A bi-proportional method applied to the Spanish Congress

V. Ramírez, F. Pukelsheim, A. Palomares, J. Martínez

Department of Applied Mathematics, University of Granada, E-18002 Granada, Spain
Institute for Mathematik, University of Augsburg, D-86135 Augsburg, Germany

A bi-proportional divisor method is applied to allocate the seats of the 2004 Spanish Congress, thus achieving proportionality relative to the population counts in the fifty-two districts, as well as proportionality relative to the vote counts for the political parties. Also, advantages and disadvantages of the method are discussed.

1. Introduction

For the election of their national parliaments, many countries subdivide the grand electoral region into smaller electoral districts, and assign seats to parties separately in each district. The idea is to bring representatives nearer to the electors. However, experience shows that separate district apportionments generally do not entail an equitable overall representation of political parties. It may happen that the overall proportion of seats of a party significantly deviates from their overall proportion of votes, be it to their advantage or to their disadvantage. For example, this has been continuously happening to the IU-party (Izquierda Unida) in Spain, since their total number of deputies consistently falls short of their proportional share of votes. The likelihood of such biases increases when there are many electoral districts that only have a small number of seats to fill.

In order to avoid this problem, some countries have taken recourse to a mixed-member proportional system. For instance, Germany elects 299 members of the Bundestag in single-seat districts. Yet the overall apportionment of the 598 regular Bundestag seats turns out to be very proportional. Here, proportionality applies to those parties that turn out to be eligible to participate in the apportionment process, that is, their vote share amounts to at least five percent of the overall ballot count. In the German system proportionality is achieved by first allocating the 598 seats in proportion to the parties’ vote counts. The seats that a party thus obtains are filled with this party’s direct-seat winners, while the remaining seats are filled from this party's candidate list. A related proposal for the Spanish Congress based on the vote total and seat total is discussed in [2], with the additional twist to reward big parties in order to improve upon governability.

In 1989 Balinski and Demange [3] proposed a bi-proportional apportionment method permitting a subdivision into several electoral districts, while at the same time securing overall proportionality with respect to vote counts. In [4] they propose an algorithm to obtain this bi-proportional apportionment solution. Other algorithms have been proposed by Pukelsheim and Ramírez. A total of thirteen algorithms is implemented in the software Bazi [5] that is made freely available by the Augsburg research group.

Section 2 explains the general idea underlying bi-proportional apportionment methods. In Section 3, we apply the bi-proportional divisor method with rounding down (Jefferson/D’Hondt/Hagenbach-Bischoff) to the 2004 election of the Spanish Congress, thus achieving proportionality relative to the population counts in the fifty-two districts, as well as proportionality relative to the vote counts for the political parties. Also, advantages and disadvantages of the method are discussed.

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Table 1
Input data and output data

<table>
<thead>
<tr>
<th>District magnitudes</th>
<th>Overall party seats</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_1$</td>
</tr>
<tr>
<td>$i$</td>
<td>$M_i$</td>
</tr>
<tr>
<td>$k$</td>
<td>$M_k$</td>
</tr>
</tbody>
</table>

In district $i = 1, \ldots, k$ with district magnitude $M_i$, party $j = 1, \ldots, \ell$ with overall party seats $P_j$, wins $v_{ij}$ votes and gets $s_{ij}$ seats.

Table 2
Votes in Aragón 2004

<table>
<thead>
<tr>
<th>District magnitudes</th>
<th>Overall party seats</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PSOE</td>
</tr>
<tr>
<td>Huesca</td>
<td>3</td>
</tr>
<tr>
<td>Teruel</td>
<td>3</td>
</tr>
<tr>
<td>Zaragoza</td>
<td>7</td>
</tr>
</tbody>
</table>

There are 13 seats to be apportioned, for a total of 701513 votes. Thus 61500 votes yield an initial weight of $13 \times 61500/701513 = 1.140$, for the first cell of Table 3.

Spanish Congress. In our sample calculation, all valid votes are considered eligible to participate in the apportionment process. The final Section 4 summarizes some features of the bi-proportional technique.

2. The bi-proportional method

Suppose that the electoral region is subdivided into electoral districts $i = 1, \ldots, k$, with district magnitudes $M_i$. The district magnitude $M_i$ signifies the number of seats to fill in district $i$. Furthermore we assume that there are parties $j = 1, \ldots, \ell$ campaigning in the electoral region. We assume that they are allocated $P_j$ seats, in proportion to their overall vote totals. Clearly, the sum of the district magnitudes must be equal to the sum of the overall party seats, $\sum_{i=1}^{k} M_i = \sum_{j=1}^{\ell} P_j = H$, where $H$ is the house size of the national parliament. Let $v_{ij}$ be the number of votes obtained in district $i$ by party $j$. A typical display of the data is shown in Table 1.

