Finite Geometries

First Irsee Conference

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Existence conditions for ovoids in $PG(3,q)$

Simeon Ball

Queen Mary College, University of London

An *ovoid* $\mathcal{O}$ of the symplectic generalised quadrangle $W(q)$ is a set of q^2+1 points with the property that no two points of $\mathcal{O}$ are collinear. When q is even the classification of ovoids of $W(q)$ is equivalent to the classification of ovoids of $PG(3,q)$ and the classification of inversive planes of even order.

Let q be even and let $W(q)$ be defined from the symplectic form

$$(x,y) = x_0y_1 + y_0x_1 + x_2y_3 + y_2x_3.$$

In [1], Glynn shows that an ovoid is given by a polynomial

$$f(X,Y) = XY + s(X,Y)^2$$

where the points of the ovoid can be assumed to be

$$\{\langle 0,1,0,0\rangle\} \cup \{\langle 1, f(x,y), x, y \mid x,y \in GF(q)\},$$

and $f(0,0)=0$. Also, he shows that every non-zero term X^iY^j of an ovoid polynomial $f(X,Y)$ has $0 \leq i+j \leq q-2$.

In fact the coefficient of X^iY^j in f is zero if $\binom{q-2-i}{j} = 0$ modulo 2, that is, if in the binary expansions of i and j they both have a 1 in the same place.

Necessary and sufficient conditions will be given that determine whether f is an ovoid polynomial or not. These are very much like Glynn's conditions for ovals given in [2].

More usable conditions can also be otained by combining the "Plane Equivalent Theorem" again given by Glynn (and independently Penttila) in [3] and Rédei polynomials.

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Cyclic caps with $O(3q^2)$ points in $AG(4,q)$ in characteristic 2

Jürgen Bierbrauer

Michigan Technological University

(Joint work with Yves Edel)

The construction of large caps in $PG(4,q)$ or $AG(4,q)$ is a difficult problem. The best known asymptotic result is a family of caps of size $O(2.5q^2)$ in $PG(4,q)$ in odd characteristic. In characteristic 2 it had hitherto not been possible to construct families of caps in $PG(4,q)$ with more than $O(2q^2)$ points. This asymptotic value is trivial to reach.

We show that for $q = 2^f,\ f$ odd, the extended q-ary dual BCH-codes of length $3(q^2+1)$ defined by exponents $0, 1$ form a $(3q^2+4)$-cap $\mathcal{K}_q \subset AG(4,q)$; equivalently, the BCH-codes satisfy $d \geq 4$.

The caps $\mathcal{K}_q$ are complete in $PG(4,q)$ and generate a good 5-dimensional code. In fact, the maximum hyperplane intersection ι_q of $\mathcal{K}_q$ is $3(q+1+t)$, where t is the largest integer $< 2\sqrt{q}$, which is $\equiv 3 \pmod 4$ provided $q > 32$, whereas $\iota_2 = 8,\ \iota_8 = 32,\ \iota_{32} = 120$. We succeed in determining the weight distribution of the q-ary code generated by $\mathcal{K}_q$. This is done by exhibiting a close link with the binary $2f$-dimensional Kloosterman code, the dual Melas code, whose weight distribution is known (Schoof, van der Vlugt 1991).

Perspectivities in irreducible collineation groups

Arrigo Bonisoli

Università di Modena e Reggio Emilia

A collineation group of a projective plane is said to be irreducible if it fixes no point, line or triangle. The importance of this concept in the study of collineation groups of finite projective planes derives primarily from the contribution of Hering in the late 70's: he was able to give a satisfactory description of such groups if the existence of central collineations is assumed.

In the past few years I have studied collineation groups of finite projective planes fixing an oval or a hyperoval. The irreducibility of the group is sometimes a consequence of other assumptions, typically involving the action of the group on the oval, such as 2–transitivity for instance. On the other hand, the existence of central collineations is generally not guaranteed by the sole assumption of irreducibility. Furthermore, ovals admitting interesting collineation groups which are not irreducible have been investigated.

In this talk I want to discuss these themes by illustrating some cases which are currently being studied. Central collineations always play a crucial role. Computational information is sometimes also important, especially in the treatment of single cases forming exceptions within classification results.

Embedding of attenuated spaces $H_q^{n,1}$ in Grassmann spaces $\mathcal{G}_{1,n,q}$

Giovanna Bonoli

Seconda Università degli Studi di Napoli

(Joint work with Sveva Freni)

Debroey [1] characterized all the semi-partial geometries with parameters (s, t, α, μ) satisfying the dual of the Veblen–Young axiom (VY*) and such that $\mu = \alpha(\alpha+1)$ or $\mu = \alpha^2$. These geometries are respectively the attenuated spaces $H_q^{n,1}$ and the Grassmann spaces $\mathcal{G}_{1,n,q}$. In this paper, starting from a semi-partial geometry with parameters $(s, t, \alpha, \mu = \alpha(\alpha+1))$, $4 \leq \alpha \neq t$, and satisfying (VY*), we construct a semi-partial geometry with parameters $(s' = \alpha^2+\alpha, t' = t, \alpha' = \alpha+1, \mu' = (\alpha')^2)$ and satisfying (VY*). In this way we reconstruct the known embedding of attenuated spaces $H_q^{n,1}$ in Grassmann spaces $\mathcal{G}_{1,n,q}$ using synthetic arguments only.

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Generalized quadrangles and regularity

Matthew Brown

University of Adelaide

A *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint, non-empty sets of objects called *points* and *lines*, and for which $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following:

(i) each point is incident with $1 + t$ lines ($t \geq 1$) and two disjoint points are incident with at most one line;

(ii) each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) if X is a point and ℓ a line not incident with X, then there is a unique pair $(Y, m) \in (\mathcal{P}, \mathcal{B})$ for which $X \, I \, m \, I \, Y \, I \, \ell$.

In this talk I will give an overview of concepts of regularity in GQs. If X is a point of a GQ $\mathcal{S}$ with collinearity relation $\sim$, then $X^\perp = \{Z \in \mathcal{P} : Z \sim X\}$, and if Y is a second point of $\mathcal{S}$, then $\{X, Y\}^\perp = X^\perp \cap Y^\perp$ and $\{X, Y\}^{\perp\perp} = (\{X, Y\}^\perp)^\perp$. We say $\{X, Y\}$ is *regular* if $|\{X, Y\}^{\perp\perp}|$ has the maximum possible value $t + 1$, and X is regular if $\{X, Y\}$ is regular for all $Y \in \mathcal{P} \setminus \{X\}$. These concepts apply dually for lines of $\mathcal{S}$. If $\mathcal{S}$ has a regular point, then $s \geq t$. The GQs $W(q)$, $T_2(\mathcal{O})$, s even, $H(3, q^2)$, dual TGQs and flock GQs all have at least one regular point.

If $\mathcal{S}$ has a regular point X, then there is an associated net $\mathcal{N}_X$ with order s and degree $t+1$. I will discuss the connections between the geometry of $\mathcal{N}_X$ and the geometry of $\mathcal{S}$, including spreads, ovoids, subquadrangles and collineations of $\mathcal{S}$. I will also discuss representing a GQ with a centre of symmetry (a regular point with a full symmetry group) by using cohomology over the net, and more generally representing a GQ with a regular point using permutations on a set with t elements associated with the net.

New spherical designs in dimension 4

Philippe Cara

Vrije Universiteit Brussel

(Joint work with N.J.A. Sloane)

A finite set X of points on the unit sphere S^{n-1} of the Euclidean space of dimension n is called a *spherical t-design in dimension n* if, for all polynomials f of degree at most t, the average of f over the set X is equal to the average of f over the sphere. Formally we have

$$\int_{S^{n-1}} f(\mathbf{x}) d\mu(\mathbf{x}) = \frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x})$$

where μ is the normalized Lebesgue measure on the sphere.

These spherical designs were introduced in 1977 by Delsarte, Goethals and Seidel. Good spherical designs are those which provide a high *strength t* with a small number of points. Often these good spherical designs arise from nice geometrical configurations with plenty of symmetry. For example the spherical 4-design with the smallest number of points in dimension 3 is the icosahedron. In dimension 4 it is well known that the 24-cell is the smallest spherical 5-design.

We present some new spherical designs of strength $t \in \{4, 5, 6, 7\}$ in dimension 4 and explain the geometry behind them.

