similar to Approval Voting (with different limitations). If greater proportionality in the Senate is desired (or in the universities elections), it would be preferable, for a behavior of the voters like the one mentioned previously, to use as method of social election a Borda-type method with the following weights: $1, \ldots, 1/3, \ldots$.

This paper sheds some critical light on several electoral systems and practices that can be seen in Spain (the constituencies size, the advantage of the main regional parties over the similar national parties, the election of the Senators, the higly manipulable electoral system to determinig university representatives or Juntas). Notwithstanding all these drawbacks, the electoral processes of the Congress, the Senate, and the municipal, regional or European elections do function in a positive sense in that they are applicable in all cases.

On the other hand, I introduce a new property for the proportionality: *Limited loss of seats in coalitions.* We put forth that a method has a limited loss of seats in the case of coalitions: the fusion of $2r$ or $2r+1$ parties does not entail a loss of more than r seats. Then, a necessary condition, for a divisor method, to imply a limited loss of a seat is that

$$d(s) \in \left[s + \frac{1}{2}; s+1\right].$$

(If $d(s) = s+t$ or the fusion is of $2r$ parties, the previous condition is also sufficient).

In accordance with this property and the properties of the parametric methods [5], I think that the most reasonable option is to use divisor methods of the parametric family from Webster to Jefferson in approaching problems of proportional allotment.

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BAZI — A Java Program for Proportional Representation

Friedrich Pukelsheim

BAZI is a freely available JAVA-Program, permitting the user to experiment with various apportionment methods, and to assess their relative merits on the basis of real data rather than abstract theory.

The pertinent theory is available in the seminal monograph [4] by Balinski and Young. Among all possible apportionment methods, the authors single out two important subclasses. The first class consists of divisor methods, the second of quota methods. BAZI features just two quota methods, the method of greatest remainders (Hamilton, Hondt, Hagenbach-Bischoff), and the Droop method.

However, a central message of the Balinski/Young monograph is that divisor methods are generally more appropriate for the apportionment problem. Of these, BAZI offers two parametric families, the divisor methods with stationary roundings, and the divisor methods with powermean roundings; for details see [5, p.357].

The powermean methods are more important from a historical point of view, comprising the five traditional methods of Adams, Dean, Hill, Webster, and Jefferson. In contrast, the stationary methods are more amenable to a mathematical analysis. BAZI relies on an algorithm [5, p. 378] whose computational complexity is minimum [6, p. 154].

On the computer screen, BAZI comes up with the graphical user interface split into three panels, the input field to the left, the methods field in the middle, and the output field on the right.

The input field invites the user to key in data of his or her own, or to read in a data file that the user has created, or to load data from the extensive data base.

In the methods field the user can select a house size (district magnitude) and, in particular, one or more apportionment methods.

Whenever the user chooses a divisor method, BAZI outputs the resulting apportionment along with a pertinent divisor. This way the user may easily confirm the results with paper and pencil (or a pocket calculator), rather than being forced to believe what the machine says.

A particular feature of BAZI is that it offers three options for multiple electoral districts. The user may choose between (1) separate evaluations for each district, (2) biproportional apportionments using divisor methods, and (3) a variant of the latter that is specifically tailored to the needs of the new Zurich electoral law of 2003.

For these matrix apportionments BAZI uses an algorithm akin to the one reported by Balinski and coauthors in [1],[2] and [3]. More precisely, BAZI implements a discrete variant of the iterative proportional fitting procedure, also known as alternating scaling. A paper to report on the specific properties of the BAZI algorithm is under preparation.

The BAZI homepage and download site is

www.uni-augsburg.de/bazi.

The site also includes the pseudocode of the program, a detailed description of the district options (1)–(3) mentioned above, and an extensive list on the Proportional Representation literature.

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Seat Biases of Apportionment Methods for Proportional Representation

Mathias Drton
(joint work with K. Schuster, F. Pukelsheim and N. R. Draper)

In proportional representation systems, apportionment methods are used to translate the electoral votes into specific seat allocations. The seat allocations are of course integer numbers, and the votes are almost continuous quantities, by comparison. One of the pertinent problems is to measure the effect of the use of a given apportionment method. Whereas previous studies have made inferences about the proportionality of apportionment methods from empirical data, this paper (Schuster *et al.* [6]) derives the
information deductively.

We concentrate on the three most popular apportionment methods (cf. Balinski/Young [1], Kopfermann [4]):

(H) the quota method of greatest remainders (Hamilton, Hare),
(W) the divisor method with standard rounding (Webster, Sainte-Laguë),
(J) and the divisor method with rounding down (Jefferson, Hondt).

Assuming repeated applications of each method, we evaluate the seat biases of the various parties. These seat biases are averages, over all possible electoral outcomes, of the differences between the (integer) seats actually apportioned, and the (fractional) ideal share of seats that would have been awarded, had fractional seats been possible.

More formally, we consider ℓ parties, numbered $1, \ldots, \ell$, with respective vote counts $v_1, \ldots, v_\ell$. In proportional representation, the number of seats allocated to a party ought to be proportional to the relative weight of their vote counts. Hence, if $V = \sum_{k=1}^{\ell} v_k$ is the total number of votes cast, there is no loss of generality to convert the vote counts into vote ratios, or *weights*, $w_k = v_k/V$, $1 \le k \le \ell$. Assuming that the weights $w_1, \ldots, w_\ell$ follow a uniform distribution over the set of any ℓ non-negative numbers summing to one, we calculate the average behavior of the seat allocations. This distributional assumption can be traced back to Pólya [5].

The *district magnitude*, that is, the total number of seats to be apportioned is denoted by M. The numbers $w_1 M, \ldots, w_\ell M$ are the *ideal shares of seats of parties* $1, \ldots, \ell$. These would be the "fractional numbers of seats" to which, ideally, each party would be entitled if that were possible. In real life, the parties are apportioned an integral number of seats $m_1, \ldots, m_\ell$, using the apportionment method in the applicable electoral law.

A common approach for evaluating the goodness of an apportionment method is to compare, for each party k, their actual seat allocation m_k with their ideal share of seats $w_k M$. This results in the *seat excess* $m_k - w_k M$ of party k. We are interested in whether an apportionment method systematically favors larger over smaller parties. Hence, we condition the averaging process on the event that party 1 is largest, party 2 is second-largest, etc., where "largeness" refers to party weights. Under this condition $w_1 \geq \ldots \geq w_\ell$, we

study the expected value of the seat excess $m_k - w_k M$ as a function of the district magnitude M. The resulting quantity

$$B_k(M) = E\big[m_k - w_k M \big| w_1 \geq \ldots \geq w_\ell\big],$$

is called the *seat bias of the k-th largest party*. The standard statistical term "bias" indicates an expected difference between all possible observable values of a quantity and its ideal value. The main results of our paper are formulas for the seat biases, for each party k, under a given apportionment method.

For the quota method of greatest remainders (Hamilton, Hare), the seat biases $B_k^{\mathrm{H}}(M)$ turn out to be identical and slightly positive, for parties $k = 1, \ldots, \ell - 1$ from the largest down to the second-smallest:

$$B_k^{\mathrm{H}}(M) = \frac{\ell + 1}{24M} + O\left(\frac{1}{M^2}\right), \tag{1}$$

$$B_\ell^{\mathrm{H}}(M) = -(\ell - 1)\frac{\ell + 1}{24M} + O\left(\frac{1}{M^2}\right). \tag{2}$$

The ℓ-th, smallest party carries the deficit that balances the positive accumulation. Even though the special role of the smallest party may appear disconcerting, its seat bias remains so small numerically as to be invisible in practice. Thus the quota method of greatest remainders is practically unbiased.

For the divisor method with standard rounding (Webster, Sainte-Laguë), the seat biases of the largest $\ell - 1$ parties $k = 1, \ldots, \ell - 1$ are given in (3), while the seat bias of the ℓ-th, smallest party is given in (4):

$$B_k^{\mathrm{W}}(M) = \frac{\ell + \frac{2}{\ell}}{24M} + \frac{\ell + 2}{24M}\left\{\left(\sum_{j=k}^{\ell-1}\frac{1}{j}\right) - 1\right\} + O\left(\frac{1}{M^2}\right), \tag{3}$$

$$B_\ell^{\mathrm{W}}(M) = -(\ell - 1)\frac{\ell + \frac{2}{\ell}}{24M} + O\left(\frac{1}{M^2}\right). \tag{4}$$

Here a certain amount of balancing goes on between the $\ell - 1$ largest parties alone. The accumulated contribution of the terms $(\ell + \frac{2}{\ell})/(24M)$ is evened out by the negative seat bias of the smallest party. However, all these theoretical seat biases are so small numerically that we do not consider them practically relevant. That is, the Webster seat allocations are practically unbiased.

For the divisor method with rounding down (Jefferson, Hondt) the situation changes dramatically. The leading term in the seat-bias is independent of the district magnitude M:

$$B_k^{\mathrm{J}}(M) = \frac{1}{2}\left\{\left(\sum_{j=k}^{\ell}\frac{1}{j}\right) - 1\right\} + O\left(\frac{1}{M}\right). \tag{5}$$

The remainder term, bounded of order $1/M$, appears to be practically irrelevant.

Now, the largest party clearly enjoys a positive seat bias and can expect

seats in excess of their ideal share. The seat biases become successively smaller, as we pass from the largest party ($k = 1$) to the smallest party ($k = \ell$). The biasedness of Jefferson's method has been observed over many years on the basis of empirical data, but our formulas permit specific calculations about the numerical sizes of the seat biases. For example, the largest party in a three-party system can expect five extra seats per twelve elections in excess to their ideal share, under the Jefferson method.

