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And Round the World Away

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When rounding a finite set of probabilities to integral multiples of $1/n$, any multiplier method guarantees that the rounded probabilities again sum to one. For multiplier methods that are stationary, we discuss the expected discrepancy and calculate unbiased multipliers, under the assumption of uniformly distributed probabilities.

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When all the world is young, lad,
And all the trees are green;
And every goose a swan, lad,
And every lass a queen;
Then hey for boot and horse, lad,
And round the world away:
Young blood must have its course, lad,
And every dog his day.
CHARLES KINGSLEY

1. INTRODUCTION

Rounding errors are commonly experienced in many practical problems. How drastically they are felt depends on the context. For instance, standard rounding to percentages of the 1975 world population leaves a discrepancy of -2 percent. Thus it misses out on more than 80 million people! While this does not quite round the world away, it *is* enough to do away with the present authors and the rest of Germany. See Table 1.

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Table 1. 1975 World Population

Continent	*Population*	*Proportion*	*Percent*
Asia (without SU)	2 295 000 000	0.57289	57
Europe (with SU)	734 000 000	0.18323	18
Americas	540 000 000	0.13480	13+
Africa	417 000 000	0.10409	10
Australia	20 000 000	0.00499	0+
Total	4 006 000 000	1.00000	98

Standard rounding leaves a discrepancy of −2 percent of the world population and thus loses 80 million people, see Kopfermann (1991, page 109). The plus signs indicate the correction for the Webster method.

Table 2 shows the 1996 US presidential vote state-by-state. Using standard rounding, the absolute counts for the three candidates are rounded to the tenth of a percent, i.e., to the form $n_i/1000$. The last column gives the discrepancy, $D = n_1 + n_2 + n_3 - 1000$. It is distributed as follows:

Discrepancy	−1	0	1
Observed frequency	5	39	8
Theoretical frequency	7	39	6

The theoretical distribution is $\mathrm{P}(D = -1) = 1001/8000$, $\mathrm{P}(D = 0) = 6000/8000$, $\mathrm{P}(D = 1) = 999/8000$. This is one of the results of the probabilistic analysis pioneered by Mosteller, Youtz and Zahn (1967), a seminal paper with plenty of empirical evidence. Our approach follows their lead.

More precisely, we denote by $r_{1/2}(x)$ the standard rounding of the positive number x to the closest integer if the fractional part of x is distinct from 1/2 (and the closest integer is unique), and to the closest even integer if the fractional part of x is equal to 1/2 (and there is a tie). See Wallis and Roberts (1956, page 175), or Bronstein and Semendjajew (1991, Section 2.1.1.2). Let $(W_1, \ldots, W_c)$ be a random vector that is uniformly distributed in the probability simplex of $\mathbb{R}^c$. Diaconis and Freedman (1979) show that then

$$\lim_{n\to\infty} \mathrm{P}\left(\sum_{i\le c} \frac{r_{1/2}(nW_i)}{n} = 1\right) = O\left(\frac{1}{\sqrt{c}}\right).$$

There is nothing built into standard rounding to preserve a linear side condition such as summing to one.

