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A majorization comparison of apportionment methods in proportional representation

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Abstract. From the inception of the proportional representation movement it has been an issue whether larger parties are favored at the expense of smaller parties in one apportionment of seats as compared to another apportionment. A number of methods have been proposed and are used in countries with a proportional representation system. These apportionment methods exhibit a regularity of order, as discussed in the present paper, that captures the preferential treatment of larger versus smaller parties. This order, namely majorization, permits the comparison of seat allocations in two apportionments. For divisor methods, we show that one method is majorized by another method if and only if their signpost ratios are increasing. This criterion is satisfied for the divisor methods with power-mean rounding, and for the divisor methods with stationary rounding. Majorization places the five traditional apportionment methods in the order as they are known to favor larger parties over smaller parties: Adams, Dean, Hill, Webster, and Jefferson.

1 Introduction

Payment in proportion to usage, or payment in proportion to services rendered is a well-established and accepted principle. In the political context, the counterpart is proportional representation. One instance is the apportionment of a number of seats to each party proportionally to the number of votes received; another, the apportionment of a number of seats to each state proportionally to the population counts. In the case of monetary payments there appears to be little discourse on methodology methods. In contrast, electoral apportionment has led to political controversy and bitter battles.

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Table 1. An example for six parties and 36 seats (Balinski and Young 2001, p. 96)

Votes	Adams m^A	Dean m^D	Hill m^H	Webster m^W	X m^X	Jefferson m^J
27 744	10	10	10	10	10	11
25 178	9	9	9	9	10	9
19 951	7	7	7	8	7	7
14 610	5	5	6	5	5	5
9 225	3	4	3	3	3	3
3 292	2	1	1	1	1	1
100 000	36	36	36	36	36	36

The apportionment in any column leads to the apportionment in the next column by the transfer of one seat from a smaller party to a larger party, as is indicated by the arrows.

From its inception in the Constitutional Congress of 1787 in the United States, and from the proportional representation movement in Europe that came into existence before 1900, alternative methods for electoral apportionment have been proposed. Why is there a problem? For monetary payments money is considered a practically infinitely divisible commodity, and we are able to allocate arbitrary fractions. This is not the case for electoral apportionments. Each seat is a single entity, and the gain or loss of an individual seat is usually considered of significant importance by the political antagonists.

From the very beginning there has been the issue whether, of two competing apportionment methods, one favors larger parties at the expense of smaller parties more than the other. The rival apportionment methods are associated with well-known names – *Thomas Jefferson, Alexander Hamilton, John Quincy Adams, Daniel Webster*, to name but a few. For an excellent introduction to the history and mathematical formulation of the subject, we recommend the seminal monograph by Balinski and Young (2001).

In order to set the stage for the exposition that follows, we refer to the example exhibited in Table 1. A perusal of this example shows a regularity in the ordering from apportionment m^A to apportionment m^J, capturing the preferential treatment of larger parties versus smaller parties. Apportionment m^A consistently favors smaller parties, in comparison with apportionment m^J which favors larger parties; the other apportionments lie in-between.

What is clear from Table 1 is that there is a movement uphill from apportionment m^A to apportionment m^J. At each step there is a transfer of one seat. The question is how to capture the structural implications of these transfers. Different descriptions of the move from one column to the next could be conceived. The ordering proposed in the present paper is called *majorization*. It has the advantage of providing a complete characterization, and has its roots in studies of equality and inequality. For a review of its history and its formal properties see Marshall and Olkin (1979); an earlier influential forerunner is the book on inequalities by Hardy et al. (1934). In the electoral literature, Raschauer (1971), Pennisi (1998) and Grilli di Cortona et al. (1999) are the only sources we know of that mention the notion of majorization.

The majorization ordering has been a helpful tool in many fields of science, including mathematics, statistics, chemistry, physics, and others. It should be emphasized, though, that the ordering has an independent and early origin in the social sciences. The political science and economics approach dates to Dalton (1920, 1935) who was led to the majorization ordering in his study of inequality of incomes.

Dalton's starting point was the simple idea that if a portion of income is transferred from a poor person to a rich person, then inequality is increased. Thus, in this ordering the case where each person has the average is the most equal, and the case where one person has all the wealth is the most unequal. Formally, if an initial vector m of incomes is altered by a transfer from poor to rich to obtain a vector m', then m' represents higher income inequality than does m. For example, when 3 units of a good must be shared by three individuals, then the apportionment $m = (1, 1, 1)$ is less unequal than the apportionment $m' = (2, 1, 0)$ which, in turn, is less unequal than $m'' = (3, 0, 0)$. This type of comparison generates the majorization ordering.