A characterisation of two classes of (α, β)-geometries

Sara Cauchie

University of Ghent

An (α, β)-*geometry* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial linear space of order (s, t), for some s and t, such that for any $x \in \mathcal{P}$ and any $L \in \mathcal{L}$, with x not incident with L, there are either α or β points on L which are collinear with x. If $\alpha = \beta$, then $\mathcal{S}$ is a *partial geometry.* An (α, β)-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be *fully embedded* in a projective space $\mathrm{PG}(n, q)$ if $\mathcal{P}$ is a subset of the point set of $\mathrm{PG}(n, q)$, if $\mathcal{L}$ is a subset of the line set of $\mathrm{PG}(n, q)$, and if I is the incidence inherited from $\mathrm{PG}(n, q)$ and $s = q$.

F. De Clerck and J.A. Thas proved that the only non-trivial partial geometry that is fully embeddable in a projective space is H_q^n [3]. Here, H_q^n has points the points of $\mathrm{PG}(n, q)$ not contained in an $(n-2)$-dimensional subspace H of $\mathrm{PG}(n, q)$, while its lines are the lines of $\mathrm{PG}(n, q)$ that are skew to H. In [2], a characterisation of H_q^n is given, using the axiom of Veblen and Young.

In [1], we described two classes of $(q, q+1)$-geometries that are very similar to H_q^n, namely $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$. The $(q, q+1)$-geometry $\mathrm{H}_q^{n,m}$ has points the points of $\mathrm{PG}(n, q)$ not contained in an m-dimensional subspace H of $\mathrm{PG}(n, q)$, $1 \leq m \leq n-3$, and lines the lines of $\mathrm{PG}(n, q)$ that are skew to H. The $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$ has the same point set as $\mathrm{H}_q^{n,m}$, while its lines are the lines that are skew to H and that are not contained in an element of a set $\Sigma = \{\sigma_1, \ldots, \sigma_d\}$ of m'-dimensional spaces σ_i through H, for $m + 2 \leq m' \leq n - 1$, such that the union of the point set of $\sigma_i \setminus \mathrm{H}$ $(i = 1, \ldots, d)$, is a partition of $\mathrm{PG}(n, q) \setminus \mathrm{H}$. In this talk we will give a characterisation of $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$, that generalizes the existing characterisation of H_q^n.

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Quantum jump codes

Chris Charnes

University of Melbourne

It is desirable to develop quantum error-correcting strategies which possibly correct a restricted class of errors only, but which tend to minimize both redundancy and the number of recovery operations. Recently, Alber et al. took the the first steps in this direction by defining a new class of one *detected jump-error correcting quantum codes* which are capable of stabilizing distinguishable qubits. Such codes are called d-detected-jump-error correcting quantum codes, *jump codes* for short, and are naturally connected with d-designs. There are also connections with other combinatorial structures, such as iso-dual codes. Of the known constructions of jump codes, group actions play an important role. Some jump codes have a suggestive combinatorial interpretation. For example, the logical states of the $(6, 2, 2)_3$ code can be thought of as superpositions of the faces of the tetrahedron.

We survey the combinatorial constructions of jump codes and prove a necessary conditions for their existence. Representation theory shows that the Pauli group supports a two error correcting jump code. However, we do not not yet understand the connection between jump codes and quantum codes protecting against Pauli errors.

Flocks of cones: star flocks

W. E. Cherowitzo

University of Colorado at Denver

This talk describes the second in a series of articles devoted to providing a foundation for a theory of flocks of arbitrary cones in $PG(3, q)$. The desire to have such a theory stems from a need to better understand the very significant and applicable special case of flocks of quadratic cones in $PG(3, q)$. Flocks of quadratic cones have connections with several other geometrical objects, including certain types of generalized quadrangles, spreads, translation planes, hyperovals (in even characteristic), ovoids, inversive planes and quasi-fibrations of hyperbolic quadrics. This rich collection of interconnections is the basis for the strong interest in such flocks. The author has attempted incremental generalizations of flocks of quadratic cones and the similarity of the results in these investigations indicated the existence of a still more general framework. The incremental approach led to more and more difficult algebraic considerations that ultimately made this approach untenable. By jumping to the most general situation and changing our point of view (as we do in this series of articles) we can transcend those algebraic difficulties and hopefully gain a clearer perspective on the subject.

Here, we study *star flocks*, flocks of arbitrary cones in $PG(3, q)$ that share a common point. This class of flocks includes the important class of *linear flocks*, those flocks whose planes share a common line. However, we shall be primarily concerned with the *proper* star flocks, those which are not linear. Star flocks have not received much attention since J. Thas has proved that, in the quadratic cone case, star flocks are either linear (either characteristic) or Knuth-Kantor flocks (odd characteristic). In the more general setting, star flocks are seen to occupy a more significant niche in the theory.

We shall investigate the consequences of the herd spaces of proper star flocks only having (algebraic) dimension two. A connection between star flocks and Rédei blocking sets in a Desarguesian plane permits a classification of star flocks of cones which are "big enough." Finally, we look at the relationship between star flocks and a special type of conic blocking set.

Permutable polarities and a class of ovoids of the Hermitian surface

A. Cossidente

University of Basilicata

(Joint work with G.L. Ebert)

A non–degenerate Hermitian variety, defined as the set of all self–conjugate points of a non–degenerate unitary polarity of $PG(r, q^2)$, provides an important example of a finite classical polar space. The combinatorial properties of non–degenerate Hermitian varieties have been studied together with those of quadrics and linear complexes in the general theory of polar spaces. In this setting important objects are the ovoids. An *ovoid* $\mathcal{O}$ of a non–degenerate Hermitian variety $\mathcal{H}(r, q^2)$, $r \geq 3$, is a set of points of $\mathcal{H}(r, q^2)$ which has exactly one common point with every generator of $\mathcal{H}(r, q^2)$. Here, *generator* means a maximal totally singular projective subspace of $\mathcal{H}(r, q^2)$. In even dimensions r, J.A. Thas proved that $\mathcal{H}(r, q^2)$ has no ovoid. In odd dimensions r, the existence problem is still open for $r > 3$, apart from some special cases settled with a negative answer by A. Blokhuis and G.E. Moorhouse. We are interested in ovoids of $\mathcal{H}(3, q^2)$. Lines lying on $\mathcal{H}(3, q^2)$ are the generators of $\mathcal{H}(3, q^2)$, and the size of an ovoid is $q^3 + 1$.

There are so many mutually inequivalent ovoids of $\mathcal{H}(3, q^2)$ known that their classification seems to be possible only under some extra condition(s). For instance, in a recent paper jointly with G. Korchmàros all transitive ovoids of $\mathcal{H}(3, q^2)$, q even, have been classified. Essentially they are of two types, classical or of *Singer type*. The construction of the latter ovoid depends on the existence of a spread of $\mathcal{H}(2, q^2)$ left invariant under the Singer group of $\mathcal{H}(2, q^2)$.

Here we are concerned with the construction of a class of ovoids of $\mathcal{H}(3, q^2)$, q odd, which admit the linear group $PGL_2(q)$ in their automorphism group. Our construction mainly relies on the Segre's theory of quadrics that are permutable with a Hermitian surface $\mathcal{H}(3, q^2)$ of $PG(3, q^2)$.

The smallest minimal blocking sets of $\mathrm{Q}(2n,q)$, for small odd q

Jan De Beule

Ghent University

(Joint work with Leo Storme)

In [2] we used results on the size of the smallest minimal blocking sets of $\mathrm{Q}(4,q)$, q even (from [1]) and projection arguments the find the following characterization of the smallest minimal blocking sets of $\mathrm{Q}(6,q)$, q even, $q \geqslant 32$.

Theorem 1 *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{Q}(6,q)$, different from an ovoid of $\mathrm{Q}(6,q)$, $|\mathcal{K}| \leqslant q^3+q$. Then there is a point $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ with the following property: $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(4,q)$ in an ovoid $\mathcal{O}$, minus the point p itself, and $|\mathcal{K}| = q^3+q$.*

The results of [1] for $\mathrm{Q}(4,q)$, q even, can be extended to $q = 3,5,7$. Then using the same projection arguments we proved the above characterization for $q = 3,5,7$. Using inductive arguments we can find analogous results for $\mathrm{Q}(2n,q), q = 3,5,7$. The situation is now dependent on q, since $\mathrm{Q}(6,3)$ has an ovoid, but $\mathrm{Q}(6,q)$, $q = 5,7$, does not. For $q = 5,7$, we proved the following characterization.