Table 2. US Presidential Vote of 5 November 1996

State	*Clinton*: %	*Dole*: %	*Perot*: %	*Discrepancy*
Alabama	664 503 : 43.2	782 029 : 50.8	92 010 : 6.0	0
Alaska	66 508 : 35.1	101 234 : 53.5	21 536 : 11.4	0
Arizona	612 412 : 47.4	576 126 : 44.5	104 712 : 8.1	0
Arkansas	467 888 : 54.5	323 622 : 37.7	66 913 : 7.8	0
California	4 639 935 : 53.2	3 412 563 : 39.1	667 702 : 7.7	0
Colorado	670 854 : 45.9	691 291 : 47.3	99 509 : 6.8	0
Connecticut	712 603 : 53.5	481 047 : 36.1	137 784 : 10.3+	−1
Delaware	140 209 : 52.4	98 906 : 36.9	28 693 : 10.7	0
District of Columbia	152 031 : 88.3	16 637 : 9.7	3 479 : 2.0	0
Florida	2 533 502 : 48.3	2 226 117 : 42.5	482 237 : 9.2	0
Georgia	1 045 552 : 46.1	1 076 875 : 47.5	145 445 : 6.4	0
Hawaii	205 012 : 59.2	113 943 : 32.9	27 358 : 7.9	0
Idaho	165 545 : 34.2	256 406 : 52.9	62 506 : 12.9	0
Illinois	2 299 476 : 54.5−	1 577 930 : 37.4	344 311 : 8.2	+1
Indiana	874 668 : 41.9	995 082 : 47.6	218 739 : 10.5	0
Iowa	615 525 : 50.8+	490 809 : 40.5	104 421 : 8.6	−1
Kansas	384 399 : 36.4	578 572 : 54.8+	92 093 : 8.7	−1
Kentucky	635 804 : 46.2	622 339 : 45.2	118 768 : 8.6	0
Louisiana	928 983 : 52.7	710 240 : 40.3	122 981 : 7.0	0
Maine	311 000 : 53.5	185 062 : 31.8	85 268 : 14.7	0
Maryland	924 284 : 54.7	651 682 : 38.6	113 684 : 6.7	0
Massachusetts	1 567 223 : 62.4	717 622 : 28.6	225 394 : 9.0	0
Michigan	1 911 553 : 52.5−	1 413 812 : 38.8	319 095 : 8.8	+1
Minnesota	1 096 355 : 52.2	751 971 : 35.8	252 986 : 12.0	0
Mississippi	385 005 : 44.2	434 547 : 49.9	51 500 : 5.9	0
Missouri	1 024 817 : 48.1	889 689 : 41.7	217 103 : 10.2	0
Montana	167 169 : 41.7	178 957 : 44.6	55 017 : 13.7	0
Nebraska	231 906 : 34.9	355 665 : 53.6	76 103 : 11.5	0
Nevada	203 388 : 45.6	198 775 : 44.6	43 855 : 9.8	0
New Hampshire	245 260 : 50.0+	196 740 : 40.1	48 140 : 9.8	−1
New Jersey	1 599 932 : 54.5−	1 080 041 : 36.8	257 979 : 8.8	+1
New Mexico	252 215 : 51.1−	210 791 : 42.7	30 978 : 6.3	+1
New York	3 515 027 : 60.0	1 862 344 : 31.8	482 770 : 8.2	0
North Carolina	1 099 132 : 44.3	1 214 399 : 49.0	165 301 : 6.7	0
North Dakota	106 360 : 40.4	124 507 : 47.3−	32 566 : 12.4	+1
Ohio	2 100 690 : 47.8	1 823 859 : 41.5	470 680 : 10.7	0
Oklahoma	488 102 : 40.6	582 310 : 48.5	130 788 : 10.9	0
Oregon	326 099 : 49.8−	256 100 : 39.1	73 265 : 11.2	+1
Pennsylvania	2 206 241 : 49.8	1 793 568 : 40.5	430 082 : 9.7	0
Rhode Island	220 592 : 61.5	98 325 : 27.4	39 965 : 11.1	0
South Carolina	495 458 : 44.1	564 387 : 50.3	63 300 : 5.6	0
South Dakota	139 295 : 43.4	150 508 : 46.9	31 248 : 9.7	0
Tennessee	905 599 : 48.4	860 809 : 46.0	105 577 : 5.6	0
Texas	2 455 735 : 44.1	2 731 998 : 49.1	377 530 : 6.8	0
Utah	220 197 : 34.1	359 394 : 55.7	66 100 : 10.2	0
Vermont	138 400 : 55.5	80 043 : 32.1	30 912 : 12.4	0
Virginia	1 070 990 : 45.6	1 119 974 : 47.7−	158 707 : 6.8	+1
Washington	899 645 : 52.9	639 743 : 37.6	161 642 : 9.5	0
West Virginia	324 394 : 51.7	231 908 : 37.0	70 853 : 11.3	0
Wisconsin	1 071 859 : 50.0	845 172 : 39.4	227 426 : 10.6	0
Wyoming	77 897 : 37.3−	105 347 : 50.4	25 854 : 12.4	+1
Candidate's Total	45 597 228 : 49.9	37 841 817 : 41.4+	7 862 865 : 8.6	−1

State-by-state standard rounding generates five times the discrepancy −1 and eight times +1. Trailing signs indicate the Webster discrepancy finish. Data from International Herald Tribune, 7 November 1996.

When rounded probabilities do sum to one, one may wish to seek some explanation. Diaconis and Freedman (1979) investigate the leading digit data of Benford (1938, page 553). It seems likely that the data were beautified in order to better fit the hypothetical model. Heiligers and Schneider (1992), in their Table 1, present weights that sum to one. The reason is (personal communication) that they first calculated all figures but one, and in a final step fitted the last figure to force a sum of one. All numbers in Pukelsheim (1993) sum to one when appropriate, by using a nondefective method of rounding.

There are plenty of rounding methods that do preserve the side condition of adding up to one. They have been proposed and investigated by politicians and political scientists, in the study of apportionment problems for electoral bodies. A fascinating court case of current date is documented by Ernst (1993). Balinski and Young (1982) is the authoritative monograph on the subject, and a gem of mathematical writing. They prove that among all rounding methods only divisor methods are not affected by severe deficiencies. These "paradoxes" are illustrated by reference to pertinent precedences in the history of the USA. Therefore we restrict our investigation to divisor methods of rounding which, for our purposes, we prefer to call *multiplier methods*.