Our seat bias results depend on the assumption of uniformly distributed weights. However, Schuster *et al.* [6] confirm the theoretical findings via empirical data from the German State of Bavaria, the Swiss Canton Solothurn, and the U.S. House of Representatives. Furthermore, Schuster *et al.* [6] give illustrations of the seat biases and provide details on their interpretation. Mathematical details are provided in Drton and Schwingenschlögl [2, 3].

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Negative Weights of Votes and Overhang Seats in the German Federal Electoral Law

Martin Fehndrich

In Elections to the German Bundestag, internal overhang seats cause an effect — negative weight of votes — where a party can get more seats if loosing some votes, or loose seats because it wins some additional votes [1],[2]. This effect is demonstrated in the federal German election 2002, where 1000 votes less for the

SPD in one federal state would have caused an additional seat for this party. In the talk, an overview over the German electoral system is given.

The reasons for Overhang Seats in general are traced back to two mechanisms: many won constituency seats and few party votes. These two mechanisms allow to describe the effect of every parameter of an electoral system on overhang seats. The possible treatments of overhang seats are presented with a view of their effect on disproportionality and additional seats. To prevent disproportionality and an increase of the house size, respectively, a rule must be defined of not awarding some of the overhanging constituency seats. Awarding all won constituency seats, one has to make tradeoffs between disproportionality and increasing house size. The biggest increase of house size with no or only a small disproporionality would be reached by awarding additional balance seats (as done in most German federal states), the biggest disproportionality but no increase of parliament by reducing the number of seats for the not overhanging parties (as in the Scottish parliamentary elections), while just awarding the overhang like in the German Bundestag stays somewhere in the middle. An additional possibility is given in systems with internal overhang seats, like the German system, where a party can have overhang seats in one federal state, but still list seats in other federal states. In this case an internal compensation could be used, where proportional seats are at first awarded to justify the constituency seats and than are awarded to a partys lists.

Negative votes are votes in a party election, without ranking, only one ballot and no second ballot.

One simple example for an electoral system allowing votes with a negative weight of votes is the quota system with largest remainder (named after Hamilton or Hare-Niemeyer), with a 5%-barring clause and 21 Seats. In an 4-party example with A, B 4400 votes, respectively, C 700 votes and D 500 votes, an additional vote for C (coming from nonvoters or D), would actually reduce the number of seats for C. Another example for negative votes is the house monotone quota system, described by Balinski and Young [3, Table A7.1/A7.2 p. 140].

A more serious problem with negative votes occurs in the German Bundestag elections. Here a reduction of the votes for the SPD in the federal state of Brandenburg by 1000 votes in the 2002 election would have caused an additional seat for this party. The effect is connected with the occurrence of internal overhang seats. Loosing votes in Brandenburg will cause a shift in the proportional seats within the party's federals state lists. Brandenburg would lose a seat in favour of Bremen. But since in Brandenburg there are enough constituency seats, this does not hurt Brandenburg's SPD-list, where then an overhang seat occurs, and in the end there is an additional seat for the SPD. The effect is independent from the rounding rule and can occur with Hamilton, Jefferson, Webster or other methods. It occurred in the elections with Jefferson until 1983, before the change to the Hamilton system. Even if we think about fractional seats, a vote for an overhanging federal list would cause the loss of a fractional part of a seat. The effect is sometimes that repeating and predictable that it becomes the best strategy under game theoretical aspects to vote for the disfavoured and overhanging party rather than voting for the favoured party. Even in other cases it is a better strategy to vote for a second choice party and not for the probably overhanging favoured list. The occurrence of this effect in an electoral system is critiqued, because it is against the rule of a direct election and some seats are justified by not given votes rather than given votes. There is a qualitative change exceeding the point of disproportionalty, if a votes weight is not just lower than others, but becoming smaller than zero. An election under this circumstances seems more a case for game theorists than an election. There is no reason in sight which could justify this effect as a trade-off against other favourable properties of an electoral system (as opposed to social choice, where a voter can rank or give more than one vote, allowing similar effects like the no show paradox).

As a solution for the German Electoral System an internal compensation rule is recommended, which prevents internal overhang seats and with that negative votes. To reduce some paradoxes one should also change from Hamilton to the Webster (Sainte-Laguë) system in the party distribution and sub-distribution.

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The Role of the Mean and the Median in Social Choice Theory

William Zwicker

A *center* is a function C that assigns, to each finite set S of points of $\mathbb{R}^n$, a central point $C(S)$ of the distribution. The mean is the most familiar center, but there are others. In particular, the *mediancentre* (the point minimizing the sum of the distances to members of S) seems attractive; it is one of several generalizations of the median to the multivariable context.

Extending work of Saari and Merlin, we show that many familiar voting systems — including Borda count, Condorcet's method of pairwise majorities, and the Kemeny Rule — have alternate descriptions as follows:

(1) Plot the vote v of each voter as a point $A(v)$ in n-space (where the choice of plotting function A depends on the particular voting system at hand).
(2) Take the mean location q of all points $A(v)$ (counting multiplicity).
(3) The outcome is the vote v_0 for which $A(v_0)$ is closest (in the l_2-metric) to q.

In particular, the plot function for the Borda count places rankings at vertices of the permutation polytope, or "permutahedron," while the Condorcet procedure and Kemeny rule each use the "pairwise comparison cube" discussed by Saari. The result for the Kemeny rule is particularly surprising, as the original description employs a type of median based on the Hamming distance between rankings, whereas the new characterization uses the mean on standard, Euclidean distance.

Several properties shared by these voting systems can now be traced to their common dependence on the mean.

If we replace the mean with the mediancentre in step (2) of any system, the result is typically a new system. For example, the *Mediancentre Borda* seems interesting; while it fails to have the consistency property, it is less manipulable than the standard Borda count, and has the interesting property that when a majority of the voters rank candidates similarly, their favorite will win. These differences can largely be explained by axiomatic differences between the mean and the mediancentre. In particular, the mean satisfies the property that

$$C(S+T) \;=\; C(S+kC(T)),$$

where S and T are multisets of points in $\mathbb{R}^n$ (several points may have the same spatial location), $S+T$ is the union counting multiplicity, T has k points counting multiplicity, and $kC(T)$ is the multiset having k points, each located at $C(T)$. In fact the mean is characterized by this property together with some symmetry and the requirement that $C(S)$ is uniquely defined for all nonempty multisets S of points of $\mathbb{R}^n$.

The corresponding axiom for the mediancentre seems to be

$$C(S+\{p\}) \;=\; C(S+\{p'\}),$$

where p is any point not located at $C(S+\{p\})$, and p' is any point on the one-sidedly infinite ray from $C(S+\{p\})$ through p (with $p' = C(S+\{p\})$ allowed). This property, together with some symmetry and the requirement that $C(S)$ be uniquely defined for all multisets S of points of $\mathbb{R}^n$, except for multisets S containing an even number of collinear points, implies a spatial majority rule property: $C(S) = p$ whenever either a strict majority of points are located at p, or exactly half the points are at p and the other half are not all located at some common different location. These same three axioms characterize the median in $\mathbb{R}^1$, but we do not know whether the same is true for $\mathbb{R}^n$.

Formal Analysis of the Results of Elections

Fuad Aleskerov

Four main issues are presented in the paper:

(1) Patterning of electoral outcomes,
(2) Polarization of electoral outcomes,
(3) Disproportionality of a parliament,
(4) Power distribution in Russian parliament during 1994–2003.

In the first issue I deal with the following problem: is it possible to find a similarity of electoral outcomes over several elections, and can we describe the notion of stability of electoral behavior being based on such similarity?

The approach uses the clustering algorithm applied to all data available on election outcomes. An important new feature of the algorithm (which is called a clustering of curves algorithm) is that it uses the relations among outcomes, not the numerical values themselves. The obtained clusters are called patterns, and one can analyze how the districts change their patterns over years. Then one can call the electoral behavior in a district as a stable one if there are no changes of patterns over years.

Using this very approach, Prof. Hannu Nurmi and I have studied the patterns of party competition in British general elections in 1992, 1997 and 2001 over 529 constituencies in England, 70 constituencies in Scotland, and 40 constituencies in Wales. Only 13 patterns of support distribution are obtained for English constituencies, and only 6 of them are sufficient to describe the electoral preferences distribution in more than 90% of the constituencies. Concerning the stability of electoral outcomes, it has been shown that almost 38% of constituencies have not changed their preferences over those three general elections. Almost 48% of constituencies changed their preference after 1992 elections and then kept stable. In other words, almost 86% of constituencies can be called stable or semi-stable in terms of their electoral outcomes. Approximately the same results are observed for Scotland and Wales. Next we have studied the stability of electoral outcomes during last seven municipal elections from 1976 to 2000 in Finland over 452 constituences. Naturally, the deviation from the stability is much higher when such long period is studied. However, 14% of constituencies are absolutely stable since they have not changed their electoral patterns during those 25 years. 51% of constituencies can be called semi-stable since they have experienced not more than one or two changes of patterns over this period, and only 1% of constituencies are completely unstable, i.e., they have experienced seven changes of patterns over these elections. These results are very illustrative for the use of this very powerful method of patterning electoral outcomes.