More specifically, majorization provides an ordering between two vectors $m = (m_1, \dots, m_\ell)$ and $m' = (m'_1, \dots, m'_\ell)$, with ordered elements $m_1 \geq \cdots \geq m_\ell$ and $m'_1 \geq \cdots \geq m'_\ell$, and with an identical component sum $m_1 + \cdots + m_\ell = m'_1 + \cdots + m'_\ell = M$. The ordering states that all partial sums of the k largest components in m are dominated by the sum of the k largest components in m', that is,

$$\begin{aligned}
m_1 &\leq m'_1, \\
m_1 + m_2 &\leq m'_1 + m'_2, \\
&\vdots \\
m_1 + \cdots + m_k &\leq m'_1 + \cdots + m'_k, \\
&\vdots \\
m_1 + \cdots + m_{\ell-1} &\leq m'_1 + \cdots + m'_{\ell-1}, \\
m_1 + \cdots + m_\ell &= M = m'_1 + \cdots + m'_\ell.
\end{aligned} \tag{1}$$

We denote this ordering by $m \prec m'$, and say that *m is majorized by* m', or equivalently, that m' *majorizes m*.

In Table 1, this ordering applies as one moves from the first apportionment column step by step to the last apportionment column. Thus the subtotal of seats assigned to a set of large parties is growing (or remains constant), and in this precise sense larger parties are increasingly better off.

It seems worthwhile to emphasize the descriptive power of the majorization concept. For instance, Nohlen (2000, p. 106) compares two apportionments, similar to the Webster apportionment m^W and the Jefferson apportionment m^J in our Table 1. He makes a point that a transition from m^W to

m^J may result in allocating an additional seat to any one of the parties except the smallest. Conversely, the loss of a seat could occur to any one of the parties except the largest. This sounds as if the transfer of a seat occurs in a random fashion. This is not so, and there is the systematic structure of transferring a seat from a smaller party to a larger party. Majorization provides the appropriate language to capture the structural properties. The total number of seats of the k largest parties in the apportionment m^W is less than or equal to the corresponding total in apportionment m^J. Equivalently, the total number of seats of the k smallest parties in apportionment m^W is larger than or equal to what those small parties total in apportionment m^J.

Although our focus is on the five traditional apportionment methods named in the United States after *John Quincy Adams, James Dean, Josef A. Hill, Daniel Webster*, and *Thomas Jefferson*, similar deliberations occurred in Europe (United Kingdom, Belgium, France, Germany, and others). We briefly give some biographical details, and refer to Kopfermann (1991) for additional information.

Baron Edward Hugh John Neal Dalton (*26 August 1887, †13 February 1962) was born in Wales and educated in Cambridge (Marshall and Olkin 1997, p. 522). He was on the faculty of the London School of Economics, and later served as Parliamentary Undersecretary at the Foreign Office and as Chancellor of the Exchequer under Prime Minister *Clement Attlee*.

Victor d'Hondt (*20 November 1841, †30 May 1901) was professor of tax law and civil rights at the University of Ghent (Carlier 1901; Beatse 1913). *Hondt*, an activist in the *Association Réformiste Belge*, published widely on the apportionment method that, in Europe, was named after him; see, for instance, Hondt (1885). In the USA, his method is associated with the name of Thomas Jefferson.

Eduard Hagenbach-Bischoff (*20 February 1833, †23 December 1910) was a physics professor at the University of Basel (Huber 1960). As a member of the Canton legislature he became a proponent of the method d'Hondt, and simplified the calculations to obtain its apportionments. Of his many publications on the subject we mention the booklet (1905).

André Sainte-Laguë (*20 April 1882, †18 January 1950) was a professor of applied mathematics at the *Conservatoire national des arts et métiers* in Paris (Chastenet 1994). Early in his career, while teaching at the *Lycée* in Douai, he published two papers (1910a,b) analyzing the optimality properties of apportionment methods. Sainte-Laguë gave special attention to the divisor method with standard rounding, which in Europe then was named after him whereas in the USA it originated with Daniel Webster. Another apportionment method that Sainte-Laguë considered is the divisor method with geometric rounding, which is the method currently in use for the apportionment of seats in the US House of Representatives (method of equal proportions, Hill method).

George Pólya (*13 December 1887, †7 September 1985) was one of the eminent mathematicians of the last century (Olkin and Pukelsheim 2001). His Collected Papers comprise four volumes of 2430 pages; the Pólya (1987)

Picture Album is a fascinating document of the scientific history of his century. Pólya authored five papers (1918, 1919a–d) scrutinizing the various apportionment methods then in use in Switzerland.

Section 2 discusses another relation from the literature that is closely related to majorization. Section 3 extends the notion of majorization from apportionment vectors to apportionment methods. In Sect. 4 we describe divisor methods of apportionment, and the signpost sequences that determine the methods. Section 5 contains our principal results, providing necessary and sufficient conditions for majorization among divisor methods. Section 6 serves to explicate the results. An Appendix provides proofs of the three propositions in Sect. 5.

2 A relation akin to majorization

A key feature of majorization is that it is a partial ordering, that is, it is reflexive, transitive, and antisymmetric (Marshall and Olkin 1979, p. 13). Balinski and Young (2001, p. 118) and Balinski and Rachev (1997, p. 15) discuss the following relation. An apportionment is said *to give up to* another apportionment if, in every pairwise comparison of a larger party i with a smaller party j, party i gains seats or party j loses seats. That is, it cannot happen that the larger party i loses seats and at the same time the smaller party j gains seats.