Our notation is the following. We have c categories (Mosteller, Youtz, and Zahn 1967; Diaconis and Freedman 1979), political candidates (Balinski and Young 1982), or support points of an experimental design (Pukelsheim 1993). The accuracy (precision, house size, number of observations) is designated by n. Given a set of weights $(w_1, \dots, w_c)$ (the exact probabilities in a table, the proportion of votes per candidate, the weights of an approximate design), the *rounding problem* consist of finding integers $(n_1, \dots, n_c)$ so that

$$\frac{n_i}{n} \approx w_i \qquad \text{and} \qquad \sum\nolimits_{i \le c} n_i = n.$$

In Section 2 we review the basic properties of *rounding rules* R, following Balinski and Young (1982). For $x \ge 0$, $R(x)$ is not single-valued, but a one-element or two-element set. The associated *rounding functions* r are characterized by the condition $r(x) \in R(x)$. The two most important families are the q-stationary roundings, and the p-mean roundings.

Section 3 is devoted to multiplier methods. Given a *multiplier* $\nu > 0$, the weights w_i are rounded to $r(\nu w_i) \in R(\nu w_i)$. The sum $\sum_{i \le c} r(\nu w_i)$ can then be augmented or reduced by varying the multiplier ν, according as the *discrepancy*

$$d = \Big(\sum\nolimits_{i \le c} r(\nu w_i) \Big) - n$$

is negative or positive. The discrepancy vanishes for some value of ν. However, that value depends on the specific set of weights $(w_1, \dots, w_c)$ being rounded.

The algorithm that we propose in Section 4 starts with an initial guess for the multiplier ν. The first step is called the *multiplier start* and gets close to a result, but may leave a nonzero discrepancy. The second step, the *discrepancy finish*, consists of a few corrective iterations, to augment some of the rounded weights if there is a negative discrepancy, or to reduce some of them if the discrepancy is positive. The initial guess for ν is crucial and should work reasonable well over the whole possible range of weights.

Therefore, in Section 5, we assume the weights $(W_1, \ldots, W_c)$ to be uniformly distributed on the probability simplex of $\mathbb{R}^c$. We establish the existence of a unique multiplier η for which the expected value of the discrepancy vanishes. Unfortunately, we know of no simple closed form expression for η.

For more explicit results, Section 6 restricts attention to multiplier methods based on rounding rules that are stationary. The expected value of the discrepancy turns out to be a sum of powers. Some brief historical comments on sums of powers are gathered in Section 7.

In Section 8 we find unbiased multipliers for q-stationary rounding rules. Unbiasedness is understood in the asymptotic sense that the expected discrepancy vanishes for an increasing accuracy n. The resulting multiplier depends on the number of categories c, the stationarity parameter $q \in [0,1]$, and the accuracy n,

$$\nu_{c,q,n} = n + c\left(q - \frac{1}{2}\right). \tag{1}$$

With these multipliers the expected discrepancy stays bounded of order $1/n$ as n tends to infinity. See also Happacher (1996), and Happacher and Pukelsheim (1996).

In summary, the *method of Adams* ($q = 0$, i.e., rounding up) has multiplier $\nu_{c,0,n} = n - c/2$, as recommended by Pukelsheim and Rieder (1992), and Pukelsheim (1993, Section 12.4). The other extreme is the *method of Jefferson* ($q = 1$, i.e., rounding down), with $\nu_{c,1,n} = n + c/2$. For the *method of Webster* ($q = 1/2$, i.e., standard rounding) the multiplier $\nu_{c,1/2,n} = n$ is the one that would also be suggested by the Rule of Three.

Standard rounding is just the same as the starting multiplier step for the method of Webster. *The reason for its frequent failure to add to one is that it misses out on the discrepancy finish of the algorithm.* Or the other way round: Standard rounding followed by the discrepancy finish is a viable method, the method of Webster, which indeed is the one most pronouncedly advocated by Balinski and Young (1982). In Tables 1 and 2 we use a trailing plus sign or minus sign to indicate the corrective action of the Webster discrepancy finish.

2. ROUNDING RULES

Balinski and Young (1982, page 99) base the definition of a rounding rule R on a *signpost sequence* $s(k) \in [k, k+1]$, for $k = 0, 1, \ldots$. The signposts are assumed to be strictly increasing, in order to avoid three-way ties. When $x = s(k)$ coincides with a signpost, there is a two-way tie and $R(x)$ is defined to be the two-element set $\{k, k+1\}$. When $x \geq 0$ lies between two signposts, $x \in \big(s(k-1), s(k)\big)$, then $R(x) = \{k\}$ is a singleton; for the starting value $k = 0$ we adjoin $s(-1) = -1$. Formally, we define

$$R(x) = \begin{cases} \{k, k+1\}, & \text{if } x = s(k); \\ \{k\}, & \text{if } x \in \big(s(k-1), s(k)\big). \end{cases}$$

Alternatively, the signpost sequence and the rounding rule fulfill the basic relation

$$k \in R(x) \qquad \Longleftrightarrow \qquad s(k-1) \leq x \leq s(k), \tag{2}$$

for all $k = 0, 1, \ldots$ and for all $x \geq 0$.