In the political studies literature one can find very few attempts to study a polarization of society on the basis of electoral outcomes. Such attempt was made by my B.S. student M. Golubenko and myself. We construct a polarization index using an analogy from physics which is called central momentum of forces with respect to the center of gravity. We consider the parties being positioned over the left-right position axes, and in each position the mass (percentage of votes for that party) is concentrated. Then by evaluating the polarization index one can conclude to which extent the electoral preferences are polarized. If there are only two parties with 50% of votes given to each of them, and these parties are located in the extreme opposite positions of the left-right spectrum, then the polarization is maximal and equal to 1. On the other hand, if there are several parties positioned at the same place on the left-right scale, never mind where this place is, the value of polarization index is equal to 0. We have evaluated the distribution of polarization over the regions of Russia using electoral outcomes of 1995, 1999 and 2003 general elections.

There are several well-known indices to evaluate the disproportionality of a parliament, e.g., Maximum Deviation index, Rae index, Gallagher index, Loosemore-Hanby index, etc. However, none of them take into account the turnout of elections and the percentage of votes "against all", which is allowed in Russia. My M.S.

student V. Platonov and I have proposed a disproportionality index which is a modification of Loosemore-Hanby's index and takes into account these additions. We have introduced a new index of disproportionality, that of relative representation. The index shows a percentage of seats in a parliament which a party receives for 1% of votes. The evaluation made for several countries (Russia, Finland, Sweden, Ukraine, Lithuania, Turkey) show that the countries of the former Soviet block are characterized with higher degree of disproportionality.

The last topic in my paper deals with the study of power distribution in the Russian parliament from 1994 to 2003. We studied Banzhaf and Shapley-Shubik indices on a monthly basis using the MPs' voting data. The indices have been evaluated for different scenarios of coalition formation. The model of coalition formation uses the index of groups positions consistency showing to which extent two groups (fractions) of MPs vote similarly. In the first scenario all evident opponents are excluded from coalitions, in the second scenario all evident and potential opponents are excluded, and in the third scenario coalitions only with evident allies are allowed. The first scenario is most close to the real coalition formation in the Russian parliament. The analysis shows, in particular, that due to the absence of intention to coalesce, the Communist Party during almost all period under study has had power near to 0, although there were periods when this party controlled more than 30% of seats. The dependence in the changes of the power indices distribution is compared with respect to political events during this period.

Procedure-Dependence of Electoral Outcomes

Hannu Nurmi

The theoretical literature abounds examples in which the voting outcomes — winners or the ranking of candidates — depends not only on the revealed preferences of the voters but also on the method used in determining the result. From the late 18th century, two main intuitive notions have played a prominent role in the literature, *viz.* one which maintains that in order to qualify as the winner, a candidate has to defeat, in pairwise comparisons, all other candidates, and the other which looks for the winner among those candidates that are placed highest on the voters' preference rankings. It is well-known that these two intuitive notions are not equivalent: the candidate that defeats all others in pairwise contests may not be best in terms of positions in the voters' preference rankings. But how often do these two notions conflict in real world elections?

The British parliamentary elections were studied by Colman and Pountney (1978) from the view point of estimating the probability of the Borda effect. This effect occurs whenever the elected candidate would be defeated by some other candidate in a pairwise comparison by a majority of votes. The British first-past-the-post (FPTP) system makes it possible that such instances occur. The problem is to know how often. Colman and Pountney used the interview data collected by the British polling organization MORI to construct preference profiles for the entire electorate. From these they then computed the likelihood of instances of the Borda effect. This paper replicates Colman and Pountney's study using the data on the 2001 British parliamentary elections. To get a wider perspective on the variability of electoral outcomes, we used Saari's (1995) geometric methodology to determine the range of all positional voting outcomes in the 2001 elections in all British constituencies. It turns out that — under the same assumptions as those made in the Colman and Pountney's study — in 12 constituencies the ranking of candidates could have been completely reversed depending on the voting rule used. Much more numerous were constituencies, 68 in number, where the actual winner would have been ranked last by another positional voting procedure. The first and second ranked candidates would have been different depending on voting rule in 49 constituencies.

The second aim of the study is to determine the pattern of party competition prevailing in British constituencies. In a study conducted together with Aleskerov we found that the optimal number of party support patterns needed to characterize the 500+ English constituencies over three most recent parliamentary elections is just 13. Moreover, about one-third of the constituencies were characterized by the same support patter over the period of three elections. Less than 10% of the constituencies were completely volatile in the sense of moving from one pattern to another in each election. In Scotland, nearly two-thirds of the constituencies experience no change in support pattern in the three elections. Similar study was conducted on Welsh constituencies. It shows that in terms of support stability, Wales is located between England and Scotland.

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The Mathematical Source of Voting Paradoxes

Donald G. Saari

The social choice literature has many articles describing certain properties of decision rules: often these properties are obtained via the so-called "axiomatic approach." The thrust of

this talk was to 1) show why the way the "axiomatic approach" is used in the social choice literature often has very little, if anything, to do with "axioms" or the "axiomatic approach," 2)

explain a way, motivated by the mathematics of "chaotic dynamics," to identify all possible consistency properties and paradoxes — both positive and negative — of positional voting methods (and all other rules based on these methods), and 3) identify the source of all possible properties of these voting rules. I had intended to also discuss how to find all possible strategic settings, who can be strategic, and the possible strategies, for any specified voting rule, but time ran out.

As for the axiomatic approach, I gave some

examples to show how the properties called "axioms" often are merely particular properties that happen to uniquely identify a particular decision rule. "Uniquely identifying" and

"characterizing via axioms" are very different. As an illustration, the two properties "Finnish-American heritage" and a particular "DNA structure" uniquely identify me, but they are not

"axioms," they do not characterize me, and they do not tell you "what you are getting," which is the usual claim for the axiomatic approach.

The second part described a way to characterize all possible outcomes. This work was motivated by the clever paradoxical example found by, for example, Brams, Fishburn, Nurmi and many others. The point is that a "paradox" identifies an unexpected property of a voting rule. For example, the profile where 6 prefer ACB, 5 prefer BCA, and 4 prefer CBA leads to the plurality ranking of ABC, and the conflicting pairwise rankings of CA, BA, CB. These rankings define the *plurality word* (ABC, BA, CA, CB), and the word identifies the plurality property that the plurality winner can be the Condorcet loser, while the plurality loser can be the Condorcet winner. In other words, each list of

rankings — each word — that CAN occur defines a property of the voting rule. On the other hand, it turns out that this same list (ABC, BA, CA, CB) can never occur with the Borda Count; it can

never be a Borda word. This means that a Borda property is that the Condorcet winner can never be Borda bottom ranked and the Condorcet loser cannot be Borda top ranked. Namely a listing that cannot occur — that cannot be a word — also defines a property of a voting rule. Consequently, to find all possible ranking properties of all possible positional methods over all possible subsets of candidates, we want to find all possible listings of rankings that

could occur over all possible profiles; we want to find all possible words. Doing so directly may be impossible, but by use of notions from chaotic dynamics, this has been done, and the results are discouraging; e.g., for most collections of voting rules (one for each subset of candidates), anything can happen. Namely, any

listing is a word. The unique voting rule that minimizes (significantly!) the number and kinds of listings that can be words is the Borda Count. Thus, this rule has the largest number

(significantly so) of positive ranking properties.

The third topic showed how to construct all possible examples that can occur with a voting procedure, how to explain all of the "paradoxes", etc. The way this is done is to emphasize the profiles rather than the voting outcomes. This is done by finding configurations of preferences

where it is arguable that the outcome is a tie. The conjecture, which turned out to be true, is that all possible differences among voting rules can be explained (and examples constructed) simply by knowing these configurations of preferences where procedures do, or do not, have a complete tie. As an illustration, all possible

properties, differences in outcomes, etc. among three candidate positional voting occur because of the different ways voting rules handle the "reversal configurations" such as (ABC, CBA). Here, only the Borda count gives a tie: all other positional methods either favor $A = C$ over B, or B over $A = C$. Indeed, the above example was created by starting with 1 person preferring ACB and 4

preferring CBA, where the CBA outcome holds for all positional pairwise outcomes. To create the paradox, 5 units of (ACB, BCA) were added: this adding of the reversal components is what caused

the plurality outcome to differ from the pairwise outcomes. Similarly, all possible differences in procedures using pairwise outcomes arise because of "Condorcet profile components" of the (ABC, BCA, CAB) type. Positional rankings are not affected, but these components change the pairwise tallies: for any number of candidates, it causes all problems with tournaments, agendas,

problems with methods using pairwise outcomes such as the Borda Count and the Kemeny method, etc... The two configurations of preferences completely describe all possible differences among three candidate decision rules that use pairwise and/or positional methods; e.g., it explains all possible differences between the Condorcet and Borda winners. Comments were made about results for $n > 3$ candidates.

On the Closeness Aspect of Three Voting Rules: Borda, Copeland and Maximin

Christian Klamler

The purpose of this paper is to provide a comparison of three different voting rules, Borda's rule, Copeland's rule and the maximin rule. Borda (1784) suggested assigning points to the m alternatives in the individual preferences, namely *m-1* points for the top ranked alternative, *m-2* points for the second ranked alternative, down to *0* points for the bottom ranked alternative. Then, for every alternative, one adds up those points over all individuals. The more points an alternative receives the higher ranked it is in the social preference. Copeland (1951) suggested calculating for each alternative the difference between the number of alternatives it beats and the number of alternatives it looses against. Again, the larger the derived number the higher ranked is the alternative in the social preference. Finally the maximin rule is based on the idea that alternatives should be ranked higher in the social preference the more minimal support they enjoy, i.e. the higher the minimal support over every other alternative.

