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PREDICTABLE CRITERIA FOR ABSOLUTE CONTINUITY AND SINGULARITY OF TWO PROBABILITY MEASURES

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Abstract. Predictable criteria for absolute continuity and singularity are established based on the multiplicative increments of the likelihood ratio process of the hypothesis relative to the alternative and not, as is usual, of the alternative relative to the hypothesis. This approach avoids any change of probability measures, disposes of an assumption on local absolute continuity, and allows for an arbitrary root of the likelihood ratio process rather than distinuguishing the square root case.

1. Introduction

The Kakutani [5] dichotomy for product measures was generalized to spaces with discrete filtrations by Kabanov, Liptser and Shiryayev [4]. Their result is stated in terms of probabilities under the alternative of an event given through conditional expectations under the hypothesis. The passage from hypothesis to alternative means a change of the underlying probability measure, and a careful treatment of what then happens to conditional expectations is essential. Here we circumvent such

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changes, the one and only probability measure under which all computations are carried out being the alternative. This approach deviates notably from the usual statistical habit of preferring the hypothesis.

Thus our exposition is based on the likelihood ratio $\bar{L}$ of the hypothesis P relative to the alternative $\bar{P}$. Theorem 1 of Section 2 characterizes absolute continuity and singularity through the $\bar{P}$-probability of the set where $\bar{L}$ is positive. In the presence of a discrete filtration, $\bar{L}$ may be represented as the product of its multiplicative increments $\nabla\bar{L}_j$. According to Lemma 1 the latter are on the average bounded by 1, as are their p-th roots. Theorem 2 of Section 3 contains our main result: Whatever the choice of $p \in (1,\infty)$, we have

$$\bar{P} \ll P \Leftrightarrow \bar{P}(\Sigma_{j\in N}\bar{E}[1 - \nabla\bar{L}_j^{1/p}|F_{j-1}] < \infty) = 1,$$

$$\bar{P} \perp P \Leftrightarrow \bar{P}(\Sigma_{j\in N}\bar{E}[1 - \nabla\bar{L}_j^{1/p}|F_{j-1}] < \infty) = 0.$$

Three corollaries show the relation with the results available in the literature all of which focus on the square root case $p = 2$. The extension to p-th roots is made feasible through inequalities given in Section 4.

2. Likelihood Ratio Processes

On a sample space Ω with sigmaalgebra F let P and $\bar{P}$ be two probability measures, called *hypothesis* and *alternative*, respectively. The Lebesgue decomposition of the hypothesis P relative to the alternative $\bar{P}$ determines P- and $\bar{P}$-uniquely an F-measurable function $\bar{L}$, called likelihood ratio, with values in $[0,\infty]$ such that

$$P(F) = \int_F \bar{L}d\bar{P} + P(F\{\bar{L} = \infty\}), \quad \text{for all } F\in F.$$

As an example illustrating the relevance of $\bar{L}$ we may characterize equality of $\bar{P}$ and P through

$$\bar{P} = P \Leftrightarrow \bar{P}(\bar{L} = 1) = 1.$$

Absolute continuity and singularity may be described similarly through the $\bar{P}$-measure of the set where the likelihood ratio remains positive. We take the following theorem as our starting point, as to Chatterji and Mandrekar [1], page 171.

THEOREM 1.

a) $\bar{P} \ll P \iff \bar{P}(\bar{L} > 0) = 1.$

b) $\bar{P} \perp P \iff \bar{P}(\bar{L} > 0) = 0.$

Proof. By passing to the complement we must study the zeroes of $\bar{L}$, i.e. whether $\bar{P}(\bar{L} = 0)$ equals 0 or 1. The Lebesgue decomposition entails $P(\bar{L} = 0) = 0$, and this yields the direct part of a) and the converse part of b). Further a P-nullset must be contained in the set $\{\bar{L} = 0\}$ $\bar{P}$-almost surely, implying $\bar{P}(F) \leq \bar{P}(\bar{L} = 0)$. This establishes the converse part of a) and the direct part of b). □