We concentrate on *q-stationary rounding rules*, for some fixed value $q \in [0, 1]$. By definition, they are given by the signpost sequence

$$s_q(k) = k + q \qquad \text{for all } k = 0, 1, \ldots . \tag{3}$$

They appear implicitly in Diaconis and Freedman (1979, equation (3.2)), with a view towards equivariance. Our terminology is inspired by Balinski and Rachev (1993). Kopfermann (1991, page 124) calls the induced apportionment methods "linear". Saari (1994) considers this family in his equation (4.3.13).

The treatise of Balinski and Young (1982) shows that the *p-mean rounding rules*, with $p \in [-\infty, \infty]$, play a greater historical role. The defining signpost sequences are

$$\widetilde{s}_p(k) = \left(\frac{k^p + (k+1)^p}{2}\right)^{1/p} \qquad \text{for all } k = 0, 1, \ldots \tag{4}$$

when $p \in (-\infty, \infty)$. The extreme cases $\widetilde{s}_{-\infty}(k) = k = s_0(k)$ and $\widetilde{s}_{\infty}(k) = k + 1 = s_1(k)$ coincide with the extreme members among the stationary signpost sequences from (3). The p-mean rounding rules are nonstationary, except for $p = -\infty, 1, \infty$.

Both families contain the *classical rounding rules:* rounding up, standard rounding, and rounding down. For fixed $p \in (-\infty, \infty)$, as k tends to infinity, we have

$$\widetilde{s}_p(k) = k + \frac{1}{2} + O\left(\frac{1}{k}\right).$$

Table 3. The Three Classical Rounding Rules

	Rounding up	*Standard rounding*	*Rounding down*
Signposts	k	$k + \frac{1}{2}$	$k + 1$
q in (3)	0	1/2	1
p in (4)	$-\infty$	1	∞
Method	Adams	Webster	Jefferson

Standard rounding and rounding up or down are members of the q-stationary and p-mean rounding rules. The corresponding rounding methods are associated with historical names, see Balinski and Young (1982).

This is of special interest for our investigations, because we round numbers of the form nw_i, and we are interested in the behavior as n tends to infinity. From this asymptotic viewpoint, the family of p-mean roundings shrinks to the classical rounding rules listed in Table 3. Hence the stationary rounding rules appear to form the richer family.

The fact that a rounding rule R is a set-valued mapping is a bit cumbersome computationally. Therefore we also introduce *rounding functions r that are compatible with R*, by demanding

$$r(x) \in R(x) \qquad \text{for all } x \geq 0.$$

Hence r is an increasing, piecewise constant function, with jumps at $s(k)$ where it takes the value k or $k+1$. Evidently a rounding rule R induces many rounding functions r, of which traditionally some are more often used than others.

Standard rounding, $q = 1/2$, is usually carried out with the rounding function $r_{1/2}$ as described in Section 1. For rounding up, $q = 0$, the counterpart is the *ceiling function* $r_0(x) = \lceil x \rceil = \min\{k : k \geq x\}$. For rounding down, $q = 1$, a convenient rounding function is the *floor function* or integer part $r_1(x) = \lfloor x \rfloor = \max\{k : k \leq x\}$.

Rounding functions apply to an individual, single argument. When it is a set of weights that is under consideration, the point is to subject each individual term in exactly the same way to the given rounding function r. In plain words, as far as the rounding function r is concerned, it ought to treat each term *in fairness and justice*. It is a second, separate step to ensure that the rounded quantities combine to yield the required total. This is what multiplier methods accomplish.

3. MULTIPLIER METHODS

Any rounding rule R has a multiplier method that comes with it. The multiplier methods that correspond to the classical rounding rules of rounding up, standard rounding, or rounding down are named after Adams, Webster and Jefferson (Balinski and Young 1982). See Table 3.

Multiplier methods introduce a new, continuous variable, the *multiplier* $\nu \geq 0$. This additional degree of freedom is used to fit the side condition that rounded weights sum to one. It is convenient to assemble the weights into a vector $\mathbf{w} = (w_1, \ldots, w_c)$. Without loss of generality we assume $w_i > 0$ for all $i = 1, \ldots, c$. For a given integer $n \geq 1$, the goal is to round w_i to a rational number of the form n_i/n, that is, to find appropriate numerators n_i. The condition $\sum_{i \leq c} n_i/n = 1$ turns into $\sum_{i \leq c} n_i = n$.

Rounding rules do not resolve two-way ties, nor do multiplier methods. Hence a set of possible numerators is proposed, according to the definition

$$M_R(\mathbf{w}, n) = \left\{ (n_1, \ldots, n_c) : \exists \nu \geq 0 \forall i \leq c \qquad n_i \in R(\nu w_i) \quad \text{and} \quad \sum\nolimits_{i \leq c} n_i = n \right\}.$$

In the rare, special case when $s(0) = 0$ and $0 \leq n < c$, we define $n_i = 1$ or $n_i = 0$ according as w_i is among the n largest weights or not. In general we adopt the convention $0/w_i < 0/w_j$ for $w_i > w_j$.