The task is how to obtain the number of seats $s_{ij}$, to be allocated in district $i$ to party $j$. Row sums and column sums are required to achieve the pre-specified district magnitudes and overall party seats. That is, the seat numbers in district $i$ must add up to $M_i$, and the seat numbers for party $j$ must add up to the overall party seats $P_j$.

The main issue is to determine the seat numbers $s_{ij}$ in such a way that they turn out to be proportional, in some sense or other, to the vote counts $v_{ij}$. It may be tempting to scale the vote count $v_{ij}$ by some common constant, and then round the resulting quotient to a neighboring integer. It turns out that this approach does not assure that the pre-specified row and column sums are met correctly. A single common constant for re-scaling the vote matrix is insufficient to achieve the desired goal.

Instead, the double proportional methods proposed by Balinski and Demange use two sets of constants, namely, district divisors and party divisors. The divisors are found by an iterative procedure (their existence is guaranteed in [3]). Once they are obtained, the seat apportionment matrix is most easily verified, by dividing the vote count $v_{ij}$ by the corresponding district divisor and the corresponding party divisor. We take the space to demonstrate the approach by example.

A three-district example. For the sake of simplicity we consider just three Spanish provinces with their actual district magnitudes for the Spanish Congress, Huesca with 3 deputies, Teruel with 3 deputies, and Zaragoza with 7 deputies. In line with the 2004 election we assume that the 13 seats are shared between three parties: PSOE 6, PP 5, and CHA 2.

Table 3 illustrates that individual rounding does not result in a valid seat apportionment. To this end we subdivide all vote counts by the vote total (701573) and multiply by the seat total (13). This calculation results in a fractional number that must be rounded to an integer value before it can be interpreted as a number of seats. Table 3 applies standard rounding, wherein a fractional number gets rounded to the nearest integer. The resulting apportionment is infeasible. It sums to a total of 14 seats instead of 13. Huesca and Teruel each get only 2 seats instead of 3, while Zaragoza is assigned 10 seats instead of 7. The PP party is allocated 6 seats instead of 5.

Since scaling the weights with a single common constant turns out not to be feasible, we instead proceed somewhat more sensitively row by row. Thus the first row is scaled by 1.60, whence the new weights 1.823:1.497:0.256 round to 2:1:0 and achieve the pre-specified district magnitude, 3. The multiplier 1.60 is not unique; in fact, any number in the range from
The rounding of these weights does indeed meet the
constitute the
but the last two rows do not fit.
The columns of the weights(#1) are scaled by the given multipliers, to obtain weights(#2) and their roundings. Row sums match the district magnitudes, but column sums still miss the overall party seats.
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The columns of the weights(#1) are scaled by the given multipliers, to obtain weights(#2) and their roundings. Column sums match the overall party seats, but the last two rows do not fit.

Multiplying the last two rows of weights(#2) produces weights(#3). Since their roundings obey the pre-specified marginals, they represent the end result.

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With the divisors displayed in the margins, the apportionment is easy to verify. For instance, PSOE in Huesca gets 2 seats since 61 500/1.37000 = 45.04.

1.32 to 1.60 would do. We choose the maximum multiplier whenever the weights have to be scaled up, and the minimum multiplier whenever they need to be scaled down. See Table 4.

The row-wise adjustments lead to an intermediate seat allocation obeying the pre-specified district magnitudes. But the overall party seats are not met: PSOE is awarded 7 seats instead of 6, and CHA gets 1 seat instead of 2. In order to correct the column sums, the weights(#1) are column-wise re-scaled to obtain weights(#2), and then rounded. The intermediate seat apportionment meets the overall party seats, but simultaneously creates a new imbalance between Teruel and Zaragoza. See Table 5.

It transpires that another re-scaling of rows is needed to obtain the final result. By multiplying the second row by 1.1 and the third row by 0.995, we obtain the weights(#3) shown in Table 6. The rounding of these weights does indeed meet the pre-specified district magnitudes, as well as the overall party seats. Hence the rounded numbers in Table 6 constitute the seat apportionment of the bi-proportional divisor method with standard rounding.