Now suppose that $(F_k)_{k \in N}$ is a discrete filtration of F, i.e. an increasing family of subsigmaalgebras starting from $F_0 = \{\emptyset, \Omega\}$ and generating F. Let P_k and $\bar{P}_k$ be the restrictions of P and $\bar{P}$ to the smaller sigmaalgebra F_k, with $\bar{L}_k$ being the corresponding likelihood ratio of P_k relative to $\bar{P}_k$. The *likelihood ratio process* $(\bar{L}_k)_{k \in N}$ then is a $\bar{P}$-supermartingale and a P-submartingale. Thus P- and $\bar{P}$-almost surely the states 0 and ∞ are absorbing and the limit $\lim_k \bar{L}_k$ exists and satisfies $\lim_k \bar{L}_k = \bar{L}$.

The criteria we aim at are expressed through the likelihood ratio process using its *multiplicative increments* $\nabla \bar{L}_j = \bar{L}_j (\bar{L}_{j-1})^-$, where $Z^- = 1/Z$ for $Z \in (0,\infty)$, $Z^- = 0$ for $Z = 0$, and $Z^- = \infty$ for $Z = \infty$. The absorption property of the states 0 and ∞ justifies the equality $\bar{L}_k = \Pi_{j \leq k} \nabla \bar{L}_j$. Multiplicative increments very much behave like conditional likelihood ratios; however, all we need is the following.

LEMMA 1. $\bar{E}[\nabla \bar{L}_j | F_{j-1}] \leq 1$, $\bar{P}$-almost surely.

Proof. Finiteness of $\bar{L}_j$ under $\bar{P}$, the definition of $\nabla \bar{L}_j$, and the supermartingale inequality $\bar{E}[\bar{L}_j | F_{j-1}] \leq \bar{L}_{j-1}$ yield, $\bar{P}$-almost surely,

$$\bar{E}[\nabla\bar{L}_j|F_{j-1}] = \bar{E}[1(\bar{L}_{j-1} = 0)\nabla\bar{L}_j + 1(\bar{L}_{j-1} > 0)\nabla\bar{L}_j|F_{j-1}]$$
$$= 1(\bar{L}_{j-1} > 0)\bar{E}[\bar{L}_j|F_{j-1}]/\bar{L}_{j-1}$$
$$\leqq 1(\bar{L}_{j-1} > 0) \leqq 1. \quad \square$$

We are now ready to turn to predictable criteria for equality, absolute continuity, and singularity.

3. Predictable Criteria

We shall consider roots of order p, for some arbitrary $p \in (1,\infty)$. The notation $\nabla\bar{L}_j^{1/p}$ is unambiguous since $(\nabla\bar{L}_j)^{1/p} = \nabla(\bar{L}_j^{1/p})$. Lemma 1 and Jensen's inequality show that the predictable process

$$\Sigma_{j\leq k}\bar{E}[1 - \nabla\bar{L}_j^{1/p}|F_{j-1}], \quad k \in N,$$

is nonnegative and increasing and hence converges in $[0,\infty)$ or diverges to ∞, $\bar{P}$-almost surely. Following Vostrikova [9] it is not difficult to establish by direct arguments that equality is characterized by

$$\bar{P} = P \iff \bar{P}(\Sigma_{j\in N}\bar{E}[1 - \nabla\bar{L}_j^{1/p}|F_{j-1}] = 0) = 1.$$

The convergence set of the series actually coincides with the set where the terminal likelihood ratio $\bar{L}$ is positive and hence characterizes absolute continuity and singularity, as we shall now show in Theorem 2.