In terms of the signposts $s(k)$ that determine the rounding rule R an alternative characterization is as follows.

Theorem 1 (Max–Min Inequality). Let $n_1, \ldots, n_c$ be integers with $\sum_{i \leq c} n_i = n$. Then $(n_1, \ldots, n_c)$ is a member of $M_R(\mathbf{w}, n)$ if and only if

$$\max_{i \leq c} \frac{s(n_i - 1)}{w_i} \leq \min_{i \leq c} \frac{s(n_i)}{w_i}. \tag{5}$$

Proof. The basic relation (2) now reads $s(n_i - 1) \leq \nu w_i \leq s(n_i)$ for all $i = 1, \ldots, c$. Division by w_i establishes the result. □

Starting out from an arbitrary member $(n_1, \ldots, n_c)$ in $M_R(\mathbf{w}, n)$ and changing the accuracy n, we can step up to a member of $M_R(\mathbf{w}, n+1)$ or step down to a member of $M_R(\mathbf{w}, n-1)$ without recalculating any multipliers. Let $\mathcal{J}$ and $\mathcal{K}$ be the set of those subscripts that attain the minimum and maximum in (5),

$$\mathcal{J} = \left\{ j \le c : \frac{s(n_j)}{w_j} = \min_{i \le c} \frac{s(n_i)}{w_i} \right\},$$
$$\mathcal{K} = \left\{ k \le c : \frac{s(n_k - 1)}{w_k} = \max_{i \le c} \frac{s(n_i - 1)}{w_i} \right\}.$$

The next two theorems state that $\mathcal{J}$ consists of the *augmentation candidates* and $\mathcal{K}$ of the *reduction candidates*, and that these sets also facilitate an enumeration of the set $M_R(\mathbf{w}, n)$.

Theorem 2 (Augmentation, Reduction). Let $(n_1, \ldots, n_c)$ *be a member of* $M_R(\mathbf{w}, n)$. *Then we have*

$$j \in \mathcal{J} \iff (n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_c) \in M_R(\mathbf{w}, n+1),$$
$$k \in \mathcal{K} \iff (n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_c) \in M_R(\mathbf{w}, n-1).$$

Proof. The direct part of the proof verifies condition (5) of Theorem 1, see Balinski and Young (1982, Proposition 3.3), or Pukelsheim (1993, Theorem 12.5b). For the converse direction Theorem 1 implies $s(n_j)/w_j \le s(n_i)/w_i$ for all $i = 1, \ldots, c$. □

There always exists a multiplier ν that can be used in the definition of $M_R(\mathbf{w}, n)$. This follows by induction from the augmentation part of Theorem 2. As is implied by Theorem 1, the set of multipliers ν that work for $\mathbf{w}$ form a compact interval, with lower and upper endpoint taken from (5).

Theorem 3 (Enumeration). Let $(n_1, \ldots, n_c)$ *be a member of* $M_R(\mathbf{w}, n)$. *Then the set* $M_R(\mathbf{w}, n)$ *is a singleton if and only if strict inequality holds in* (5). *Otherwise equality holds in* (5) *and there are* $\binom{a+b}{a}$ *roundings in* $M_R(\mathbf{w}, n)$, *where* a *is the number of augmentation candidates in* $\mathcal{J}$ *and* b *is the number of reduction candidates in* $\mathcal{K}$.

Proof. The proof uses similar arguments that establish Theorem 12.7 in Pukelsheim (1993). For details see Theorem 1 in Happacher and Pukelsheim (1996). □

4. ROUNDING ALGORITHM

We can now be more precise about our multiplier method algorithm that was mentioned in Section 1. An Emacs Lisp implementation of the algorithm is proposed by Dorfleitner, Happacher, Klein and Pukelsheim (1996).

The algorithm is initialized by choosing a rounding function r that is compatible with the rounding rule R, and by picking a multiplier ν that is thought to work reasonably well for the given accuracy n.

- The first step, the multiplier start, rounds the weights w_i to n_i/n with $n_i = r(\nu w_i)$.
- The second step, the discrepancy finish, evaluates the discrepancy $d = \left(\sum_{i\le c} n_i\right) - n$. While $d \neq 0$, we loop to augment or reduce $n_1, \ldots, n_c$ according to Theorem 2.

Upon termination the set $M_R(\mathbf{w}, n)$ may be enumerated using Theorem 3.

For standard rounding with multiplier $\nu = n$, the result of Mosteller, Youtz and Zahn (1967), and Diaconis and Freedman (1979) says that the algorithm does *not* terminate with the first step, with probability one as n and c tend to infinity. This statement should not be construed as evidence against the multiplier start. Instead it emphasizes the need to continue on into the discrepancy finish.

The initial choice of the multiplier ν depends on the distribution of the weight vectors $\mathbf{w}$ that are fed into the algorithm. Specific applications may suggest specific distributions. Lacking such specifications, we take any point $\mathbf{w}$ in the probability simplex of $\mathbb{R}^c$ to be equally probable.