In conclusion, the bi-proportional apportionment that goes along with the input data from Table 2 is displayed in Table 7 in a compact form documenting the input data as well as exhibiting the output data. The pre-specified district magnitudes and overall party seats are printed in italics, to the left and at the top. The district divisors and the party divisors are also shown in italics, to the right and at the bottom. The divisors are not unique, but may vary in small intervals, as long as
Table 8
Seat apportionments for the 2004 Spanish Congress, based on the nationwide vote totals

<table>
<thead>
<tr>
<th>Party</th>
<th>Seats</th>
<th>Sainte-Laguë</th>
<th>D'Hondt</th>
<th>Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSOE</td>
<td>152</td>
<td>158</td>
<td>164</td>
<td></td>
</tr>
<tr>
<td>PP</td>
<td>135</td>
<td>139</td>
<td>148</td>
<td></td>
</tr>
<tr>
<td>IU</td>
<td>18</td>
<td>18</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>CiU</td>
<td>9</td>
<td>11</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>ERC</td>
<td>9</td>
<td>9</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>PNV</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>CC</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>BNG</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>PA</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>CHA</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>EA</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>NA-BAI</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>EV</td>
<td>1</td>
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<td>0</td>
<td></td>
</tr>
<tr>
<td>PSM...</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>CENB</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>ARALAR</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>LV-E</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>PAR</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>CDS</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>EV-AE</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>350</td>
<td>350</td>
<td>350</td>
<td></td>
</tr>
<tr>
<td>Divisors</td>
<td>72 400</td>
<td>69 760</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the Sainte-Laguë apportionment, the votes are divided by 72 400 and the resulting quotients are rounded in a standard fashion to obtain the seat numbers shown. For the D'Hondt apportionment, the divisor is 69 760 and all quotients get truncated to their integer parts. The D'Hondt method comes closest to the current electoral law, whence this method is also used in the bi-proportional apportionment in Table 9.

3. The 2004 election to the Spanish Congress

We now apply the bi-proportional divisor method with rounding down (Jefferson/D'Hondt/Hagenbach-Bischoff) to the election of the Spanish Congress in March 2004. There are 52 electoral districts (circumscriptions), the 50 provinces plus the autonomic cities of Ceuta and Melilla. The Spanish Congress comprises 350 seats. The district magnitudes used are those from the 2004 election.

The first step is to obtain the overall party seats, across all of Spain, irrespective of the subdivision into the electoral districts. Table 8 shows the 20 parties that each received more than 30 000 nationwide votes, and their vote counts. The apportionment of the 350 congressional seats proportionally to these vote counts results in the columns labelled

- “Sainte-Laguë” when the divisor method with standard rounding (Webster/Sainte-Laguë) is used,
- “D'Hondt” when the divisor method with rounding down (Jefferson/D'Hondt/Hagenbach-Bischoff) is used, and
- “Current” when the apportionment from the 2004 electoral law is used.

The total number of all valid votes, including those of over seventy minor parties not listed in Table 8, was 25 483 504. Note that party PAR with 36 540 votes represents less than 0.2% of all valid votes. On the other hand, a nationwide five percent threshold as in Germany would exclude all parties with fewer than 1 274 176 votes, and thus leave only the first three top runners. Rather than entering into a discussion which apportionment would result from a five, four, three, etc. percent threshold, we rely in the following on the divisor method with rounding down (Jefferson/D'Hondt/Hagenbach-Bischoff).
which comes closest to the current results that are entailed by the pertinent electoral law. For the sake of demonstration, we then also use this method for the bi-proportional calculations. Nevertheless, in order to facilitate governability, it could be possible to give a bonus to the most voted-for party [2].

In Table 8 we can compare the total number of seats the parties receive under D'Hondt with the number of seats they currently have. In any case, the PSOE is the party that obtains the plurality of seats. In order to build an absolute majority, it would need the support of other parties. In the current allotment, the support of any two parties amongst IU, CIU, ERC or PNV produces a majority. With D'Hondt, the support of IU would suffice.

The D'Hondt apportionment, though close to the current apportionment, is seen to be much more concordant with the actual vote counts. IU wins about twice as many votes as ERC, and is allocated twice as many seats. In contrast, the current vote counts. IU wins about twice as many votes as ERC, and is allocated twice as many seats. In contrast, the current...
allocation gives IU fewer seats than ERC. PA wins twice as many votes as CHA and D'Hondt allocates two seats as compared to one. The current law denies PA any representation in Congress, yet rewards CHA with one seat.

The bi-proportional divisor method with rounding down yields the seat numbers as shown in Table 9. For this data set, with 52 rows and 11 columns, five row scalings and four column scalings are needed to obtain the result.