THEOREM 2.

a) $$\bar{P} \ll P \iff \bar{P}(\Sigma_{j\in N}\bar{E}[1 - \nabla\bar{L}_j^{1/p}|F_{j-1}] < \infty) = 1.$$

b) $$\bar{P} \perp P \iff \bar{P}(\Sigma_{j\in N}\bar{E}[1 - \nabla\bar{L}_j^{1/p}|F_{j-1}] < \infty) = 0.$$

Proof. It is convenient to introduce the nonpositive decreasing process

$$\bar{A}_k = \Sigma_{j\leq k}\bar{E}[p(\nabla\bar{L}_j^{1/p} - 1)|F_{j-1}], \quad k \in N.$$

We shall prove that $\{\bar{L} > 0\} = \{\bar{A} > -\infty\}$, $\bar{P}$-almost surely. Assertions a) and b) then follow immediately from Theorem 1. The argument is broken up into the three steps of (I) truncation, using with some cut-off point

$c > 0$ the function $u_c(x) = x1(|x| \leq c) + \text{sign}(x)c1(|x| \geq c)$, (II) compensation, based on the predictable characterization of the convergence set of submartingales with bounded additive increments, and (III) transformation into the desired terms, employing appropriate inequalities.-

I. Since $\bar{P}(\bar{L} < \infty) = 1$ we have, $\bar{P}$-almost surely,

$$\{\bar{L} > 0\} = \{\Sigma \log \nabla\bar{L}_j > -\infty\} = \{\Sigma u_p(\log \nabla\bar{L}_j) > -\infty\}.$$

Because of $u_p(\log x) \leq x - 1$, Lemma 1 leads to $\bar{E}[u_p(\log \nabla\bar{L}_j)|F_{j-1}] \leq \bar{E}[\nabla\bar{L}_j - 1|F_{j-1}] \leq 0$. Thus $\bar{C}_k = \Sigma_{j\leq k} u_p(\log \nabla\bar{L}_j)$ defines a $\bar{P}$-supermartingale whose additive increments are bounded by p. Moreover $\{\bar{L} > 0\} = \{\bar{C}_\infty > -\infty\}$, $\bar{P}$-almost surely.

II. Theorem 5 of Kabanov, Liptser and Shiryayev [4], i.e. Theorem VII.5.5 of Shiryayev [8], now applies to the $\bar{P}$-submartingale $(-\bar{C}_k)_{k\in N}$. Therefore the convergence set of $(\bar{C}_k)_{k\in N}$ $\bar{P}$-almost surely coincides with the set $\{\bar{B}_\infty > -\infty\}$ where the nonpositive decreasing predictable process $(\bar{B}_k)_{k\in N}$ is given by

$$\bar{B}_k = \Sigma_{j\leq k}\bar{E}[u_p(\log \nabla\bar{L}_j) - u_p^2(\log \nabla\bar{L}_j)|F_{j-1}], \quad k \in N.$$

Thus $\{\bar{C}_\infty > -\infty\} = \{\bar{B}_\infty > -\infty\}$, $\bar{P}$-almost surely.

III. According to Lemma 4 of Section 4 we can find some constant $b > 1$ such that, for all $x \geq 0$,

$$bp(x^{1/p}-1) - (b-1)(x-1) \leq u_p(\log x) - u_p^2(\log x) \leq p(x^{1/p}-1).$$

Inserting $\bar{L}_j$ for x and taking conditional expectations, Lemma 1 entails, $\bar{P}$-almost surely, $b\bar{A}_k \leq \bar{B}_k \leq \bar{A}_k$. Hence $\{\bar{B}_\infty > -\infty\} = \{\bar{A}_\infty > -\infty\}$, $\bar{P}$-almost surely, and the proof is complete. □

<u>COROLLARY</u> 2.1.

a) $\bar{P} \ll P \iff \bar{P}(\Pi_{j\in N}\bar{E}[\nabla\bar{L}_j^{1/p}|F_{j-1}] > 0) = 1.$

b) $\bar{P} \perp P \iff \bar{P}(\Pi_{j\in N}\bar{E}[\nabla\bar{L}_j^{1/p}|F_{j-1}] > 0) = 0.$

<u>Proof.</u> For $x \in [0,1]$ we have $bp(x^{1/p} - 1) \leq u_p(\log x) \leq p(x^{1/p} - 1) \leq 0$.