5. UNIFORMLY DISTRIBUTED WEIGHTS

In the sequel we assume that the weight vector $(W_1, \ldots, W_c)$ is random, with a uniform distribution on the probability simplex of $\mathbb{R}^c$. The number of categories, c, remains fixed. Let R be a rounding rule based on the signposts $s(k)$.

The event that for a multiplier $\nu > 0$ a component hits a signpost, $\bigcup_{i\le c}\bigcup_{k\ge 0}\{\nu W_i = s(k)\}$, has probability zero. Hence, almost surely, $R(\nu W_i)$ is a singleton, and any two rounding functions r and $\tilde{r}$ compatible with R satisfy $R(\nu W_i) = \{r(\nu W_i)\} = \{\tilde{r}(\nu W_i)\}$, for every multiplier $\nu > 0$. Thus we lose much of the discrete charm of the deterministic version of the problem, but are free to choose an arbitrary rounding function r provided it is compatible with the rounding rule R.

Given a multiplier $\nu > 0$ we define the *total*

$$T_{c,r,\nu} = \sum_{i \le c} r(\nu W_i). \tag{6}$$

This is an integer-valued random variable that by choice of ν we would like to bring close to n, in order to achieve a small discrepancy $T_{c,r,\nu} - n$. Indeed, there is a unique multiplier $\eta_{c,r,n}$ that makes the expected total equal to n.

Theorem 4 (Existence). For $\nu > 0$ we introduce $\ell = \max\{k \ge -1 : s(k) \le \nu\}$. Then we have

$$\mathrm{E}[T_{c,r,\nu}] = \frac{c}{\nu^{c-1}} \sum_{k=0}^{\ell} \Big(\nu - s(k)\Big)^{c-1}.$$

If $s(0)$ is positive, then for all $n \ge 0$ there exists a unique multiplier $\eta_{c,r,n} \ge s(0)$ that satisfies $\mathrm{E}[T_{c,r,\eta_{c,r,n}}] = n$. If $s(0)$ is zero, then for all $n \ge c$ there exists a unique multiplier $\eta_{c,r,n} \ge s(1)$ that satisfies $\mathrm{E}[T_{c,r,\eta_{c,r,n}}] = n$.

Proof. Define the integer-valued random variable $N_1 = r(\nu W_1)$. By exchangeability we get $\mathrm{E}[T_{c,r,\nu}] = c\,\mathrm{E}[N_1]$. For $k = 0, 1, \ldots$ we have $\{N_1 > k\} = \{W_1 > s(k)/\nu\}$. This yields $\mathrm{P}(N_1 > k) = (1 - s(k)/\nu)^{c-1}$ for $k \le \ell$, and $\mathrm{P}(N_1 > k) = 0$ for $k > \ell$. From $\mathrm{E}[T_{c,r,\nu}] = c \sum_{k=0}^{\infty} \mathrm{P}(N_1 > k)$ we now obtain the expression for the expected total.

The function $f(\nu) = \mathrm{E}[T_{c,r,\nu}]$ is continuous on $(0, \infty)$. If $s(0)$ is positive then f vanishes on $(0, s(0)]$, if $s(0)$ is zero then f equals c on $(0, s(1)]$; in either case f afterwards strictly increases to infinity. Therefore the equation $f(\nu) = n$ has a unique solution $\eta_{c,r,n} \ge s(0)$ or $\eta_{c,r,n} \ge s(1)$ according as $s(0)$ is positive or zero. □

6. STATIONARY ROUNDING METHODS

From now on we restrict attention to a q-stationary rounding function r_q, with signpost sequence (3), and denote the total from (6) by

$$T_{c,q,\nu} = \sum_{i \le c} r_q(\nu W_i). \tag{7}$$

The basic relation (2) almost surely yields $\nu W_i - q < r_q(\nu W_i) < \nu W_i - q + 1$, and

$$\nu - cq < T_{c,q,\nu} < \nu - cq + c.$$

With $\nu_{c,q,n} = n + c(q - 1/2)$ from (1) we almost surely obtain symmetry around n,

$$n - \frac{c}{2} < T_{c,q,n+c(q-1/2)} < n + \frac{c}{2}.$$

In case of $c = 2$ categories, the integer-valued random variable $T_{2,q,n+2q-1}$ strictly lies between $n - 1$ and $n + 1$. Hence it degenerates to a constant,

$$T_{2,q,n+2q-1} = n \qquad \text{almost surely.}$$

In particular, we have $\eta_{c,r_q,n} = n + 2q - 1$ in Theorem 4. Thus the discrepancy vanishes almost surely when a q-stationary rounding rule is applied to two categories with multiplier $n + 2q - 1$. For standard rounding this is already pointed out by Mosteller, Youtz and Zahn (1967, page 850). In plain words, two candidates never create a rounding problem.

In case of three or more categories we can still be more explicit about the expected total that appears in Theorem 4.