This relation fails to be transitive, and hence does not qualify as a partial ordering. To clarify the notion of transitivity in this context, consider three apportionments of 21 seats:

Party	m	m'	m''
1	10	11	11
2	6	5	5
3	3	3	4
4	2	2	1
	21	21	21

(2)

Moving from m to m', party 2 gives up one seat to party 1. From m' to m'', party 4 gives up one seat to party 3. But comparing m with m'', party 2 loses a seat whereas party 3 gains a seat, whence the transitivity property does not generally hold.

The following lemma proves that the relation of one apportionment giving up to another one implies majorization. The converse is not generally true, as evidenced in (2). Thus majorization orders more apportionments, just enough so as to achieve transitivity.

Lemma. *Consider two apportionments $m_1 \geq m_2 \geq \cdots \geq m_\ell$ and $m'_1 \geq m'_2 \geq \cdots \geq m'_\ell$. If, for all $i < j$, we have $m_i \leq m'_i$ or $m_j \geq m'_j$ then m is majorized by m'. The converse is not generally true.*

Proof. The proof is indirect. Suppose that m is not majorized by m', then for some i we have

$$\begin{aligned} m_1 &\le m_i', \\ m_1 + m_2 &\le m_1' + m_2', \\ &\vdots \\ m_1 + \cdots + m_{i-1} &\le m_1' + \cdots + m_{i-1}', \\ m_1 + \cdots + m_{i-1} + m_i &> m_1' + \cdots + m_{i-1}' + m_i'. \end{aligned}$$

Consequently, we must have $m_i > m_i'$. However, the total sums are equal, so that it must be that $m_j < m_j'$ for some $j > i$. For the converse part, we refer to (2) where, although m is majorized by m'', we have $m_2 = 6 > 5 = m_2''$ and $m_3 = 3 < 4 = m_3''$. The proof of the Lemma is complete.

Balinski and Young (2001, p. 118) prove that, for divisor methods, monotonicity of the signpost ratios as demanded in Proposition 1 below is sufficient for their relation to apply. Similarly, Saari (1994, p. 307; 1995, p. 271) finds that signpost ratio monotonicity is sufficient so that "one method favors large states more than another method" without, however, providing a formal definition for his relation. Here, we show that monotonicity of the signpost ratios is a necessary and sufficient condition for two divisor methods to be comparable in the majorization partial ordering.

Thus we hope that the present paper offers a technical as well as a conceptual contribution. Technically, signpost ratio monotonicity transpires to be not only sufficient but also necessary; this could have been formulated entirely relative to the giving up-relation. Conceptually, we much prefer to proceed to the majorization partial ordering which has proved extremely powerful in many other instances where the issue is to assess fairness of competing allocations.

3 Majorization of two apportionment methods

In proportional representation electoral systems involving ℓ parties, an apportionment is calculated from given vote counts $v_1, v_2, \ldots, v_\ell$, for a given district magnitude M. Of course, the vote counts v_i are whole numbers. However, there are other applications where the proportional allocation of M items is based on nonnegative *weights* v_i, see Balinski and Young (2001, p. 96). In general, then, we assume that we are given ℓ weights $v_i \in [0, \infty)$, and that these weights are ordered from largest to smallest, $v_1 \ge v_2 \ge \cdots \ge v_\ell$.

A procedure that governs the apportionment calculations is called an *apportionment method*. Let A be the apportionment method to be used. The apportionment result then consists, practically almost always, of a single apportionment vector $m = (m_1, m_2, \ldots, m_\ell)$. However, a general method must also accommodate tied situations, for instance when ℓ parties with identical weights share $\ell + 1$ seats. Balinski and Young (2001, p. 96) discuss such ties in detail. The set of all apportionment vectors that A associates with a weight vector $v = (v_1, v_2, \dots, v_\ell)$ is denoted by $A(v)$.

For two specific apportionment vectors m and m', the majorization relation (1) presupposes vectors with decreasingly ordered elements. Therefore we restrict attention to apportionment methods that guarantee this property. A method A is said to be *weakly weight monotone* if

$$v_1 > v_2 > \cdots > v_\ell \Rightarrow m_1 \geq m_2 \geq \cdots \geq m_\ell$$

for all $(m_1, m_2, \dots, m_\ell) \in A(v)$, see Balinski and Young (2001, p. 147). We can now extend the notion of majorization from (1), to also apply to two apportionment *methods*.