Theorem 2 *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{Q}(2n+2,q)$, $n \geqslant 2$, $|\mathcal{K}| \leqslant q^{n+1}+q^{n-1}$. Then there is an $(n-2)$-dimensional space π, $\pi \subset \mathrm{Q}(2n+2,q)$, $\pi \cap \mathcal{K} = \emptyset$, with the following property: $T_\pi(\mathrm{Q}(2n+2,q)) \cap \mathrm{Q}(2n+2,q) = \pi\mathrm{Q}(4,q)$ and $\mathcal{K}$ is a cone with vertex π and base $\mathcal{O}$, where $\mathcal{O}$ is an ovoid of $\mathrm{Q}(4,q)$, minus the points of the vertex π, and $|\mathcal{K}| = q^{n+1}+q^{n-1}$.*

For $q = 3$ we proved a characterization using ovoids of $\mathrm{Q}(6,3)$.

Theorem 3 *Let $\mathcal{K}$ be a minimal blocking set of $\mathrm{Q}(2n+2,3)$, $n \geqslant 3$, $|\mathcal{K}| \leqslant q^{n+1}+q^{n-2}$. Then there is an $(n-3)$-dimensional space π, $\pi \subset \mathrm{Q}(2n+2,3)$, $\pi \cap \mathcal{K} = \emptyset$, with the following property: $T_\pi(\mathrm{Q}(2n+2,3)) \cap \mathrm{Q}(2n+2,3) = \pi\mathrm{Q}(6,3)$ and $\mathcal{K}$ is a cone with vertex π and base $\mathcal{O}$, where $\mathcal{O}$ is an ovoid of $\mathrm{Q}(6,3)$, minus the points of the vertex π, and $|\mathcal{K}| = q^{n+1}+q^{n-2}$.*

We will discuss several aspects of the theorems and the difficulties which arise for other values of q.

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On the number of finite near hexagons with three points on each line

Bart De Bruyn

Ghent University

A graph Γ of diameter n is called a *near* $2n$*-gon* if it satisfies the property that every maximal clique contains a unique point nearest to any given point. Since any two adjacent vertices of such a graph are contained in a unique maximal clique, a partial linear space $\mathcal{S}_\Gamma$ is obtained if one regards the maximal cliques of Γ as lines. Since the graph Γ can be retrieved from the geometry $\mathcal{S}_\Gamma$, we have a bijective correspondence between a class of graphs and a class of partial linear spaces. We call a partial linear space a *near polygon* if its collinearity graph is a near polygon. From this (equivalent) point of view, a near 0-gon is a point, a near 2-gon is a line and the class of near quadrangles coincides with the class of (possibly degenerate) generalized quadrangles. In [1] it was shown that there are only 11 (possibly infinite) near hexagons with three points on each line satisfying the following condition:

(C1) every two points at distance 2 have at least two common neighbours.

In my talk, I will show that there are only finitely many finite near hexagons satisfying the following condition:

(C2) for every point x there exists a point y at distance 3 from x.

In fact, (C1) $\Rightarrow$ (C2). Condition (C2) is not very restrictive since the structure of near hexagons not satisfying (C2) is known.

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Ovals in nets of small degree

David A. Drake

University of Florida

A *k-arc* in an r-net (a net with r parallel classes) is a set of k points, no three collinear, so that each pair of the k points is joined by a line of the net. A k-arc is called an *oval* if $k = r$; a *hyperoval* if $k = r + 1$. The Desarguesian affine plane $\Pi(D)$ holds a 5-net with oval if and only if the division ring D contains a root of $x^2 - x - 1$ (proved in the finite case by Hirschfeld) and a 7-net with oval if and only if D contains a root of $x^3 - x^2 - 2x + 1$; $\Pi(D)$ holds a 6-net with oval if and only if either $\gcd(6, \operatorname{char} D) = 1$ or $\mathrm{GF}(4) \leq D$.

The plane $\Pi(D)$ holds an r-net with hyperoval for $r = 3$ if and only if $\operatorname{char} D = 2$, for $r = 5$ if and only if $\mathrm{GF}(4) \leq D$, and for $r = 7$ if and only if $\operatorname{char} D = 2$ and $|D| \geq 8$.

Let O_r and H_r denote the respective sets of all integers n for which there exists an r-net of order n with an oval and with a hyperoval. Then H_r is the empty set if r is even, and $H_r \subseteq O_r$ for all r. The set of integers $n \geq 7$ is contained in $H_3 \cap O_4$; $\{n \mid n \geq 16\} \subseteq H_5$; $\{n \mid n \geq 40\} \subseteq O_6$; $\{n \mid n \geq 63\} \subseteq H_7$; $\{n \mid n \geq 83\} \subseteq O_8$.

A classification theorem for two-intersection sets with respect to hyperplanes in finite projective spaces

Nicola Durante

University of Naples

A two-intersection set with respect to hyperplanes in $\mathrm{PG}(d,q)$ is a set of points of $\mathrm{PG}(d,q)$ intersecting every hyperplane either in m or in n $(m < n)$ points. Two-intersection sets with respect to hyperplanes in $\mathrm{PG}(d,q)$ have been studied in the last thirty years by many authors. Although a lot of examples have been found there are still very few classification theorems.

We present a classification theorem, for small m, of such sets in $\mathrm{PG}(d,q)$, $d \geq 3$. The proof does not use the fact that q is a prime power and also holds in finite d-dimensional $(d \geq 3)$ locally projective linear spaces.

Odd order flag-transitive affine planes

G. L. Ebert

University of Delaware

With the exception of Hering's plane of order 27, all known odd order flag-transitive affine planes are one of two types: admitting a cyclic transitive action on the line at infinity, or admitting a transitive action on the line at infinity with two equal-sized cyclic orbits.

In this talk we concentrate on the cases when the dimension over the kernel for these planes is two or three, showing that then the known examples are the only possibilities for either of these two types. Moreover, subject to a relatively mild *gcd* condition, one of these two actions must occur. Hence, subject to this *gcd* condition, all odd order flag-transitive affine planes of dimension two or three over their kernel have been classified. In the two-dimensional setting caps in $PG(3,q)$ are used in the classification. In the three-dimensional setting several seemingly disparate ideas are used: the classification of pencils of quadrics in the plane, Baer subplane partitions, the theory of linearized polynomials, and a little bit of cyclotomy theory. The higher dimensional settings are much trickier, but perhaps now seem within reach.

Arithmetic of cubic curves

Andreas Enge

École Polytechnique and INRIA Futurs

(Joint work with A. Basiri, J.-C. Faugère, and N. Gürel)

Before being able to deploy a discrete logarithm based cryptosystem on top of some class of algebraic curves, it is necessary to make the arithmetic of these curves explicit. Computations take in fact place in the Jacobian, an associated class group, obtained as the quotient of the divisor group by principal divisors. A divisor can hereby be seen as a formal sum of points defined over the algebraic closure of the field of constants. Usual rationality arguments show that one can then carry out all computations over the field of constants itself, the most important case being that of a finite field.

Jacobian elements on curves with a totally ramified infinite prime can be canonically represented by the *reduced divisor*, i.e., the positive divisor of minimal degree in a given class. The degree of a reduced divisor is bounded above by the genus g of the curve. Conversely, on quadratic, i.e. hyperelliptic curves, essentially all divisors of degree at most g are reduced. We examine this question in the context of cubic curves, by which we understand curves whose degree in *one* variable is 3. For genus 3 and 4, we provide a complete geometric characterisation of the non-reduced low-degree divisors.

We present two algorithms for the arithmetic of such cubic curves. The first one, inspired by Cantor's algorithm for hyperelliptic curves, is easily implemented with a few lines of code, making use of a polynomial arithmetic package. Its correctness relies on the reducedness criteria of the previous paragraph. The second approach, quite general in nature and applicable to further classes of curves, uses the FGLM algorithm for switching between Gröbner bases for different orderings. Carrying out the computations symbolically, we obtain explicit reduction formulae in terms of the input data.

On finite matroids with two more hyperplanes than points

Eva Ferrara Dentice

Seconda Università degli Studi di Napoli

It is known that every finite simple matroid of rank $n+1$ is either a geometric lattice of height $n+1$ or a linear space of dimension n. Thus, in particular, some questions about n-dimensional linear spaces are useful tools to approach the study of simple matroids.

A characteristic problem for finite matroids is to derive geometric properties from algebraic relations among arithmetic parameters of the matroid. A fundamental result in this sense is the Theorem of Basterfield and Kelly (1968), saying that the number of hyperplanes of a finite matroid is greater than or equal to the number of points, with equality in the case of generalized projective spaces.