Theorem 5 (Trisection). For $q \in [0,1]$ and $\nu > 0$, we introduce $\ell = \lfloor \nu - q \rfloor$ and $\epsilon = \nu - q - \ell \in [0,1]$. Then we have $\nu = \ell + q + \epsilon$, and

$$\mathrm{E}[T_{c,q,\nu}] = \frac{c}{\nu^{c-1}} \sum_{k=0}^{\ell} (k + \epsilon)^{c-1}.$$

Proof. Since $s_q(k) = k + q$, the result follows from Theorem 4, by replacing $\nu - k - q$ by $\ell - k + \epsilon$ and reversing the order of summation. □

For stationary rounding rules the expected total thus is a sum of powers, a prominent subject in former centuries.

7. SUMMA POTENTATIS

Formulas for sums of squares already appear in Fibonacci's *Liber Abbaci* in the thirteenth century, see Lüneburg (1993, page 132). The idea of expressing $\sum_{k \le \ell} k^{c-1}$ as a polynomial in ℓ of degree c is presented in Faulhaber's book *Miracula Arithmetica*, published 1622 in Augsburg. Schneider (1993) and Hawlitschek (1995) tell of the man and his time.

Johannes Faulhaber (1580–1635) was a *Rechenmeister* in the town of Ulm on the Danube river. Like other craftsmen, the *Rechenmeister* kept their knowledge a privilege that was not laid open to the public outside the profession's guild. Following this tradition, Faulhaber advertised his arithmetical abilities by publishing a book of problems he claimed he could solve. He was deeply hurt and felt impaired in his business when a colleague from Nürnberg put out a solution manual soon after.

However, Faulhaber's publications testify to the change of scientific spirit that evolved during that time. His later writings do explain the solution methods and delineate the underlying systematic insight. He even took up the new custom of referencing the sources used. Faulhaber also demonstrated his genius by predicting Judgment Day, repeatedly though not successfully. This and other dubious mythical speculations may have distracted from his mathematical achievements that eventually fell into oblivion.

8. UNBIASED MULTIPLIERS

Elementary calculus gives a feeling for the polynomial representation of the sum of powers that appears in Theorem 5:

$$c\sum_{k=0}^{\ell}(k+\epsilon)^{c-1} \approx c\int_{-1/2}^{\ell+1/2}(x+\epsilon)^{c-1}\,dx \approx \left(\ell+\frac{1}{2}+\epsilon\right)^c = \left(\nu-q+\frac{1}{2}\right)^c.$$

Geometrically, the addition of $1/2$ serves as a continuity correction. Numerically, a polynomial in $\ell+1/2+\epsilon$ approximates the sum much better than a polynomial in $\ell+\epsilon$, in that the exponents drop off in steps of two, see Burrows and Talbot (1984). This enables us to evaluate the asymptotic behavior of the expected total of Theorem 5.

Theorem 6 (Expectation). For $q \in [0,1]$ and $\nu > q$ we have, with $\ell = \lfloor \nu - q \rfloor$ and $\epsilon = \nu - q - \ell \in [0,1]$ as in Theorem 5,

$$\begin{aligned} \mathrm{E}[T_{c,q,\nu}] &= \frac{(\nu-q+\frac{1}{2})^c}{\nu^{c-1}}\left\{1-\frac{1}{12}\binom{c}{2}\frac{1}{(\nu-q+\frac{1}{2})^2}+\frac{7}{240}\binom{c}{4}\frac{1}{(\nu-q+\frac{1}{2})^4}\right. \\ &\qquad \left.-\frac{31}{1344}\binom{c}{6}\frac{1}{(\nu-q+\frac{1}{2})^6}+\frac{127}{3840}\binom{c}{8}\frac{1}{(\nu-q+\frac{1}{2})^8}\mp\cdots\right\}+\frac{\pi_c(\epsilon)}{\nu^{c-1}} \\ &= \nu - c\left(q-\frac{1}{2}\right)+\rho_c(\nu), \end{aligned}$$

with a polynomial π_c in ϵ of degree c in the first representation, and a remainder term $\rho_c(\nu) = O(1/\nu)$ as $\nu \to \infty$ in the second representation. For even c the sum in the first representation terminates with last binomial coefficient being equal to $\binom{c}{c-2}$.

Proof. Section 2 of Burrows and Talbot (1984) carries over to the shifted summands $k+\epsilon$ that appear in Theorem 5, provided they start the summation at $k=0$ rather than $k=1$. An analysis of their formula (2.11) provides the first representation given above. The second representation follows from the binomial expansion of $(\nu-q+1/2)^c$. □