Definition. *Given two weakly weight monotone apportionment methods A and A', we say that A is majorized by A', denoted by $A \prec A'$, if either they are equal or, for every number ℓ of participating parties and for all weights $v_i > v_2 > \cdots > v_\ell \geq 0$ and for each district magnitude M, every apportionment $m \in A(v)$ is majorized by every apportionment $m' \in A'(v)$.*

In the set of all apportionment methods, this relation is a partial ordering. That is, it is reflexive ($A \prec A$), transitive ($A \prec A'$ and $A' \prec A''$ implies $A \prec A''$), and antisymmetric ($A \prec A'$ and $A' \prec A$ implies $A = A'$). Naturally, there is no necessity that any two arbitrary apportionment methods A and A' be comparable in the majorization ordering. The main result of the present paper is to establish a necessary and sufficient condition for determining majorization, under the assumption that the two apportionment methods are divisor methods.

A brief comment may be in order why the notion of weak weight monotonicity concentrates on strictly ordered weights $v_1 > v_2 > \cdots > v_\ell$, thereby neglecting any tie $v_i = v_j$. For example, consider the weight vector $v = (45, 25, 25, 5)$, and choose $M = 10$. Due to the tie $v_2 = v_3 = 25$, the divisor method with rounding down (Jefferson, Hondt) results in two apportionment vectors,

$$m = (5, 3, 2, 0), \quad \tilde{m} = (5, 2, 3, 0),$$

of which only the first appears in decreasing order as needed in (1). Requiring only weakly ordered weights $v_1 \geq v_2 \geq \cdots \geq v_\ell$ would thus exclude standard methods from further consideration.

4 Divisor methods and signpost sequences

A *divisor method of apportionment* is defined through numbers $s(k)$ in the interval $[k, k+1]$ such that the sequence $s(0), s(1), \dots$ is strictly increasing. Balinski and Young (2001, p. 64) picture an individual number $s(k)$ as a "signpost" or "dividing point" splitting the interval $[k, k+1]$ into a left part where numbers are rounded down to k, and a right part where numbers are rounded up to $k+1$. For $s(k)$ itself, there is the option to round down to k or to round up to $k+1$, thus possibly generating multiplicities.

The numbers rounded this way are the quotients of the weights and a divisor, $v_1/d, v_2/d, \ldots, v_\ell/d$, for some choice of divisor $d > 0$ common to all weights. If party i gets m_i seats, then necessarily $s(m_i - 1) \leq v_i/d \leq s(m_i)$. The divisor d is adjusted so that the sum of all seats becomes equal to the district magnitude, $m_1 + m_2 + \cdots + m_\ell = M$. Clearly, every divisor method is weakly weight monotone.

Alternatively, the apportionment m can be found by treating a divisor method as a rank-index method (Balinski and Young 2001, p. 142). That is, the M largest ratios $v_i/s(k)$ for $i = 1, \ldots, \ell$ and $k = 0, 1, \ldots$ are determined, and for each occurrence of party i it gets one seat.

To illustrate these ideas, let A be a divisor method with initial signposts $s(0) = 0.5$ and $s(1) = 1.4$. If two parties have vote counts $v_1 = 75$ and $v_2 = 25$, then two seats are apportioned according to $m = (2, 0)$. With divisor $d = 51$, this is readily checked:

$$\frac{v_1}{d} = \frac{75}{51} = 1.47 > 1.4 = s(1) \Rightarrow m_1 = 2,$$

$$\frac{v_2}{d} = \frac{25}{51} = 0.49 < 0.5 = s(0) \Rightarrow m_2 = 0.$$

Suppose A' is another divisor method with initial signposts $s'(0) = 0.5$ and $s'(1) = 1.6$. With the same vote counts as before, the two seats are now apportioned according to $m' = (1, 1)$. With divisor $d = 49$, we obtain

$$\frac{v_1}{d'} = \frac{75}{49} = 1.53 < 1.6 = s'(1) \Rightarrow m'_1 = 1,$$

$$\frac{v_2}{d'} = \frac{25}{49} = 0.51 > 0.5 = s'(0) \Rightarrow m'_2 = 1.$$

The growth of the signpost $s(1)$ from 1.4 to 1.6 makes it increasingly difficult for the larger party to secure as many seats as before.

Two common signpost sequences are the *power-mean signposts*

$$s_1(k, p) = \left(\frac{k^p}{2} + \frac{(k+1)^p}{2}\right)^{1/p}, \quad -\infty \leq p \leq \infty; \tag{3}$$

and the *stationary signposts*

$$s_2(k, q) = k + q = (1 - q)k + q(k + 1), \quad 0 \leq q \leq 1. \tag{4}$$

In (3), the three exceptional values $p = -\infty, 0, \infty$ need special mentioning. The case $p = -\infty$ has $s_1(k, -\infty) = k$, the other extreme is $s_1(k, \infty) = k + 1$. For $p = 0$ we obtain the geometric mean, $s_1(k, 0) = \sqrt{k(k+1)}$. We also note that $p = 1$ gives the arithmetic mean, and $p = -1$ the harmonic mean.

The power-mean signpost sequences (3) with $p = -\infty, -1, 0, 1, \infty$ yield, in turn, the five traditional apportionment methods: the Adams method (divisor method with rounding up), the Dean method (divisor method with harmonic rounding), the Hill method (divisor method with geometric rounding, method of equal proportions), the Webster method (divisor method with standard rounding, method of Sainte-Laguë), and the Jefferson method (divisor method with rounding down, method d'Hondt, method of Hagenbach-Bischoff).