Usual comparisons of such voting rules focus on non-binary aspects (Laffond et al., 1995), e.g. comparing the actual choices of such voting rules for different preference profiles, or calculating the probabilities of voting rules leading to the same choices (e.g. Gehrlein and Fishburn, 1978, and Tataru and Merlin, 1997). Nurmi (1988, p. 207) provides a possible interpretation of such results by stating that "the estimates concerning the probabilities that two procedures result in different choice sets can be viewed as distances between the intuitions." Moreover he adds that "...the fact that the Condorcet extension methods (Copeland's and the max-min method) are pretty close to each other was to be expected." "Closeness" in this sense means the probability of two voting rules choosing the same winner at the same preference profile. In contrast, "closeness" could also be reasonably interpreted with respect to the distance between the outcomes of the different voting rules, i.e. the difference between the rankings derived from two voting rules. To be more precise, assume a set of alternatives X and two social preferences $\succeq, \succeq'$ on X. We will consider two social preferences $\succeq, \succeq'$ as opposed if for all $x, y \in X$, $x \succeq y \Leftrightarrow y \succeq' x$ and for some $x, y \in X$, $x \succ y \Leftrightarrow y \succ' x$. I.e. opposed social preferences are exactly opposite to each other. This paper shows, that in contrast to the conclusions drawn from using a probabilistic approach, "closeness" in the sense of comparing social preferences is neither guaranteed for Copeland's and the maximin method nor for the Borda and the maximin method. It is proved that there exist preference profiles for which the Copeland ranking and the Borda ranking are exactly the opposite of the maximin ranking. That the Copeland ranking and the Borda ranking are opposed has been shown by Saari and Merlin (1996). Similar comparisons exist for Borda's rule and simple majority rule. It is well known that the Condorcet winner (the alternative that beats every other alternative by a simple majority) is never bottom ranked in the Borda ranking and the Condorcet loser (the alternative beaten by every other alternative) is never top ranked in the Borda ranking (Saari, 1995). Hence, even in cases where the winning alternatives are different, we can ensure a minimal degree of consistency between the rules. However, several recent results (e.g. Ratliff, 2001, 2002 and Klamler 2002) show that such a relationship does not exist for many other pairs of voting rules.

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Selecting Committees Without Complete Preferences

Thomas Ratliff

In many ways, the Condorcet criterion is the most natural way to compare candidates: if one candidate is preferred to every other candidate in head-to-head elections, then it is plausible to argue that this candidate should be the winner. When choosing a committee of size m, we can apply a similar criterion.

Definition 1. Given a profile with n candidates $A_1, A_2, \ldots, A_n$, define the *Condorcet committee* of size m to be the set M of m candidates such that A_i is preferred to A_j in pairwise elections for all $A_i \in M$ and all $A_j \notin M$.

As we know very well, the Condorcet winner may not exist since there may be a cycle among the top-ranked candidates, and a cycle involving all candidates would preclude the existence of a Condorcet committee. Notice that we are merely partitioning the candidates into two disjoint groups: those on the committee and those off. We do not care whether we have cycles within the disjoint groups, but only that those on the committee are preferred to those not on the committee.

When there is no Condorcet winner, Charles Dodgson (aka Lewis Carroll) proposed in 1874 picking the candidate that is "closest" to being a Condorcet winner by choosing the candidate that requires the fewest adjacent switches in the voters' preferences to become the Condorcet winner. Since he is selecting a single winner, Dodgson does not care if there is a cycle among the remaining candidates; requiring a complete transitive ranking forces more structure than Dodgson views as necessary. We can easily adapt Dodgson's method to measure how far a set of m candidates is from being the Condorcet committee.

Definition 2. In an election with n candidates, define the *Dodgson Committee*, denoted $\mathcal{DC}_m$, to be the set of size m that requires the fewest adjacency switches so that A_i is preferred to A_j in pairwise elections for all $A_i \in \mathcal{DC}_m$ and all $A_j \notin \mathcal{DC}_m$.

There are, however, several anomalous results that can arise:

- The Condorcet winner may be excluded from $\mathcal{DC}_m$.
- If $j \neq k$, then $\mathcal{DC}_j$ and $\mathcal{DC}_k$ may be disjoint or may have any number of candidates in common.

These results can be found in "Some startling inconsistencies when electing committees", T. Ratliff, *Social Choice and Welfare* **21** 3 (2003), 433–454.

In addition to these inconsistencies, a fundamental objection to selecting a committee based on the rankings of individual candidates is that this may not actually capture the voters' preferences. Voters are often concerned with the overall composition of the committee and consider how the individual members will interact. For example, a voter may prefer two candidates in their top-ranked committee because they represent contrasting viewpoints, but would not want one candidate on the committee without the other. A strict listing of the individual candidates could not detect such a preference without additional information.

The motivation for considering this issue arose in the spring of 2003 at Wheaton College in Massachusetts during the selection of three faculty to serve on the search committee for the next president of the college. When Wheaton had last conducted a presidential search in 1992, three men were elected as the faculty representatives on the committee, which was very controversial on the campus. Wheaton has a long standing commitment to gender balance and awareness, partially based upon its history as a women's college (Wheaton began admitting men in 1988). The faculty was almost evenly divided between women and men, and the election of three men was acceptable to almost no one, including those who were selected to serve on the committee. The selection was a result of a process that only considered voters' preferences for individual candidates and not their preferences for the overall composition of the representatives.

The goal was to select one faculty representative from each of the three academic divisions of the college. An initial ballot used approval voting to reduce the field of possible candidates to six, two from each of the divisions, and the final ballot allowed the faculty to select their preferred candidate in each division. This approach seems very reasonable on the surface. However, by decomposing the voters' preferences of the overall composition of the committee into choices on individual candidates, the procedure selected candidates that were individually preferred by a majority, but the overall composition was nearly unanimously unacceptable.

We should not divorce the voters' opinions of the overall group into opinions of individual candidates. This can be viewed as analogous to some of the objections that are raised to the binary independence axiom in Arrow's Theorem: If complete transitive rankings of candidates are broken down into comparisons on pairs and then reassembled to gain an overall ranking, then vital information is lost.

Because of the experience with the selection process in 1992, the faculty at Wheaton were open to adopting another voting method in 2003. The faculty committee responsible for all faculty elections (of which the author is a member) proposed a different method for the final ballot. An approval voting nominating ballot was used as in 1992 to reduce the field to two faculty members from each of the three divisions. Since the requirement was that there be one faculty member from each division selected, this left a total of eight possible groups of faculty representatives. The final ballot asked the voters to rank the eight possible groups, and the Borda Count was used to select the winning group.

There are several interesting observations in this election.

- Of the 71 ballots received, only three were disqualified because the voter failed to rank all eight groups.
- The group selected by the Borda Count was also the Condorcet winner.
- The voters' rankings indicate that their preferences are more complex than could be detected by a simple listing of the candidates or by simple yes/no

votes on the individual candidates. For approximately half of the voters (35 out of 68), their first place and last place committees were not disjoint. For seven of these voters, their first and last place committees differed by a single candidate.
- There are very few rankings that appear more than once; there are 64 distinct rankings from the 68 voters. Even if we restrict to the top three groupings in each ranking, there are still 45 distinct rankings, and the largest duplicate ranking had only five voters.

Overall, the Wheaton faculty were very pleased with the process and the outcome. However, several faculty commented that they would have had a difficult time ranking more than eight options. In general, it will often be impractical to expect the voters to rank all possible committees since the number of possible committees can be extremely large even for a small number of candidates. For example, there are 210 possibilities when selecting a committee of size four from a group of ten candidates.

We define an intermediate approach for selecting a committee that is based upon each voter ranking their top k committees, for some fixed value of k. From this partial ranking, we want to detect overlap within the ranked committees and to extract groups of candidates that the voters believe would work well together.

Definition 3. Assume that there are n possible candidates for a committee of size m and that each of the N voters ranks their top k committees.

Build a weighted graph G with n vertices corresponding to the n candidates. We form a complete graph with edges connecting every pair of vertices, and also include n loops, one for each vertex. Initially assign a weight of zero to every edge in G, and then determine the weights of the edges by examining the rankings of each of the N voters as follows:

- For a voter's top ranked committee, add k to each edge connecting candidates listed in the committee, including the loop that connects each candidate to itself.
- Apply the same technique to the second ranked committee, except in this case we add $k-1$ to each edge.
- In general, for the jth ranked committee, add $k-j+1$ to the edges corresponding to this committee.

The (not necessarily unique) winning committee $\mathcal{C}_m$ is the subgraph of G with m vertices of maximal weight.

Note that the reason for including the loops is to recognize overlaps in voters' preferences for single candidates as well the overlap in groups of candidates. Also notice that we can easily represent G as a symmetric $n \times n$ matrix M where the (i,j) entry corresponds to the weight of the edge connecting candidates i and j.

An objection to this approach is that it only detects an overlap in voters' preferences of single candidates or of pairs of candidates but places no additional weight on triples or quartets of candidates. A natural extension would be to form a hypergraph that includes all subsets up to size m and weight the hypergraph similarly.

A Question for Mathematicians: Would Disputed Elections Be (Sufficiently) Less Probable If U.S. Presidents Were Directly Elected?

Jack H. Nagel

In the 2000 U.S. presidential election, the very close vote in the pivotal state of Florida led to an agonizing recount that was ended by a highly controversial decision of the U.S. Supreme Court. The debacle provoked renewed calls for abolition of the Electoral College (E.C.). However, defenders of that institution countered that the E.C. system, by confining disputed results to just one state (or a small set of states), renders the problems of disputed outcomes much less severe than it would be if a national direct vote resulted in an extremely close vote, thus touching off a nightmarish Florida-style recount nationwide.