The remainder terms $\rho_c(\nu)$ for $c = 2, 3, 4$ categories are as follows:

$$\rho_2(\nu) = \frac{q(q-1) - \epsilon(\epsilon-1)}{\nu},$$

$$\rho_3(\nu) = 3\frac{1/6 + q(q-1)}{\nu} - \frac{q(q-\frac{1}{2})(q-1) + \epsilon(\epsilon-\frac{1}{2})(\epsilon-1)}{\nu^2},$$

$$\rho_4(\nu) = 6\frac{1/6 + q(q-1)}{\nu} - 4\frac{q(q-\frac{1}{2})(q-1)}{\nu^2} + \frac{q^2(q-1)^2 - \epsilon^2(\epsilon-1)^2}{\nu^3},$$

$$\rho_c(\nu) = \binom{c}{2}\frac{1/6 + q(q-1)}{\nu} + O\left(\frac{1}{\nu^2}\right) \qquad \text{for all } c \geq 3.$$

From this it is easy to obtain the asymptotic order,

$$|\rho_2(\nu)| \leq \frac{1}{4\nu},$$

$$|\rho_3(\nu)| \leq \frac{1}{2\nu} + \frac{1}{10\nu^2},$$

$$|\rho_4(\nu)| \leq \frac{1}{\nu} + \frac{1}{5\nu^2} + \frac{1}{16\nu^3}.$$

In case $c = 2$ the multiplier $\eta_{c,r_q,n} = n + 2q - 1$ yields $\rho_2(n + 2q - 1) = 0$, see Section 5. For three or more categories, Theorem 6 has a companion result for the variance.

Theorem 7 (Variance). For $c \geq 3$ categories and $q \in [0, 1]$ we have

$$\mathrm{V}[T_{c,q,\nu}] = \frac{c}{12} + \frac{2}{3}\binom{c}{2}\frac{q(q-\frac{1}{2})(q-1)}{\nu} + O\left(\frac{1}{\nu^2}\right) \qquad \text{as } \nu \to \infty.$$

Proof. Straightforward, though lengthy calculations establish the result. For details see Happacher (1996). □

The preceding formulas emphasize the three classical rounding methods of Adams, Webster and Jefferson. For instance, in general the variance equals $c/12$ plus a remainder term that is bounded of order $1/\nu$. For $q = 0, 1/2, 1$, however, the order improves to $1/\nu^2$. The term $c/12$ in the variance points towards the convolution of rectangular distributions governing the asymptotic distribution theory in Mosteller, Youtz and Zahn (1967), and Diaconis and Freedman (1979). See also Happacher (1996).

Table 4. Expected Discrepancy $\mathrm{E}[D_{c,q,n}]$ *for the Three Classical Rounding Methods*

c	*Adams* ($q=0$)	*Webster* ($q=1/2$)	*Jefferson* ($q=1$)
3	$\frac{1}{2n-3}$	$-\frac{1}{4n}$	$\frac{1}{2n+3}$
4	$\frac{1}{n-2}$	$-\frac{1}{2n}$	$\frac{1}{n+2}$
5	$\frac{10}{3(2n-5)}-\frac{4}{3(2n-5)^3}$	$-\frac{5}{6n}+\frac{7}{48n^3}$	$\frac{10}{3(2n+5)}-\frac{4}{3(2n+5)^3}$
6	$\frac{5}{2(n-3)}-\frac{1}{2(n-3)^3}$	$-\frac{5}{4n}+\frac{7}{16n^3}$	$\frac{5}{2(n+3)}-\frac{1}{2(n+3)^3}$
7	$\frac{7}{2n-7}-\frac{28}{3(2n-7)^3}+\frac{16}{3(2n-7)^5}$	$-\frac{7}{4n}+\frac{49}{48n^3}-\frac{31}{192n^5}$	$\frac{7}{2n+7}-\frac{28}{3(2n+7)^3}+\frac{16}{3(2n+7)^5}$
8	$\frac{14}{3(n-4)}-\frac{7}{3(n-4)^3}+\frac{2}{3(n-4)^5}$	$-\frac{7}{3n}+\frac{49}{24n^3}-\frac{31}{48n^5}$	$\frac{14}{3(n+4)}-\frac{7}{3(n+4)^3}+\frac{2}{3(n+4)^5}$

The expectation of the asymptotically unbiased discrepancy $D_{c,q,n}$ *is bounded of order* $1/n$. *Depending on the categories, c, and the method, q, the constants exhibit a strange symmetry between* $q=0$ *and* $q=1$.

Finally we return to the discrepancy $T_{c,q,\nu}-n$ that instigated the study of the totals in (7). With $\nu_{c,q,n}$ from (1) we define the *asymptotically unbiased discrepancy*,

$$D_{c,q,n}=T_{c,q,n+c(q-1/2)}-n.$$

Theorem 6 verifies the asymptotic claim that is implied by the name,

$$\mathrm{E}[D_{c,q,n}]=\nu_{c,q,n}-c\left(q-\frac{1}{2}\right)+O\left(\frac{1}{\nu_{c,q,n}}\right)-n=O\left(\frac{1}{n}\right)\quad\xrightarrow{n\to\infty}\quad 0.$$

For $c\le 8$ categories and the classical methods, the exact expected values of $D_{c,q,n}$ are shown in Table 4. The exact distribution of $D_{c,q,n}$ is derived by Happacher (1996).

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