Absolute regularity of stochastic processes and fields – *Mixing* is a structural property of the probability law of a family of dependent random variables (e.g., stochastic processes or fields; cf. **Stochastic process**; **Random field**) which describes the degree of (weak) dependence between subfamilies of random variables defined on distant parts of the index set. First, to avoid confusion, one has to distinguish between **mixing** properties of invariant (not necessarily finite or probability) measures used in **ergodic theory** and mixing conditions in **probability theory** based on appropriate mixing coefficients measuring the dependence between σ-algebras generated by random variables on disjoint index subsets. An essential feature of these coefficients is, in contrast to ergodic theory, that they provide uniform bounds of dependence over all events of the σ-algebras; see [2] for a survey. This is the main difficulty in verifying a specific mixing condition of a given stochastic process. On the other hand, this uniformity is the key to proving bounds of the covariances in terms of the mixing coefficient and the moments of the marginal distributions. In turn, from these covariance estimates one can obtain central limit theorems, other weak and strong limit theorems for sums of random variables, functional limit theorems, etc.

An important, and in many situations quite natural, *measure of dependence between two arbitrary sub-σ-fields* $\mathcal{A},\mathcal{B}\subset\mathcal{F}$ is the *absolute regularity* (or *β-mixing* or *weak Bernoulli*) *coefficient*

$$\beta(\mathcal{A},\mathcal{B}) = \mathsf{E}\sup_{B\in\mathcal{B}}\left|\mathsf{P}(B\mid\mathcal{A})-\mathsf{P}(B)\right|,$$

which was first studied by V.A. Volkonskiĭ and Yu.A. Rozanov [6], who attributed this measure of dependence to A.N. Kolomogorov. As has been shown in [6], $\beta(\mathcal{A},\mathcal{B})$ can be described in a different way. Let $\mathsf{P}_{\mathcal{A}\otimes\mathcal{B}}$ be the restriction to the product-σ-field $\mathcal{A}\otimes\mathcal{B}$ of the measure on $\Omega\times\Omega$ induced by P and the diagonal mapping $\omega\mapsto(\omega,\omega)$ and let $\mathsf{P}_{\mathcal{A}}$, $\mathsf{P}_{\mathcal{B}}$ denote, respectively, the restrictions to $\mathcal{A}$ and $\mathcal{B}$ of P. Then

$$\beta(\mathcal{A},\mathcal{B}) = \sup_{C\in\mathcal{A}\otimes\mathcal{B}}\left|\mathsf{P}_{\mathcal{A}\otimes\mathcal{B}}(C)-(\mathsf{P}_{\mathcal{A}}\times\mathsf{P}_{\mathcal{B}})(C)\right| =$$

$$=\frac{1}{2}\sup\sum_{i=1}^{I}\sum_{j=1}^{J}\left|\mathsf{P}(A_i\cap B_j)-\mathsf{P}(A_i)\,\mathsf{P}(B_j)\right|,$$

where the supremum is taken over all pairs of finite partitions $\{A_1,\dots,A_I\}$ and $\{B_1,\dots,B_J\}$ of Ω such that $A_i\in\mathcal{A}$, $i=1,\dots,I$, and $B_j\in\mathcal{B}$, $j=1,\dots,J$. Absolute regularity of a stochastic process $\{X_t : t\in\mathbf{Z}^1\}$ is then defined by $\lim_{s\to\infty}\beta_X(s)=0$, where

$$\beta_X(s)=\sup_t\beta\left(\sigma\{X_u : u\le t\},\sigma\{X_u : u\ge t+x\}\right).$$

Here, $\sigma\{X_u : u\in I\}$ denotes the σ-algebra generated by the random variables between the braces. Note that β-mixing is one of the weakest mixing conditions, but is indeed slightly stronger than α-mixing; see [1] and references therein. There are a lot of examples of β-mixing random processes in discrete and continuous time; see [1], [2]. Further examples are renewal processes (see [3] and **Renewal process**), and solutions of stochastic differential equations (see [5] and **Stochastic differential equation**). Various versions of the **central limit theorem** for β-mixing random variables were proved under a certain rate of decay of $\beta_X(s)$ as $s\to\infty$ and the existence of $\mathsf{E}|X_t|^{2+\delta}$ for some $\delta>0$, see [7]. Roughly speaking, the asymptotic theory of β-mixing random variables is quite similar to that of α-mixing random variables.

It should be mentioned that absolute regularity of random fields $\{X_z : z\in\mathbf{Z}^d\}$ (like other strong mixing properties) is very distinct from absolute regularity

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Mixing
Stochastic process
Random field
mixing
ergodic theory
probability theory
weak→ weak limit theorem
strong limit theorems→ strong limit theorem
measure of dependence between two arbitrary sub-σ-fields
absolute regularity→ absolute regularity coefficient
β-mixing→ β-mixing coefficient
weak Bernoulli→ weak Bernoulli coefficient
coefficient→ absolute regularity coefficient→ β-mixing coefficient→ weak Bernoulli coefficient
V.A. Volkonskiĭ
Yu.A. Rozanov
A.N. Kolomogorov
mixing conditions→ mixing condition
α-mixing
Renewal process
Stochastic differential equation
central limit theorem
β-mixing random variables
α-mixing random variables

of random processes, see [2]. It is not adequate to require that the β-mixing coefficient tends to zero when two half-spaces of index sets are separated by a distance tending to infinity. This is only the case for m-dependent fields (cf. **m-dependent process**). At least one of the index sets must be bounded and the content of its boundary is important in order to derive limit theorems for random fields. A central limit theorem is proved in [4]. There are many examples of spacial random structures in **stochastic geometry**, such as point processes, germ-grain models, or random tesselations, which could be shown to be β-mixing. In other words, for random fields associated with such random sets, sharp bounds of the corresponding β-mixing coefficients are available, see [4].

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m-dependent process
stochastic geometry