The three largest parties, PSOE, PP, and IU, campaign in all districts. It so happens that no district features more than two other parties. These other parties, whose national totals range from about 800,000 (CIU) down to 80,000 (EA), are compactly displayed in just one column where a superscript number indicates their identity, as follows:

1. CIU in Catalunya (Barcelona, Girona, Lleida and Tarragona) has party divisor 0.97;
2. ERC in Catalunya has party divisor 0.87;
3. PNV in the Basque Country (Álava, Guipúzcoa and Vizcaya) has party divisor 1.1;
4. CC in the Canary Islands (Las Palmas and Santa Cruz de Tenerife) has party divisor 1.2;
5. BNG in Galicia (A Coruña, Lugo, Ourense and Pontevedra) has party divisor 1;
6. PA in Andalucía (Almería, Cádiz, Córdoba, Granada, Huelva, Jaén, Málaga and Sevilla) has party divisor 0.515;
7. CHA in Aragón (Huesca, Teruel and Zaragoza) has party divisor 1; and
8. EA in the Basque Country has party divisor 0.7.

A peculiar effect of any bi-proportional apportionment is the possible occurrence of discordant seat assignments. By definition, we speak of a discordant seat assignment in two cells of Table 9 whenever one cell features more votes but fewer seats than the other. This is particularly irritating within the same district. In the current system seats are assigned just within that district and with no regard to the rest of the nation, which makes discordant seat assignments within districts impossible. (But the current system has to pay the price. Securing more homogeneity within districts aggravates the heterogeneity between districts, and on a national level.)

Here are some examples. In Asturias, PSOE is weaker than PP (305,240:307,977), but wins more seats (4:3). Other discordant seat assignments between PSOE and PP occur in Castellón, Madrid, Toledo, and Valadolid. Within the PSOE party, a discordant seat assignment occurs between A Coruña and Álava (287,324:561,374 votes versus 4:1 seats).

Such frictions are unavoidable since a bi-proportional apportionment mediates between two goals that, at times, are conflicting. One goal is to achieve proportionality as pre-specified by the district magnitudes, the other, proportionality as pre-specified by the overall party seats. Since the turn-out in the 52 districts is not identical, and hence creates different proportional weightings than those based on the population and the district magnitudes, global balance cannot be achieved without local adjustments. In fact, it is this global view that represents the distinguished meritorious feature of a bi-proportional method. The method achieves proportionality among electoral districts, relative to population counts, as well as proportionality among parties, relative to vote counts. As such it brings about a “nationalization” of the electoral process for the major national institution, the Congress.

4. Conclusions

The bi-proportional method is a new recent technique to solve a proportional representation problem that comes in a rectangular table of data, imposing restrictions in the direction of rows as well as in the direction of columns. For application to political elections, the table is made up of the vote counts that various political parties receive in a number of electoral districts. The restrictions are the district magnitudes, and the overall party seats.

Usually the district magnitudes are determined in the middle of a legislative period, proportionally to the population’s census data supplied by the statistical offices. In contrast, the overall party seats are calculated on the eve of election day, proportionally to the nationwide vote totals of the parties. This “super-apportionment” honors the popular vote irrespective of the subdivision into various districts. This guarantees that all voters contribute to the final result in an equal manner, without being advantaged or disadvantaged when casting their vote in a small, rural district or in a large, municipal district.

The bi-proportional method then proceeds to a sub-apportionment to obtain the number of seats of parties per districts. The principle of proportional representation persists, in that the results within a district are scaled by a common factor, the district divisor, as well as that the results within a party are scaled by a common divisor, the party divisor. However, since the method serves two goals, as dictated by district magnitudes and overall party seats, the interaction of the two sets of divisors is occasionally counter-intuitive.

The high degree of proportionality that is thus achieved on a national level would suggest to introduce a threshold of a minimum vote percentage before a party becomes eligible to participate in the apportionment process. Otherwise, since all votes contribute towards the final result, parties are well advised to adopt strategies of presenting themselves in all electoral districts. Therefore a bi-proportional method induces a “nationalization” of the election of a national political body such as the Congress.

A computer is needed to calculate a bi-proportional seat apportionment. However, once the final result is made public, verification is much easier than it used to be with the old system. All a voter has to do is to take the vote count of his or her party in his or her district, and divide it by the divisors that are published with the final apportionment. The increased transparency for the individual citizen goes along well with the increased equality on a national level [12].
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References