Two theorems of Bridges (1972) and Ferrara Dentice (1996) characterize finite matroids with one more hyperplane than points.

Denoted by $\mathcal{M}^2_{n+1}$, a finite matroid of rank $n+1$ containing two hyperplanes more than points, the cases $\mathcal{M}^2_3$ and $\mathcal{M}^2_4$ were solved by De Witte (1976) and De Vito and Lo Re (1996), respectively.

In this talk I present a characterization of $\mathcal{M}^2_{n+1}$ for a general $n \geq 4$.

New results on minihypers following from characterizations of t-fold $(n-k)$-blocking sets

Sandy Ferret

Ghent University

(Joint work with L. Storme, P. Sziklai and Zs. Weiner)

A *t-fold $(n-k)$-blocking set* in $PG(n,q)$ is a set B of points of $PG(n,q)$ intersecting every k-dimensional subspace of $PG(n,q)$ in at least t points. A t-fold $(n-k)$-blocking set in $PG(n,q)$ is called *minimal* if no proper subset of it is still a t-fold $(n-k)$-blocking set.

An *$\{f,m;N,q\}$-minihyper* is a pair (F,w), where F is a subset of the point set of $PG(N,q)$ and w is a weight function $w : PG(N,q) \to \mathbb{N}$, written $x \mapsto w(x)$, satisfying the following:

(1) $w(x) > 0 \Leftrightarrow x \in F$;

(2) $\sum_{x \in F} w(x) = f$;

(3) $\min\{\sum_{x \in H} w(x) \mid H \in \mathcal{H}\} = m$, where $\mathcal{H}$ denotes the set of hyperplanes.

Minihypers were introduced to study linear codes meeting the Griesmer bound [2], and are t-fold $(n-k)$-blocking sets satisfying certain conditions. We apply characterizations on minimal t-fold blocking sets in $PG(n,q)$ [1], to obtain new results on minihypers.

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On finite projective planes in Lenz-Barlotti class at least I.3

Dina Ghinelli

Università di Roma "La Sapienza"

(Joint work with D. Jungnickel)

We clarify the connections between several notions that have appeared in the literature:

- finite projective planes admitting a collineation group of Lenz-Barlotti type I.3 or I.4;
- partially transitive planes of type (3) in the sense of Hughes;
- planes admitting a quasiregular collineation group of type (g) in the Dembowski–Piper classification;
- a certain type of difference set relative to disjoint subgroups in the sense of Hiramine which we will call a "neo-difference set", as the abelian case corresponds to neofields.

In particular, we establish that Lenz-Barlotti type I.4 is equivalent to (necessarily abelian) quasiregular groups of type (g) and to abelian neo-difference sets. Following this, we survey the known restrictions and prove some new ones for planes of Lenz-Barlotti class I.4, using the setting of neo-difference sets. This allows us not only to avoid neofields for the major part of our exposition (using the standard machinery of group rings instead), but also to give simpler and more transparent proofs in many cases, stressing the analogy to planar and affine difference sets. As a side result, we also obtain a new synthetic construction for projective triangles in Desarguesian planes.

Some results on sets of lines and quantum codes

David Glynn

Quantum Error Correction Project Aotearoa

(Joint work with Rey Casse, Aaron Gullliver, Manish Gupta, Johannes Maks)

A quantum set of n lines of the binary projective space $PG(n-k-1,2)$ $(k \geq 0)$ has any subspace of dimension $n-k-3$ skew to an even number of lines of the set. It gives a set of points on the Grassmannian of lines with the sum of the point coordinates being zero. These quantum sets come from the additive quantum codes $[[n,k,d]]$. For example, a code's distance and purity can be explained geometrically. In the self-dual case $k=0$, the isotropic systems of Boucher are equivalent, as are the simple graphs on n vertices up to an equivalence relation called local or vertex neighbourhood complementation (vnc). So a quantum state comes from any graph.

The hyperoval of $PG(2,4)$ is equivalent to an optimal linear $[[6,0,4]]$ quantum code of distance 4. The same holds for a quantum set of 6 lines of $PG(5,2)$, every 3 of which are independent, and for two types of graphs in the vnc orbit on 6 vertices: the wheel with 5 vertices on a rim connected by spokes to a point in the centre, and the graph on 6 vertices that is the triangular prism.

There is a general property of any set of n lines in binary space of dimension $n-1$. A transversal of such a set is a sequence of n points, one on each line. The lines may coincide or intersect, and so some of these points may be the same. A partial transversal can have less than n points. A basis transversal has all the points of the transversal independent.

Theorem *Let S be a set of n lines in $PG(n-1,2)$ and let P be a partial transversal contained in a transversal T. Then T is a basis transversal if and only if there is an odd number of basis transversals containing P but skew to $T \setminus P$.*

By labelling the points of each of the lines 1,2,3 we get a 3^n−hypercube H from S, with the 1's in H corresponding to basis transversals. Since the hyperdeterminant $\det_2$ of H is 0 there will be an even number of ways to find 3 mutually skew basis transversals on any set of n lines of $PG(n-1,2)$.

Curtis–Phan–Tits theory

Ralf Gramlich

Technische Universität Darmstadt

(Joint work with C.D.Bennett, C.Hoffman, S.Shpectorov)

In [1] the authors describe a uniform geometric approach to the similar-in-appearance theorems by Curtis and Tits and by Phan. Since the Curtis–Tits theorem is directly related to (twin) buildings, there already exist a number of different approaches, for example, by Mühlherr [4] and by Timmesfeld. The current goal is to provide a similar machinery for Phan-type theorems, which are directly related to some sort of twisted (twin) buildings. In this talk we will discuss a general description of Phan-type theorems by studying automorphisms acting on groups with a twin BN-pair.

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Are there elation quadrangles with st not a prime power?

Dirk Hachenberger

University of Augsburg

Most of the known classes of finite generalized quadrangles (GQs) belong to the class of elation GQs. By definition, these admit a certain type of (rich) automorphism group and therefore may be constructed as a coset geometry involving a particular family of subgroups, the Kantor (or 4-gonal) family. A conjecture of Kantor states that the parameters, s and t, of an elation GQ necessarily have to be powers of the same prime. One intention of my talk is to summarise some of the results which are focused on the group theoretical part of the problem and which support Kantor's conjecture. A second intention is to draw attention to a group theoretical constellation, which, if not ruled out, gives rise to elation GQ s, where st has exactly two distinct prime divisors.

On $(k, (q+1)/2)$ arcs in $PG(2, q)$

Ray Hill

University of Salford

(Joint work with Chris Love and Gabor Korchmáros)

The problem of finding large (k, n) arcs in $PG(2, q)$ is in general difficult because, for most values of n, there are no natural geometric constructions. Two exceptions are the cases $n = 2$, where we have conics and ovals, and $n = (q+1)/2$, which is the case we consider here. The problem of finding large $(k, (q+1)/2)$ arcs is attractive because

(i) it seems natural to ask what is the largest size of a set of points in a projective plane such that the set contains no more than half the points of every line;

(ii) there are three natural constructions of $(k, (q+1)/2)$ arcs of with the size $k = (q^2 - q + 2)/2$, one due to Barlotti and two recently discovered variations of the Barlotti arc.

As well as giving these three constructions, we consider the questions of whether $(q^2 - q + 2)/2$ is in general the largest size of a $(k, (q+1)/2)$ arc and, if so, whether there might be any other such arcs of this size.

Ovoids of the Hermitian surface in odd characteristic

Gábor Korchmáros

Università di Basilicata

(Joint work with L. Giuzzi)

An ovoid $\mathcal{O}$ of the polar space arising from the Hermitian surface $\mathcal{H}(3,q^2)$ of $PG(3,q^2)$ is a set of q^3+1 points which meets every generator (that is, every line contained in $\mathcal{H}(3,q^2)$) in exactly one point. The intersection of $\mathcal{H}(3,q^2)$ with any non–tangent plane provides an ovoid, namely, the *classical* ovoid of $\mathcal{H}(3,q^2)$. Existence of non–classical ovoids of $\mathcal{H}(3,q^2)$ was pointed out by S.E. Payne and J.A. Thas in 1994, who constructed a non–classical ovoid $\mathcal{O}$ from the classical one $\mathcal{O}$ by replacing the $q+1$ points of $\mathcal{O}$ lying on a chord ℓ by the common points of $\mathcal{O}$ with the polar line ℓ' of ℓ.