Both (3) and (4) represent averages of k and $k+1$, with $s_1(k,p)$ being the mean of power p, and $s_2(k,q)$ the arithmetic mean with weights $1-q$ and q. These two families have some member sequences in common. For example, $p=-\infty$ and $q=0$ yield the value k; whereas $p=1$ and $q=1/2$ yield $k+1/2$; finally $p=\infty$ and $q=1$ yield $k+1$. In general, however, different values of the parameters p and q generate different signpost sequences.

For large values of k, the divisor methods with power-mean rounding reduce to three methods only, the Adams method ($p=-\infty$), the Webster method ($p=1$), and the Jefferson method ($p=\infty$). This is due to the limiting relationship

$$\lim_{k\to\infty}(s_1(k,p)-k)=\begin{cases}1 & \text{for } p=\infty,\\ 1/2 & \text{for } -\infty<p<\infty,\\ 0 & \text{for } p=-\infty.\end{cases}$$

The limit is obtained using l'Hospital's rule, as $x=1/k$ tends to zero in $s_1(k,p)-k=[\{(1+(1+x)^p)/2\}^{1/p}-1]/x$. Thus, in the intervals $[k,k+1]$ with k large, the signposts move to the midpoints $k+1/2$ when p is finite, whereas they coincide with the left endpoints k or the right endpoints $k+1$ when p is infinite. In contrast, the stationary signpost family (4) maintains its richness also for large values of k.

The two signpost sequences (3) and (4) generate "compromises" between k and $k+1$. However, other such sequences can be constructed. For example, the signpost $s_3(k,q)=k^{1-q}(k+1)^q$, for $0\le q\le 1$, is a weighted geometric mean of k and $k+1$ (Dorfleitner and Klein 1999, p. 151). The associated family of divisor methods includes the Adams method ($q=0$), the Hill method ($q=1/2$), and the Jefferson method ($q=1$).

The three one-parameter signpost families s_1, s_2, s_3 can be embedded in the single two-parameter family

$$s_0(k,p,q)=((1-q)k^p+q(k+1)^p)^{1/p},\quad -\infty\le p\le\infty,\quad 0\le q\le 1.$$

As with (3), the exceptional values $p=-\infty,0,\infty$ require separate definitions, namely $s_0(k,-\infty,q)=k$, and $s_0(k,0,q)=k^{1-q}(k+1)^q$, and $s_0(k,\infty,q)=k+1$. This yields $s_1(k,p)=s_0(k,p,1/2)$, and $s_2(k,q)=s_0(k,1,q)$, and $s_3(k,q)=s_0(k,0,q)$. Yet another family is generated by $s_4(k,p)=\log((e^{pk}+e^{p(k+1)})/2)^{1/p}$, for $-\infty\le p\le\infty$, beginning with the Adams method ($p=-\infty$), passing through the Webster method ($p=0$), and ending with the Jefferson method ($p=\infty$).

5 Principal results for divisor methods

We first show that majorization among two divisor methods requires a monotonicity relationship involving the signpost sequences that define the two methods.

Proposition 1. *Let A be a divisor method with signpost sequence $s(0), s(1), s(2), \ldots$ and let A' be another divisor method with a different signpost sequence $s'(0), s'(1), s'(2), \ldots$. Then method A is majorized by method A' if and only if the signpost ratios $s(k)/s'(k)$ are strictly increasing in k.*

The proof of Proposition 1 is deferred to the Appendix. It is always the case that, as k tends to infinity, the signpost ratios $s(k)/s'(k)$ is bounded from below and from above by $k/(k+1) \leq s(k)/s'(k) \leq (k+1)/k$. Hence the sequence of signpost ratios converges to the limit one, as k tends to infinity. Therefore, under Proposition 1, the sequence converges to one from below. This entails $s(k) < s'(k)$ for all k, meaning that a transition from method A to A' moves all signposts to larger values. Only $k = 0$ is an exception; when $s'(0) = 0$, we set $s(0)/0 = 0$ for $s(0) = 0$ and $s(0)/0 = \infty$ for $s(0) > 0$.

We now return to the specific divisor methods defined by the power-mean signposts (3), and by the stationary signposts (4).

Proposition 2. *The divisor method with power-mean rounding of order p is majorized by the divisor method with power-mean rounding of order p' if and only if $p \leq p'$.*

Proposition 3. *The divisor method with stationary rounding of shift q is majorized by the divisor method with stationary rounding of shift q' if and only if $q \leq q'$.*

We again defer the proofs to the Appendix. Proposition 2 puts the five traditional divisor methods into the majorization ordering

Adams $\prec$ Dean $\prec$ Hill $\prec$ Webster $\prec$ Jefferson.

That the apportionment results of the traditional methods are ordered by majorization is plainly visible in the congressional apportionments for the US censuses 1791–2000 provided in Balinski and Young (2001, pp. 158–176). Proposition 3 is already implicit in Theorem 2.8 of Balinski and Rachev (1997, p. 15).

