{g_2}$ does not minimize the variance in $G_2$. For $g_2$ we obtained

$$ g_2 = \sigma^{*4} (0.4843 Z_4^1 + 0.4841 Z_2^4 + 0.5956 Z_3^4 - 0.3997 Z_4^1) - 3.4639, $$

and

$$ \text{Var } g_2 = 46.49 < 47.17 = \text{Var } g_2. $$

Thirdly, let us introduce design-condition III [cf., Anscombe, pp. 6,20] of up to permutations equal covariances of the residuals, or precisely: the rows of $M$ are equal up to permutations of their entries. Then design-conditions II and III force $\overline{g_2}$ to have minimal variance in $G_2$: in Theorem 2 f), $(\text{Diag} M) M^{*2} m = M_{11}^4 M^{*2} 1_n$ is a multiple of $1_n$, and the same is true of $M^{*4} 1_n$. This is an example with a nontrivial $\rho$.

Fourthly, we replace in $\overline{g_2}$ the generally unknown $\sigma^{*4}$ by an estimate, and compare our $g_2$, obtained along these lines, with Anscombe's $g_2$. As before, define

$$ \nu := \text{rank } M = n - \text{rank } X; \quad d := \sum_{i,j=1}^n M_{ij}^4. \text{Estimate } \sigma^2 \text{ by } s^2 = v^{-1} \sum_{i=1}^n Z_i^4, \text{ as usual.} $$

Replacing $\sigma^2$ by $s^2$ generates a bias even under normality:

$$ \mathbb{E} \sigma^{*4} 1_n^t Z^{*4} - 3 m^t m = -6 (\nu + 2)^{-1} m^t m. $$

Correcting for this bias by changing the constant yields

$$ \tilde{g}_2 := \frac{1}{d} (s^{-4} \sum_i Z_i^4 - 3 (\nu + 2)^{-1} \nu m^t m). $$

Anscombe [1961, p. 7] uses a different divisor:

$$ g_2 = \frac{D}{D} \tilde{g}_2, \quad D := d - 3 [\nu (\nu + 2)]^{-1} \left[ \sum_{i=1}^n M_{ii}^4 \right]^2. $$

Since $d/D > 1$, the variance of $g_2$ always exceeds that of $\tilde{g}_2$. Under normality, both estimators are unbiased for $\gamma_2 = 0$. The asymptotic properties are the same: if (*) sup \{rank $X_n$ $|$ $n = 1, 2, \ldots$ $\} \leq p$, then

$$ 0 \leq 1 - D/d = 3 (1_n^t m^{*2})^2 / [\nu (\nu + 2) \sum M_{ij}^4] $$

$$ \leq 3n \sum M_{ii}^4 / [\nu (\nu + 2) (\sum M_{ii}^4 + \sum_{i \neq j} M_{ij}^4)] $$

$$ \leq 3n / [(n - p) (n - p + 2)] \to 0 $$
as $n \to \infty$. For the model $Y \sim (1_n, \mu; \sigma^2 I_n)$, Anscombe’s $g_2$ specializes to Fisher’s kurtosis statistic [cf., Cramér, eq. (29.3.8)], whereas $\tilde{g}_2$ does not.

Finally note that under condition (*) all variances in Theorems 1 d) and 2 d) tend to zero, whence the proposed estimators are consistent. □

We remark that for the estimation of the kurtosis one may utilize yet another derived model, namely,

$$Z^{\otimes 4} \sim \left[ \sum_{i=1}^{n} (Me_i)^{\otimes 4} : 3 \pi_{2} (\vec{M})^{\otimes 2} \right] \left( \begin{array}{c} \sigma_2^4 \gamma_2 \\ \sigma_2^4 \end{array} \right) ; \sigma_2^{8} \pi_{2} A_{\Gamma} \pi_{2}.$$ 

The approach chosen above reveals more clearly the analogy of estimating skewness and kurtosis, and also conforms to a greater extent with Anscombe’s paper [1961].

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