While conceding that a national recount would be worse than one confined to a single state, I conjecture that the E.C. structure makes the *probability* of such a dispute substantially higher than it would be with a national direct vote. As I am a political scientist and not at all a mathematician, I pose the following problem to my mathematical colleagues: Is it the case that

$$\mathrm{Prob}[(V_{1,N} - V_{2,N}) < T_N] \quad \ll \quad \begin{array}{l}\mathrm{Prob}[\{s_j\} : \{s_j\} \text{ is critical to a winning} \\ \text{candidate's Electoral College victory and} \\ (V_{1,j} - V_{2,j}) < T_j \text{ for all } s_j],\end{array}$$

where $V_{1,N}$ and $V_{2,N}$ are the popular vote totals nationwide of the leading candidate and the runner-up; $\{s_j\}$ is a set of states with one or more members; $V_{1,j}$ and $V_{2,j}$ are the candidates popular vote totals in state j; and T_N and T_j are vote margins that would "trigger" (either in a mandatory or in a permissive sense) recounts nationally and in state j, respectively.

Currently, 28 states specify recount triggers — 15 for mandatory (automatic) recounts and 13 for margins below which candidates are permitted to request recounts. The U.S. has never held a nationwide direct election, so T_N must be set somewhat arbitrarily. The analysis might be carried out with several possible values (e.g., 10K, 50K, 100K, and 250K). A key premise, however, is that although recount triggers as absolute numbers may grow with the size of the electorate, as a percentage of the total vote, they decrease. That principal already exists in the laws of 10 states, which specify that recounts in statewide elections require a margin that is a lower percentage of the vote than recounts in smaller election districts. In addition, when comparing across states, numerical triggers are only modestly associated with state populations, and they rise at a decreasing rate. There are, however, two exceptions, both extreme outliers with remarkably liberal triggers for recount requests (Texas and Illinois). Besides examining legal requirements, I am also attempting to find out the vote margins in elections where recounts were actually held, especially in the larger states and particularly in Texas and Illinois.

In addition to seeking the help of a mathematician or statistician for a comparison of the two probabilities based on a priori assumptions, I also plan a historical analysis based on actual nationwide and state-by-state popular vote margins.

Postscript: I am very pleased to report that the Workshop will result in just the sort of collaborative effort for which I hoped. Another participant, Professor Vincent Merlin of the University of Caen, has already conducted closely related analyses concerning the likelihood of "wrong-winner" elections (also known as the majority paradox or the referendum paradox) under district-based election structures like the Electoral College. He has taken an interest in the problem I pose, and we plan to work together on it in June 2004, during a visit he has scheduled to the U.S.

Probability Models for the Analysis of Voting Rules in a Federal Union
Vincent Merlin

In an election between two parties (A and B, Left and Right, Yes and No) it might be that a party wins in a majority of districts (or states, constituencies, etc...) while it gets less votes than its opponent in the whole country. In Social Choice Theory, this situation is known as the compound majority paradox, or the referendum paradox. Although occurrences of such paradoxical results have been observed worldwide in political elections (e.g. United States, United Kingdom, France), no study evaluates theoretically the likelihood of such situations. We propose three probability models in order to tackle this issue. The first two models have been used for a long time in social choice theory to compute the theoretical likelihood of discrepancies among voting rules in the three candidate case. The Impartial Culture (IC) assumption states that each voter picks randomly and independently his party affiliation with probability one half. The Impartial Anonymous Culture (IAC) assume that in each district every result is equally likely: Party A has the same probability to get 45%, 65% or 100% of the vote in a given constituency. However, if the number of districts is large enough, the distribution of the votes in favor of A in the country will follow a normal distribution centered around the point 50%. Thus, both IC and IAC models assume that the competition between A and B is close in the whole country. The third model introduce a bias (or shift) in favor of a party. The Biased and Rescaled Impartial Anonymous Culture (BRIAC) assumes that the percentage of votes for A in a given district is drawn uniformly on the interval $[\frac{1}{2} - D + E, \frac{1}{2} + D + E]$. The bias in favor of A is measured by E and the dispersion by D. In fact, the only key parameter of this model is $p = E/D$. The value $E = 0$ gives back the IAC model, up to scaling factor.

For the case where each district has the same (large) population and the IC model, our results prove that the likelihood of this paradox is 16.2% in the three-district case and computer simulations show that it rapidly tends to 20,5% when the number of districts increases. The same pattern is observed under the IAC assumption: The probability of the paradox is 12.5 % with three districts and we estimates that it tends to 16.5% as the number of districts increases. This probability decreases with the number of states when a candidate receives significatively more vote than his opponent over the whole country (parameter p of the BRIAC model). However, p needs to be larger than 0.1 (e.g. 1% bias for $\pm$ 10% dispersion) to get a significative result.

In the case of unequal population state, a new question arises : what is the apportionment method which minimizes the probability of the paradox under a given probability model? Let $m = (m_1, \dots, m_i, \dots m_N)$ be the vector of the distribution of the population on the N districts, and a_i be the number of mandates for district i. More precisely, we assume that $a_i = m_i^\beta$ for the IC case and $a_i = m_i^\alpha$ for the IAC case. We then run several computer simulations for different values of the vector m in order to find the optimal values of α and β. In each case, we find out that the minimal value for the paradox has been obtained around $\alpha = 1$ for the IAC model, and around $\beta = 0.5$ for the IC model. This last result with the IC case is consistent with some previous results by Felsenthal and Machover on the value of β that minimizes the mean majority deficit. Computers simulations also tend to show that the probability of the paradox slightly increases as the inequalities among the states in term of population increase.

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Foundations of Behavioral Social Choice Research

Michel Regenwetter

This presentation consists of two parts. In the first part, I provide an overview of a forthcoming book with the same title, co-authored with Bernard Grofman (University of California at Irvine), A.A.J. Marley (University of Victoria) and Ilia Tsetlin (INSEAD). This book is a synergetic summary of several underlying journal articles [1, 2, 3, 4, 5, 6, 8, 9, 7]. We provide a mathematical modeling and statistical inference framework

that is tailored towards developing descriptive (as opposed to normative) theories of social choice behavior and towards testing them against empirical data. We believe that our work provides a first systematic attempt towards a formal behavioral theory of social choice behavior, in the spirit of behavioral economics and of behavioral decision theory (a la Kahneman and Tversky). Our empirical work on majority rule decision making demonstrates that some influential strands of theoretical research (the impartial culture assumption, domain restriction conditions such as Sen's value restriction and Black's single peakedness) are descriptively invalid. I now highlight our six most important contributions.

(1) We argue for the limited theoretical relevance and demonstrate the lack of empirical evidence for cycles in mass electorates by replacing "value restriction" and similar classic domain restriction conditions, as well as the "impartial culture" assumption, with more realistic assumptions about preference distributions. We show that our behaviorally plausible conditions, which we validate on empirical data, predict that majority rule decision making is extremely unlikely to generate cycles (among sincere preferences) for realistic distributions in mass electorates. A major implication is that majority rule provides a 'solution' (in practice) to Arrow's impossibility theorem.
(2) In order to better integrate social choice research with the other decision sciences, we expand the classical domains of permissible preference states by allowing for more general binary preference relations than linear or weak orders and by considering probabilistic representations of preference and utility, including a broad range of random utility models.
(3) We develop methodologies to (re)construct preference distributions from incomplete data, i.e., data which do not provide either complete rankings or complete sets of pairwise comparisons.
(4) We highlight the dependence of social choice results on assumed models of preference or utility.
(5) We develop a statistical sampling and Bayesian inference framework that usually places tight upper and lower bounds on the probability of any majority preference relation (cycle or not). We also discuss how such statistical considerations of social choice processes dramatically alter the focus of what are important research questions: For instance, finding the correct winner is often more important than worrying about cycles. Statistical and empirical considerations can also reverse some famous policy implications: For instance, high turnout, not low turnout, as often argued, is desirable when using majority rule.
(6) We demonstrate that in situations where sampling may be involved, misrepresentation (i.e., erroneous evaluations) of the majority preferences is a far greater (and much more probable) threat to democratic decision making than majority cycles.

In the second part of the presentation, I give an overview of results from a systematic analysis of American Psychological Association election data under the single transferable vote (STV), for four elections, each with five candidates and nearly 20,000 voters. This is collaborative work with graduate students Arthur

Kantor (University of Illinois at Urbana-Champaign) and Aeri Kim (University of Illinois at Urbana-Champaign). A full report on this work will be submitted for publication in a major research journal. STV is a particularly interesting paradigm for behavioral social choice research because the ballots provide partial or full preference rankings from the voters. To summarize our main findings: We use several methods to infer majority, Borda, plurality, STV, and other social welfare orders from the ballot data. We can report with high statistical confidence that there were no majority cycles, that the social welfare orders under majority rule and Borda were essentially identical, and that STV generates outcomes that are consistent with both of these classical criteria. Our findings are robust across multiple methods of data analysis. We also discuss the fact that some real world STV elections are tallied in a probabilistic fashion and we compare the probabilistic tally to the deterministic 'genuine STV' tally as defined by the *British Electoral Reform Society.*

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Matchings and Allocations
Michel Balinski
(joint work with Mourad Baïou)

There are two distinct finite sets of agents, the *row-agents* I ("employees") and the *column-agents* J ("employers"). Each agent has a strict preference order over the agents of the opposite set. The preferences are collectively called Γ. Each employee $i \in I$ has $s(i)$ units of work to offer, each employer $j \in J$ seeks to obtain $d(j)$ units of work, and $\pi(i,j) \geq 0$ is the maximum number of units that $i \in I$ may contract with $j \in J$. Accordingly, a *stable allocation problem* [3] is specified by a quadruple (Γ, s, d, π) where Γ is a set of preferences, $s > 0$ a vector of $|I|$ reals, $d > 0$ a vector of $|J|$ reals, and $\pi \geq 0$ an $|I|$ by $|J|$ matrix of reals.