A straightforward generalisation of this procedure consists in replacing more than one chord of $\mathcal{O}$, each with its own polar line. The condition that the resulting set is an ovoid is easily stated: the chords must pairwise intersect outside of $\mathcal{O}$. The above procedure will be called *derivation* or *multiple derivation* accordingly as one or more chords are used. Our purpose is to construct an ovoid of $\mathcal{H}(3,q^2)$ for every odd $q \geq 5$ which cannot be obtained from the classical one either by derivation or by multiple derivation.

The essential idea of the construction relies on a nice property of $\mathbf{F}_{q^2}$-maximal curves: the $\mathbf{F}_{q^2}$–rational points of an $\mathbf{F}_{q^2}$–maximal curve naturally embedded in a Hermitian variety are pairwise non–conjugate under the associated unitary polarity. The automorphism group Γ of such an ovoid has a normal cyclic subgroup Φ of order $\frac{1}{2}(q+1)$ such that $\Gamma/\Phi \cong PGL(2,q)$. Furthermore, Γ has three orbits on the ovoid, one of size $q+1$ and two of size $\frac{1}{2}q(q-1)(q+1)$.

A model for the generalised hexagon $H(q)$, q odd

Elisabeth Kuijken

Ghent University

A generalised hexagon is a point-line incidence structure which has an incidence graph with diameter 6, girth 12 and valency at least 3 in every vertex. The points and lines of the split Cayley hexagon $H(q)$ of order q, q a prime power, are usually defined as all points and certain lines of the parabolic quadric $Q(6, q)$ in $\mathrm{PG}(6, q)$. To construct the lines explicitly, one needs equations or coordinates in some form. In [1], Bader and Lunardon describe an alternative model for the dual of $H(q)$, q odd and not a power of 3. It starts from a twisted cubic in $\mathrm{PG}(3, q)$ and lives in the symplectic polar space $W(5, q)$. We propose a modification of this model which makes it work if q is a power of 3 as well.

The twisted triality hexagon $H(q^3, q)$ of order (q^3, q), q a prime power, is defined on the hyperbolic quadric $Q(7, q^3)$ in $\mathrm{PG}(7, q^3)$ using a triality. It contains $H(q)$ as a subhexagon. An alternative description for the dual of $H(q^3, q)$, q odd, is given by Lunardon in [2]. It lives in the symplectic polar space $W(9, q)$ and starts from a particular partial ovoid of the symplectic polar space $W(7, q)$. If q is not a power of 3, the dual of $H(q)$ is obtained as a subhexagon of the dual of $H(q^3, q)$ by intersection with an appropriate five-dimensional subspace. By adapting the model for the dual of $H(q^3, q)$ in a similar way as the other model, we can extend this construction to the case where q is a power of 3.

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Desarguesian spreads, and good eggs in $\mathrm{PG}(4n-1,q)$, q odd

Michel Lavrauw

Università degli Studi di Napoli Federico II

(Joint work with Guglielmo Lunardon)

An $(n-1)$-*spread* of $\mathrm{PG}(kn-1,q)$, $n \geq 1$, is a set $\mathcal{S}$ of $(n-1)$-dimensional subspaces such that every point of $\mathrm{PG}(kn-1,q)$ is contained in exactly one element of $\mathcal{S}$. Suppose $k \geq 3$. A spread $\mathcal{S}$ is called *Desarguesian* if and only if every $(2n-1)$-space spanned by two elements of $\mathcal{S}$ contains exactly q^n+1 elements of $\mathcal{S}$. This implies that the elements of $\mathcal{S}$ correspond to the points of $\mathrm{PG}(k-1,q^n)$ and the $(2n-1)$-spaces spanned by two elements of $\mathcal{S}$ correspond to lines of $\mathrm{PG}(k-1,q^n)$. This correspondence can be explained in a geometrical way, following [B. Segre, 1964], by embedding $\mathrm{PG}(kn-1,q)$ as a subgeometry in $\mathrm{PG}(kn-1,q^n)$, and the existence of a $(k-1)$-space π in $\mathrm{PG}(kn-1,q^n)$, such that every point P of π is *imaginary*, i.e., the subspace $L(P)$ of $\mathrm{PG}(kn-1,q^n)$ spanned by P and its conjugates with respect to the n-th extension $\mathrm{GF}(q^n)$ over $\mathrm{GF}(q)$ has dimension $n-1$. In this talk we give a construction of such a $(k-1)$-space π, together with a particular embedding of $\mathrm{GF}(q^n)$ in $V(n,q^n)$, and give an application to the theory of good eggs in $\mathrm{PG}(4n-1,q)$, q odd.

Residually weakly primitive sporadic geometries

Dimitri Leemans

Université Libre de Bruxelles

For about eight years now, a team, consisting mainly of Francis Buekenhout, Michel Dehon, Philippe Cara and myself, has been busy classifying residually weakly primitive geometries for "small" simple groups. In the last few years, I concentrated mainly on classifying such geometries for sporadic groups. Currently six sporadic groups are completely analyzed : M_{11}, M_{12}, J_1, J_2, M_{22} and M_{23}. In this talk, I will present some geometries obtained and give geometric constructions for them. Most geometries I will show have a non-linear diagram.

Maximal arcs in Desarguesian planes

Ka Hin Leung

National University of Singapore

We study the problem of determining the largest d of a non-Denniston maximal arc of degree 2^d generated by a $\{p, 1\}$-map in $\mathrm{PG}(2, 2^m)$ via a recent construction of Mathon. On one hand, we show that there are $\{p, 1\}$-maps that generate non-Denniston maximal arcs of degree $2^{\frac{m+1}{2}}$, where $m \geq 5$ is odd. Together with Mathon's result in the m even case, this shows that there are always $\{p, 1\}$-maps generating non-Denniston maximal arcs of degree $2^{\lfloor \frac{m+2}{2} \rfloor}$ in $\mathrm{PG}(2, 2^m)$. On the other hand, we prove that the largest degree of a non-Denniston maximal arc in $\mathrm{PG}(2, 2^m)$ constructed using a $\{p, 1\}$-map is less than or equal to 2^{m-3}. We conjecture that this largest degree is actually $2^{\lfloor \frac{m+2}{2} \rfloor}$.

Constructions of caps and complete caps in PG$(n, 2)$

Petr Lisonek

Simon Fraser University

Several new combinatorial constructions of caps and complete caps in PG$(n, 2)$ are given, mostly as one- or two-parameter families of caps. We show that for every $m \geq \lceil 2n/3 \rceil$ there exists a complete cap in PG$(n, 2)$ that intersects every m-flat of this space. We find, for each $\alpha \in [1.89, 2]$, a sequence of complete caps in PG$(n, 2)$ whose sizes grow roughly as α^n.

We also discuss the relevance of our constructions to the problem of finding the least dependent caps of a given cardinality in a given dimension, where the dependency of a cap is the number of linearly dependent quadruples of points in it. We recall the applications of caps with low dependency in coding theory and experiment design. Finally we report the results of a complete classification of caps in PG$(6, 2)$.

Some of this is joint work with Mahdad Khatirinejad Fard.

Translation ovoids of orthogonal polar spaces

G. Lunardon

Università di Napoli

(Joint work with O. Polverino)

Denote by $\mathbb{P}$ both the polar space associated with a non-singular quadric of $PG(2n,q)$ $(n \geq 2)$ and the polar space associated with a non-singular quadric of $PG(2n+1,q)$ with $n \geq 1$. An ovoid O of $\mathbb{P}$ is a set of points which has exactly a common point with each totally singular subspace of $\mathbb{P}$. An ovoid O is a *translation* ovoid with respect to a point x of O if there is a collineation group of $\mathbb{P}$ fixing x linewise and acting sharply transitively on $O \setminus \{x\}$.

Examples of translation ovoids of $Q^+(3,q)$ are the conics contained in it. The ovoids of the Klein quadric $Q^+(5,q)$ correspond to line spreads of $PG(3,q)$ and translation ovoids are equivalent to semifield spreads. Hence, $Q^+(5,q)$ has ovoids and translation ovoids for all values of q. If $Q(4,q) = H \cap Q^+(5,q)$ is a non-singular quadric, where H is a hyperplane of $PG(5,q)$, then ovoids of $Q(4,q)$ are equivalent to symplectic spreads of $PG(3,q)$ and translation ovoids are equivalent to symplectic semifield spreads. The interest of translation ovoids is also due to their relationship with translation generalized quadrangles of order (q^2,q) because it has been proved that, for q odd, semifield flocks of the quadratic cone $PG(3,q)$ and translation ovoids of $Q(4,q)$ are equivalent objects.