Define $\nabla\bar{D}_j = \bar{E}[\nabla\bar{L}_j^{1/p}|F_{j-1}]$. Inserting $\nabla\bar{D}_j^p$ for x, we obtain $b\bar{A}_\infty \leq \Sigma u_p(\log \nabla\bar{D}_j^p) \leq \bar{A}_\infty$. Then $\bar{P}$-almost surely, $\{\bar{A} > -\infty\} = \{\Sigma u_p(\log \nabla\bar{D}_j^p) > -\infty\} = \{\Sigma \log \nabla\bar{D}_j^p > -\infty\} = \{\Pi\nabla\bar{D}_j > 0\}$. □

Introduce the processes

$$\bar{S}_k = \Pi_{j\leq k} \nabla\bar{L}_j^{1/p}(\bar{E}[\nabla\bar{L}_j^{1/p}|F_{j-1}])^-, \quad k\in N;$$

$$\bar{D}_k = \Pi_{j\leq k} \bar{E}[\nabla\bar{L}_j^{1/p}|F_{j-1}], \quad k\in N.$$

Then $(\bar{S}_k)_{k\in N}$ is a $\bar{P}$-supermartingale, and $(\bar{D}_k)_{k\in N}$ is a decreasing non-negative predictable process, cf. Lemma 11 of Liptser, Pukelsheim and Shiryayev [6]. We have the multiplicative decomposition $\bar{L}_k^{1/p} = \bar{S}_k\bar{D}_k$, and since all terms converge individually this extends to $\bar{L}^{1/p} = \bar{S}_\infty\bar{D}_\infty$. Thus when $\bar{D}$ becomes zero so does $\bar{L}$. Moreover, it follows from the proof of Corollary 2.1 that the zeroes of $\bar{L}$ and $\bar{D}_\infty$ in fact coincide $\bar{P}$-almost surely, cf. Eagleson and Gundy [2].

We single out some special cases. In the case of product measures we have $\Omega = \times\Omega_j$, $F = \otimes B_j$, $P = \otimes Q_j$, $\bar{P} = \otimes\bar{Q}_j$, and $F_k = (\otimes_{j\leq k}B_j)\times(\times_{j>k}\Omega_j)$, where Q_j and $\bar{Q}_j$ are probability measures on the measurable space (Ω_j,B_j). The marginal likelihood ratios $\bar{\ell}_j$ of Q_j relative to $\bar{Q}_j$ may attain the values 0 and ∞ independently of each other, whence their product may not be defined on all of Ω. However, we have

$$\bar{E}[\nabla\bar{L}_j^{1/p}|F_{j-1}] = 1(\bar{L}_{j-1} > 0)\bar{E}\left[\frac{\bar{L}_j^{1/p}}{\bar{L}_{j-1}^{1/p}}\,|F_{j-1}\right]$$
$$= 1(\bar{L}_{j-1} > 0)\bar{E}[\bar{\ell}_j^{1/p}].$$

Therefore we also consider the set $\{\cap\bar{L}_j > 0\}$ where the measure $\bar{P}$ does not separate from P in finite time.

<u>COROLLARY</u> 2.2. In the product case of $P = \otimes Q_j$ and $\bar{P} = \otimes\bar{Q}_j$ we have:

a) $\bar{P} \ll P \iff \bar{P}(\cap_{j\in N}\{\bar{L}_j > 0\}) = 1$ and $\Sigma_{j\in N}\bar{E}[1 - \bar{\ell}_j^{1/p}] < \infty$.

b) $\bar{P} \perp P \iff \bar{P}(\cap_{j\in N}\{\bar{L}_j > 0\}) = 0$ or $\Sigma_{j\in N}\bar{E}[1 - \bar{\ell}_j^{1/p}] = \infty$.