Notation. $i' >_j i$ means that agent $j \in J$ prefers i' to i in I, and similarly, $j' >_i j$ means that agent $i \in I$ prefers j' to j in J. If either $i \in I$ or $j \in J$ refuses to work with the other, then $\pi(i,j) = 0$. The set $(i, j^>) \overset{\text{def}}{=} \{(i,l) : l >_i j\}$ identifies all agents $l \in J$ that are strictly preferred by row-agent i to column-agent j; and $(i, j^\geq) \overset{\text{def}}{=} \{(i,l) : l \geq_i j\}$ all that are strictly preferred as well as j itself. The sets $(i^>, j)$ and $(i^\geq, j)$ are defined similarly. In general, if S is a set, $(r, S) \overset{\text{def}}{=} \{(r,s) : s \in S\}$, and similarly for (S, r); moreover, if $y(s), s \in S$, is a real number, then $y(S) \overset{\text{def}}{=} \sum_{s\in S} y(s)$.

An *allocation* $x = \big(x(i,j)\big)$ of a problem (Γ, s, d, π) is a set of real-valued numbers satisfying

$$x(i,J) \leq s(i), \text{ all } i \in I,$$
$$x(I,j) \leq d(j), \text{ all } j \in J,$$
$$0 \leq x(i,j) \leq \pi(i,j), \text{ all } (i,j) \in \Gamma,$$

called, respectively, the *row*, the *column* and the *entry* constraints. It may be assumed that $\pi(i,j) \leq \min\big\{s(i), d(j)\big\}$. An allocation x is *stable* if for every $(i,j) \in \Gamma$,

$$x(i,j) < \pi(i,j) \quad \text{implies} \quad x(i,j^\geq) = s(i) \quad \text{or} \quad x(i^\geq, j) = d(j).$$

The *recruitment* or *university admissions problem* is an allocation problem where the $\pi(i,j) = 0$ or 1, the $s(i)$ are positive integers, and the $d(j) = 1$; and the *stable marriage problem* is a recruitment problem where in addition the $s(i) = 1$. There is an extensive literature on these problems (see in particular the book [4]).

In general the set of stable allocations form a nonempty distributive lattice, and the cardinality of the set may be exponential. However, generically, if the reals s, d, and π are chosen at random, then there exists exactly one stable allocation.

In the presence of many stable solutions it is of interest to determine a specific rule for choosing one. A rule is *I-monotonic* if when some agent $i \in I$ goes up in the rankings of one or several of the agents J then i may only receive a better allocation. A rule is *I-strategy-proof* if no subset of agents of I can alter their preferences (that is, falsify them) and thereby obtain better allocations for themselves. Exactly one and the same rule is characterized by either of these two properties [2]. These characterizations have practical applications to recruitment and admissions problems (for an expository account see [1]).

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Analysis of QM Rules in the Draft Constitution for Europe Proposed by the European Convention, 2003

Moshé Machover

We analyse and evaluate the qualified majority (QM) decision rules for the Council of Ministers of the EU that are included in the Draft Constitution for Europe proposed by the European Convention. We use a method similar to the one we used in our paper on the Nice Treaty (Felsenthal and Machover 2001). However, we put a special stress on the power of a voter — in this case a minister representing a Member State on the CM — to block a proposed bill (Colemans "power to prevent action").

We make a detailed comparison between the decision rule proposed by the Draft Constitution and that included in the Nice Treaty. We show that the former is much less equitable than the latter. On the other hand, the former achieves a radical — perhaps too radical — increase in effectiveness by means of a great (but uneven) reduction in blocking powers.

The criteria we use in our evaluation are grouped under two main headings: *democratic legitimacy* and *effectiveness*.

Democratic legitimacy. Here we view the CM as the upper tier of a composite two-tier decision-making system. Assuming that each minister votes at the CM according to the majority opinion in his/her country, the indirect voters of this composite system are the citizens of the EU, acting via their respective ministers. We use two main criteria for assessing democratic legitimacy.

First, *equitability.* (Slogan: *One Person, One Vote!*) According to Penroses Square-Root Rule (PSQRR), all EU citizens have equal (indirect) voting power iff the voting powers of the Member States at the CM are proportional to the square root of the size their respective electorates.[3]

[3]For a proof, see Felsenthal and Machover (1998, pp. 667). Throughout, by *voting power* we mean voting power as quantified by the Penrose measure, aka "the absolute Banzhaf index". We take population size as proxy for the size of the electorate.

We measure global deviation of a given QM rule from PSQRR using the index of distortion D — variously attributed to Loosemore and Hanby (1971) or to Duncan and Duncan (1955) — between the distribution of the relative voting powers (measured by the normalized Banzhaf index) under the given rule, and the distribution prescribed by PSQRR. Individual deviations are measured by $d := q - 1$, where q is the ratio obtained by dividing the relative voting power of a given Member State under a given rule by the relative power it ought to have under PSQRR. In the table below, D as well as $\max |d|$ (the maximal value of d) and $ran(d)$ (the range of d, i.e., the difference between the greatest and smallest values of d) are given in percentage terms.

Second, *majoritarianism.* (Slogan: *Majority Rule!*) In any non-trivial two-tier system such as the one under consideration, it is possible that the decision at the upper tier may go against the majority of the indirect voters at large. When this happens, the margin by which the majority of citizens opposing the decision exceeds the minority supporting it is the *majority deficit* of the decision. (If the decision is not opposed by a majority of the citizens, the majority deficit is 0.) Assuming random voting (independent flipping of true coins), the majority deficit is a non-negative random variable. Its expected (mean) value Δ — the *mean majority deficit* — is a measure of the deviation of the given QM rule from majoritarianism.

A third putative criterion of democratic legitimacy — maximization of the sum of the citizens voting powers (slogan: *Power to the People!*) — turns out to be redundant. This is because this sum, denoted by Σ, satisfies the identity $\Sigma = \Sigma_{\max} 2\Delta$, where $\Sigma_{\max}$ is the maximal value of Σ, obtained under majority rule EU-wide direct referendum.[4] In fact, our calculation of Δ uses this very identity.

Effectiveness. Here we view the CM as a decision-making body in its own right, ignoring its role in the two-tier system. The main measure of effectiveness (or compliance) of a decision rule is Colemans index A ("ability of the collectivity to act").[5] A is the a priori probability that a bill will be approved (rather than blocked) by the CM. It is given by $A := \omega/2^n$, where ω is the number of divisions whose outcome is positive (= the number of so-called winning coalitions) and n is the number of voters — in our case, Member States. Equivalently, but more suggestively, we measure resistance to approving a bill in terms of a priori betting odds against a bill being approved.

Results. Some of our results are summarized in the following table.

As we can see, C_{27} is much less equitable than N_{27} and even than the present rule. The present rule has two egregious individual deviations: Germany with 20.1% less than its equitable share of voting power, and Luxembourg with 124.1% more than it "deserves; but the overall deviation from equitability, as measured by D, is much worse in the case of C_{27}. The latter over-endows the four largest and six smallest Member States and under-endows all the rest. The two most egregious cases are Malta (118.2% too much) and Greece (20.8% too little). On the other

[4]For a proof, see Felsenthal and Machover (1998, pp. 6061).

[5]Coleman (1971).

Rule	D	max $\lvert d\rvert$	$ran(d)$	Δ	A	Odds
Present	5.1903	124.1	144.2	5519	0.078	12:1
N_{27}	4.8227	77.6	99.7	7937	0.020	49:1
C_{27}	8.7090	118.2	139.0	3761	0.219	7:2
Rule B	0.2490	1.2	2.1	3882	0.198	4:1

Table 1. In this table, "Present" denotes the current QM rule, for the present 15-member CM. N_{27} is the QM rule prescribed in the Nice Treaty (signed 26 February 2001) for a 27-member CM (the existing 15 members plus the ten scheduled to join the EU in May 2004, plus Romania and Bulgaria). C_{27} is the QM rule (for the same 27 members) included in the Draft Constitution for Europe proposed by the European Convention. Rule B is a benchmark rule which we regard as optimal: it is a weighted rule in which weights are proportional to the square root of **population** sizes and the quota is 60% of the total weight. For the meaning of the column headings, see text above. The odds given in the last column are approximate.

hand, the Nice rule has a dangerously high resistance to passing a bill: the a priori odds against it are 49:1 as compared with the present 12:1 (and 9:1 in the previous period, before the 1995 enlargement). This threatens to paralyse the CM. The proposed rule C_{27} goes in the opposite direction and reduces the resistance dramatically. In our view, it goes a bit too far, as there are good arguments for privileging the status quo to some extent against proposed changes. The detailed figures (not reproduced here) show that C_{27} achieves this greater compliance at the cost of considerable reduction in the blocking powers of the Member States as compared with the Nice rule. Moreover this reduction is very uneven: the four largest and six smallest Member States stand to lose relatively little blocking power, while the others lose a substantial amount. The most egregious cases are Germany (14.5% loss) and Portugal and Belgium (62.4% loss).