It is shown that each $GF(s)-$linear set is either a canonical subgeometry or the projection of a canonical subgeometry. We also prove that a translation ovoid of an orthogonal polar space $\mathbb{P}$ is equivalent to a $GF(s)-$linear set disjoint from a non-singular quadric and that if $\mathbb{P}$ has a translation ovoid, then $\mathbb{P}$ is one of $Q^+(3,q)$, $Q(4,q)$, $Q^+(5,q)$.

Semipartial geometries from 1-systems of $W_5(q)$

Deirdre Luyckx

Ghent University

(Joint work with J. A. Thas)

A 1-*system* of $W_5(q)$ is a set $\mathcal{M}$ of q^3+1 lines $L_0, L_1, \ldots, L_{q^3}$ of $W_5(q)$ such that every generator of $W_5(q)$ which contains a line $L_i \in \mathcal{M}$, is disjoint from all lines $L_j \in \mathcal{M} \setminus \{L_i\}$.

Let $\mathcal{M}$ be a 1-system of $W_5(q)$ in $\mathsf{PG}(5,q)$ and embed $\mathsf{PG}(5,q) := H$ as a hyperplane in $\mathsf{PG}(6,q)$. Let $\mathcal{P}$ be the set of all points of $\mathsf{PG}(6,q) \setminus H$, let $\mathcal{L}$ be the set of planes of $\mathsf{PG}(6,q)$, not in H, which meet H in a line of $\mathcal{M}$, and let I be the natural incidence of $\mathsf{PG}(6,q)$. Then $\mathsf{SPG}(\mathcal{M}) := (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = q^2-1$, $t = q^3$, $\alpha = q$ and $\mu = q^2(q^2-1)$.

A *strong regulus* of $W_5(q)$ is a regulus of lines of $W_5(q)$, the opposite regulus of which entirely consists of lines of $W_5(q)$. If q is odd, then $W_5(q)$ does not have strong reguli; so let q be even. A 1-system $\mathcal{M}$ of $W_5(q)$, q even, is *locally hermitian* at a line $L \in \mathcal{M}$ if and only if for every line $M \in \mathcal{M} \setminus \{L\}$, the unique strong regulus containing L and M is contained in $\mathcal{M}$.

A class of locally hermitian 1-systems of $W_5(q)$ is known to exist for all q even with $q > 2$; see [1]. Concerning their semipartial geometries, the following theorem can be shown.

Theorem 1 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two locally hermitian 1-systems of $W_5(q)$, q even and $q > 2$. Then $\mathsf{SPG}(\mathcal{M}_1)$ and $\mathsf{SPG}(\mathcal{M}_2)$ are isomorphic if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic for the stabilizer of $W_5(q)$ in $\mathsf{P\Gamma L}(6,q)$.*

From every locally hermitian 1-system of $W_5(q)$, q even, one can construct a new 1-system, which is not locally hermitian, by replacing a strong regulus by its opposite regulus. In this context, the following result holds.

Theorem 2 *Let $\mathcal{M}$ be a 1-system of $W_5(q)$, q even, which is locally hermitian at $L \in \mathcal{M}$. Let $\mathcal{M}'$ be obtained from $\mathcal{M}$ by replacing a strong regulus of $\mathcal{M}$ through L by its opposite regulus. Then $\mathsf{SPG}(\mathcal{M})$ and $\mathsf{SPG}(\mathcal{M}')$ are not isomorphic.*

From the above theorems, the existence of several new examples of semipartial geometries follows.

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Maximal arcs from quadrics

Rudolf Mathon

University of Toronto

(Joint work with N. Hamilton)

A maximal $\{q(n-1)+n; n\}$-arc in a projective plane of order q is a subset of $q(n-1)+n$ points such that every line meets the set in 0 or n points for some $2 \leq n \leq q$. For such a maximal arc n is called the degree. Recently, new maximal arcs in Desarguesian projective planes have been constructed [1] including the previously known maximal arcs of Denniston type. The new construction is based on the concept of closed sets of conics and generates a hierarchy of maximal arcs similar to that of subspaces in a projective space.

The main theorem describes two families of maximal arcs based on closed sets of conics induced by maximal totally singular subspaces of non-degenerate elliptic and hyperbolic quadrics in some subspace of the corresponding projective spaces.

Theorem *In* $\mathrm{PG}(2, q)$, $q = 2^{2m}$ *or* $q = 2^{2m+1}$, $m \geq 2$, *there exist* $b_0, b_1, b_2 \in \mathrm{GF}(q)$, $b_2 \neq 0$, *and* $A \subset \mathrm{GF}(q)$, *such that the set*

$$\{\ x^2 + xy + (b_0 + b_1\lambda + b_2\lambda^3)y^2 + \lambda z^2 = 0\ :\ \lambda \in A\ \}$$

is a closed set of conics giving rise to a maximal arc of degree 2^{m+1}.

Finally, we show that the number of isomorphism classes of maximal arcs of degree $\sqrt{2q}$ in $\mathrm{PG}(2, q)$, $q = 2^{2m+1}$ that are not of Denniston type is at least $2q^{m/4-10}$.

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Maximal arc partitions of designs

Vassili C. Mavron

University of Wales Aberystwyth

Rahilly established a link between affine designs of class number 4 and Hadamard 2-designs possessing a spread of lines, each of maximum size 3. A k-arc in a 2-design is a non-empty subset of points that meets every block in 0 or k points. Observing that a line of size 3 in a Hadamard 2-design is a 1-arc in the complementary design, we can extend this idea more generally to affine designs of class number m and certain Hadamard 2-designs whose complements possess spreads of $(m/2)$-arcs.

SDP designs and spin models

Gary McGuire

National University of Ireland, Maynooth

(Joint work with Carl Bracken)

Spin models arose in topology in connection with invariants of oriented links in 3-space. Recently Bannai and Sawano (following work of Guo and Huang) showed that the existence of a "four-weight spin model with exactly two values on W_2" is equivalent to the existence of a symmetric design with certain additional properties. One of these properties states that the symmetric design should be a quasi-3 design, that is, that there should be only two triple intersection sizes for blocks. The $(16, 6, 2)$ SDP design has these properties, and they asked whether all SDP designs would correspond to spin models. We answer this question in this talk. Our main theorem provides a characterisation of the SDP designs that satisfy these additional properties.

Theorem *An SDP design D corresponds to a four-weight spin model with exactly two values on W_2 if and only if D is equivalent to the symplectic SDP design.*

Counting with ovoids and friends

Alan Offer

Universiteit Gent

(Joint work with H. Van Maldeghem)

A *distance-j ovoid* of a generalized polygon Γ is a set $\mathcal{O}$ of points mutually at distance at least $2j$ from each other and such that for every element x of Γ there exists some point $p \in \mathcal{O}$ whose distance from x is at most j. For instance, a distance-m ovoid of a generalized $2m$-gon is just what is usually called an *ovoid*.

It is known that, if a generalized hexagon with order (s, t) admits an ovoid or spread, then $s = t$. This was proved with a simple counting argument. Now doing essentially the same counts for other distance-j ovoids of finite generalized polygons leads to similar results about when these structures may exist. To exploit the approach a little more, some other sets, similar to distance-j ovoids, are introduced and subjected to the same sorts of counts as well.

Classification of codes exploiting an invariant

Fernanda Pambianco

University of Perugia

(Joint work with S. Marcugini and A. Milani)

We consider the problem of computing the equivalence classes of a set of linear codes. This problem arises, for example, using computer-based extension processes that construct new codes of dimension d_1 starting from codes of dimension d_2, with $d_2 < d_1$. When extending a code in this way, several equivalent copies of the same code are obtained. A classification step allows the computation of the set of nonequivalent codes, but, when the number of examples to classify is high, some strategy has to be adopted to reduce the computational complexity of this phase.

A pre-classification phase based on the use of an invariant can reduce the computational complexity of the subsequent classification phase: a partition of the set S of the codes to classify can be computed defining S_i as the subset of S containing the codes having the same value i of the invariant. Codes belonging to different sets of the partition are not equivalent. Then it is sufficient to classify separately the codes in each S_i; the set of representatives of the equivalence classes in S is the union of the sets of representatives of the equivalence classes in S_i. If each S_i contains only one equivalence class, the computational complexity of the classification step is $O(|S|)$.