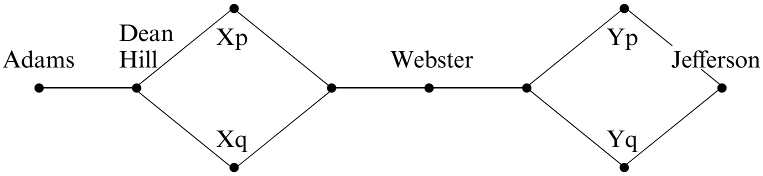
6 Some examples

Table 2 provides another example. The total number of votes is 100 000, as it is in Table 1. Hence in both examples, the entries in the first column can be read as a count of votes (42 919 etc.), or they can be interpreted as weights giving the proportion of votes (0.42919 etc.). It is interesting to discuss how the examples in Tables 1 and 2 compare.

Table 1 presents the complete series of apportionments obtained from the power-mean divisor methods, and from the stationary divisor methods; they happen to coincide. Table 2 is an example where the two series differ. In both cases, the series starts with the Adams apportionment, passes through the Webster apportionment, and terminates with the Jefferson apportionment. Of course, there exist other apportionments than the five traditional ones of

Table 2. An example for ten parties and 100 seats (Balinski and Rachev 1997, p. 14)

Votes	Apportionments obtained from the power-mean divisor methods							
42 919	41	42	43	43	43	44	44	45
13 048	13	13	13	13	13	13	13	13
10 879	11	11	11	11	11	11	11	11
10 581	10	10	10	11	11	11	11	11
9 547	10	9	9	9	10	9	10	10
5 708	6	6	6	6	6	6	6	5
2 502	3	3	3	3	2	2	2	2
1 898	2	2	2	2	2	2	1	1
1 461	2	2	2	1	1	1	1	1
1 457	2	2	1	1	1	1	1	1



Votes	Apportionments obtained from the stationary divisor methods							
42 919	41	42	42	43	43	44	44	45
13 048	13	13	13	13	13	13	13	13
10 879	11	11	11	11	11	11	11	11
10 581	10	10	11	11	11	11	11	11
9 547	10	9	9	9	10	9	10	10
5 708	6	6	6	6	6	6	5	5
2 502	3	3	3	3	2	2	2	2
1 898	2	2	2	2	2	2	2	1
1 461	2	2	2	1	1	1	1	1
1 457	2	2	1	1	1	1	1	1

The apportionment series from the power-mean divisor methods (*top*), and from the stationary divisor methods (*bottom*) need not coincide, as in this example. However, both apportionment series start with the Adams apportionment, proceed by transferring a seat from a smaller party to a larger party, and end in the Jefferson apportionment. Within each series, every apportionment is majorized by its successor.

Adams, Dean, Hill, Webster and Jefferson, such as X in Table 1. In Table 1 all seat transfers occur over minimum distance, between pairs of contiguous parties. Table 2 shows that this need not be so in general; in the top part from the Hill apportionment to apportionment Xp a seat is transferred over maximum distance, from the smallest party to the largest party.

In the present examples, the power-mean series and the stationary series happen to comprise an equal number of apportionments (six in Table 1, and eight in Table 2). In other examples, not quoted here, these numbers differ. Furthermore, in those instances where the two series in Table 2 yield distinct results, the stationary apportionment happens to be majorized by the power-mean apportionment ($\mathrm{Xq} \prec \mathrm{Xp}$, and $\mathrm{Yq} \prec \mathrm{Yp}$). It is a consequence of Proposition 1 that this need not hold in general.

In order to verify that Tables 1 and 2 present the complete series of apportionments obtainable from the power-mean divisor methods and from the stationary divisor methods, we argue as follows.

Generally, let $s(k)$ denote the signposts defining a divisor method, and let the vote counts $v_1, v_2, \ldots, v_\ell$ be given. Assume that party i is apportioned m_i seats, and party j is apportioned m_j seats. A transfer of a seat from party j to party i changes the respective allocations to $m_i + 1$ and $m_j - 1$ seats, and is possible only if there is a tie,

$$\frac{v_i}{d} = s(m_i), \quad \frac{v_j}{d} = s(m_j - 1). \tag{5}$$

In a tied situation such as (5) there is the option for parties i and j to be allocated m_i and m_j seats, or $m_i + 1$ and $m_j - 1$ seats, respectively. Elimination of the divisor d in equations (5) yields a single equation,

$$\frac{s(m_i)}{s(m_j - 1)} = \frac{v_i}{v_j}. \tag{6}$$

In a parametric family of signposts, equation (6) turns into a formula that determines the parameter value giving rise to a tie.