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The Treaty of Nice and the Council of Ministers: A Mathematical Analysis of the Distribution of Power

Werner Kirsch

The treaty of Nice (EU-summit, December 2000) contains a complicated three-step-procedure for decision making in the Council of Ministers of the EU. For a proposal to pass, the first step requires a majority of countries (8 out of 15 states, resp. 13 out of 25). The second step consists of a weighted voting with qualified majority rule. This means that the states are assigned a certain number of votes (voting weights), which were the result of negotiations on the summit. For example, the four biggest states (Germany, France, United Kingdom and Italy) got 29 votes each, Spain and Poland, the next biggest countries, 27 votes each. To approve a proposal, more than (about) 70% of the votes are required, the exact threshold depending on the current number of members of the Union. While these weights are monotone in the countries population, they are pretty arbitrary otherwise, as Germany has more than 82 million citizens, France slightly less than 60 million and Poland about 38 million. Mainly to appease Germany, the Nice treaty contains a third voting step for the Council in which each state has a number of votes proportional to its population. In this step, a support of 62% of the population (as represented by their governments) is required. With its three steps, a far from transparent way to assign voting weights to the member states and strange looking thresholds, the Nice procedure is certainly one of the most complicated voting systems in history. For example, without a careful analysis, it is not at all clear to which extent the third step ("population voting") affects the power of the members in the Council. It is, however, easy to see that the first step of the voting is completely redundant as a qualified majority in the second step can only be achieved with a majority of countries. We use the *Banzhaf index* to quantify the power distribution in the Council after Nice. If we neglect the "population voting", the four big states obviously have the same power in the council. It turns out that the third step has virtually no effect on the power of the big states relative to each other. For example, the Banzhaf index of Germany for the three-step-voting is only by 10^{-7} bigger than the one of France, although the population of Germany is by more than one third bigger than the one of France. In other words: on average Germany will take advantage of its bigger population in one out of ten million votes in the Council! The voting system for the Council as provided by the Nice treaty seems to be very complicated and based on "smoky backroom negotiations" rather than on rational criteria. Moreover, with a 25 or 27 member Union the threshold is so high, that an effective work of the Council seems impossible.

Assignments of Seats as a Modelling Example in the Classroom of Upper Secondary Schools

Thomas Jahnke

In schools and even — to a certain extent — in universities, mathematics is often taught in a bureaucratic manner: getting and exercising and remembering standard procedures and algorithms. Teaching and learning mathematics should be based on principles like

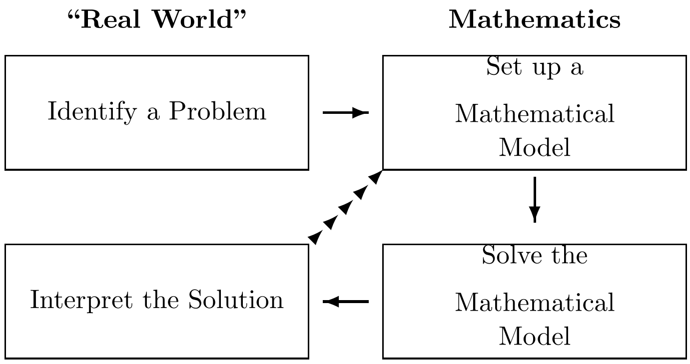
- posing questions instead of answering them,
- active investigations and
- exploring and discovering.

Teaching and learning modelling could enrich the usual syllabus by its contents as well as its methods. While application could be seen as looking from mathematics to the "real world," science and technology modelling emphasises the other direction: looking from the real world towards mathematics in order to solve problems by the use of mathematical knowledge and methods. We cannot define modelling but we can characterise this activity by some essential points [2]:

- Mathematical modelling consists of applying your mathematical skills to obtain useful answers to real problems.
- Learning to apply mathematical skills is very different from learning mathematics itself.
- Models are used in a very wide range of applications, some of which initially do not appear to be mathematical by nature.
- Models often allow quick and cheap evaluations of alternatives, leading to optimal solutions which are not otherwise obvious.
- There are no precise rules in mathematical modelling and no 'correct'answers.
- Modelling can be learned only by doing.

While applications of mathematics often turn out to be a very straightforward approach from a problem to its unique solution, mathematical modelling is done step by step in a specific circle.

The Process or Circle of Modelling



Not all modelling examples are suitable for a classroom. A good modelling example should be

- relevant (nor only for mathematicians),
- realistic,
- motivating (not only for mathematicians),
- rich (in its mathematical aspects),
- allowing different approaches,
- enlightening,
- accessible,
- not too open and too closed and
- mathematically dense.

The problem of the assignment of parliament seats after elections in a representative democracy satisfies these demands. In the German constitution, there are no special rules or procedures for the assignments of seats stated. This is provided in special election laws. The following methods are used:

- *Hare/Niemeyer* (i.e. Hamilton) in elections for the Bundestag and the elections for the Landtag in Bayern, Berlin, Brandenburg, Bremen, Hamburg, Hessen, Mecklenburg-Vorpommern, Nordrhein-Westfalen, Rheinland-Pfalz, Sachsen-Anhalt and Thüringen;
- *D'Hondt* (i.e. Jefferson) in elections for the Landtag in Baden-Wrtemberg, Niedersachsen, Saarland, Sachsen and Schleswig-Holstein;
- *Sainte-Laguë* (i.e. Webster) in elections for the Landtag in Bremen and to choose the heads and the members for the differently sized committees of the Bundestag.

In the classroom, we start by providing the students with the results of a Bundestag election and the texts of the election laws. Later we give them data to discover the Alabama paradox and to construct the majority paradox. The discussion of the concept of the *success value* of a vote leads from the Hare/Niemeyer method to the DHondt method, whose principle is to make the seats as expensive as possible. On the other hand, DHondt violates the upper quota condition and is biased in favour of the bigger parties.

So far, the students have worked on elections laws, procedures, paradox results and principles. The floor is now open to see the assignment problem in a more general way:

> Given votes $v_1 + v_2 + v_3 + \ldots + v_n = v$ and seats $s_1 + s_2 + s_3 + \ldots + s_n = s$, make the set $(v_1; v_2; v_3; \ldots; v_n)$ as similar as possible to the set $(s_1; s_2; s_3; \ldots; s_n)$.

Now, the students are prepared and able to set up their own research program:

(1) Which principles are realising justice the best?
(2) Test the principles and discuss their consequences.
(3) Find for every principle one or several algorithms to realise it.
(4) Construct a procedure fulfilling positive or negative conditions.
(5) Find visualisations for the different procedures.

Beside more formal questions like

> assign the seats to the parties in a way that
> - $\min\{v_i/s_i\}$ is maximal,
> - $|v_i/s_i - v/s|$ is minimal,
> - $|v_i/s_i - v_j/s_j|$ is minimal,

the students discuss the whole modelling process and are asked to write a final report presenting their results. They learn modelling and get deep insights about the problem as well as mathematics itself.

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A Weighted Voronoi Diagram Approach to Political Districting

Bruno Simeone
(joint work with Isabella Lari and Frederica Ricca)

Soon after modern democracies were established, gerrymandering practices, consisting of partisan manipulation of electoral district boundaries, began to occur in several states and countries. In order to oppose such practices, researchers started thinking of automatic procedures for political districting, designed so as to be as neutral as possible. Commonly adopted criteria are:

- *Integrity:* The territory to be subdivided into districts consists of territorial units (wards, townships, counties, etc.) and each unit cannot be split between two or more districts.
- *Contiguity:* The units of each district should be geographically contiguous, that is, one can walk from any point in the district to any other point of it without ever leaving the district.

- *Population equality* (or *population balance*): Under the assumption that the electoral system is majoritarian with single-member districts, all districts should have roughly the same population (*one man – one vote* principle).
- *Compactness:* Each district should be compact, that is, "closely and neatly packed together" (Oxford Dictionary). Thus, a round-shaped district is deemed to be acceptable, while an octopus- or an eel-like one is not.

A broad survey of political districting algorithms is given in (Grilli di Cortona et al., 1999). Later work focuses on local search (e.g., Ricca and Simeone, 2000; Bozkaya, Erkut, and Laporte, 2003). It is also worth mentioning the branch-and-price approach in (Mehrotra, Johnson, and Nemhauser, 1998). Here we propose a novel approach based on weighted Voronoi regions (or diagrams). This notion is not new in the literature, especially in the area of computational geometry (see, e.g., Aurenhammer and Edelsbrunner, 1984). What we believe to be new, besides the specific application to political districting, is our iterative updating of node weights to achieve population balance.

The input to our procedure is the following:

- a *contiguity graph* $G = (V, E)$, whose nodes represent the territorial units and there is an edge between two nodes if the two corresponding units are neighboring;
- a positive integer r, the number of districts;
- a subset $S \subset V$ of r nodes, called *centers* (all remaining nodes will be called *sites*);
- positive integral node weights $p_i, i \in V$, representing territorial unit *populations*;
- positive real distances $d_{i,s}$ for all sites i and all centers s.

We denote by $\bar{P}$ the mean district population (= (total population)/r).

The integrity criterion dictates that a district must be a subset of nodes; according to the contiguity criterion, such a subset must be connected.

A *district map* is a partition of V into r connected subsets (the districts), each containing exactly one center. Given any district map, we denote by D_s the unique district containing center s. We look for a district map such that, informally speaking, the district population imbalance is small and the districts are compact enough.