To have a partition with subsets containing just a few equivalence classes, starting from an invariant simple to compute (the minimum weight in our case), we define, for a code C, a new invariant as the sum of the values of the original invariant for all the codes obtained by truncating C in all possible ways.

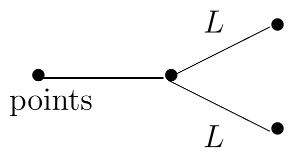
In our practical applications we have finished with almost all subsets S_i containing one or just a few equivalence classes. There is an additional cost, namely, the computation of the invariant for the codes of S and for several truncated codes; but this remains negligible with respect to the cost of the classification phase. Using this technique, $[13, 5, 8]_7$, $[14, 5, 9]_8$ and $[15, 4, 11]_9$ codes have been classified.

From local to global truncations

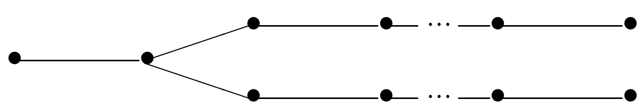
Antonio Pasini

University of Siena

When characterizing certain diagram geometries of rank $n > 2$ (e.g. certain buildings of rank n) as point-line geometries, one starts from a point-line system satisfying suitable conditions and constructs some of the missing objects (e.g. singular subspaces, symplecta etc.) as distinguished substructures of the point-line system. However, it often happens that not all objects of the geometry to characterize can easily be produced in a direct way and we only end up with a geometry Γ of rank less than n that is locally the truncation of the geometry Δ that we aimed to get; that is, the residue of a point of Γ is isomorphic to a truncation of the residue of the corresponding element of Δ, but we do not know if Γ itself, as a whole, is a truncation of Δ. At that stage it is very helpful to have a theorem stating that, under suitable conditions, a 'locally truncated' geometry like Γ is indeed a truncation of a rank n geometry like Δ. Earliest theorems of this kind are due to Ronan (1986) and to Brouwer and Cohen (1986). Further progress and improvements of those earliest results have later been obtained by Kasikova and Shult (2002) and Ceccherini and Pasini (2001). We continue that investigation, considering the following situation: Γ has rank 4, it belongs to the following diagram:



and is locally a truncated projective geometry. We prove that, under suitable but not too strong conditions, Γ is the truncation of a quotient of a building belonging to a diagram as follows:



In fact, we will obtain our result from a more general theorem. More applications of the latter will be discussed.

The complete Adelaide oval stabilizer

Stan Payne

University of Colorado at Denver

(Joint work with J. A. Thas)

W. E. Cherowitzo, C. M. O'Keefe and T. Penttila have given a general recipe for constructing a family of flock generalized quadrangles (GQ) with parameters (q^2, q), $q = 2^e$. They show that three previously known infinite families arise along with one new family to which they give the name Adelaide. For each $q = 2^e$ with e even, there arises an Adelaide flock GQ which belongs to none of the other known infinite families if $e > 4$ and which was previously unknown if $e > 16$.

Associated with each flock GQ are several other types of geometrical objects, including a family of ovals in $PG(2, q)$. In the Adelaide case, for each $q = 2^e$ with e even there is, up to projective equivalence, only one oval, which is called the Adelaide oval. The work of Cherowitzo, et al. shows that the collineation group of the Adelaide GQ induces a cyclic group of order 2e that acts on $PG(2, q)$ and stabilizes the Adelaide oval. However, they leave open the problem of determining whether or not this cyclic group of order $2e$ is the complete group of the oval. It is the main goal of the present work to show that this is indeed the case.

First, we construct an absolutely irreducible homogeneous polynomial $G(x, y, z)$ of degree 6 and show that the plane algebraic curve C derived from G consists of exactly the points of (one model of) the Adelaide oval together with just one additional point that turns out to be the unique singular point of C. Then using some algebraic geometry we are able to show that we have the complete stabilizer of the oval. Moreover, this cyclic stabilizer of order $2e$ also turns out to be the complete stabilizer of the associated Adelaide hyperoval.

Constant weight codes, divisible designs and finite Laguerre geometries

Ralph–Hardo Schulz

Freie Universität Berlin

Constant weight codes are codes over the alphabet $A = \{1, 2, \ldots, s\}$ with words of equal length and equal Hamming weight k (number of non-zero entries).

One can get codes of this kind by using divisible $t-$designs. These are incidence structures whose point set is divided into classes of equal size and whose block set consists of $k-$sets of points such that, with any transversal t-set of points, there is incident a constant number λ_t of blocks.

In our talk, we want to describe a construction of divisible 3-designs from finite Laguerre geometries; these are chain geometries on the point set of the projective line over the (local) ring of dual numbers over a finite field GF(q). We use the fact that the automorphism group of such a geometry operates transitively on transversal point triples.

Problems by the wayside

Ernie Shult

Kansas State University

The talk concerns a number of open issues that the speaker either stumbled over, glossed over, or ignored in his occasional sorties into finite geometry. Here is a sampling of problem areas that will be discussed:

1. spaces whose intersecting lines generate specified geometries;
2. odd cocliques revisited;
3. orthogonal (C^*, C)-geometries revisited;
4. ovoids: pairwise disjoint ovoids and other variations;
5. $(0, 2)$-sets in $H(3, q^2)$ and 1-systems of $Q + (7, q)$;
6. hyperplanes of dual polar spaces.

Plane algebraic curves with Singer automorphisms

Alessandro Siciliano

University of Basilicata

(Joint work with A. Cossidente)

Non-singular plane algebraic curves $\mathcal{X}_q$ over $\mathbb{F}_q$ with a Singer cyclic group of $\mathrm{PGL}(3,q)$ in their automorphism group are classified. Our main result states that if the Singer group is an automorphism group of $\mathcal{X}$ then either $\deg(\mathcal{X}) = q+2$ or $\deg(\mathcal{X}) \geq q^2+q+1$. In the former case $\mathcal{X}$ is projectively equivalent to the curve $\mathcal{X}_q$ with equation $X^{q+1}Y+Y^{q+1}+X = 0$ studied recently by Ruud Pellikaan.

The configuration of the $\mathbb{F}_{q^3}$–rational points of $\mathcal{X}_q$ has a nice combinatorial property. In fact, apart from three distinguished points, the set of $\mathbb{F}_{q^3}$-rational points of such a curve can be partitioned into $2-(q^2+q+1, q+1, 1)$ designs each isomorphic to the finite projective plane $\mathrm{PG}(2,q)$.

Finally, the full automorphism group $\mathrm{Aut}(\mathcal{X}_q)$ of $\mathcal{X}_q$ is determined. It turns out that $\mathrm{Aut}(\mathcal{X}_q)$, $q > 2$, is the normalizer in $\mathrm{PGL}(3,q)$ of a Singer cyclic group of $\mathrm{PGL}(3,q)$, which is a metacyclic group of order $3(q^2+q+1)$.

Classification of t-fold $(n-k)$-blocking sets in $PG(n,q)$

Leo Storme

Ghent University

(Joint work with S. Ferret, P. Sziklai and Zs. Weiner)

A *t-fold $(n-k)$-blocking set* in $PG(n,q)$ is a set B of points of $PG(n,q)$ intersecting every k-dimensional subspace in at least t points. A t-fold $(n-k)$-blocking set B of $PG(n,q)$ is called *minimal* when no proper subset of B is still a t-fold $(n-k)$-blocking set.

A 1-fold $(n-k)$-blocking set of $PG(n,q)$ is also simply called an *$(n-k)$-blocking set* of $PG(n,q)$.

We present new classification results on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$.

We first extend the following $1 \pmod p$ result of Szőnyi and Weiner [1] to a $t \pmod p$ result on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$.

Theorem 1 *Let B be a minimal $(n-k)$-blocking set in $PG(n,q)$, $q=p^h$, $p>2$ prime, $h \geq 1$, of size less than $3(q^{n-k}+1)/2$. Then every subspace that intersects B in at least one point, intersects B in $1 \pmod p$ points.*

Theorem 2 *Let B be a minimal t-fold $(n-k)$-blocking set in $PG(n,q)$, $q=p^h$, $p>2$ prime, $h \geq 1$, of size less than $(t+3/2)(q^{n-k}+1)$. Then every k-dimensional subspace intersects B in $t \pmod p$ points, and any subspace of dimension less than k intersects B in $0,1,\ldots,t \pmod p$ points.*

Such $1 \pmod p$ results, or more general $t \pmod p$ results, also appear with respect to other substructures in finite projective spaces, and are very useful for obtaining classification theorems on these substructures.