Proof. We obtain $\Sigma\bar{E}[1 - \bar{L}_j^{1/p} | F_{j-1}] = \Sigma 1(\bar{L}_{j-1} = 0) + \Sigma 1(\bar{L}_{j-1} > 0)\bar{E}[1 - \bar{\ell}_j^{1/p}]$. Put $F = \{\cap \bar{L}_j > 0\}$. As 0 is absorbing, the series $\Sigma 1(\bar{L}_{j-1} = 0)$ has value 0 on F, and ∞ on $\Omega - F$. Thus $\bar{P}(\Sigma\bar{E}[1 - \nabla\bar{L}_{j-1}^{1/p} | _{j-1}] < \infty) = \bar{P}(F\{\Sigma\bar{E}[1 - \bar{\ell}_j^{1/p}] < \infty\})$, whence follow a) and b). □

The property $\bar{P}(\cap_{k\in N}\bar{L}_k > 0) = 1$ is equivalent to $\bar{P}_k \ll P_k$ for all $k \in N$, i.e. $\bar{P}$ is *locally absolutely continuous* relative to P. Imposing this property, Corollary 2.2 simply turns into the dichotomy of Kakutani [5]. More generally, a change of the underlying probability measure becomes feasible and leads to Theorem 1 of Kabanov, Liptser and Shiryayev [4], as follows.

COROLLARY 2.3. In the general case of Theorem 2, when $\bar{P}$ is locally absolutely continuous relative to P, we have:

a) $\qquad \bar{P} \ll P \iff \bar{P}(\Sigma_{j\in N}E[1 - \nabla L_j^{1/p} | F_{j-1}] < \infty) = 1.$

b) $\qquad \bar{P} \perp P \iff \bar{P}(\Sigma_{j\in N}E[1 - \nabla L_j^{1/p} | F_{j-1}] < \infty) = 0.$

Proof. Lemma 7 of Kabanov, Liptser and Shiryayev [4] provides the formula $\bar{E}[\eta | F_{j-1}] = E[\eta/\nabla\bar{L}_j | F_{j-1}]$, $\bar{P}$-almost surely. In order to apply Theorem 2 we insert $\eta = \nabla\bar{L}_j^{1/p}$ and get $\eta/\nabla\bar{L}_j = \nabla\bar{L}_j^{-1/q} = \nabla L_j^{1/q}$, where $1/p + 1/q = 1$ and $\nabla L_j = 1/\nabla\bar{L}_j$ is the multiplicative increment of the likelihood ratio process $L_k = 1/\bar{L}_k$, $k \in N$, of the alternative $\bar{P}$ relative to the hypothesis P. □

The conditional expectations $\bar{E}[\nabla\bar{L}_j^{1/p} | F_{j-1}]$ may be written as conditional Hellinger integrals of order 1/p. For when relative to some probability measure μ the probability measures P and $\bar{P}$ are absolutely continuous with densities Z and $\bar{Z}$, respectively, then we have the equality, $\bar{P}$-almost surely,

$$\bar{E}[\nabla\bar{L}_j^{1/p} | F_{j-1}] = E_\mu[\nabla Z_j^{1/p}\ \nabla\bar{Z}_j^{1-1/p} | F_{j-1}].$$

Hellinger integrals are employed by Mémin and Shiryayev [7], and Jacod [3]. Those authors also study inequalities which are related to the ones to be established in the final Section 4, cf. § 2.3 and § 5.4 of Mémin and Shiryayev [7], and equation (3.4) and Theorem 4.1 of Jacod [3].

4. Some Inequalities

In Lemma 2 we shall make no appeal to any truncation.

LEMMA 2. Suppose $p > 1$ and $b > 1$. Then the function

$$g(x) = \log x - (\log x)^2 - bp(x^{1/p} - 1) + (b-1)(x-1)$$

has a zero x_0 in the open intervall $(0,1)$ if and only if $b > 3p/(p-1)$; and in this case $g(x) > 0 = g(x_0) = g(1)$, for all $x > x_0$, $x \neq 1$.