Specifically, we first consider the stationary signposts s_2, and start with the Adams apportionment m^A. Inserting $s_2(m_i^A, q)$ and $s_2(m_j^A - 1, q)$ from (4) into (6), we obtain the formula for q:

$$q^A(i,j) = \frac{m_i^A v_j - (m_j^A - 1)v_i}{v_i - v_j}. \tag{7}$$

Because of Proposition 3 we know that a transfer from party j to party i is possible only when $i < j$, that is, when party i is larger than party j. Thus, among ℓ parties, there are $\ell(\ell - 1)/2$ pairs to be considered. For each pair $i < j$, formula (7) provides a solution $q^A(i, j)$. Let q^A be the smallest of these numbers. In other words, as q increases from zero upwards, of all the ties that are possible the one at q^A materializes first.

For the $\ell = 6$ parties of Table 1 there are $(6)(5)/2 = 15$ pairings, and 15 comparisons of formula (7) are required. The minimum q^A is between parties 5 and 6, for which

$$q^A = q^A(5, 6) = \frac{(3)(3292) - (1)(9225)}{9225 - 3292} = \frac{651}{5933} = 0.109\,725.$$

At this value q^A, the Adams apportionment m^A is tied with the Dean apportionment m^D. Similarly, the value $q^D = q^D(4, 5) = 2295/5385 = 0.426\,184$ is calculated where the Dean apportionment is tied with the Hill apportionment, and so on. The rounding step of any divisor method is rather sensitive to determining the correct value of q, so that a large number of decimals is usually required.

For the power-mean signposts s_1 from (3), Eq. (6) takes the form

$$\left(\frac{m_i^p + (m_i + 1)^p}{(m_j - 1)^p + m_j^p}\right)^{1/p} = \frac{v_i}{v_j},$$

which does not admit a closed form solution in p. However, because the left hand side is monotone in p, the solution is readily obtained numerically. For instance, using the computer program Maple for the data in Table 1, we obtain $p^A = p^A(5,6) = -3.363\,395$, and $p^D = p^D(4,5) = -0.265\,628$.

Appendix: Proofs

Proof of Proposition 1. For the direct part, let A and A' be two distinct divisor methods satisfying $A \prec A'$. We need to show that $s(k)/s'(k) < s(k+1)/s'(k+1)$ for all k. Our proof is indirect, assuming the contrary,

$$\frac{s(k+1)}{s'(k+1)} \le \frac{s(k)}{s'(k)} \quad \text{for some integer } k \ge 0. \tag{8}$$

The left hand side of (8) is bounded from below by $(k+1)/(k+2) > 0$, whence $s(k) > 0$. Strict monotonicity of the signpost sequence entails $a = s(k+1)/s(k) > 1$. Now the interval

$$I = \left[\frac{s(k+1)}{s(k)}, \frac{s'(k+1)}{s'(k)}\right] \tag{9}$$

is nonempty, by (8), and its left endpoint a satisfies $1 < a < \infty$. If the interval is nondegenerate we can choose two integers v_1 and v_2 such that v_1/v_2 lies in its interior. Because of $v_1/v_2 \ge a > 1$, we get $v_1 > v_2$. If the interval degenerates, $I = \{a\}$, we can still define two weights $v_1 = a/(1+a) > v_2 = 1/(1+a) > 0$, with $v_1/v_2 = a \in I$. This construction provides us with a situation of two parties, with respective weights $v_1 > v_2 > 0$. We choose a district magnitude $M = 2k+2$.

We claim that $m = (k+2, k)$ is an apportionment under method A. We establish our claim by verifying the max-min inequality of Balinski and Young (2001, p. 100), according to which m is an apportionment under method A if and only if

$$\max\left\{\frac{v_1}{s(k+2)}, \frac{v_2}{s(k)}\right\} \le \min\left\{\frac{v_1}{s(k+1)}, \frac{v_2}{s(k-1)}\right\}. \tag{10}$$

That is, we need to check four inequalities,

$$\frac{v_1}{s(k+2)} \le \frac{v_1}{s(k+1)}, \tag{10a}$$

$$\frac{v_1}{s(k+2)} \le \frac{v_2}{s(k-1)}, \tag{10b}$$

$$\frac{v_2}{s(k)} \le \frac{v_1}{s(k+1)}, \tag{10c}$$

$$\frac{v_2}{s(k)} \le \frac{v_2}{s(k-1)}. \tag{10d}$$

But (10a) follows from $s(k+1) < s(k+2)$, (10b) from $v_1/v_2 \le$

$s'(k+1)/s'(k) \leq s(k+2)/s(k-1)$, (10c) from $s(k+1)/s(k) \leq v_1/v_2$, and (10d) from $s(k-1) < s(k)$. If (8) is fulfilled with $k=0$ then the inequality in (10) has right hand side simply equal to $v_1/s(1)$, whence (10b, d) become irrelevant.