We present how this $t \pmod p$ result implies new classification results on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$. The most general results are obtained when q is square. We give classification results on minimal t-fold $(n-k)$-blocking sets in $PG(n,q)$, q square, containing subspaces $PG(k,q)$, but also, possibly projected, subgeometries over $GF(\sqrt{q})$.

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Generalized quadrangles with elation points and the classification of translation generalized quadrangles

Koen Thas

Ghent University

One of the great results obtained in the theory of finite generalized quadrangles (GQ's) is that of the classification of finite Moufang quadrangles by P. Fong and G. M. Seitz, as a corollary of a result on finite split BN-pairs of rank 2. It can be shown that, in a finite Moufang quadrangle, for each point there is a group of automorphisms fixing the point linewise and acting regularly on the points not collinear with it (such a point is called an *elation point*; if the group is abelian, it is called a *translation point*). Also, for each line there is a group of automorphisms fixing the line pointwise and acting regularly on the lines not concurrent with it; so each line is an *elation line*.

Let $\mathcal{S}$ be a thick GQ which has at least one elation point. Then we have the following possibilities for $\mathcal{S}$:

(a) $\mathcal{S}$ has precisely one elation point;

(b) $\mathcal{S}$ has a line of elation points;

(c) each point of $\mathcal{S}$ is an elation point.

Recently, K. Thas and H. Van Maldeghem have completely classified GQ's of Type (c), thus solving one of the oldest problems in the automorphism theory of finite GQ's.

In our talk, we want to go somewhat further, and report on a first step in the classification of GQ's of Type (b). We also plan to report on some relevant recent results culminating from joint work with J. A. Thas on the classification of translation generalized quadrangles, that is, those GQ's having translation points, and with H. Van Maldeghem on generalized Moufang conditions for generalized quadrangles.

Constructions of difference systems of sets

Vladimir D. Tonchev

Michigan Technological University

(Joint work with Vladimir I. Levenshtein)

A collection of q disjoint subsets $Q_i,\ i = 1, ..., q$, of $N_n = \{1, 2, \ldots, n\}$ is called a *difference system of sets* (DSS) if for each number $s,\ s = 1, ..., n-1$, the equation

$$x - y = s \bmod n \tag{1}$$

has at least λ solutions such that $x \in Q_i,\ y \in Q_j,\ i, j = 1, ..., q,\ i \neq j$. DSS were introduced by V. Levenshtein in 1971 and have been used for the construction if quasi-linear codes that provide synchronization in the presence of errors.

The *redundancy* of a DSS is defined as $r = \sum_{i=1}^{q} |Q_i|$. Let $r_q(n, \lambda)$ be the minimum redundancy of a DSS with parameters n, q, and λ. A DSS is *optimal* if its redundancy is equal to $r_q(n, \lambda)$. A DSS is *perfect* if for every number $s,\ 1 \leq s \leq n-1$, the equation (1) has exactly λ solutions. A DSS is *regular* if all subsets Q_i are of the same size. We use the notation DSS-(n, m, q, λ) for a regular DSS with q subsets of size m on the set N_n and parameter λ; its redundancy equals $r = qm$. Any cyclic (v, k, λ) difference set is a perfect regular DSS-$(v, 1, q, \lambda)$. Thus DSS generalize cyclic difference sets.

Levenshtein (1971) proved the following lower bound on the minimum redundancy $r_q(n, \lambda)$:

$$r_q(n, \lambda) \geq \sqrt{\frac{q\lambda(n-1)}{q-1}}, \tag{2}$$

with equality if and only if the DSS is perfect and regular.

In this paper, constructions of DSS based on cyclic difference sets and finite geometry are discussed. In particular, it is shown that the existence of certain spreads in $PG(2s, p)$ implies that that, for every prime power p and every integer $s > 1$, there exists a perfect regular DSS with parameters

$$n = \frac{p^{2s+1} - 1}{p - 1},\ m = p + 1,\ q = \frac{p^{2s} - 1}{p^2 - 1},\ \lambda = \frac{p^{2s-1} - p}{p - 1}.$$

Characterizations of finite classical hexagons

Hendrik Van Maldeghem

Ghent University

(Joint work with J. De Kaey and A. Offer)

Some new characterizations of the finite split Cayley hexagons are obtained. More exactly, given a generalized hexagon Γ of order s, and a subhexagon Γ' of order $(1, s)$, we define for each point x of Γ not incident with any line of Γ' the *sphere in* Γ' *subtended by* x. Such a sphere is a set of flags in the projective plane defined by Γ'. We use these spheres to characterize the split Cayley hexagons. The notion of a *classical sphere* is obtained from the classical situation, and we prove the following result.

Theorem *Let* Γ *be a generalized hexagon of order* s *and* Γ' *a subhexagon of order* $(1, s)$ *which is the double of a classical projective plane* $\mathcal{P}$. *If all subtended spheres are classical, or if every elation of* $\mathcal{P}$ *extends (through* Γ'*) to a collineation of* Γ *(not necessarily unique), then* Γ *is a split Cayley hexagon.*

We discuss some generalizations and intermediate steps, leading to other, more geometric, characterizations.

Characterization of projective systems related to linear codes

Rita Vincenti

University of Perugia

(Joint work with E. Montanucci)

An $[n,k]_q$-*projective system* $\mathcal{X}$ of the $(k-1)$-projective geometry $P^{k-1} = PG(k-1,q)$ over $GF(q)$ is a collection of n not necessarily distinct points. It is called *non-degenerate* if these n points are not contained in any hyperplane. There exists a natural 1–1 correspondence between the equivalence classes of a non-degenerate $[n,k]_q$-*projective system* $\mathcal{X}$ and of a linear non-degenerate $[n,k]_q$-*code* C such that if $\mathcal{X}$ is an $[n,k]_q$-projective system and C is the corresponding code, then the non-zero codewords of C correspond to hyperplanes of P^{k-1}, up to a non-zero factor, the correspondence preserving the parameters n, k, d_t. Consequently, the higher weights of C are given by $d_t = d_t(C) = n - \max\{|\mathcal{X} \cap S| : S < P^{k-1},\ \operatorname{codim} S = t\}$. The *spectrum* of the code C (or of the corresponding projective system $\mathcal{X}$) is defined by the numbers $A_i^{(s)} = |\{S < P^{k-1} : \operatorname{codim} S = s,\ |S \cap \mathcal{X}| = n - i\}|$ for all $i = 1, 2, ..., n, \quad s = 1, 2, ..., k-2$.

In [2] it was shown that if we start, in $PG(k-1,q)$, from a projective system $\mathcal{X}$ that is a variety having an automorphism group A, then A induces a linear automorphism group of the related code C in such a way it is easier to decode C by using the strategy of building a PD-set to decode it.

Here, we characterize a projective system $\mathcal{X}$ as an *n-set with characters* in terms of the parameters of the related code C.

Theorem (i) *The projective system $\mathcal{X}$ is an n-set of rank k and characters J_1, J_2, ..., J_{k-2} with respect to hyperplanes, $(k-3)$-dimensional subspaces, ..., planes, lines, respectively.*

(ii) *For dimension $k = 4$, if the parameters of C are $n = q^2+1$, $k = 4$, $d_1 = q^2 - q$, $d_2 = q^2 - 1$ then $\mathcal{X}$ is an ellipsoid of P^3; if $n = (q+1)^2$, $k = 4$, $d_1 = q^2$, $d_2 = q^2 + q$, $A_{d_1}^{(1)} = (q+1)^2$, $A_{d_2}^{(2)} = 2(q+1)$ then $\mathcal{X}$ is a hyperbolic quadric of P^3 .*

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Surfaces in $PG(3, q)$

José Felipe Voloch

University of Texas

We will discuss some geometric properties of surfaces in three-dimensional projective spaces over finite fields, in particular asymptotic lines and flecnodal points, with applications to counting rational points on such surfaces. Our main result is the following theorem.

Theorem *Let X be a smooth surface in $PG(3, q)$ of degree d, defined over the field $GF(q)$ with q prime and assume that $2 < d < q$. Let m be the number of lines contained in X. Then*

$$\#X(GF(q)) \leq d(d + q - 1)(d + 2q - 2)/6 + m(q + 1).$$

In particular,

$$\#X(GF(q)) \leq d(d + q - 1)(d + 2q - 2)/6 + d(11d - 24)(q + 1).$$