Proof. I. The second derivative $g''(x) = \{2(\log x) - 3 + b(p-1)x^{1/p}/p\}/x^2$ vanishes at the unique solution $x'' \in (0,\infty)$ of $2 \log x = 3 - b(p-1)x^{1/p}/p$. The behaviour of this equation in $x = 1$ leads to

$$x'' \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right. 1 \Longleftrightarrow b \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right. 3p/(p-1); \; g''(x) \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right. 0 \Longleftrightarrow x \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right. x'' \; .$$

II. The first derivative $g'(x) = b - 1 - \{2(\log x) - 1 + bx^{1/p}\}/x$ thus is strictly decreasing for $x \leqq x''$ and strictly increasing for $x \geqq x''$, with values $+\infty$, 0, $b-1$ at $x = 0, 1, \infty$, respectively. Since x'' is the only critical point of g', the zero $x = 1$ of g' must be paired by a second zero $x' \in (0,\infty)$ whose location $x' < x''$, $x' = x''$, or $x' > x''$ is determined through $x'' < 1$, $x'' = 1$, or $x'' > 1$, respectively. Hence g' is negative between x' and 1, vanishes in these two points, and is positive otherwise.

III. In case $x' < x'' < 1$ the function g then increases strictly from $-\infty$ through a zero $x_0 \in (0,x')$ to a positive local maximum at x', falls through the point of inflection x'' down to a local minimum at 1 where it vanishes, and strictly increases to $+\infty$. In case $x' > x'' > 1$ the behaviour of g is similar, in case $x' = x'' = 1$ we find that 1 is a saddle point of g. □

We now determine b for prescribed zero $x_0 = e^{-p}$ of g. As it turns out this works for an arbitrary cut-off point $c > 0$, rather than just $c = p$. To this end define for $p \in (1,\infty)$ and $c \in (0,\infty)$ the number

$$b_{c,p} = \frac{c + c^2 + e^{-c} - 1}{e^{-c} - 1 - p(e^{-c/p} - 1)} .$$

In the course of the following proof we show that $b_{c,p} > 3p/(p-1)$.

LEMMA 3. When $p > 1$, $c > 0$, and $b \geq b_{c,p}$ we have, for all $x \geq 0$,

$$bp(x^{1/p} - 1) - (b-1)(x-1) \leq u_c(\log x) - u_c^2(\log x).$$

Proof. I. The factor $p(x^{1/p} - 1) - (x-1)$ accompanying b is negative, and so it suffices to choose $b = b_{c,p}$. We claim that $b_{c,p} > 1$. Indeed, the denominator $N(c) = e^{-c} - 1 - p(e^{-c/p} - 1)$ strictly increases from $N(0) = 0$ to $N(\infty) = p - 1$. Hence $b_{c,p} > 1$ if and only if the difference $D(c) = c + c^2 + p(e^{-c/p} - 1)$ between numerator and denominator is positive. But $D(0) = 0$, and $D'(c) = 1 + 2c - e^{-c/p} > 0$.
II. By construnction, $x_0 = e^{-c}$ is a zero of g. Lemma 2 gives $b_{c,p} > 3p/(p-1)$ and, for all $x \geq e^{-c}$,

$$b_{c,p}p(x^{1/p} - 1) - (b_{c,p} - 1)(x-1) \leq \log x - (\log x)^2.$$

As the left-hand side is increasing for $x \leq 1$, truncation to $u_c(\log x) - u_c^2(\log x)$ extends the inequality to the interval $[0,e^{-c}]$, while on the intervall $[e^c,\infty]$ the existing inequality is made even "bigger". □

The complementary inequality $u_c(\log x) - u_c^2(\log x) \leq p(x^{1/p} - 1)$ is determined through the behaviour at $x = 0$, and holds true when $p \leq c + c^2$. Thus we may summarize as follows.

LEMMA 4. When $p > 1$, $c \geq (\sqrt{4p+1} - 1)/2$, and $b \geq b_{c,p}$ we have, for all $x \geq 0$,

$$bp(x^{1/p} - 1) - (b-1)(x-1) \leq u_c(\log x) - u_c^2(\log x) \leq p(x^{1/p} - 1).$$

In the square root case $p = 2$ we may choose $c = 1$, the factor $(x-1) - 2(x^{1/2} - 1)$ accompanying $-b$ turns into $(1 - \sqrt{x})^2$, cf. Kabanov, Liptser and Shiryayev [4], and we may take $b = 9$. In general, the cut-off point c depends on the order p of the root under consideration. The feasible choice c = p, as in the proof of Theorem 2, seems to be the simplest way to make this dependence visible.

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