If one takes as districts the ordinary Voronoi regions w.r.t. the distances $d_{i,s}$, a good compactness is usually achieved, but a poor population balance might ensue. In order to re-balance district populations, one would like to promote site migration out of "heavier" districts (populationwise) and into lighter ones. Then the basic idea is to consider weighted distances $d'_{i,s} = w_s \cdot d_{i,s}$, where each weight w_s is proportional to P_s, the population of district D_s; and to perform a Voronoi iteration w.r.t. the biased distances $d'_{i,s}$. Do this iteratively: at iteration k, $k = 1, 2, \ldots$, two different recursions may be taken into consideration, namely, a *static* one,

$$d^k_{i,s} = \frac{P^{k-1}_s}{\bar{P}} d^0_{i,s} \quad , i \in V \setminus S, s \in S$$

and a *dynamic* one,

$$d^k_{i,s} \;=\; \frac{P^{k-1}_s}{\bar{P}} d^{k-1}_{i,s} \quad , i \in V \setminus S, s \in S \; ,$$

where, in both cases, $d^0_{i,s} = d_{i,s}$, P^0_s is the population of the (ordinary) Voronoi region containing center s, and P^k_s is the population of D_s after iteration $k = 1, 2, \ldots$; stop as soon as the districts become stable.

The above sketched algorithm will be called a *full transfer* one. One may also consider a *single transfer* version of it, by letting sites migrate from one district to another one at a time. Here too, one may adopt either the static or the dynamic recursion defined above. So one gets altogether four variants of the weighted Voronoi algorithm (static/dynamic recursion; full/single transfer).

One possible implementation of the single transfer algorithm is the following: at iteration k, site i is a candidate for migrating from D_q to D_t if:

(1) $P^{k-1}_t = \min\{P^{k-1}_s \,:\, s = 1, \ldots, r\}$
(2) $d^k_{i,t} = \min\{d^k_{j,t} \,:\, j \notin D_t\}$
(3) $d^k_{i,t} < d^k_{i,q}$
(4) $P^k_t < P^{k-1}_t$

The algorithm stops when the set of candidates is empty.

Next, we define four desirable properties to be met by weighted Voronoi algoritms — or at least by some variants of them.

(i) *Order invariance:* $d^k_{i,s} < d^k_{j,s} \iff d_{i,s} < d_{j,s}, \;\; s \in S; i, j \in V \setminus S.$
(ii) *Re-balancing:* At iteration $k = 1, 2, \ldots$, site i migrates from D_q to D_t only if $P^{k-1}_q > P^{k-1}_t$.

Definition 1. Given a graph G and any two nodes i, j of G, a *geodesic* between i and j is any shortest path between i and j in G when all edge-lengths are equal to 1. The *geometric distance* between i and j in G is the number of edges in any geodesic between i and j.

(iii) *Geodesic consistency:* At any iteration, if node j belongs to district D_s and node i lies on any geodesic between j and s, then i also belongs to D_s.

One can show that geodesic consistency implies contiguity, but the converse does not necessarily hold. Moreover, geodesic consistency holds when the input distances are geometric distances on the contiguity graph G. On the other hand, some counterexamples show that for arbitrary input distances contiguity might not hold.

(iv) *Finite termination.*

Finite termination in general is not guaranteed, as shown by counterexamples. However, one can prove that the single transfer dynamic weighted Voronoi algorithm, under conditions (1) – (4) above, enjoys finite termination.

The following table shows the results we have obtained so far.

Property	Static		Dynamic	
	Full transfer	**Single transfer**	**Full transfer**	**Single transfer**
Order invariance	√	√	√	√
Re-balancing			√	√
Geodesic consistency				√[1]
Finite termination				√[2]

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Voting in Social Choice Theory

Maurice Salles

My purpose is to outline the difference between aggregation functions

directly based on individual preferences and those based on voting games. In doing this, the role of individual indifferences is shown to be crucial. The most studied practical aggregation function is majority rule. An option x is socially preferred to an option y if the number of individuals who prefer x to y is greater than the number of individuals who prefer y to x. In this case, when individual preferences are conveniently restricted, we know since

Duncan Black that the social preference is transitive. Taking as a basis a voting game where "powerful" coalitions are a priori defined, Dummett and Farquharson have demonstrated that Black's type of conditions could be extended. When there are no restrictions on individual preferences (supposed to be complete preorders on a finite set of options), Nakamura provided an existence theorem for the core, given a list of individual preferences, based on a

comparison between the number of options and a number given by the structure of the voting game. Unfortunately this comparison is very restrictive and accordingly other solution concepts have been studied. The stability set due to Rubinstein is never empty when individual preferences are linear orders. However,

[1]under the assumption that distances are geometric ones in G

[2]under conditions (1)–(4) above

this property does not hold when individual preferences are complete preorders. Le Breton and Salles then obtained results using the same number as the

number used by Nakamura. Some other solution concepts are based on binary relations that are transitive so that obtaining maximal elements is not a problem when the set of options is finite. The interesting mathematical aspects in this case are due to the consideration of sets of options having a geometrical or topological structure (the problems we are then facing are related to the absence of continuity properties of the relevant social preference

relation). When the space of options is a compact part of the Euclidean space, the core is non-empty when the dimension of the Euclidean space is conveniently restricted (this was shown by Greenberg, and extended by Saari).

Dividing the Indivisible: Procedures for Allocating Cabinet Ministries to Political Parties in a Parliamentary System

Steven J. Brams
(joint work with Todd R. Kaplan)

How coalition governments in parliamentary democracies form and allocate cabinet ministries to political parties is the subject of a substantial empirical and theoretical literature. By and large, a rule of proportionality, whereby parties are given more ministries or more prestigious ministries (e.g., finance, foreign affairs, or defense) in proportion to their size, is followed. However, small centrist parties that are pivotal in coalitions (e.g., the Free Democrats in Germany) have successfully bargained for larger-than-proportional allocations.

This task is complicated when less-than-compatible parties, like the Christian Democrats and the Greens, join the same coalition. While fiscal conservatism and protecting the environment are often at odds, these parties may still be accommodated if, for instance, the Christian Democrats are given the finance ministry and the Greens the environmental-protection ministry, and each has major influence over policies in its area.

To facilitate the allocation of cabinet ministries to political parties, we propose procedures that take into account both party interests and party size. This mechanism shifts the burden of making cabinet choices from the prime minister designate, or *formateur*, who is usually the leader of the largest party in a coalition government, to party leaders that join the government. Thereby these procedures give party leaders primary responsibility for the make-up of the coalition government.

We assess the fairness of this procedure, based on different criteria of fairness. Our analysis is inspired by an apportionment method used in Northern Ireland in 1999 to determine the *sequence* in which parties made ministry choices (it has also been used in Danish cities and the German Bundestag). This method works such that the largest party in a coalition gets first choice; presumably, it would choose the position of prime minister. After that, the apportionment method determines the order of choice.

For example, suppose there are three parties, ordered by size $A > B > C$, and there are six ministries to be allocated. If the sequence is $ABACBA$, A will receive three ministries, B two ministries, and C one ministry. But beyond these numbers, the sequence says that A is entitled to a second choice before C gets a first choice, and C gets a first choice before B gets a second choice.

If parties have complete information about each others preferences, we show that it may not be rational for them to choose *sincerely* — that is, to select their most-preferred ministry from those not yet chosen. Rather, a party (e.g., A) may do better postponing a sincere choice and, instead, selecting a less-preferred ministry if (i) that ministry might be the next choice of a party that follows it in the sequence (e.g., B or C) and (ii) As sincere choice is not in danger of being selected by B or C before As turn comes up again. Such *sophisticated choices*, which take into account what other parties desire, can lead to very different allocations from sincere ones.

If there are only two parties, sophisticated choices and sincere choices both yield Pareto-optimal allocations: No parties, by trading ministries, can do better, based on their ordinal rankings of ministries. However, this is not true if there are three or more parties that make sophisticated choices, which was first demonstrated for sequential choices made in professional sports.

What we show here for the first time is the problem of *nonmonotonicity*: A political party may do worse by choosing earlier in a sequence, independent of the Pareto-optimality of the sophisticated choices. Hence, the apparent advantage that a partys size gives it by placing it early in a sequence can, paradoxically, work to its disadvantage — it may actually get more preferred choices by going later.

Like Pareto-nonoptimal allocations, nonmonotonicity *cannot* occur if parties are sincere. Thus, we are led to ask how sincere choices might be recoveredor induced in the first place if the parties know they "cannot get away with" insincere choices. While there is an allocation mechanism that makes sincerity optimal for two parties, there are difficulties in extending it to more than two parties.

By putting the choice of ministries in the hands of party leaders, these leaders are made responsible for their actions. Ultimately, we believe, party leaders will be more satisfied making their own choices rather than having to bargain for them. Moreover, this greater satisfaction should translate into more stable coalition governments, which is a subject that has been extensively studied by a many scholars.

The allocation procedures we analyze could go a long way toward minimizing the horse trading that typically ensues when a *formateur* bargains with party leaders over the ministries they will be offered. By cutting down on the rents extracted in the bargaining process, a coalition government is likely to form more expeditiously and be less costly to maintain.

This is not to say that the procedures we discuss solve all problems. Because ministries are indivisible, there will not generally be a perfect match of the claims of each party and its allocation. Furthermore, there are certain problems that are ineradicable, whatever allocation procedure is used. For example, it may not be possible to eliminate envy among equally entitled parties. Nevertheless, we believe the procedures that we discuss offer a promising start to attenuating conflicts that have plagued the formation of coalition governments and, not infrequently, led to their downfall.

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