We next claim that $m' = (k+1, k+1)$ is an apportionment under method A'. For this to hold true the max-min inequality takes the form

$$\max\left\{\frac{v_1}{s'(k+1)}, \frac{v_2}{s'(k+1)}\right\} \leq \min\left\{\frac{v_1}{s'(k)}, \frac{v_2}{s'(k)}\right\}. \tag{11}$$

Again we need to check four inequalities,

$$\frac{v_1}{s'(k+1)} \leq \frac{v_1}{s'(k)}, \tag{11a}$$

$$\frac{v_1}{s'(k+1)} \leq \frac{v_2}{s'(k)}, \tag{11b}$$

$$\frac{v_2}{s'(k+1)} \leq \frac{v_1}{s'(k)}, \tag{11c}$$

$$\frac{v_2}{s'(k+1)} \leq \frac{v_2}{s'(k)}. \tag{11d}$$

Now (11a) follows from $s'(k) < s'(k+1)$, (11b) from $v_1/v_2 \leq s'(k+1)/s'(k)$, (11c) from $s'(k)/s'(k+1) \leq s(k+1)/s(k) \leq v_1/v_2$, and (11d) from $s'(k) < s'(k+1)$.

In summary, the methods A and A' produce the apportionments $m = (k+2, k)$ and $m' = (k+1, k+1)$ where, evidently, m is not majorized by m'. This contradicts the assumption $A \prec A'$, thus invalidating (8).

For the converse part, we follow the lines of argument in Balinski and Young (2001, p. 118), and Balinski and Rachev (1997, p. 15). Let the signpost ratios be strictly increasing. For some vote counts $v_1, v_2, \ldots v_\ell$ and district magnitude M, let m be an apportionment under method A and m' an apportionment under A'. We prove, for all $v_i > v_j$, that $m_i \leq m'_i$ or $m_j \geq m'_j$; this forces m to be majorized by m', see the Lemma in Sect. 2. Otherwise, there exist two weights $v_i > v_j$ satisfying

$$m_i > m'_i \quad \text{and} \quad m_j < m'_j. \tag{12}$$

In view of the above mentioned max-min inequality there are divisors d for A and d' for A' such that

$$\frac{v_i}{d} \geq s(m_i - 1), \quad \frac{v_j}{d} \leq s(m_j); \quad \frac{v_i}{d'} \leq s'(m'_i), \quad \frac{v_j}{d'} \geq s'(m'_j - 1).$$

This leads to the first and last inequalities in

$$\frac{v_i}{v_j} \leq \frac{s'(m'_i)}{s'(m'_j - 1)} \leq \frac{s'(m_i - 1)}{s'(m_j)} < \frac{s(m_i - 1)}{s(m_j)} \leq \frac{v_i}{v_j}. \tag{13}$$

The second inequality follows from (12), whereas the strict inequality holds by assumption on the monotonicity of the signpost ratios. But (13) is a contradiction, whence (12) cannot hold true. The proof is complete.

Proof of Proposition 2. In (3), consider the power-mean signposts $s_1(k,p)$ and $s_1(k,r)$ for $p < r$. We aim to establish monotonicity of the signpost ratios $s_1(k,p)/s_1(k,r)$.

In case $k = 0$ and $p \leq 0$, we have $s_1(0,p) = 0$ and the convention $0/0 = 0$ from Sect. 5 secures

$$s_1(0,p)/s_1(0,r) = 0 < s_1(1,p)/s_1(1,r).$$

In all other cases, that is when $k > 0$ or $p > 0$, we show that

$$g(r) = \frac{s_1(k+1,r)}{s_1(k,r)} < \frac{s_1(k+1,p)}{s_1(k,p)} = g(p),$$

namely, the function $g(r)$ is strictly decreasing in r. Upon setting $x_1 = k+2$, $x_2 = k+1$ and $y_1 = k+1$, $y_2 = k$, we may rewrite $g(r)$ in the form

$$g(r) = \frac{((k+1)^r + (k+2)^r)^{1/r}}{(k^r + (k+1)^r)^{1/r}} = \left(\frac{\sum_{i=1}^2 x_i^r}{\sum_{j=1}^2 y_j^r}\right)^{1/r}$$

when $r \neq 0$. The continuous continuation to $r = 0$ is the ratio of the geometric means, $g(0) = (x_1x_2)^{1/2}/(y_1y_2)^{1/2} = \sqrt{(k+2)/k}$.

Because $x_1 > x_2 > 0$ and $y_1 > y_2 \geq 0$ and $y_1/x_1 > y_2/x_2$, Proposition 5.B.3 in Marshall and Olkin (1979, p. 130) applies and states that $g(r)$ is decreasing in r. Moreover, the function g is analytic, whence if it is constant on some open interval then it is constant on the whole real line. This is not the case, as it decreases from $g(-\infty) = (k+1)/k$ down to $g(\infty) = (k+2)/(k+1)$. Hence g is strictly decreasing, and the proof is complete.

Proof of Proposition 3. In (4), consider the stationary signposts $s_2(k,q)$ and $s_2(k,r)$ for $q < r$. Straightforward calculation gives $s_2(k+1,q)s_2(k,r) - s_2(k,q)s_2(k+1,r) = r - q > 0$. Hence $s_2(k,q)/s_2(k,r)$ is strictly increasing in k. The proof is complete.

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